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Research article

Analysis on a diffusive SIS epidemic system with linear source and frequency-dependent incidence function in a heterogeneous environment

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Abstract: In this paper, we consider a diffusive SIS epidemic reaction-diffusion model with linear source in a heterogeneous environment in which the frequency-dependent incidence function is $SI/(c + S + I)$ with c a positive constant. We first derive the uniform bounds of solutions, and the uniform persistence property if the basic reproduction number $\mathcal{R}_0 > 1$. Then, in some cases we prove that the global attractivity of the disease-free equilibrium and the endemic equilibrium. Lastly, we investigate the asymptotic profile of the endemic equilibrium (when it exists) as the diffusion rate of the susceptible or infected population is small. Compared to the previous results [1, 2] in the case of $c = 0$, some new dynamical behaviors appear in the model studied here; in particular, $\mathcal{R}_0$ is a decreasing function in $c \in [0, \infty)$ and the disease dies out once c is properly large. In addition, our results indicate that the linear source term can enhance the disease persistence.

Keywords: SIS model with linear source; frequency-dependent incidence function; basic reproduction number; disease-free equilibrium and endemic equilibrium; global attractivity; uniform persistence; asymptotic profile

1. Introduction

In the study of transmission of infectious disease, people have realised that environmental heterogeneity and individual motility are significant factors that should be incorporated into the mathematical models. In recent decades, more and more research works have been devoted to the investigation of the dynamics of infectious disease modelled by reaction-diffusion systems in which the migration of population and environmental heterogeneity are taken into account. One may refer to [1, 3–9] and the references therein.

In a recent paper [1], Allen et al. investigated a frequency-dependent SIS (susceptible-infected-susceptible) epidemic reaction-diffusion model, which reads as

$$
\begin{cases}
\dfrac{\partial S}{\partial t} - d_S \Delta S = -\beta(x)\dfrac{SI}{S+I} + \gamma(x)I, & x \in \Omega,\ t > 0,\\
\dfrac{\partial I}{\partial t} - d_I \Delta I = \beta(x)\dfrac{SI}{S+I} - \gamma(x)I, & x \in \Omega,\ t > 0,\\
\dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0,\\
S(x,0) = S_0(x) \geq 0,\ I(x,0) = I_0(x) \geq, \not\equiv 0.
\end{cases}
\tag{1.1}
$$

Here, S and I stand for the density of susceptible and infected population at location x and time t respectively; the positive constants d_S and d_I represent the motility of susceptible and infected individuals, respectively; the function $\beta(x)$ is the rate of disease transmission, and $\gamma(x)$ is the recovery rate of infected individuals, all of which are positive Hölder continuous functions on $\overline{\Omega}$. The habitat $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, and the Neumann boundary conditions mean that no population flux crosses the boundary $\partial\Omega$. For realistic implication, the initial data S_0 and I_0 are assumed to be nonnegative continuous functions on $\overline{\Omega}$, and there is a positive number of infected individuals, i.e., $\int_\Omega I_0(x)dx > 0$.

It is easily seen from (1.1) that

$$
\int_\Omega (S(x,t) + I(x,t))dx = \int_\Omega (S_0(x) + I_0(x))dx, \quad \forall t > 0,
$$

which means that the total number of population is conserved.

Clearly, model (1.1) dose not account into account the birth rate of the susceptible population and the death rate induced by disease. Indeed, these factors are important in the evolution of disease transmission; see [7, 10, 11]. With such a consideration, in the paper [12], the authors studied a varying total population model in which the linear external source term $\Lambda(x) - S$ was introduced. That is, the model (1.1) becomes the following:

$$
\begin{cases}
\dfrac{\partial S}{\partial t} - d_S \Delta S = \Lambda(x) - S - \beta(x)\dfrac{SI}{S+I} + \theta\gamma(x)I, & x \in \Omega,\ t > 0,\\
\dfrac{\partial I}{\partial t} - d_I \Delta I = \beta(x)\dfrac{SI}{S+I} - \gamma(x)I, & x \in \Omega,\ t > 0,\\
\dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0,\\
S(x,0) = S_0(x) \geq 0,\ I(x,0) = I_0(x) \geq, \not\equiv 0, & x \in \Omega.
\end{cases}
\tag{1.2}
$$

where d_S, d_I, β, γ, S and I have the same epidemiological interpretation as in (1.1). The parameter $\theta \in [0, 1]$ represent the number of infected population becomes susceptible. The positive functions β, γ, Λ also be assumed to be Hölder continuous over $\overline{\Omega}$. In the linear external source term, $\Lambda(x)$ and $-S$, respectively, account for the birth rate of the susceptible population and the disease-induced death rate. It is worth mentioning that in some cases, people ignore the effect of external source on the infected population; one may see, for instance, [3, 9, 13] for related discussion.

Different from model (1.1), a new feature in (1.2) is that the total population of susceptible and infected individuals are varying with respect to time $t > 0$. On the other hand, the works in [14–17] have shown that, in certain circumstances, the frequency-dependent incidence function $\frac{SI}{S+I}$ used in models (1.1) and (1.2) may not be appropriate to describe the transmission process of disease; instead an alternate incidence function should be $\frac{SI}{c+S+I}$, where c is a positive constant. Based on model (1.2), in this paper we are led to study the following SIS epidemic model:

$$\begin{cases} \dfrac{\partial S}{\partial t} - d_S \Delta S = a(x) - \mu(x)S - \dfrac{\beta(x)SI}{c+S+I} + \gamma(x)I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial I}{\partial t} - d_I \Delta I = \dfrac{\beta(x)SI}{c+S+I} - [\gamma(x) + \mu(x)]I, & x \in \Omega,\ t > 0, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ S(x,0) = S_0(x) \geq 0,\ I(x,0) = I_0(x) \geq, \not\equiv 0, & x \in \Omega. \end{cases} \tag{1.3}$$

In model (1.3), it should be noted that we also assume that the infected population allows the same natural death rate as for the susceptible population, which is represented by the function $\mu(x)$. With such a consideration, the system (1.3) is more realistic to describe the disease transmission in some cases as suggested in [15–17]. From now on, we always assume that c is a nonnegative constant, the positive function $a(x)$ stands for the recruitment rate of the susceptible corresponding to births and immigration; $a(x)$ and $\mu(x)$ are also positive Hölder continuous functions on $\overline{\Omega}$. All the other parameters have the same assumptions as in (1.1) and (1.2).

Since $c > 0$, the term $\frac{SI}{c+S+I}$ is a smooth function of S and I in the first quadrant. By the standard theory for parabolic equations, combined with our assumption on the initial data, it is well known that (1.3) admits a unique classical solution (S, I) (namely, $S,\ I \in C^{2,1}(\overline{\Omega} \times (0, \infty))$. Moreover, it follows from the strong maximum principle and the Hopf boundary lemma for parabolic equations that both $S(x,t)$ and $I(x,t)$ are positive for $x \in \overline{\Omega}$ and $t \in (0, \infty)$.

The aim of the present paper is to provide the theoretical analysis of solution to (1.3) and its steady-state (i.e., equilibrium) problem. The extinction or persistence behavior of the infectious disease in the long run is one of our main focuses. Once the disease can persist, what we are particularly interested in is the spatial distribution of the disease when the diffusion (migration) rate (represented by d_S or d_I in our current context) is controlled to be small. Such information will be useful for decision-makers to understand and predict the pattern of disease occurrence and then to take more effective actions/measures to eradicate diseases; one may refer to Section 5 for further discussion.

We would like to mention that many research works have been devoted to the study of the related epidemic systems; one may see, for instance, [18–34].

The rest of our paper is organized as follows. In section 2, we derive the uniform bounds of the solution of (1.3) and discuss the uniform persistence in terms of the basic reproduction number. In section 3, we study the global attractivity of the disease-free equilibrium and the endemic equilibrium in some special cases. In section 4, we analyze the asymptotic profile of the endemic equilibrium (if it exists) as the diffusion coefficient d_S or d_I goes to zero. Section 5 ends the paper with some discussion of the epidemiological implications of the theoretical results obtained in this paper.

2. Uniform boundedness and uniform persistence

In this section, we will establish the uniform boundedness of solutions to (1.3), and then study the uniform persistence property. In the following, we use DFE and EE to represent the disease-free equilibrium and the endemic equilibrium, respectively.

From now on, for notational simplicity, we denote

$$F^* = \max_{x\in\overline{\Omega}} F(x) \text{ and } F_* = \min_{x\in\overline{\Omega}} F(x)$$

for any given function $F \in C(\overline{\Omega})$.

