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Research article

The effects of impulsive toxicant input on a single-species population in a small polluted environment

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Abstract: In this paper, we study a single-species population model with pulse toxicant input in a small polluted environment. The intrinsic rate of population change is affected by the environmental toxin load and toxin in the organisms which is influenced by toxin in the environment and the food chain. A new mathematical model is established. By the Pulse Compare Theorem, we find the surviving threshold of the population and obtain the sufficient conditions of persistence and extinction of the population.

Keywords: pollution; extinction; persistence; impulsive toxicant input; surviving threshold

1. Introduction

In the real world, with the rapid development of modern industry and agriculture, environmental pollution has become an increasingly serious problem. Untreated pollutants are continuously release into the environment. It causes many serious environmental problems and damages ecological system. The issue is global. Many populations have become extinct or endangered [1,2]. Therefore, it is very important to study living conditions of the population in a polluted environment.

In the 1980s, Hallam et al. (1983–1984) studied the toxicant effects in the polluted environment on a single-species population. During the discussion in their paper, they assumed that, relative to population size, the capacity of the environment is large, so the population absorption and excretion of toxins can be omitted. Many good results were obtained about the extinction and persistence of the population [3–5]. But in a relatively closed environment with a large population the effect of the populations own emission of toxins cant be omitted. He, et al. (2007, 2009) studied the survival problem of the population assuming the intrinsic rate of population change affected by the environment and the toxins of the body [6–12].

In most cases, the toxins entering into the environment are assumed to be continuous, but in the real life, discharging toxins are not always ture, and the majority of cases are often expressed as a periodic emission, such as industrial waste water or waste water discharge, agricultural pesticide spraying, etc. In these cases, the discharge time of toxins, compared with the population's life cycle, is very short, but their effect on the organism is long. Liu, et al. (2003) studied the survival effects of population on the pulse cycle toxin emissions under a fixed input quantitative toxin [13] at a fixed time. Zhang, et al. (2008) established a single population model in a polluted environment by assuming other outside toxins discharged into the environment at a fixed time. They showed that the population is extinct, when the pulse period is less than a certain threshold. On the contrary, the population is permanent. They also demonstrated that sustained living conditions can ensure existence and uniqueness of positive periodic solutions which are globally asymptotically stable [14–16]. Jiao, et al. (2009) established a single population model in which toxins in polluted environment are impulsive inputs on the basis of the hypothesis that toxins in the population are also affected by toxins in the food chain. Discussed the extinction and permanent existence of the population, and drew a conclusion that the population can be protected by changing the input toxin quantity and period [17–19].

Based on the work that has been done [16], this paper studies the survival of a single population in a less polluted space. Assuming that the population density is uniform, the input and output of the population are not considered. If toxins concentration of individuals in population $C_o(t)$ is considered to be the individual endotoxin mass divided by the individual average mass m_0. Environmental toxin concentration $C_e(t)$ is considered to contain toxin mass in environment medium divided by total mass of medium in environment m_e. Influenced by the concentration of environmental toxin $C_e(t)$ and the concentration of individual toxin $C_o(t)$, the intrinsic growth rate of population density $x(t)$, which accords with the Logistic law, is considered as the linear dose response function $r_0 - \alpha C_o(t) - \beta C_e(t)$ of the populations and environmental toxins. Concentration of individual $C_o(t)$ was mainly derived from the environmental toxin absorption rate $KC_e(t)$ and the food intake rate $fC_e(t)$. However, the individual excretion rate $gC_o(t)$, the metabolic rate $mC_o(t)$ and the death rate $d_oC_o(t)$ could reduce endotoxin concentration in population. The change rate of total environmental endotoxin is caused by the following two aspects: On the one hand, the periodic impulse emission rate of pollutants μ and the amount of toxin released to the environment $(g + d_0 + \alpha C_o(t) + \beta C_e(t))C_o(t)m_o x(t)$ by individual toxin excretion, death decomposition, individual toxin and environmental toxin population death, and the periodic impulse emission rate of pollutants; on the other hand, the individual absorption rate of the population $km_oC_o(t)x(t)$, and the reduction rate of environmental toxin concentration $hm_eC_e(t)$ caused by natural volatilization of environmental toxin, photosynthesis and bacterial degradation. Suppose $h > m$. In summary, the following models are established (1.1).

$$\begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = x(t)(r_0 - \alpha C_o(t) - \beta C_e(t) - \lambda x(t)), t \neq nT \\ \frac{\mathrm{d}C_o(t)}{\mathrm{d}t} = KC_e(t) + fC_e(t) - (g + m + d_0 - \lambda x(t))C_o(t), t \neq nT \\ \frac{\mathrm{d}C_e(t)}{\mathrm{d}t} = [-K_1C_e(t) + (g_1 + d_1 + \alpha_1 C_o(t) + \beta_1 C_e(t))C_o(t)]x(t) - hC_e(t), t \neq nT \\ \triangle x(t) = 0, \triangle C_o(t) = 0, \triangle C_e(t) = \mu, t = nT. \end{cases} \tag{1.1}$$

where

$$g_1 = \frac{gm_0}{m_e}, K_1 = \frac{Km_0}{m_e}, d_1 = \frac{d_0m_0}{m_e}, \alpha_1 = \frac{\alpha m_0}{m_e}, \beta_1 = \frac{\beta m_0}{m_e}. \tag{1.2}$$

All symbols g_1, K_1, r_0, d_0, d_1, g, m, K, h in model (1.1) are positive constants. The symbols in the model (1.1) are shown in Table 1.

Table 1. Model (1.1) of related parameters.