Lemma 2.1. *Assume that $d_S = d_I$. For any solution (S, I) of (1.3), there holds*

$$S(x,t) + I(x,t) \leq \max\{\frac{a^*}{m}, \ s_0^* + (1+\varepsilon_1)I_0^*\}, \ \ \forall x \in \overline{\Omega}, \ t \geq 0. \tag{2.1}$$

Here, ε_1 is any given positive constant so that $1 - \varepsilon_1\beta^* > 0$ and

$$m_1 = \min\{\mu_* - \varepsilon_1\beta^*, \ \tfrac{(1+\varepsilon_1)\mu_*+\varepsilon_1\gamma_*}{1+\varepsilon_1}\}.$$

Proof. Let $d_S = d_I = d$. For any given positive constant ε_1 so that $1 - \varepsilon_1\beta^* > 0$, we set

$$V_1(x,t) = S(x,t) + (1+\varepsilon_1)I(x,t).$$

In view of (1.3), we then have

$$\begin{aligned}
\frac{\partial V_1}{\partial t} - d\Delta V_1 &= a(x) - \mu(x)S + \varepsilon_1\beta(x)\frac{SI}{C+S+I} - \mu(x)I - \varepsilon_1[\gamma(x)+\mu(x)]I \\
&\leq a(x) - (\mu_* - \varepsilon_1\beta^*)S - \frac{(1+\varepsilon_1)\mu_* + \varepsilon_1\gamma_*}{1+\varepsilon_1}\cdot(1+\varepsilon_1)I \\
&\leq a(x) - m_1V_1.
\end{aligned}$$

It is easily seen that V_1 and $\max\{\frac{a^*}{m_1}, S_0^* + (1+\varepsilon_1)I_0^*\}$ is a pair of upper and lower solutions to the initial-boundary value problem:

$$\begin{cases}
\dfrac{\partial v}{\partial t} - d\Delta v = a(x) - m_1 v, & x \in \Omega, t > 0, \\
\dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\
v(x,0) = S_0(x) + (1+\varepsilon_1)I_0(x), & x \in \Omega.
\end{cases}$$

By the comparison principle for parabolic equations, we obtain

$$S(x,t) + (1+\varepsilon_1)I(x,t) \leq V_1(x,t) \leq \max\{\tfrac{a^*}{m_1}, S_0^* + (1+\varepsilon_1 I_0^*)\}, \quad \forall x \in \overline{\Omega}, t \geq 0.$$

□

For the general case, we also derive the uniform boundedness of solution to (1.3). Indeed, we can state the following result.

Lemma 2.2. *For any solution* (S, I) *of* (1.3)*, there exists a positive constant* M_1 *depending on initial data such that*

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} \le M_1, \ \forall t \ge 0. \tag{2.2}$$

Moreover, there exists some positive constant M_2 *independent of initial data such that*

$$\|S(\cdot,t)\|_{L^\infty(\Omega)} + \|I(\cdot,t)\|_{L^\infty(\Omega)} \le M_2, \ \forall t \ge T, \tag{2.3}$$

for some large $T > 0$.

Proof. For any fixed constant $\varepsilon_2 > 0$ such that $\mu_* - \varepsilon_2\beta^* > 0$, we denote

$$m_2 = \min\{u_* - \varepsilon_2\beta^*, \ \tfrac{\mu_*+\varepsilon_2(\gamma_*+\mu_*)}{1+\varepsilon_2}\}.$$

Then we set

$$V_2(t) = \int_\Omega [S(x,t) + (1+\varepsilon_2)I(x,t)]dx, \quad t \ge 0.$$

In view of (1.3), it can be easily shown that

$$\begin{aligned}
\frac{dV_2}{dt} &= \int_\Omega a(x)dx - \int_\Omega \mu(x)S\,dx + \varepsilon_2 \int_\Omega \beta(x)\frac{SI}{c+S+I}dx \\
&\quad - \int_\Omega [\mu(x) + \varepsilon_2(\gamma(x) + \mu(x)]I dx \\
&\le \int_\Omega a(x)dx - \mu_* \int_\Omega S\,dx + \varepsilon_2\beta^* \int_\Omega S\,dx - [\mu_* + \varepsilon_2(\gamma_* + \mu_*)] \int_\Omega I dx \\
&= \int_\Omega a(x)dx - (\mu_* - \varepsilon_2\beta^*) \int_\Omega S\,dx - \frac{\mu_* + \varepsilon_2(\gamma_* + \mu_*)}{1+\varepsilon_2} \cdot (1+\varepsilon_2) \int_\Omega I dx \\
&\le \int_\Omega a(x)dx - m_2 V_2,
\end{aligned}$$

from which it follows that

$$\frac{dV_2}{dt} + m_2 V_2 \le \int_\Omega a(x)dx \le |\Omega| a^*.$$

Hence, we have

$$V_2(t) \le V_2(0)e^{-m_2 t} + \frac{|\Omega|a^*}{m_2}(1 - e^{-m_2 t}), \quad t \ge 0. \tag{2.4}$$

Note that S and I are nonnegative. Thus together with (2.4), one can use [6, Lemma 2.1] (or [35]) with $\sigma = p_0 = 1$ to derive (2.2). Moreover, the inequality (2.4) infers that

$$\limsup_{t\to\infty} V_2 \le \frac{|\Omega|a^*}{m},$$

which is independent of initial data. Making use of [6, Lemma 2.1] again, we can derive (2.3). □

In what follows, we discuss the uniform persistence of solution. To this aim, we first consider the elliptic problem:

$$-d_S \Delta S = a(x) - \mu(x) S,\ x \in \Omega;\quad \frac{\partial S}{\partial \nu} = 0,\ x \in \partial\Omega. \tag{2.5}$$

Obviously, (2.5) admits a uniform solution $\hat{S} > 0$, and $(\hat{S}, 0)$ is a unique disease-free equilibrium of (1.3), which we call as DFE.

We then define the basic reproduce number $\mathcal{R}_0$:

$$\mathcal{R}_0 = \sup_{\varphi \in H^1(\Omega), \varphi \neq 0} \Big\{ \frac{\int_\Omega \frac{\beta \hat{S}}{c+\hat{S}} \cdot \varphi^2 dx}{\int_\Omega d_I |\nabla \varphi|^2 + (\gamma + \mu)\varphi^2 dx} \Big\}. \tag{2.6}$$

Indeed, one can follow the idea of next generation operators in [35] to introduce the basic reproduction number, which coincides with the value $\mathcal{R}_0$.

It should be noticed that the basic reproduction number $\mathcal{R}_0$ defined by (2.6) implicitly depends on the diffusion rate d_S of the susceptible population; this qualitatively differs from the basic reproduction number $\mathcal{R}_0$ defined in [1] and [12].

Let λ_0 be the principal eigenvalue of the following eigenvalue problem

$$\begin{cases} d_I \Delta \psi - \big(\gamma + \mu - \dfrac{\beta \hat{S}}{c + \hat{S}}\big)\psi + \lambda \psi = 0, & x \in \Omega, \\ \dfrac{\partial \psi}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{2.7}$$

Then, we have the following proposition; the proof is the same as [1, Lemma 2.3] and is omitted here.

Proposition 2.3. *The following statements hold.*

(a) $\mathcal{R}_0$ *is a monotone decreasing function of* d_I *with* $\mathcal{R}_0 \to \max_{x \in \overline{\Omega}} \frac{\beta \hat{S}}{(c+\hat{S})(\gamma+\mu)}$ *as* $d_I \to 0$ *and* $\mathcal{R}_0 \to \int_\Omega \frac{\beta \hat{S}}{(c+\hat{S})} / \int_\Omega (\gamma + \mu)$ *as* $d_I \to \infty$.

(b) *If* $\int_\Omega \frac{\beta \hat{S}(x)}{c+\hat{S}(x)} dx < \int_\Omega (\gamma(x) + \mu(x)) dx$*, and* $\frac{\beta \hat{S}}{c+\hat{S}} - (\gamma + \mu)$ *changes sign, then there exists a threshold value* $d_I^* \in (0, \infty)$ *so that* $\mathcal{R}_0 < 1$ *for* $d_I > d_I^*$ *and* $\mathcal{R}_0 > 1$ *for* $d_I < d_I^*$.

(c) *If* $\int_\Omega \frac{\beta \hat{S}(x)}{c+\hat{S}(x)} dx \geq \int_\Omega (\gamma(x) + \mu(x)) dx$*, then* $\mathcal{R}_0 > 1$ *for all* d_I.

(d) $\mathcal{R}_0 > 1$ *when* $\lambda^* < 0$*,* $\mathcal{R}_0 = 1$ *when* $\lambda^* = 0$*, and* $\mathcal{R}_0 < 1$ *when* $\lambda^* > 0$.

(e) $\mathcal{R}_0$ *is a monotone decreasing function of* c*, and* $\mathcal{R}_0 < 1$ *if* $c > c^*$ *for some* $c^* \geq 0$.

Proposition 2.4. *If* $\mathcal{R}_0 > 1$ *then the DFE* $(\hat{S}, 0)$ *is unstable, and if* $\mathcal{R}_0 < 1$*, it is stable.*

The proof of Proposition 2.4 is similar to the proof of [1, Lemma 2.4] and hence the details are omitted.

In view of (2.3) of Lemma 2.2, we can establish the uniform persistence of (1.3) when $\mathcal{R}_0 > 1$. In fact, according to the theory developed by Magal and Zhao (see [36, Theorem 4.5] or [37]), we can conclude the following theorem.

Theorem 2.5. *If* $\mathcal{R}_0 > 1$*, then there exists some real number* $\eta > 0$ *independent of the initial data, such that any solution* (S, I) *of* (1.3) *satisfies*

$$\liminf_{t \to \infty} S(x,t) \geq \eta \ \ \text{and} \ \ \liminf_{t \to \infty} I(x,t) \geq \eta \ \ \text{uniformly for } x \in \overline{\Omega},$$

and hence, the disease persists uniformly. Furthermore, (1.3) *admits at least one EE when* $\mathcal{R}_0 > 1$.

3. Global attractivity of the DFE and EE

This section is devoted to the study of the global attractivity of the DFE and EE of (1.3). In the first subsection, we obtain the global attractivity of the DFE in the the nonhomogeneous environment. In the second subsection, we will derive that the global attractivity of the EE in the homogeneous environment.

3.1. Global attractivity of the DFE

For later purpose, we need a useful lemma; see [38, Lemma 2.5.1].

Lemma 3.1. *Let a_1 and $a_2 > 0$ be any constants. Assume that $\varphi, \psi \in C^1([a_1, \infty)), \psi(t) \geq 0$ in $[a_1, \infty)$ and φ is bounded from below. If $\varphi'(t) \leq -a_2\psi(t)$ and $\psi'(t) \leq K$ in $[a_1, \infty)$ for some constant K, then* $\lim_{t\to\infty} \psi(t) = 0$.

Our main result of this subsection reads as follows.

Theorem 3.2. *The DFE $(\hat{S}, 0)$ is globally attractive if one of the following conditions holds:*

(i) $\beta(x) \leq \gamma(x) + \mu(x),\ \forall x \in \overline{\Omega}$;
(ii) $c > c^*$ *for some constant* $c^* > 0$.

Proof. We first handle case (i). To verify our result, we construct the following Lyapunov function

$$V(t) = \frac{1}{2}\int_\Omega I^2(x,t)dx, \quad \forall t \geq 0.$$

Hereafter (S, I) is the solution of (1.3).

Then some elementary calculation yields

$$\begin{aligned} V'(t) &= \int_\Omega I\,\frac{\partial I}{\partial t}dx \\ &= \int_\Omega I\Big[d_I\Delta I + \frac{\beta(x)SI}{c+S+I} - (\gamma(x)+\mu(x))I\Big]dx \\ &= -\int_\Omega |\nabla I|^2dx + \int_\Omega \frac{\beta(x)S}{c+S+I}I^2dx - \int_\Omega(\gamma(x)+\mu(x))I^2dx \\ &\leq -\int_\Omega |\nabla I|^2dx + \int_\Omega \beta(x)I^2dx - \int_\Omega(\gamma(x)+\mu(x))I^2dx \\ &\leq -(\gamma(x)+\mu(x)-\beta(x))_*\int_\Omega I^2dx \leq 0. \end{aligned}$$

This motivates us to define

$$\psi(t) = (\gamma(x)+\mu(x)-\beta(x))_*\int_\Omega I^2dx \geq 0, \quad t \geq 0.$$

Due to the Lemma 2.2, we know that both $\|S(\cdot,t)\|_{L^\infty(\Omega)}$ and $\|I(\cdot,t)\|_{L^\infty(\Omega)}$ are bounded. Thus, by [39, Theorem A2], we have

$$\|S(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})} + \|I(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_0, \quad \forall t \geq 1, \tag{3.1}$$

for some constant $C_0 > 0$. In addition, using the second equation of (1.3), we can see that $\psi'(t)$ is bounded from above for $t \in [1, \infty)$. In view of Lemma 3.1 (by taking $\varphi(t) = V(t)$), we can conclude that

$$I(\cdot, t) \longrightarrow 0 \ \text{ in } L^2(\Omega), \ \text{ as } t \to \infty. \tag{3.2}$$

Moreover, in light of (3.1), it is clear that the set $\{I(\cdot, t) : \ t \geq 1\}$ is compact in $C^2(\overline{\Omega})$. Combining this fact with (3.2), we assert that

$$I(\cdot, t) \longrightarrow 0 \ \text{ in } C^2(\overline{\Omega}), \quad \text{as } t \to \infty.$$

Thus, for any small $\epsilon > 0$, it follows that

$$I(x, t) \leq \epsilon, \ \ \forall x \in \overline{\Omega}, \ t \geq T,$$

for some large T.

Using the above fact, it is easy to find that S is a lower solution of the following parabolic problem:

$$\begin{cases} \dfrac{\partial \overline{w}}{\partial t} - d_S \Delta \overline{w} = a(x) - \mu(x)\overline{w} + \varepsilon\gamma^*, & x \in \Omega, \ t > T, \\ \dfrac{\partial \overline{w}}{\partial \nu} = 0, & x \in \partial\Omega, \ t > T, \\ \overline{w}(x, T) = S(x, T), & x \in \Omega. \end{cases} \tag{3.3}$$

Let w_1 be the solution of (3.3). Then by the comparison principle, we have

$$S(x, t) \leq w_1(x, t), \quad \forall x \in \overline{\Omega}, \ t \geq T.$$

Similarly, we can find that S is an upper solution to

$$\begin{cases} \dfrac{\partial \underline{w}}{\partial t} - d_S \Delta \underline{w} = a(x) - \mu(x)\underline{w} - \varepsilon\beta^*, & x \in \Omega, \ t > T, \\ \dfrac{\partial \underline{w}}{\partial \nu} = 0, & x \in \partial\Omega, \ t > T, \\ \underline{w}(x, T) = S(x, T), & x \in \Omega. \end{cases} \tag{3.4}$$

Thus, by letting w_2 be the solution of (3.4), we have

$$S(x, t) \geq w_2(x, t), \ \ \forall x \in \overline{\Omega}, \ t \geq T.$$

It is standard to show that problem (3.3) and problem (3.4) exist a unique positive steady state, denoted by $\hat{S}_-(\varepsilon, x)$ and $\hat{S}_+(\varepsilon, x)$, respectively. Moreover, we have

$$w_1(x, t) \to \hat{S}_-(\varepsilon, x) \text{ and } w_2(x, t) \to \hat{S}_+(\varepsilon, x) \text{ uniformly on } \overline{\Omega}, \text{ as } t \to \infty,$$

On the other hand, it is easy to check that

$$\hat{S}_-(\varepsilon, x), \ \hat{S}_+(\varepsilon, x) \to \hat{S}(x) \text{ uniformly for } x \in \overline{\Omega}, \text{ as } \varepsilon \to 0.$$

According to the arbitrariness of ε, we conclude that

$$S(x,t) \to \hat{S}(x) \text{ uniformly on } \overline{\Omega}, \text{ as } t \to \infty.$$

This proves our assertion in case (i).

We next consider case (ii). First of all, one can check the proof of Lemma 2.2 and claim that

$$S(x,t) \le C_0, \quad \forall x \in \overline{\Omega}, \ t \ge 0, \tag{3.5}$$

for some positive constant C_0, depending on the initial data but independent of $c \ge 0$. Hence, by (3.5), it is easily noticed that I is a lower solution of the following parabolic problem:

$$\begin{cases} \dfrac{\partial \overline{w}}{\partial t} - d_I \Delta \overline{w} = \dfrac{\beta(x)C_0}{c + C_0}\overline{w} - [\gamma(x) + \mu(x)]\overline{w}, & x \in \Omega, \ t > 0, \\ \dfrac{\partial \overline{w}}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ \overline{w}(x,0) = I_0(x), & x \in \Omega. \end{cases} \tag{3.6}$$

Let w_3 be the solution of (3.6). Since C_0 does not depend on c, one can take c to be large so that

$$\frac{\beta(x)C_0}{c + C_0} - [\gamma(x) + \mu(x)] < 0, \quad \forall x \in \overline{\Omega}.$$

Then, a simple analysis, together with the parabolic comparison principle, shows that

$$I(x,t) \le w_3(x,t) \to 0 \text{ uniformly for } x \in \overline{\Omega}, \text{ as } t \to \infty.$$

Now, in view of the above assertion, a similar argument as in case (i) allows us to conclude that

$$S(x,t) \to \hat{S}(x) \text{ uniformly on } \overline{\Omega}, \text{ as } t \to \infty.$$

This proves our assertion in case (ii). The proof is complete.

□

3.2. Global attractivity of the EE

In this subsection, we will consider the global attractivity of the EE by assuming that all of the parameters a, β, μ and γ are positive constant (that is, the environment is spatially homogeneous).

In this situation, in view of (3.1), we can see that the unique DFE is given by $(\hat{S}, 0) = (\frac{a}{\mu}, 0)$. And the unique EE $(\tilde{S}, \tilde{I})$ exists if and only if $\mathcal{R}_0 = \frac{\beta\frac{a}{\mu}}{(\gamma+\mu)(c+\frac{a}{\mu})} > 1$, where

$$\tilde{S} = \frac{(\gamma+\mu)(c+\frac{a}{\mu})}{\beta} = \frac{1}{\mathcal{R}_0} \cdot \frac{a}{\mu}, \quad \tilde{I} = \frac{a}{\mu} - \frac{1}{\beta}(\mu+\gamma)(c+\frac{a}{\mu}) = \frac{(\gamma+\mu)(c+\frac{a}{\mu})}{\beta}(\mathcal{R}_0 - 1).$$

Our result is stated as follows.