Parameter	Description	Unit
$X(t)$	Single population biomass in a given space at time t	Quantity / litre
$C_0(t)$	Average mass toxin concentration of individuals in population at time t	
$C_e(t)$	Toxin concentration in environmental mass medium at time t	
$r_0(t)$	Intrinsic growth rate of the population without toxicant	1 day^{-1}
α	Individual toxins inhibit population growth rate	1 day^{-1}
β	Environmental toxins inhibit population growth rate	1 day^{-1}
λ	Intra specific competition of the population	litre/Quantity· day
d_0	The death rate of the population without toxicant	1 day^{-1}
m_0	Average mass of Individuals in a Population	kg
m_e	The total mass of the medium in the environment	kg
μ	The toxicant input amount at every time	
T	The period of the impulsive effect about the exogenous input of toxicant	day
K	Individual absorption rate of environmental toxicant	1 day^{-1}
f	Individual intake rate of toxin in environmental food	1 day^{-1}
g	Excretion rate of individual toxin	1 day^{-1}
m	Purification rate of individual toxin	1 day^{-1}
h	Toxicant loss rate from the environment itself by volatilization and so on	1 day^{-1}

In this paper, we study the dynamic behavior of model (1.1). In section 2, we prove that the model (1.1) has the non-negative solutions and they are ultimately bounded by inequality scaling method, thus the survival upper bound of the population is found. In section 3, by the Pulse Compare Theorem, we get the solution of model (1.1), which has a non-negative lower bound and derive the sufficient condition of persistent survival of the population. In section 4, we obtain the sufficient condition for extinction of the population. In section 5, numerical conclusions are obtained by

MATLAB. Some summaries are given in the last section.

2. Non-negative and boundedness of solutions

In order to prove the persistence of solutions for model (1.1), we need to show that they are non-negative and have upper and lower bounds. First we prove there exists the positive solution.

We set initial values of model (1.1) as follows:

$$x(0) > 0, 0 \leq C_o(0) \leq 1, 0 \leq C_e(0) \leq 1. \tag{2.1}$$

First, we have the following conclusion regarding to the positive property of solutions of model (1.1).

Theorem 2.1. *The solution $(x(t), C_o(t), C_e(t))$ of model (1.1) with initial conditions (2.1) is non-negative.*

Proof. Integrating the first function of model (1.1) from 0 to t gives

$$x(t) = x(0)exp\left(\int_0^t (r_0 - \alpha C_o(\tau) - \beta C_e(\tau) - \lambda x(\tau))d\tau\right),$$

So, if $x(0) > 0$, we have $x(t) > 0$ for $t \geq 0$.

Next, we prove $C_o(t) > 0$, $C_e(t) > 0$.

Since $C_e(0) \geq 0$, $\triangle C_e(t) = \mu > 0$, it is obvious $C_e(0^+) > 0$. However, for $C_o(0)$, we have two cases as follows.

Case I: $C_o(0) = 0$.

As $\triangle C_o(t) = 0$, from the second and third functions of model (1.1), we have

$$\left.\frac{\mathrm{d}C_o(t)}{\mathrm{d}t}\right|_{t=0^+} = KC_e(0^+) + fC_e(0^+) - (g + m + d_0 - \lambda x(0^+))C_o(0^+) = (K + f)C_e(0^+) > 0,$$

$$\left.\frac{\mathrm{d}C_e(t)}{\mathrm{d}t}\right|_{t=0^+} = -K_1C_e(0^+)x(0^+) - hC_e(0^+) < 0.$$

Hence there must exist a positive number ε such that $C_o(t) > 0, C_e(t) > 0$ for $t \in (0, \varepsilon)$.

Then, when $t > 0$, we have

$$C_o(t) > 0, C_e(t) > 0. \tag{2.2}$$

If it is not, there must exist a positive number t^* such that $C_o(t^*) \cdot C_e(t^*) = 0$ for $t \in ((n-1)T, nT]$ and $C_o(t) > 0, C_e(t) > 0$ for $t \in (0, t^*)$. Therefore there are only three situations at the endpoint t^*:

For the first situation: $C_o(t^*) = 0$, $C_e(t^*) > 0$.

If $C_o(t) > 0$ is true, then it is obvious $\frac{\mathrm{d}C_o(t^*)}{\mathrm{d}t} \leq 0$ for $t \in (0, t^*)$. But from the second function of model (1.1), we have

$$\left.\frac{\mathrm{d}C_o(t)}{\mathrm{d}t}\right|_{t=t^*} = (K + f)C_e(t^*) > 0.$$

There is a contradiction, so the first situation does not hold.

For the second situation: $C_o(t^*) > 0$, $C_e(t^*) = 0$.

If $C_e(t) > 0$ is true, then it is obvious $\frac{dC_e(t^*)}{dt} \le 0$ for $t \in (0, t^*)$. But from the third function of model (1.1), we have

$$\left.\frac{dC_e(t)}{dt}\right|_{t=t^*} = [g_1 + d_1 + \alpha_1 C_o(t^*)]C_o(t^*)x(t^*) > 0.$$

There is a contradiction, thus the second situation is not true.

For the third situation: $C_o(t^*) = 0$, $C_e(t^*) = 0$.

It is obvious that $(x(t), 0, 0)$ is the solution of model (1.1). At the same time, it is also the solution of model (1.1) with initial values $x(t^*) > 0, C_o(t^*) = 0, C_e(t^*) = 0$. The uniqueness theorems of solution yields $C_o(t) \equiv 0$, $C_e(t) \equiv 0$ for $t > 0$. This is also a contradiction. Hence the third situation doesn't hold. We conclude that $C_o(0) = 0$.