Theorem 3.3. *Assume that the parameters a, β, μ and γ are positive constant and $d_S = d_I$. If $\mathcal{R}_0 > 1$, then the EE $(\tilde{S}, \tilde{I})$ is globally attractive.*

Proof. By setting $d_S = d_I = d$, we construct the following Lyapunov functional

$$W(t) = \int_\Omega M(S(x,t), I(x,t))dx, \quad \forall t > 0,$$

where

$$\begin{aligned} M(S,I) = (S - \tilde{S}) + (I - \tilde{I}) - (c + \tilde{S} + \tilde{I}) \ln \frac{c + S + I}{c + \tilde{S} + \tilde{I}} \\ + \frac{2\mu(c + \tilde{S} + \tilde{I})}{\beta(c + \tilde{I})} \left(I - \tilde{I} - \tilde{I} \ln \frac{I}{\tilde{I}} \right). \end{aligned}$$

For simplicity, we note

$$g_1(S,I) = a - \mu S - \beta \frac{SI}{c + S + I} + \gamma I, \quad g_2(S,I) = \beta \frac{SI}{c + S + I} - (\gamma + \mu) I.$$

Then simple calculation gives

$$\begin{aligned} W'(t) &= \int_\Omega [M_S(S,I)S_t + M_I(S,I)I_t]\, dx \\ &= d \int_\Omega [M_S(S,I)\Delta S + M_I(S,I)\Delta I]\, dx \\ &\quad + \int_\Omega [M_S(S,I)g_1(S,I) + M_I(S,I)g_2(S,I)]\, dx. \end{aligned}$$

Moreover, integrating by parts, we have

$$\int_\Omega M_S(S,I)\Delta S\, dx = -\int_\Omega \left[M_{SS}(S,I) |\nabla S|^2 + M_{SI}(S,I) \nabla S \cdot \nabla I \right] dx,$$

$$\int_\Omega M_I(S,I)\Delta I dx = -\int_\Omega \left[M_{IS}(S,I) \nabla S \cdot \nabla I + M_{II}(S,I) |\nabla I|^2 \right] dx.$$

It is easy to see that

$$M_{SS} = M_{SI} = M_{IS} = \frac{c + \tilde{S} + \tilde{I}}{(c + S + I)^2}, \quad M_{II} = \frac{c + \tilde{S} + \tilde{I}}{(c + S + I)^2} + \frac{2\mu(c + \tilde{S} + \tilde{I})\tilde{I}}{\beta(c + \tilde{I})I^2}.$$

Thus

$$\begin{aligned} & d \int_\Omega [M_S(S,I)\Delta S + M_I(S,I)\Delta I]\, dx \\ &= -d \int_\Omega \left\{ \frac{c + \tilde{S} + \tilde{I}}{(c + S + I)^2} |\nabla S|^2 + 2 \frac{c + \tilde{S} + \tilde{I}}{(c + S + I)^2} \nabla S \cdot \nabla I \right. \\ &\quad \left. + \left[\frac{c + \tilde{S} + \tilde{I}}{(c + S + I)^2} + \frac{2\mu(c + \tilde{S} + \tilde{I})\tilde{I}}{\beta(c + \tilde{I})I^2} \right] |\nabla I|^2 \right\} dx \\ &= -d \int_\Omega \left\{ \frac{c + \tilde{S} + \tilde{I}}{(c + S + I)^2} |\nabla(S + I)|^2 + \frac{2\mu(c + \tilde{S} + \tilde{I})\tilde{I}}{\beta(c + \tilde{I})I^2} |\nabla I|^2 \right\} dx \end{aligned}$$

$$\leq 0. \tag{3.7}$$

In addition, by direct computations, we have

$$\begin{aligned}
& M_S(S,I)g_1(S,I) + M_I(S,I)g_2(S,I) \\
= & \Big(1 - \frac{c+\tilde{S}+\tilde{I}}{c+S+I}\Big)g_1 + \Big[1 - \frac{c+\tilde{S}+\tilde{I}}{c+S+I} + \frac{2\mu(c+\tilde{S}+\tilde{I})}{\beta(c+\tilde{I})}\Big(1-\frac{\tilde{I}}{I}\Big)\Big]g_2 \\
= & \Big(1 - \frac{c+\tilde{S}+\tilde{I}}{c+S+I}\Big)(g_1+g_2) + \frac{2\mu(c+\tilde{S}+\tilde{I})}{\beta(c+\tilde{I})}\Big(1-\frac{\tilde{I}}{I}\Big)g_2 \\
= & \Big(1 - \frac{c+\tilde{S}+\tilde{I}}{c+S+I}\Big)(a-\mu S-\mu I) + \frac{2\mu(c+\tilde{S}+\tilde{I})}{\beta(c+\tilde{I})}\Big(1-\frac{\tilde{I}}{I}\Big)\Big[\frac{\beta SI}{c+S+I} - (\gamma+\mu)I\Big] \\
= & \frac{(S-\tilde{S})+(I+\tilde{I})}{c+S+I}[-\mu(S-\tilde{S})-\mu(I-\tilde{I})] + \frac{2\mu(c+\tilde{S}+\tilde{I})}{\beta(c+\tilde{I})}(I-\tilde{I})\Big(\frac{\beta S}{c+S+I} - \frac{\beta\tilde{S}}{c+\tilde{S}+\tilde{I}}\Big) \\
= & -\mu\frac{[(S-\tilde{S})+(I+\tilde{I})]^2}{c+S+I} + \frac{2\mu(c+\tilde{S}+\tilde{I})}{c+\tilde{I}}(I-\tilde{I})\Big(\frac{S}{c+S+I} - \frac{\tilde{S}}{c+\tilde{S}+\tilde{I}}\Big) \\
= & -\mu\frac{(S-\tilde{S})^2}{c+S+I} - \frac{2\mu(S-\tilde{S})(I-\tilde{I})}{c+S+I} - \frac{\mu(I-\tilde{I})^2}{c+S+I} \\
& + \frac{2\mu(c+\tilde{S}+\tilde{I})}{c+\tilde{I}}(I-\tilde{I})\frac{(c+\tilde{I})(S-\tilde{S})-\tilde{S}(I-\tilde{I})}{(c+S+I)(c+\tilde{S}+\tilde{I})} \\
= & -\mu\frac{(S-\tilde{S})^2}{c+S+I} - \frac{2\mu(S-\tilde{S})(I-\tilde{I})}{c+S+I} - \frac{\mu(I-\tilde{I})^2}{c+S+I} + \frac{2\mu(S-\tilde{S})(I-\tilde{I})}{c+S+I} - \frac{2\mu\tilde{S}(I-\tilde{I})^2}{(c+\tilde{I})(c+S+I)} \\
= & -\mu\frac{(S-\tilde{S})^2}{c+S+I} - \frac{\mu(I-\tilde{I})^2}{c+S+I} - \frac{2\mu\tilde{S}(I-\tilde{I})^2}{(c+\tilde{I})(c+S+I)} \leq 0.
\end{aligned}$$

Here, we have used the fact:

$$a = \mu(\tilde{S}+\tilde{I}), \quad \gamma+\mu = \frac{\beta\tilde{S}}{c+\tilde{S}+\tilde{I}}.$$

According to (3.7), we thus derive that $W'(t) \leq 0$, $\forall t > 0$ along all trajectories. By a standard argument, we can obtain

$$(S(\cdot,t), I(\cdot,t)) \to (\tilde{S},\tilde{I}) \text{ in } \Big[L^2(\Omega)\Big]^2, \text{ as } t \to \infty.$$

Due to the Lemma 2.2, we know that both $\|S(\cdot,t)\|_{L^\infty(\Omega)}$ and $\|I(\cdot,t)\|_{L^\infty(\Omega)}$ are bounded. As a consequence, by [39, Theorem A2], we have

$$\|S(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})} + \|I(\cdot,t)\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_0, \quad \forall t \geq 1,$$

Thus, $\{(S(\cdot,t), I(\cdot,t)) : t \geq 1\}$ is compact in $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$. Then, combining this with the above L^2-convergence, we obtain that

$$(S(\cdot,t), I(\cdot,t)) \to (\tilde{S},\tilde{I}) \text{ in } \Big[C^2(\overline{\Omega})\Big]^2, \text{ as } t \to \infty.$$

Thus, the EE $(\tilde{S},\tilde{I})$ is globally attractive. □

4. Asymptotic profiles of the EE

In this section, we consider the asymptotic behavior of the positive solution of the following elliptic system:

$$\begin{cases} -d_S\Delta S = a(x) - \mu(x)S - \dfrac{\beta(x)SI}{c+S+I} + \gamma(x)I, & x \in \Omega, \\ -d_I\Delta I = \dfrac{\beta(x)SI}{c+S+I} - [\gamma(x)+\mu(x)]I, & x \in \Omega, \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

when d_S or d_I goes to zero.

4.1. The case of $d_S \to 0$

Via a singular perturbational argument, it is easily seen that (2.5) has a unique positive solution $\hat{S}$ which converges uniformly to $\hat{S}_1(x) = \frac{a(x)}{\mu(x)}$ as $d_S \to 0$. We also recall that λ_0 is the principal eigenvalue of problem (2.7). Then, it follows that

$$\lambda_0 \to \tilde{\lambda}, \ \ as \ \ d_S \to 0,$$

where $\tilde{\lambda}$ is the principle eigenvalue of the following eigenvalue problem

$$\begin{cases} d_I\Delta\psi - \Big(\gamma + \mu - \dfrac{\beta\hat{S}_1}{c+\hat{S}_1}\Big)\psi + \lambda\psi = 0, & x \in \Omega, \\ \dfrac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.2}$$

In order to ensure that the elliptic system (4.1) admits a positive solution, we always assume $\tilde{\lambda} < 0$ in this subsection. Then, we will investigate the asymptotic behavior of positive solution of (4.1) as $d_S \to 0$ while $d_I > 0$ is fixed.