Case II: $C_o(0) > 0$.

From $C_o(0) > 0$ and the continuity of $C_o(t)$, for any $\varepsilon_1 > 0$, we have $C_o(t) > 0$ for $t \in (0, \varepsilon_1)$.

Furthermore, $C_e(0^+) > 0$, whatever $\left.\frac{dC_e(t)}{dt}\right|_{t=0^+}$ is positive or negative, we can promise that, for any $\varepsilon_2 > 0$, we have $C_e(t) > 0$ for $t \in (0, \varepsilon_2)$.

So let $\varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0$, there is $C_o(t) > 0, C_e(t) > 0$ for $t \in (0, \varepsilon)$.

Next we prove, when $t > 0$, there is

$$C_o(t) > 0, C_e(t) > 0.$$

Then the proof of Case II is similar to that of Case I, the result still holds. □

Next we prove that all positive solutions of model (1.1) have upper bounds.

Theorem 2.2. *For model (1.1), if* $\frac{fr_0}{\lambda} + \frac{mg}{g_1} < \frac{hg}{g_1}$*, for any* $t \in R^+$*, there must exist a positive number* M*, such that*

$$\limsup_{t\to\infty} x(t) \le M, \limsup_{t\to\infty} C_o(t) \le M, \limsup_{t\to\infty} C_e(t) \le M.$$

Proof. From Theorem 2.1 and the first equation of model (1.1), we have

$$\frac{dx(t)}{dt} \le x(t)(r_0 - \lambda x(t)).$$

Standard comparison theorem produces

$$\limsup_{t\to\infty} x(t) \le \frac{r_0}{\lambda} \triangleq M_1. \tag{2.3}$$

Defining $V(t) = C_o(t)x(t) + \frac{g}{g_1}C_e(t)$ and using (1.2), one can obtain

$$D^+V(t) + mV(t) = fC_e(t)x(t) - \frac{g}{g_1}(h - m)C_e(t).$$

Expression (2.3) and $\frac{fr_0}{\lambda} + \frac{mg}{g_1} < \frac{hg}{g_1}$, when $t \ne nT$, gives

$$D^+V(t) + mV(t) \le \left(\frac{fr_0}{\lambda} - \frac{(h-m)g}{g_1}\right)C_e(t) < 0.$$

When $t = nT$, there is $V(nT^+) = V(nT) + \frac{\mu g}{g_1}$, so for $t \in (nT, (n+1)T]$, pulse inequality (Lemma 2.2) in [20] gives

$$V(t) \le V(0)e^{-mt} + \frac{\mu g}{g_1}\frac{e^{-m(t-T)}}{1-e^{-mT}} + \frac{\mu g}{g_1}\frac{e^{mT}}{e^{mT}-1},$$

Hence $V(t)$ is uniformly bounded, which is

$$\limsup_{t\to\infty} V(t) \le \frac{\mu g}{g_1}\frac{e^{mT}}{e^{mT}-1} \triangleq M_2.$$

According to the definition of $V(t)$ and Theorem 2.1, one can derive

$$\limsup_{t\to\infty} C_o(t)x(t) \le M_2, \limsup_{t\to\infty} C_e(t) \le \frac{g_1}{g}M_2. \tag{2.4}$$

The second function of model (1.1) and (2.4) leads to

$$\limsup_{t\to\infty} C_o(t) \le \frac{(K+f)g_1M_2 + \lambda g M_2}{g(g+m+d_0)} \triangleq M_3.$$

Let $M = \max(M_1, \frac{g_1}{g}M_2, M_3)$, so there is

$$\limsup_{t\to\infty} x(t) \le M, \limsup_{t\to\infty} C_o(t) \le M, \limsup_{t\to\infty} C_e(t) \le M.$$

□

From Theorem 2.1 and Theorem 2.2, there is a invariant set in mode (1.1), that is $\Omega = \left\{ (x(t), C_o(t), C_e(t)) \middle| 0 \le x(t) \le M, 0 \le C_o(t) \le M, 0 \le C_e(t) \le M \right\}$.

let

$$R_0 = \frac{\mu}{r_0 T}\left(\frac{\alpha(K+f)}{h(g+m+d_0)} + \frac{\beta}{h}\right).$$

3. Persistence survival of population

In Theorem 2.2, we know that the solutions of model (1.1) have upper bounds. In this section, in order to investigate the survival of the population, we will prove the model (1.1) has a non-negative lower bound. Now we can analyze the model (1.1) by the impulsive differential equations comparison theorem to find the lower bound as follows.

Theorem 3.1. *For model (1.1), if $R_0 < 1$, then the population $x(t)$ will be uniformly persistent.*

Proof. From Theorem 2.2, we know that is $x(t)$ ultimately bounded. Hence, in order to prove that the population $x(t)$ is uniformly persistent, we can only need to show that $x(t)$ has the lower bound. If not, for any $\delta > 0$, when $t > 0$, there is

$$x(t) < \delta, \tag{3.1}$$

Considering the last two equations of model (1.1)

$$\begin{cases} \frac{dC_o(t)}{dt} = KC_e(t) + fC_e(t) - (g + m + d_0 - \lambda x(t))C_o(t), t \ne nT \\ \frac{dC_e(t)}{dt} = [-K_1C_e(t) + (g_1 + d_1 + \alpha_1 C_o(t) + \beta_1 C_e(t))C_o(t)]x(t) - hC_e(t), t \ne nT \\ \triangle C_o(t) = 0, \triangle C_e(t) = \mu, t = nT. \end{cases} \tag{3.2}$$

We set up $(C_o(t), C_e(t))$ is the solution of model (3.2).

From Theorem 2.2 and (3.1), we get the pulse comparison equation corresponding equation of model (3.2):

$$\begin{cases} \frac{\mathrm{d}u(t)}{\mathrm{d}t} = Kv(t) + fv(t) - (g + m + d_0)u(t) + \lambda M\delta, t \neq nT \\ \frac{\mathrm{d}v(t)}{\mathrm{d}t} = a - hv(t), t \neq nT \\ \triangle u(t) = 0, \triangle v(t) = \mu, t = nT. \end{cases} \tag{3.3}$$

where $a = (g_1 + d_1 + \alpha_1 M + \beta_1 M)M\delta$.

Let $(u(t), v(t))$ be the solution of model (3.3) with initial values $u(0) = C_o(0), v(0) = C_e(0)$.

First we prove that model (3.3) only has a positive periodic solution $(\overline{u(t)}, \overline{v(t)})$, which is globally attractive.

In the interval $(nT, (n+1)T]$, the solution of the second function of model (3.3) is

$$v(t) = \frac{a}{h} + \left(v(nT^+) - \frac{a}{h}\right)e^{-h(t-nT)}. \tag{3.4}$$

From $\triangle v(t) = \mu$ and (3.4), we have

$$v((n+1)T^+) = \left(v(nT^+) - \frac{a}{h}\right)e^{-hT} + \frac{a}{h} + \mu, \tag{3.5}$$

which implies a stroboscopic map

$$v((n+1)T^+) = H(v(nT^+)), \tag{3.6}$$

where

$$H(y) = \left(y - \frac{a}{h}\right)e^{-hT} + \frac{a}{h} + \mu,$$

We get the only fixed point of map (3.6), that is

$$v^* = \frac{\mu}{1 - e^{-hT}} + \frac{a}{h}. \tag{3.7}$$

It is easily to show that $|H'(v^*)| = e^{-hT} < 1$, so sequence $\left\{(v(n+1)T^+)\right\}$ converges to v^*.

Using (3.4) and (3.7), we have

$$\overline{v(t)} = \frac{a}{h} + \left(v^* - \frac{a}{h}\right)e^{-h(t-nT)}, nT < t \leq (n+1)T.$$

Similarly, from the first function of model (3.3), we can get

$$\overline{u(t)} = p + \frac{q(v^* - a/h)}{\mu}e^{-h(t-nT)} + \left(u^* - p - \frac{q(v^* - a/h)}{\mu}\right)e^{-(g+m+d_0)(t-nT)},$$

$$nT < t \leq (n+1)T.$$

where

$$p = \frac{(K+f)a/h + \lambda M\delta}{g + m + d_0}, q = \frac{\mu(K+f)}{g + m + d_0 - h},$$

$$u^* = p + \frac{q(v^* - a/h)(e^{-hT} - e^{-(g+m+d_0)T})}{\mu(1 - e^{-(g+m+d_0)T})}.$$

Therefore the model (3.3) only has a positive periodic solution $(\overline{u(t)}, \overline{v(t)})$.

Now we prove the positive periodic solution $(\overline{u(t)}, \overline{v(t)})$ is also globally asymptotically stable.

If $(u(t), v(t))$ is the any solution of model (3.3), define the conversion

$$M(t) = u(t) - \overline{u(t)}, N(t) = v(t) - \overline{v(t)}. \tag{3.8}$$

Then expression (3.3) is changed as follows:

$$\begin{cases} \frac{\mathrm{d}M(t)}{\mathrm{d}t} = (K + f)N(t) - (g + m + d_0)M(t), \\ \frac{\mathrm{d}N(t)}{\mathrm{d}t} = -hN(t). \end{cases} \tag{3.9}$$

Let $M(0) = u(0), N(0) = v(0)$ be initial values of model (3.9).

From the second function of model (3.9), we have

$$N(t) = v(0)e^{-ht}. \tag{3.10}$$

First function of model (3.9) also gives

$$M(t) = \frac{(K + f)v(0)}{h - (g + m + d_0)}e^{-ht} + \left(u(0) - \frac{(K + f)v(0)}{h - (g + m + d_0)}\right)e^{-(g+m+d_0)t}. \tag{3.11}$$

Using (3.10) and (3.11), we get $\lim_{t\to\infty} N(t) = 0, \lim_{t\to\infty} M(t) = 0$. So $(\overline{u(t)}, \overline{v(t)})$ is globally attractive.

Next we prove the population $x(t)$ is uniformly persistent.

Using the Comparison Theorem [21] and the globally asymptotically stable property of $(\overline{u(t)}, \overline{v(t)})$, there exists a positive number $T_0 > 0$, for arbitrarily small $\varepsilon > 0$, and when $t > T_0$, we have

$$C_0(t) \le u(t) \le \overline{u(t)} + \varepsilon, C_e(t) \le v(t) \le \overline{v(t)} + \varepsilon. \tag{3.12}$$

Using (3.12) and the first function of model (1.1),we have

$$\begin{aligned} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= x(t)(r_0 - \alpha C_0(t) - \beta C_e(t) - \lambda x(t)) \\ &\ge x(t)\Big(r_0 - \alpha(\overline{u(t)} + \varepsilon) - \beta(\overline{v(t)} + \varepsilon) - \lambda\delta\Big). \end{aligned} \tag{3.13}$$