Theorem 4.1. *Fix $d_I > 0$ and assume that $\tilde{\lambda} < 0$. Let $d_S \to 0$, then every positive solution (S_{d_S}, I_{d_S}) of* (4.1) *satisfies (up to a subsequence of $d_S \to 0$)*

$$(S_{d_S}, I_{d_S}) \to (W_S, W_I) \ \ uniformly \ on \ \overline{\Omega},$$

where

$$\begin{aligned} W_S(x) &= J(x, W_I(x)) \\ &:= \frac{1}{2}\Bigg\{\frac{a+\gamma W_I}{\mu} - (c+W_I) - \frac{\beta W_I}{\mu} \\ &\quad + \sqrt{\Big[\frac{a+\gamma W_I}{\mu} - (c+W_I) - \frac{\beta W_I}{\mu}\Big]^2 + \frac{4(a+\gamma W_I)(c+W_I)}{\mu}}\Bigg\} \end{aligned}$$

and W_I is a positive solution of

$$\begin{cases} -d_I\Delta W_I = \dfrac{\beta(x)J(x,W_I)W_I}{c+J(x,W_I)+W_I} - [\gamma(x)+\mu(x)]W_I, & x\in\Omega,\\ \dfrac{\partial W_I}{\partial\nu} = 0, & x\in\partial\Omega. \end{cases} \tag{4.3}$$

Proof. Mentioned as before, (4.1) has at least one EE for all small d_S when $\tilde{\lambda} < 0$. In what follows, we divide the proof into three steps to derive the conclusion.

Step 1. A priori bounds for S, I. Integrating the first and the second equation of (4.1) over Ω, respectively, we have

$$\int_\Omega \Big[\mu(x)S + \beta(x)\frac{SI}{c+S+I}\Big]dx = \int_\Omega a(x)dx + \int_\Omega \gamma(x)Idx \tag{4.4}$$

and

$$\int_\Omega \beta(x)\frac{SI}{c+S+I}dx = \int_\Omega [\gamma(x)+\mu(x)]Idx. \tag{4.5}$$

Inserting (4.5) into (4.4) gives

$$\int_\Omega \mu(x)Idx + \int_\Omega \mu(x)S\,dx = \int_\Omega a(x)dx. \tag{4.6}$$

From (4.6), we get

$$\int_\Omega Idx + \int_\Omega S\,dx \le \frac{|\Omega|a^*}{\mu_*}. \tag{4.7}$$

It is obvious that the L^1-bounds of S and I are independent of both d_S and d_I.

We write the I-equation as follows

$$\begin{cases} \Delta I + \dfrac{1}{d_I}\Big[\dfrac{\beta(x)S}{c+S+I} - \gamma(x) - \mu(x)\Big]I = 0, & x\in\Omega,\\ \dfrac{\partial I}{\partial\nu} = 0, & x\in\partial\Omega. \end{cases} \tag{4.8}$$

Then, we apply the Harnack-type inequality (refer to [40] or [41, Lemma 2.2]) to (4.8) to assert that

$$\max_{\overline{\Omega}} I \le C\min_{\overline{\Omega}} I. \tag{4.9}$$

Hereafter, the positive constant C is independent of $d_S > 0$, and it may vary from place to place.

By (4.7) and (4.9), we can derive

$$I(x) \le \max_{\overline{\Omega}} I \le C\min_{\overline{\Omega}} I \le \frac{C}{|\Omega|}\int_\Omega Idx \le C, \quad \forall x\in\overline{\Omega}. \tag{4.10}$$

Step 2. Convergence of I. From (4.10), we get

$$\Big\|\frac{1}{d_I}\Big[\frac{\beta(x)S}{c+S+I}-\gamma(x)-\mu(x)\Big]I\Big\|_{L^p(\Omega)}\le C,\quad \forall\, p>1.$$

Applying the standard L^p-estimate for elliptic equations ([42]), it follows that

$$\|I\|_{W^{2,p}(\Omega)}\le C\ \text{ for any given } p>1.$$

Then, taking p to be sufficiently large and using the embedding theorem [42], we can see that

$$\|I\|_{C^{1+\alpha}(\overline{\Omega})}\le C\ \text{ for some } 0<\alpha<1.$$

Hence, there exist a subsequence of $d_S\to 0$, say $d_i:=d_{S,i}$, satisfying $d_i\to 0$ as $i\to\infty$, and a corresponding positive solution $(S_i,I_i):=(S_{d_{S,i}},I_{d_{S,i}})$ of (4.1) with $d_S=d_i$, such that

$$I_i\to W_I\ \text{ uniformly on } \overline{\Omega},\ \text{ as } i\to\infty, \tag{4.11}$$

where $W_I\in C^1(\overline{\Omega})$ and $W_I\ge 0$. By (4.9), we know that

$$\text{either } W_I\equiv 0\ \text{ on } \overline{\Omega}\ \text{ or }\ W_I>0\ \text{ on } \overline{\Omega}. \tag{4.12}$$

Suppose that $W_I\equiv 0$. That is,

$$I_i\to W_I\equiv 0\ \text{ uniformly on } \overline{\Omega},\ \text{ as } i\to\infty.$$

Then for sufficiently small ε with $0<\varepsilon<\min_{x\in\overline{\Omega}}a(x)$, we have $0\le I_i(x)\le\varepsilon,\forall x\in\overline{\Omega}$, for all large i. Combining this fact with the first equation of (4.1), for all large i, one sees that (S_i,I_i) satisfies

$$-d_i\Delta S_i\le a(x)-\mu(x)S_i+\varepsilon\gamma^*,\ x\in\Omega;\ \ \frac{\partial S_i}{\partial\nu}=0,\ x\in\partial\Omega.$$

This leads us to consider the following auxiliary system:

$$-\,d_i\Delta S_i=a(x)-\mu(x)S_i+\varepsilon\gamma^*,\ x\in\Omega;\ \ \frac{\partial S_i}{\partial\nu}=0,\ x\in\partial\Omega. \tag{4.13}$$

It is clear that (4.13) admits a unique positive solution, denoted by u_i. A simple upper and lower solution argument guarantees that

$$S_i\le u_i\ \text{ on } \overline{\Omega},\ \text{ for all large } i. \tag{4.14}$$

Similarly, for all large i, (S_i,I_i) satisfies

$$-d_i\Delta S_i\ge a(x)-\mu(x)S_i-\varepsilon\beta^*,\ x\in\Omega;\ \ \frac{\partial S_i}{\partial\nu}=0,\ x\in\partial\Omega.$$

We also consider the following auxiliary system:

$$-\,d_i\Delta S_i=a(x)-\mu(x)S_i-\varepsilon\beta^*,\ x\in\Omega;\ \ \frac{\partial S_i}{\partial\nu}=0,\ x\in\partial\Omega. \tag{4.15}$$

Let v_i denote the unique positive solution of (4.15). Similarly as before, we have

$$S_i \geq v_i \quad \text{on } \overline{\Omega}, \quad \text{for all large } i. \tag{4.16}$$

By a singular perturbation argument as in [43, Lemma 2.4], it is easy to show that

$$u_i \to \frac{a(x)+\varepsilon\gamma^*}{\mu(x)}, \;\; v_i \to \frac{a(x)-\varepsilon\beta^*}{\mu(x)} \;\text{ uniformly on } \overline{\Omega}, \;\text{ as } i \to \infty.$$

Hence, sending $i \to \infty$, by (4.14) and (4.16), we find

$$\frac{a(x)-\varepsilon\beta^*}{\mu(x)} \leq \liminf_{i\to\infty} S_i(x) \leq \limsup_{i\to\infty} S_i(x) \leq \frac{a(x)+\varepsilon\gamma^*}{\mu(x)} \text{ on } \overline{\Omega}.$$

Due to the arbitrariness of ε, we obtain that

$$S_i \to \frac{a(x)}{\mu(x)} \;\text{ uniformly on } \overline{\Omega}, \;\; \text{as } i \to \infty. \tag{4.17}$$

We now consider the second equation of (4.1) and then know that I_i satisfies

$$-d_I\Delta I_i = \frac{\beta(x)S_iI_i}{c+S_i+I_i} - [\gamma(x)+\mu(x)]I_i, \; x \in \Omega; \quad \frac{\partial I_i}{\partial \nu} = 0, \; x \in \partial\Omega. \tag{4.18}$$

Define $\hat{I}_i := \frac{I_i}{\|I_i\|_{L^\infty(\Omega)}}$. Then $\|\hat{I}_i\|_{L^\infty(\Omega)} = 1$ for all $i \geq 1$, and $\hat{I}_i$ solves

$$-d_I\Delta \hat{I}_i = [\frac{\beta(x)S_i}{c+S_i+I_i} - \gamma(x)+\mu(x)]\hat{I}_i, \; x \in \Omega; \quad \frac{\partial I_i}{\partial \nu} = 0, \; x \in \partial\Omega. \tag{4.19}$$

As before, using the standard compactness argument for the elliptic equation, after passing to a further subsequence if necessary, we assume that

$$\hat{I}_i \to \hat{I} \;\text{ in } C^1(\overline{\Omega}), \;\; \text{as } i \to \infty,$$

where $\hat{I} \in C^1(\overline{\Omega})$ with $\hat{I} \geq 0$ on $\overline{\Omega}$ and $\|\hat{I}\|_{L^\infty(\Omega)} = 1$.