Setting $n_1 \in N$ and $n_1T > T_0$, and integrating (3.13) from nT to $(n + 1)T$ $(n > n_1)$ leads to

$$\begin{aligned} x\big((n+1)T\big) &\ge x(nT)exp\left(\int_{nT}^{(n+1)T} \Big(r_0 - \alpha(\overline{u(t)} + \varepsilon) - \beta(\overline{v(t)} + \varepsilon) - \lambda\delta\Big)dt\right) \\ &= x(nT)exp\left(\int_{nT}^{(n+1)T} \Big[(r_0 - \alpha\varepsilon - \beta\varepsilon - \lambda\delta) - \alpha\overline{u(t)} - \beta\overline{v(t)}\Big]dt\right) \\ &= x(nT)exp\Big[(r_0 - \lambda\delta - \alpha\varepsilon - \beta\varepsilon)T - \alpha\Big(pT - \frac{q(e^{-hT} - 1)}{h(1 - e^{-hT})} \\ &\quad - \frac{1}{g + m + d_0}\Big(u^* - p - \frac{q}{1 - e^{-hT}}\Big)(e^{-(g+m+d_0)T} - 1)\Big) \end{aligned}$$

$$
\begin{aligned}
& -\beta\left(\frac{aT}{h}-\frac{(v^*-a/h)(e^{-hT}-1)}{h}\right)\Big]\\
= {} & x(nT)exp\Big[(r_0-\lambda\delta-\alpha\varepsilon-\beta\varepsilon)T\\
& -\alpha\left(pT+\frac{q}{h}-\frac{q}{g+m+d_0}\right)-\beta\left(\frac{aT}{h}+\frac{\mu}{h}\right)\Big]\\
= {} & x(nT)exp(\kappa)\geq x(0^+)exp(n\kappa), \qquad (3.14)
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa &= (r_0-\lambda\delta-\alpha\varepsilon-\beta\varepsilon)T-\alpha\left(pT+\frac{q}{h}-\frac{q}{g+m+d_0}\right)-\beta\left(\frac{aT}{h}+\frac{\mu}{h}\right)\\
&= \left(r_0-\lambda\delta-\alpha\varepsilon-\beta\varepsilon-\frac{\alpha(K+f)a/h+\lambda\alpha M\delta}{g+m+d_0}-\frac{a\beta}{h}\right)T\\
&\quad -\frac{\alpha^2\mu(K+f)}{h(g+m+d_0)}-\frac{\beta\mu}{h}. \qquad (3.15)
\end{aligned}
$$

From $R_0<1$, we know $r_0T>\frac{\alpha\mu(K+f)}{h(g+m+d_0)}+\frac{\beta\mu}{h}$. Hence, for given $M>0$, we choose sufficiently small $\delta>0$ and $\varepsilon>0$, such that

$$
\left(r_0-\alpha\varepsilon-\beta\varepsilon-\lambda\delta-\frac{\alpha(K+f)a/h+\lambda\alpha M\delta}{g+m+d_0}-\frac{a\beta}{h}\right)T>\frac{\alpha\mu(K+f)}{h(g+m+d_0)}+\frac{\beta\mu}{h}.
$$

From (3.15), we get $\kappa>0$. So (3.14) yields $\lim_{n\to\infty}x(nT)=\infty$. This is a contradiction with (3.1). Hence there exists a positive number $t_1\geq T_0$ such that $x(t_1)>\delta$.

Next prove, when $t\geq t_1$, we have

$$
x(t)\geq \delta exp(-\omega T), \qquad (3.16)
$$

where $\omega=\sup_{t\geq 0}\left\{\left|r_0-\alpha(\overline{u(t)}+\varepsilon)-\beta(\overline{v(t)}+\varepsilon)-\lambda\delta\right|\right\}$.

If not, there exist $t_2>t_1$ such that $x(t_2)<\delta exp(-\omega T)$. According to the continuity of $x(t)$ with t, there exists $t^*\in(t_1,t_2)$ such that $x(t^*)=\delta$ and $x(t)<\delta$ for $t\in(t^*,t_2)$. From (3.14), (3.15) and $R_0<1$, we know $\kappa>0$, that is $r_0-\alpha(\overline{u(t)}+\varepsilon)-\beta(\overline{v(t)}+\varepsilon)-\lambda\delta>0$. For any $t\in(t^*,t_2)$, we choose anon-negative integer l such that $t_2\in(t^*+lT,t^*+(l+1)T]$. Integrating (3.13) from t^* to t_2, we get

$$
\begin{aligned}
\delta exp(-\omega T) &> x(t_2)\geq x(t^*)exp\left(\int_{t^*}^{t_2}\left(r_0-\alpha(\overline{u(t)}+\varepsilon)-\beta(\overline{v(t)}+\varepsilon)-\lambda\delta\right)dt\right)\\
&= \delta exp\left(\left(\int_{t^*}^{t^*+lh}+\int_{t^*+lh}^{t_2}\right)\left(r_0-\alpha(\overline{u(t)}+\varepsilon)-\beta(\overline{v(t)}+\varepsilon)-\lambda\delta\right)dt\right)\\
&> \delta exp\left(\int_{t^*+lT}^{t_2}\left(r_0-\alpha(\overline{u(t)}+\varepsilon)-\beta(\overline{v(t)}+\varepsilon)-\lambda\delta\right)dt\right)\\
&\geq \delta exp(-\omega t).
\end{aligned}
$$

There is a contradiction. So (3.16) is right.

In summary, we have

$$\liminf_{t\to\infty} x(t) \geq \delta exp(-\omega T).$$

□

Sufficient conditions of persistence of the population are obtained from Theorem 3.1. On the contrary, if the condition of Theorem 3.1 is false, the population may be extinct. Next we study the conditions of the extinction of the population.

4. Extinction of population

Theorem 4.1. *For model (1.1), if $R_0 \geq 1$, then the population $x(t)$ becomes extinct.*

Proof. For proving the extinction of population $x(t)$, we only prove $\lim\limits_{t\to\infty} x(t) = 0$.

Using Theorem 2.2 and the last two functions of model (1.1), we have

$$\begin{cases} \frac{dC_o(t)}{dt} \geq KC_e(t) + fC_e(t) - (g + m + d_0)C_o(t), t \neq nT \\ \frac{dC_e(t)}{dt} \geq -(-K_1M + h)C_e(t), t \neq nT \\ \triangle C_o(t) = 0, \triangle C_e(t) = \mu, t = nT. \end{cases} \quad (4.1)$$

The pulse comparison equation corresponding to the model (4.1) is

$$\begin{cases} \frac{ds(t)}{dt} = Kw(t) + fw(t) - (g + m + d_0)s(t), t \neq nT \\ \frac{dw(t)}{dt} = -(-K_1M + h)w(t), t \neq nT \\ \triangle s(t) = 0, \triangle w(t) = \mu, t = nT. \end{cases} \quad (4.2)$$

Let $(s(t), w(t))$ denote the solution of model (4.2) with the initial values $s(0) = C_o(0), w(0) = C_e(0)$.

Following the same procedure as the solving process of model (3.2), model (4.2) has a positive periodic solution as follows:

$$\begin{cases} \overline{s(t)} = s^* e^{-(g+m+d_0)(t-nT)} + \frac{\mu(K+f)\left(e^{-(g+m+d_0)(t-nT)} - e^{-(K_1M+h)(t-nT)}\right)}{(K_1M+h-g-m-d_0)\left(1-e^{-(K_1M+h)T}\right)}, \\ \overline{w(t)} = w^* e^{-(K_1M+h)(t-nT)}. \end{cases}$$

where

$$\begin{cases} s^* = \frac{\mu(K+f)\left(e^{-(g+m+d_0)T} - e^{-(K_1M+h)T}\right)}{(K_1M+h-g-m-d_0)\left(1-e^{-(K_1m+h)T}\right)\left(1-e^{-(g+m+d_0)T}\right)}, \\ w^* = \frac{\mu}{1-e^{-(K_1M+h)T}}. \end{cases}$$

And the positive periodic solution $(\overline{s(t)}, \overline{w(t)})$ of (4.2) is globally asymptotically stable.

Using the Comparison Theorem [21] and the globally asymptotically stable property of $(\overline{s(t)}, \overline{w(t)})$, there exists a positive number $t_0 > 0$, for arbitrarily small $\varepsilon_0 > 0$, and when $t > t_0$, we have

$$C_o(t) \geq s(t) \geq \overline{s(t)} - \varepsilon_0, C_e(t) \geq w(t) \geq \overline{w(t)} - \varepsilon_0. \quad (4.3)$$

For proving the extinction of the population $x(t)$ of model (1.1), the contradiction method is used. Assuming that, for arbitrarily small $\eta > 0$, when $t > t_0$, there is

$$x(t) \geq \eta. \tag{4.4}$$

Using (4.3), (4.4) and the first function of model (1.1), we get, when $t > t_0$

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \leq x(t)\Big(r_0 - \alpha(\overline{s(t)} - \varepsilon_0) - \beta(\overline{w(t)} - \varepsilon_0) - \lambda\eta\Big). \tag{4.5}$$

Setting $n_2 \in N$ and $n_2T > t_0$, and integrating (4.5) from nT to $(n+1)T$ $(n \geq n_2)$ yields to

$$\begin{aligned} x((n+1)T) &\leq x(nT)exp\left(\int_{nT}^{(n+1)T}\Big(r_0 - \alpha(\overline{s(t)} - \varepsilon_0) - \beta(\overline{w(t)} - \varepsilon_0) - \lambda\eta\Big)dt\right) \\ &= x(nT)exp\Big[(r_0 + \alpha\varepsilon_0 + \beta\varepsilon_0 - \lambda\eta)T \\ &\quad -\frac{\alpha\mu(K+f)}{(g+m+d_0)(K_1M+h)} - \frac{\beta\mu}{K_1M+h}\Big] \\ &= x(nT)exp\kappa_1 \leq x(0^+)exp(n\kappa_1), \end{aligned} \tag{4.6}$$

where

$$\kappa_1 = (r_0 + \alpha\varepsilon_0 + \beta\varepsilon_0 - \lambda\eta)T - \frac{\alpha\mu(K+f)}{(g+m+d_0)(K_1M+h)} - \frac{\beta\mu}{K_1M+h}. \tag{4.7}$$

From $R_0 \geq 1$, we know $r_0T \leq \frac{\alpha\mu(K+f)}{h(g+m+d_0)} + \frac{\beta\mu}{h}$. For given $M > 0$, we choose sufficiently small $\varepsilon_0 > 0$ and $\eta > 0$ such that

$$(r_0 + \alpha\varepsilon_0 + \beta\varepsilon_0 - \lambda\eta)T \leq \frac{\alpha\mu(K+f)}{(g+m+d_0)(K_1M+h)} + \frac{\beta\mu}{K_1M+h},$$

From (4.7), we get $\kappa_1 \leq 0$.

When $\kappa_1 = 0$, (4.6) gives $x((n+1)T) \leq 0$. From Theorem 2.1, we also get $x((n+1)T) \geq 0$, which is $x((n+1)T) = 0$. It demonstrates that the population $x(t)$ is eventually extinct.