Together with the fact $I_i \to 0$ uniformly on $\overline{\Omega}$ as $i \to \infty$, from (4.17) and (4.19), one can show that $\hat{I}$ fulfills

$$-d_I\Delta \hat{I} = \Big[\frac{\beta(x)\frac{a(x)}{\mu(x)}}{c+\frac{a(x)}{\mu(x)}} - \gamma(x)+\mu(x)\Big]\hat{I}, \; x \in \Omega; \quad \frac{\partial \hat{I}}{\partial \nu} = 0, \; x \in \partial\Omega. \tag{4.20}$$

Applying the Harnack-type inequality ([40] or [41, Lemma 2.2]) to (4.20), we obtain $\hat{I} > 0$ on $\overline{\Omega}$. This implies that $\tilde{\lambda}$ of the eigenvalue problem (4.2) must be zero. It contradicts our assumption that $\tilde{\lambda} < 0$. Hence, the latter always holds in (4.12). That is

$$I_i \to W_I > 0 \;\text{ uniformly on } \overline{\Omega}, \;\; \text{as } i \to \infty. \tag{4.21}$$

Step 3. Convergence of S. Observe that S_i fulfills

$$\begin{cases} -d_i\Delta S_i = a(x) - \mu(x)S_i - \dfrac{\beta(x)S_iI_i}{c+S_i+I_i} + \gamma(x)I_i, & x \in \Omega, \\ \dfrac{\partial S_i}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases}$$

From (4.21), given any small $\varepsilon > 0$, it holds

$$0 < W_I - \varepsilon \le I_i \le W_I + \varepsilon, \quad \forall x \in \Omega, \tag{4.22}$$

for all large i. For simplicity, denote $W_{\pm,I} = W_I \pm \varepsilon$. Hence, we have

$$\begin{aligned} a - \mu S_i - \frac{\beta S_i I_i}{c + S_i + I_i} + \gamma I_i &\le a - \mu S_i - \frac{\beta S_i W_{-,I}}{c + S_i + W_{-,I}} + \gamma W_{+,I} \\ &= \mu \frac{(J_+^{1,\epsilon}(x, W_I(x)) - S_i)(S_i - J_-^{1,\epsilon}(x, W_I(x)))}{c + S_i + (W_I - \epsilon)}, \end{aligned}$$

where

$$\begin{aligned} J_\pm^{1,\epsilon}(x, W_I(x)) &= \frac{1}{2}\Bigg\{ \frac{a + \gamma W_{+,I}}{\mu} - (c + W_{-,I}) - \frac{\beta W_{-,I}}{\mu} \\ &\pm \sqrt{\Big[\frac{a + \gamma W_{+,I}}{\mu} - (c + W_{-,I}) - \frac{\beta W_{-,I}}{\mu}\Big]^2 + \frac{4(a + \gamma W_{+,I})(c + W_{-,I})}{\mu}} \Bigg\}, \end{aligned}$$

for all large i.

Notice that $J_+^{1,\epsilon} > 0$ and $J_-^{1,\epsilon} < 0$ on $\overline{\Omega}$. Then, given large i, we consider the following auxiliary elliptic problem:

$$\begin{cases} -d_i \Delta z = \mu \dfrac{(J_+^{1,\epsilon}(x, W_I(x)) - z)(z - J_-^{1,\epsilon}(x, W_I(x)))}{c + z + W_{-,I}}, & x \in \Omega, \\ \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.23}$$

It is easily observed that S_i is a lower solution of (4.23). And any sufficiently large constant $C > 0$ satisfies $S_i \le C$ is an upper solution of (4.23). Therefore, (4.23) admits at least one positive solution, denoted by Z_i, which satisfies $S_i \le Z_i \le C$ on $\overline{\Omega}$. By similar arguments as in the proof of [43, Lemma 2.4], we find that

$$Z_i \longrightarrow J_+^{1,\epsilon}(x, W_I(x)) \quad \text{uniformly on } \overline{\Omega} \text{ as } i \to \infty.$$

Since S_i is a lower solution of (4.23), we have

$$\limsup_{i \to \infty} S_i(x) \le J_+^{1,\epsilon}(x, W_I(x)) \ \text{ uniformly on } \overline{\Omega}. \tag{4.24}$$

On the other hand, from (4.22), for all large i, we have

$$\begin{aligned} a - \mu S_i - \frac{\beta S_i I_i}{c + S_i + I_i} + \gamma I_i &\ge a - \mu S_i - \frac{\beta S_i W_{+,I}}{c + S_i + W_{+,I}} + \gamma W_{-,I} \\ &= \mu \frac{(J_+^{1,\epsilon}(x, W_I(x)) - S_i)(S_i - J_-^{1,\epsilon}(x, W_I(x)))}{c + S_i + (W_I + \epsilon)}, \end{aligned}$$

where

$$J_\pm^{2,\epsilon}(x, W_I(x)) = \frac{1}{2}\Bigg\{ \frac{a + \gamma W_{-,I}}{\mu} - (c + W_{+,I}) - \frac{\beta W_{+,I}}{\mu}$$

$$\pm\sqrt{\Big[\frac{a+\gamma W_{-,I}}{\mu}-(c+W_{+,I})-\frac{\beta W_{+,I}}{\mu}\Big]^2+\frac{4(a+\gamma W_{-,I})(c+W_{+,I})}{\mu}}\Bigg\},$$

with $J_+^{2,\epsilon}(x, W_I(x)) > 0$ and $J_-^{2,\epsilon}(x, W_I(x)) < 0$ on $\overline{\Omega}$.

As before, for any given large i, we also consider the following auxiliary elliptic problem:

$$\begin{cases} -d_i\Delta z = \mu\dfrac{(J_+^{2,\epsilon}(x, W_I(x)) - z)(z - J_-^{2,\epsilon}(x, W_I(x)))}{c + z + W_{+,I}}, & x \in \Omega, \\ \dfrac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.25}$$

Observe that S_i and 0 is a pair of upper and lower solution of (4.25). Hence, we can assert that (4.25) admits at least one positive solution. Using a similar argument as before, we further get

$$\liminf_{i\to\infty} S_i(x) \geq J_+^{2,\epsilon}(x, W_I(x)) \;\text{ uniformly on } \overline{\Omega}. \tag{4.26}$$

Notice that

$$J_+^{1,0}(x, W_I(x)) = J_+^{2,0}(x, W_I(x)) = J(x, W_I(x)).$$

Due to the arbitrariness of ε, by (4.24) and (4.26), we derive that

$$S_i(x) \to J(x, W_I(x)) \;\text{ uniformly for } x \in \overline{\Omega}, \text{ as } i \to \infty.$$

In addition, by (4.18), we can easily see that W_I satisfies (4.3). The proof is complete.

□

4.2. The case of $d_I \to 0$

In this subsection, we will analyze the asymptotic behavior of positive solution of (4.1) as $d_I \to 0$ with $d_S > 0$ being fixed. According to Proposition 2.3(a) and Theorem 2.5, we need to assume that $\{\beta(x)\hat{S}(x)/c + \hat{S}(x) > \gamma(x) + \mu(x) : x \in \overline{\Omega}\}$ is nonempty so that (4.1) has positive solution for all small d_i.

As usual, we denote $g^+ = \max\{g, 0\}$. Our main result can be stated as follows.

Theorem 4.2. *Fix $d_S > 0$ and assume that $\{\beta(x)\hat{S}(x)/c + \hat{S}(x) > \gamma(x) + \mu(x) : x \in \overline{\Omega}\}$ is nonempty. Let $d_I \to 0$, then every positive solution (S_{d_I}, I_{d_I}) of (4.1) fulfills*

$$(S_{d_I}, I_{d_I}) \to (W^S, W^I) \;\; \textit{uniformly on } \overline{\Omega},$$

where W^S is the unique positive solution of

$$\begin{cases} -d_S\Delta W^S = a(x) - \mu(x)W^S - \dfrac{\beta(x)W^S W^I}{c + W^S + W^I} + \gamma(x)W^I, & x \in \Omega, \\ \dfrac{\partial W^S}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{4.27}$$

and W^I is a nonnegative function

$$W^I = \left\{\frac{[\beta(x) - \gamma(x) - \mu(x)]W^S - c[\gamma(x) + \mu(x)]}{\gamma(x) + \mu(x)}\right\}^+. \tag{4.28}$$

Proof. We divide the proof into four steps for clarity. In the following, let us denote m to be a positive constant which does not depend on $d_I > 0$ and may vary from line to line.

Step 1. Lower bound of S. Pick $x_1 \in \overline{\Omega}$ so that $S(x_1) = \min_{\overline{\Omega}} S(x)$. In light of the first equation of (4.1), it follows from [44, Proposition 2.2] that

$$a(x_1) - \mu(x_1)S(x_1) - \frac{\beta(x_1)S(x_1)I(x_1)}{c + S(x_1) + I(x_1)} + \gamma(x_1)I(x_1) \leq 0. \tag{4.29}$$

By (4.29), we obtain

$$a_* < a(x_1) + \gamma(x_1)I(x_1) \leq \mu(x_1)S(x_1) + \frac{\beta(x_1)S(x_1)I(x_1)}{c + S(x_1) + I(x_1)} \leq \mu^* S(x_1) + \beta^* S(x_1),$$

from which we further have

$$S(x) \geq S(x_1) \geq \frac{a_*}{\mu^* + \beta^*} > 0. \tag{4.30}$$

Step 2. $W^{1,q}$-bound of S for some $q \geq 1$. We now write the S-equation as

$$\begin{cases} -d_S \Delta S + [\mu(x) - \dfrac{\beta(x)I}{c + S + I}]S = a(x) + \gamma(x)I, & x \in \Omega, \\ \dfrac{\partial S}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.31}$$

From (4.7), we derive $\int_\Omega |a(x) + \gamma(x)I| dx \leq m$. Hence, by the L^1-estimate theory for elliptic equations (see [45, Lemma 2.2] or [46]), we get

$$\|S\|_{W^{1,q}(\Omega)} \leq m, \quad \forall\, q \in [1, N/(N-1)), \quad (\text{or } \forall\, 1 \leq q < \infty \text{ if } N = 1). \tag{4.32}$$

Step 3. L^p-bound of S and I. In view of (4.32), by the Sobolev embedding theorem, we can see that $W^{1,q}(\Omega)$ is compactly embedded into $L^{p_0}(\Omega)$, $\forall p_0 \in [1, N/(N-q))$. This implies that

$$\|S\|_{L^{p_0}(\Omega)} \leq m, \quad \forall\, 1 < p_0 \leq \frac{Nq}{N-q}.$$

As q is close to $N/(N-1)$, it is clear that

$$\|S\|_{L^{p_0}(\Omega)} \leq m, \quad \forall\, 1 < p_0 < \frac{N}{N-2}. \tag{4.33}$$

Notice that (4.33) holds for any $1 < p_0 < \infty$ when $N = 2$.