When $\kappa_1 < 0$, (4.6) shows $x((n+1)T) \leq x(0^+)exp(n\kappa_1) \to 0 (n \to \infty)$, which is contradiction with (4.4). So there exists $t_1 > t_0$ such that $x(t_1) < \eta$.

Now we prove that, when $t > t_1$, we have

$$x(t) < \eta exp(\omega_1 T), \tag{4.8}$$

where $\omega_1 = \sup\limits_{t\geq 0}\left\{\left|r_0 - \alpha(\overline{s(t)} - \varepsilon_0) - \beta(\overline{w(t)} - \varepsilon_0) - \lambda\eta\right|\right\}$.

If not, there exists $t_2 > t_1$ such that $x(t_2) \geq \eta exp(\omega_1 T)$. Hence there exists $t^* \in (t_1, t_2)$ such that $x(t^*) = \eta$ and $x(t) > \eta$ for $t \in (t^*, t_2)$. From (4.6), (4.7) and $R_0 \geq 1$, we know $\kappa_1 \leq 0$, that is $r_0 - \alpha\left(\overline{s(t)} - \varepsilon_0\right) - \beta\left(\overline{w(t)} - \varepsilon_0\right) - \lambda\eta \leq 0$. We choose anon-negative integer l_0 such that $t_2 \in (t^* + l_0T, t^* + (l_0+1)T]$. Integrating (4.5) from t^* to t_2 leads to

$$\eta exp(\omega_1 T) \leq x(t_2) < x(t^*)exp\left(\int_{t^*}^{t_2}\Big(r_0 - \alpha(\overline{s(t)} - \varepsilon_0) - \beta(\overline{w(t)} - \varepsilon_0) - \lambda\eta\Big)dt\right)$$

$$
\begin{aligned}
&= \eta exp\left(\left(\int_{t^*}^{t^*+l_0T} + \int_{t^*+l_0T}^{t_2}\right)\left(r_0 - \alpha(\overline{s(t)} - \varepsilon_0) - \beta(\overline{w(t)} - \varepsilon_0) - \lambda\eta\right)dt\right) \\
&\le \eta exp(-\omega_1 T).
\end{aligned}
$$

There is a contradiction. Therefore (4.8) is right. As the arbitrariness of η, we have $\lim_{t\to\infty} x(t) = 0$. □

Above we present the theoretical results of the model. Next we use MATLAB to draw the diagram of model (1.1) to verify the correctness of the theoretical results.

5. Uniform persistence of population

Pollutant regularly input towards the environment is directly related to the survival of the population $x(t)$. Theorems 3.1 and 4.1 give the sufficient conditions for survival and extinction of population $x(t)$. Using numerical simulation, we analysis the influence of T and μ on the survival of population $x(t)$. Assuming $r_0 = 0.6, \alpha = \beta = 0.1, \lambda = 0.2, K = 0.2, f = 0.1, g = m = 0.1, d_0 = 0.8, K_1 = 0.1, g_1 = \alpha_1 = \beta_1 = 0.05, d_1 = 0.1, h = 0.1, x(0) = 1, C_o(0) = 0.5, C_e(t) = 0.8$.

Let $\mu = 1, T = 3$, we can get $R_0 = 0.73 < 1$, then the conditions of Theorem 3.1 are satisfied, population $x(t)$ is survived. As shown in Figure 1.

Let $\mu = 2, T = 3$, we can get $R_0 = 1.44 \geq 1$, then the conditions of Theorem 4.1 are satisfied, population $x(t)$ is extinct. As shown in Figure 2.

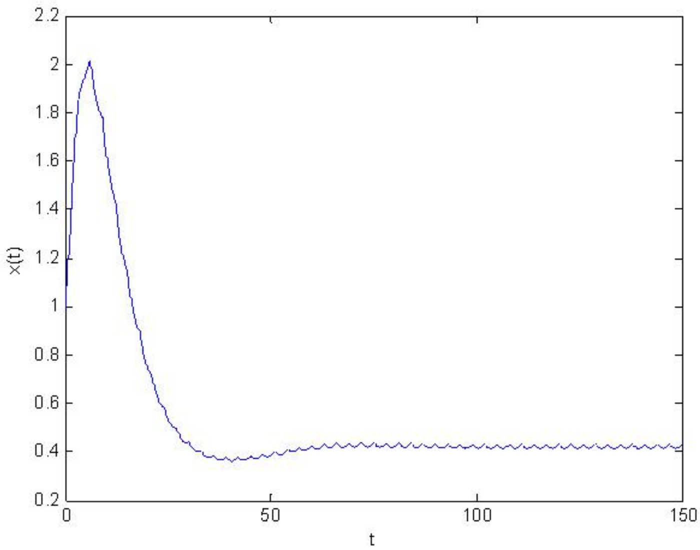

Figure 1. Existence of $x(t)$ when $\mu = 1, T = 3$.

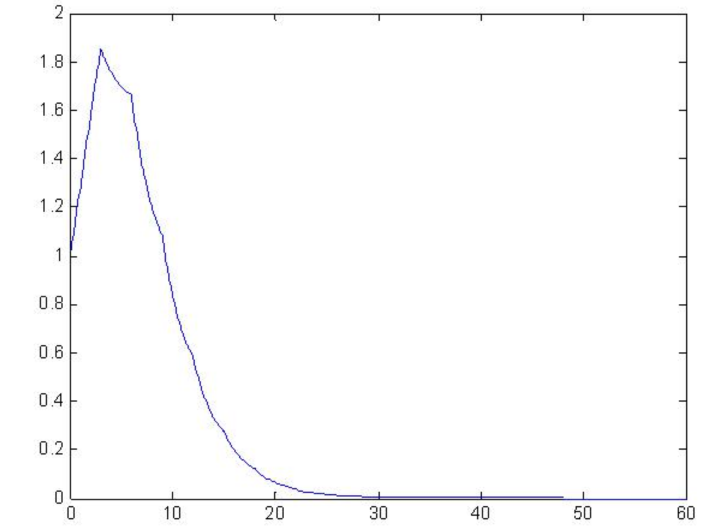

Figure 2. Extinction of $x(t)$ when $\mu = 2, T = 3$.