We now multiply the second equation of (4.1) by I^k for any fixed $k > 0$ and then integrate by parts to obtain that

$$0 \leq d_I k \int_\Omega I^{k-1} |\nabla I|^2 \, dx = \int_\Omega \frac{\beta(x)S I^{k+1}}{c + S + I} dx - \int_\Omega [\gamma(x) + \mu(x)] I^{k+1} dx.$$

By direct calculation, we get

$$[\gamma_* + \mu_*] \int_\Omega I^{k+1} dx \leq \beta^* \int_\Omega \frac{S I^{k+1}}{c + S + I} dx. \tag{4.34}$$

Taking $\frac{1}{p_0} + \frac{1}{q_0} = 1$ (notice $\frac{1}{q_0-1} = p_0 - 1$) and $k_0 = \frac{1}{q_0}$, by (4.7), (4.33), (4.34) and Hölder inequality, we have

$$[\gamma_* + \mu_*] \int_\Omega I^{k_0+1} dx \le \beta^* \int_\Omega \frac{S I^{k_0+1}}{c + S + I} dx \le \beta^* \left(\int_\Omega S^{p_0} dx \right)^{1/p_0} \left(\int_\Omega I dx \right)^{1/q_0} \le m.$$

This means that

$$\|I\|_{L^{k_0+1}(\Omega)} \le m. \tag{4.35}$$

Then, we take $k_1 = (k_0 + 1)/q_0 = 1/q_0 + 1/q_0^2$. As before, by (4.34) and Hölder inequality, together (4.33) and (4.35), we infer that

$$[\gamma_* + \mu_*] \int_\Omega I^{k_1+1} dx \le \beta^* \int_\Omega \frac{S I^{k_1+1}}{c + S + I} dx \le \beta^* \left(\int_\Omega S^{p_0} dx \right)^{1/p_0} \left(\int_\Omega I dx \right)^{1/q_0} \le m,$$

that is,

$$\|I\|_{L^{k_1+1}(\Omega)} \le m.$$

Repeating the iteration as above, we can easily see that

$$\|I\|_{L^{k_\infty+1}(\Omega)} \le m, \tag{4.36}$$

where

$$k_\infty = \frac{1}{q_0} + \frac{1}{q_0^2} + \frac{1}{q_0^3} + \cdots = \frac{1}{q_0 - 1} = p_0 - 1.$$

Notice that (4.36) can be deduced through finitely many times of iterations with the help of (4.33). Thus, we deduce

$$\|I\|_{L^{p_0}(\Omega)} \le m. \tag{4.37}$$

Combining (4.33) and (4.37), from the equation (4.31) by using the well-known L^p-theory, one can assert that

$$\|S\|_{W^{2,p_0}(\Omega)} \le m.$$

By the Sobolev embedding theorem again, $W^{2,p_0}(\Omega)$ is compactly embedded into $L^{p_1}(\Omega)$, $\forall p_1 \in (1, Np_0/(N - 2p_0))$. Observe that $\frac{Np_0}{N-2p_0} \to \frac{N}{N-4}$ as $p_0 \to \frac{N}{N-2}$ (see (4.33)). Thus, we have

$$\|S\|_{L^{p_1}(\Omega)} \le m, \quad \forall\, 1 < p_1 < \frac{N}{N-4} \text{ or } \forall\, 1 < p_1 < \infty \text{ if } N \le 4.$$

By a similar argument as in deducing (4.37), one gets

$$\|I\|_{L^{p_1}(\Omega)} \le m.$$

Making use of (4.31), the Sobolev embedding theorem and the well-known L^p-theory repeatedly, we can eventually conclude that

$$\|S\|_{L^p(\Omega)}, \|I\|_{L^p(\Omega)} \le m, \quad \forall\, 1 \le p < \infty. \tag{4.38}$$

Step 4. Convergence of S and I. According to (4.38), for the equation (4.31), it holds that

$$\|S\|_{W^{2,p}(\Omega)} \le C, \quad \forall\, 1 < p < \infty.$$

Then taking sufficiently large p, the standard embedding theorem enables us to conclude that, up to a sequence of $d_I \to 0$, denoted by $d_j := d_{I,j}$, with $d_j \to 0$ as $j \to \infty$, the corresponding positive solution sequence $(S_j, I_j) := (S_{d_{I,j}}, I_{d_{I,j}})$ of (4.1) with $d_I = d_{I,j}$ fulfills

$$S_j \to W^S \ \text{ in } C^1(\overline{\Omega}), \ \text{as } j \to \infty, \tag{4.39}$$

where $W^S \in C^1(\overline{\Omega})$ and $W^S > 0$ on $\overline{\Omega}$ due to (4.30). Observe that I_j fulfills

$$\begin{cases} -d_j \Delta I_j = \dfrac{\beta(x) S_j I_j}{c + S_j + I_j} - [\gamma(x) + \mu(x)] I_j, & x \in \Omega, \\ \dfrac{\partial I_j}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{4.40}$$

In light of (4.39) and (4.40), using a simple upper and lower solution similarly as in step 3 of Theorem 4.1, we have

$$I_j \to W^I \text{ in } C^1(\overline{\Omega}), \text{ as } j \to \infty,$$

where W^I is given by (4.28).

It is clear that W^S satisfies (4.27). Moreover, by the expression of W^I, we can see that (4.27) admits a unique positive solution (refer to [47, Lemma A.1]). Thus, we can conclude that all the above limits hold without passing to a subsequence. This proof is complete.

□

5. Discussion

In this paper, we have studied the SIS reaction-diffusion model (1.3) in which we have taken into account the natural mortality of the susceptible and infected populations. First of all, we have established the uniform bounds of solution to (1.3); see Lemma 2.1 and Lemma 2.2. Then, we define the basic reproduction number $\mathcal{R}_0$ associated with (1.3):

$$\mathcal{R}_0 = \sup_{\varphi \in H^1(\Omega), \varphi \neq 0} \left\{ \frac{\int_\Omega \frac{\beta \hat{S}}{c + \hat{S}} \varphi^2 dx}{\int_\Omega d_I |\nabla \varphi|^2 + (\gamma + \mu) \varphi^2 dx} \right\}.$$

It is worth mentioning that $\mathcal{R}_0$ depends on the diffusion rates d_S and d_I when $c > 0$, while $\mathcal{R}_0$ depends only on the diffusion rate d_I when $c = 0$. Thus, compared with the model (1.2) (i.e., $c = 0$), the parameters d_S and c play vital roles in the dynamics of the infectious disease in (1.3). In particular, we have proved that $\mathcal{R}_0$ is decreasing with respect to $c \in [0, \infty)$, and when c is larger than a value, the basic reproduction number $\mathcal{R}_0 < 1$ so that the disease dies out in the long run; see Theorem 3.2(ii).

In another special case that $\beta(x) \leq \gamma(x) + \mu(x), \ \forall x \in \overline{\Omega}$, we have proved the global stability of the disease-free equilibrium via a Lyapunov function method; see Theorem 3.2. On the other hand, when the spatial environment is homogeneous, that is, all the parameters in (1.3) are positive constants, we have shown the global stability of the endemic equilibrium provided that the diffusion rates are equal and the basic reproduction number $\mathcal{R}_0 > 1$; see Theorem 3.3. This result means that the disease will persist all the time. In the general situation of spatially heterogeneous environment, once $\mathcal{R}_0 > 1$, the uniform persistence property has been proved so that the disease exists eventually in the whole habitat;

refer to Theorem 2.5. We suspect that the uniform persistence property holds if $\mathcal{R}_0 > 1$ whereas the disease extinction occurs if $\mathcal{R}_0 \leq 1$; this is a challenging problem and deserves future investigation.

According to Theorem 2.5 as well as the above discussion, once $\mathcal{R}_0 > 1$, (1.3) has an endemic equilibrium exists and the infectious disease will uniformly persist in space, and vice versa. Therefore, it becomes important to understand how the heterogeneity of spatial environment and the mobility of population dispersal (reflected by the change of the migration rates d_I and d_S) prescribe the spatial profile of the endemic equilibrium, because this will help decision-makers to predict the pattern of disease occurrence and henceforth to conduct effective/optimal control strategies of disease eradication. This leads us to explore the asymptotic behavior of endemic equilibrium with respect to small diffusion rate d_S or d_I, which in turn will tell us the spatial distribution of susceptible and infected population

Theorems 4.1 and 4.2 show that as the mobility of the susceptible or infected population goes to zero, the infectious disease will always exist in space at least in some region. Especially, it follows from Theorem 4.1 that the disease exists in the entire habitat even if the mobility of the susceptible is restricted to be small enough. This result is in sharp contrast with that of (1.1), as shown in [1] by Allen et al, where they proved that as the mobility of the susceptible goes to zero, the density of the infected population will vanish and so the disease dies out eventually. However, Peng in [12] showed that, for the model (1.2) which also includes the linear external source term $\Lambda(x) - S$, the density of the infected population will not vanish when the mobility of the susceptible or infected population goes to zero; that is, the infectious disease will always exist.