From Figures 1 and 2, we can observe that when T is the same and μ increases, population $x(t)$ changes from survival to extinction.

Let $\mu = 1, T = 2.5$, we can get $R_0 = 0.88 < 1$, then the conditions of Theorem 3.1 are satisfied, population $x(t)$ is survived. As shown in Figure 3.

Let $\mu = 1, T = 1$, we can get $R_0 = 2.2 \geq 1$, then the conditions of Theorem 4.1 are satisfied, population $x(t)$ is extinct. As shown in Figure 4.

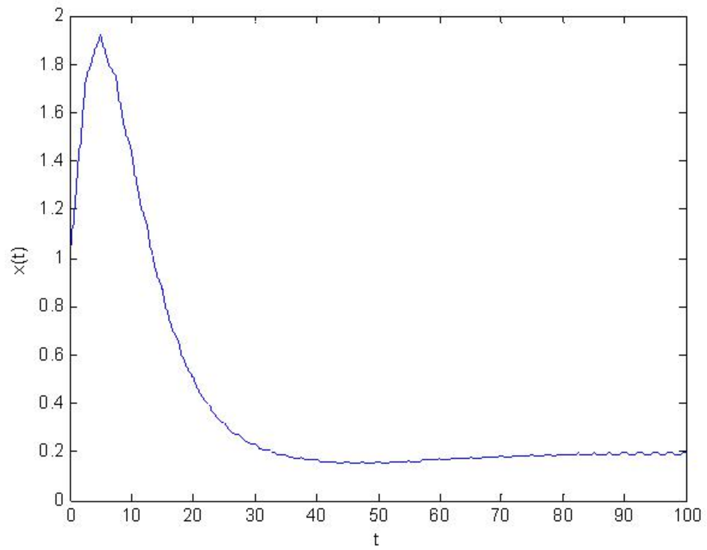

Figure 3. Existence of $x(t)$ when $\mu = 1, T = 2.5$.

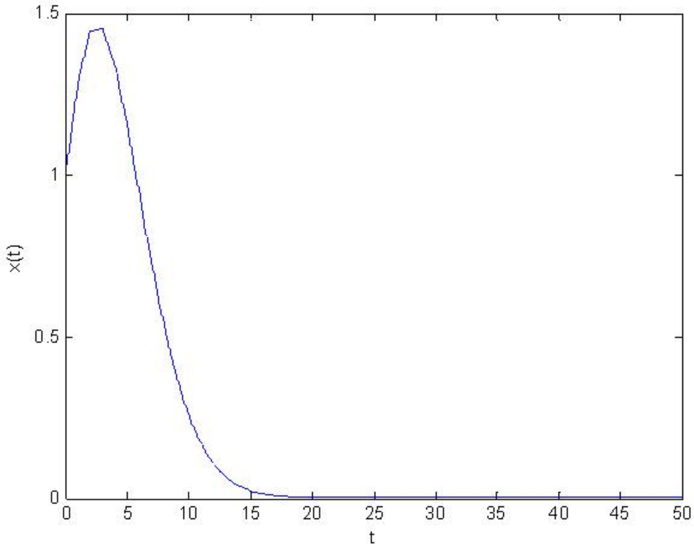

Figure 4. Extinction of $x(t)$ when $\mu = 1, T = 1$.

From Figures 3 and 4, we can observe that when μ is the same and T lessens, population $x(t)$ changes from survival to extinction.

6. Conclusion

In this paper, we study a single-population model with pulse input of environmental toxin in a small polluted environment. We obtain the conditions and a threshold of extinction and persistence of the population. The threshold is $R_0 = \mu\left(\frac{\alpha\mu(K+f)}{h(g+m+d_0)} + \frac{\beta}{h}\right)\Big/r_0T$, that is, when $R_0 < 1$, the population is persistence. When $R_0 \geq 1$, the population is extinction.

The degree of pollution of the environment is directly related to survival and extinction of the population. As the definition of threshold, if the toxicant input amount is constant, in order to ensure the survival of the population, we must extend the period of the exogenous input of toxicant. If the period of the exogenous input of toxicant discharge is unchanging, in view of ensuring the survival of population, we must decrease the toxicant input amount. At the same time, the results of numerical simulation demonstrate the influence of the period and amount of the exogenous input of toxicant on survival and extinction of populations.

Due to the limitations of the population to survive, the pollution problem in a small environment in this paper is more consistent with real problem than that in a big environment. Comparing the results of two types of environment, we can note that when the toxicant input amount is the same, the threshold of extinction and persistence of the population in a small environment is smaller and the survival condition of population weakens. So in order to make the population to survive in a small environment, we only reduce the amount of discharge toxins and extend the time of emission. In real life, when facing pollutants from the environment, young population and adult population have different reactions. Considering the population with the different age structure has more practical significance, so this issue can be studied as a follow-up research on the basis of the current research work.

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Conflict of interest

The authors declare there is no conflict of interest.

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