The above results suggest that simply controlling the migration rate of the susceptible or infected population can not eliminate the disease modelled by (1.2) and (1.3). In other words, the presence of the linear external source term $\Lambda(x) - S$ enhances the persistence of disease and the infectious disease will become more threatening and hard to control. As a consequence, in such a situation, more effective measures should be taken to eradicate diseases.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. L. J. S. Allen, B. M. Bolker, Y. Lou, et al., Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model, *Dis. Contin. Dyn. Syst. A*, **21** (2008), 1–20.
2. R. Peng, Asymptotic profiles of the positive steady state for an SIS epidemic reaction-diffusion model. Part I, *J. Differ. Equations*, **247** (2009), 1096–1119.
3. R. M. Anderson and R. M. May, Populaition biology of infectious diseases, *Nature* **280** (1979), 361–367.

4. R. Cui and Y. Lou, A spatial SIS model in advective heterogeneous environments, *J. Differ. Equations*, **261** (2016), 3305–3343.

5. R. Cui, K.-Y. Lam and Y. Lou, Dynamics and asymptotic profiles of steady states of an epidemic model in advective environments, *J. Differ. Equations*, **263** (2017), 2343–2373.

6. Z. Du and R. Peng, A priori L^∞-estimates for solutions of a class of reaction-diffusion systems, *J. Math. Biol.,* **72** (2016), 1429–1439.

7. H. W. Hethcote, Epidemiology models with variable population size, Mathematical understanding of infectious disease dynamics, 63-89, Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap., 16, World Sci. Publ., Hackensack, NJ, 2009.

8. W. O. Kermack and A. G. McKendrick, Contribution to the mathematical theory of epidemics-*I*, *Proc. Roy. Soc. London Ser. A*, **115** (1927), 700–721.

9. M. Martcheva, An introduction to mathmatical epidemiology, Springer,New York, (2015).

10. H. W Hethcote, The mathematics of infectious diseases, *SIAM Rev.*, **42** (2000), 599–653.

11. B. Li, H. Li and Y. Tong, Analysis on a diffusive SIS epidemic model with logistic source, *Z. Angew. Math. Phys.*, **68** (2017), Art. 96, 25pp.

12. H. Li, R. Peng and F. B. Wang, Varying total population enhances disease persistence: qualitative analysis on a diffusive SIS epidemic model, *J. Differ. Equations*, **262** (2017), 885–913.

13. M. E. Alexander and S. M. Moghadas, Bifurcation Analysis of an SIRS Epidemic Model with Generalized Incidence, *SIAM J. Appl. Math.*, **65** (2001), 1794–1816.

14. R. M. Anderson and R. M. May, Regulation and stability of host-parasite interactions.I. Regulatory processes, *J. Anim. Ecol.*, **47** (1978), 219–247.

15. O. Diekmann and M. Kretzschmar, Patterns in the effects of infectious diseases on population growth, *J. Math. Biol.*, **29** (1991), 539–570.

16. J. A. P. Heesterbeck and J. A. J. Metz, The saturating contact rate in marriage and epidemic models, *J. Math. Biol.*, **31** (1993), 529–539.

17. M. G. Roberts, The dynamics of bovine tuberculosis in possum populations and its eradication or control by culling or vaccination, *J. Anim. Ecol.*, **65** (1996), 451–464.

18. Y. Cai, K. Wang and W. Wang, Global transmission dynamics of a Zika virus model, *Appl. Math. Lett.*, **92** (2019), 190–195.

19. L. Chen and J. Sun, Optimal vaccination and treatment of an epidemic network model, *Physics Lett. A*, **378** (2014), 3028–3036.

20. L. Chen and J. Sun, Global stability and optimal control of an SIRS epidemic model on heterogeneous networks, *Physica A*, **410** (2014), 196–204.

21. K. Deng and Y. Wu, Dynamics of a susceptible-infected-susceptible epidemic reaction-diffusion model, *Proc. Roy. Soc. Edinburgh Sect. A*, **146** (2016), 929–946.

22. X. Gao, Y. Cai, F. Rao, et al., Positive steady states in an epidemic model with nonlinear incidence rate, *Comput. Math. Appl.*, **75** (2018), 424–443.

23. J. Ge, C. Lei and Z. Lin, Reproduction numbers and the expanding fronts for a diffusion-advection SIS model in heterogeneous time-periodic environment, *Nonlinear Anal. Real World Appl.*, **33** (2017), 100–120.

24. K. Kuto, H. Matsuzawa and R. Peng, Concentration profile of endemic equilibrium of a reaction-diffusion-advection SIS epidemic model, *Calc. Var. Partial Dif.*, **56** (2017), Art. 112, 28 pp.

25. C. Lei, F. Li and J. Liu, Theoretical analysis on a diffusive SIR epidemic model with nonlinear incidence in a heterogeneous environment, *Discrete Contin. Dyn. Syst. Ser. B*, **23** (2018), 4499–4517.

26. B. Li and Q. Bie, Long-time dynamics of an SIRS reaction-diffusion epidemic model, *J. Math. Anal. Appl.*, **475** (2019), 1910–1926.

27. H. Li, R. Peng and Z. Wang, On a diffusive SIS epidemic model with mass action mechanism and birth-death effect: analysis, simulations and comparison with other mechanisms, *SIAM J. Appl. Math.*, **78** (2018), 2129–2153.

28. H. Li, R. Peng and T. Xiang, Dynamics and asymptotic profiles of endemic equilibrium for two frequency-dependent SIS epidemic models with cross-diffusion, *Eur. J. Appl. Math.*, 2019, https://doi.org/10.1017/S0956792518000463, in press.

29. Z. Lin, Y. Zhao and P. Zhou, The infected frontier in an SEIR epidemic model with infinite delay, *Discrete Contin. Dyn. Syst. Ser. B*, **18** (2013), 2355–2376.

30. R. Peng and S. Liu, Global stability of the steady states of an SIS epidemic reaction-diffusion model, *Nonlinear Anal.*, **71** (2009), 239–247.

31. L. Pu and Z. Lin, A diffusive SIS epidemic model in a heterogeneous and periodically evolving environment, **16** (2019), 3094–3110.

32. X. Wen, J. Ji and B. Li, Asymptotic profiles of the endemic equilibrium to a diffusive SIS epidemic model with mass action infection mechanism, *J. Math. Anal. Appl.*, **458** (2018), 715–729.

33. Y. Wu and X. Zou, Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism, *J. Differ. Equations*, **261** (2016), 4424–4447.

34. M. Zhu, X. Guo and Z. Lin, The risk index for an SIR epidemic model and spatial spreading of the infectious disease, *Math. Biosci. Eng.*, **14** (2017), 1565–1583.

35. R. Peng and X. Zhao, A reaction-diffusion SIS epidemic model in a time-periodic environment, *Nonlinearity*, **25** (2012), 1451–1471.

36. P. Magal and X.-Q Zhao, Global attractive and steady states for uniformly persistent dynamical systems, *SIAM. J. Math. Anal.*, **37** (2005), 251–275.

37. X.-Q. Zhao, Dynamical Systems in Populaition Biology, Springer-Verlag, New York,(2003)

38. M. Wang, Nonlinear Partial Differential Equations of Parabolic Type, Science Press, Beijing, 1993(in chinese).

39. K. J. Brown, P. C. Dunne and R. A. Gardner, A semilinear parabolic system arising in the theory of superconductivity, *J. Differ. Equations*, **40** (1981), 232–252.

40. G. M. Lieberman, Bounds for the steady-state Sel'kov model for arbitrary p in any number of dimensions, *SIAM J. Math. Anal.*, **36** (2005), 1400–1406.

41. R. Peng, J. Shi and M. Wang, On stationary patterns of a reaction-diffusion model with autocatalysis and saturation law, *Nonlinearity*, **21** (2008), 1471–1488.

42. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equation of Second Order*, Springer, (2001).

43. Y. Du, R. Peng and M. Wang, Effect of a protection zone in the diffusive Leslie predator-prey model, *J. Differ. Equations*, **246** (2009), 3932–3956.

44. Y. Lou and W.-M. Ni, Diffusion, self-diffusion and cross-diffusion, *J. Differ. Equations*, **131** (1996), 79–131.

45. W.-M. Ni and I. Takagi, On the Neumann problem for some semilinear elliptic equations and eystems of activator-inhibitor type, *Trans. Amer. Math. Soc.*, **297** (1986), 351–368.

46. H. Brezis and W. A. Strauss, Semi-linear second-order elliptic equations in L^1, *J. Math. Soc. Japan*, **25** (1973), 565–590.

47. R. Peng and F. Yi, Asymptotic profile of the positive steady state for an SIS epidemic reaction-diffusion model: Effects of epidemic risk and population movement, *Phys. D*, **259** (2013), 8–25.