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Research article

Hopf bifurcation, stability switches and chaos in a prey-predator system with three stage structure and two time delays

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Abstract: A three stage-structured prey-predator model with digestion delay and density dependent delay for the predator is investigated. The stability of the equilibrium point and the Hopf bifurcation of the system by choosing time delay as a bifurcation parameter in five cases are considered, and the conditions for the positive equilibrium occurring local Hopf bifurcation are given in each case. Numerical results show that delayed system considered has not only periodic oscillation, stability switches but also chaotic oscillation, even unbounded oscillation. Finally, delays induced Hopf bifurcation, stability switches, complicated dynamic behaviors of the system are discussed in detail.

Keywords: Prey-predator system; time delays; Hopf bifurcation; stability switches; chaos

1. Introduction

In the real world, many species have two distinctive stages—immature and mature, of life in their lives. A delayed single-specie model with two stages is introduced by Aiello and Freedman [1, 2] in 1990. A single-specie model with stage-structured is considered by Wang and Chen [3] in 1997, and found that there exists a stable periodic solution in that model. The single-specie model with two stage-structured have been received much attentions and summarized by Liu et al. [4]. In these papers, the authors assume that the species have two different stages—immature and mature, and only the mature member can reproduce themselves. But, some species go through three different life stages—immature, mature and old. A single-specie model with delay and three different life history stages and cannibalism has investigated by Gao [5], and shown that there would be a stability switches for the positive equilibrium when time delays are increased from zero. A nonautonomous predator-prey system (1.1)

$$\begin{cases} x'(t) = x(t)[a(t) - b(t)x(t) - c(t)y_2(t) - d(t)y_3(t)], \\ y_1'(t) = \alpha(t)x(t)y_3(t-\tau) - \beta_1(t)y_1(t) - \gamma_1(t)y_1(t), \\ y_2'(t) = \gamma_1(t)y_1(t) - \beta_2(t)y_2(t) - \gamma_2(t)y_2(t) - \eta_1(t)y_2^2(t), \\ y_3'(t) = \gamma_2(t)y_2(t) - \eta_2(t)y_3(t), \end{cases} \tag{1.1}$$

with three-stage-structured and time delay has considered by Yang and Shi [6], and the conditions for the existence of the positive periodic solution are obtained.

Time delays play an important role in population dynamics, which can cause the loss of stability of the equilibrium, bifurcate various types of periodic solutions, unbounded solutions and even chaotic solutions. Time delay is common in biodynamic systems [7], and harmful delays can cause fluctuation(period solution) in population density, and which would make the system subject to chaotic oscillation, unstable oscillation and extinct [8–15], even the time delay is very small.

Recently, a prey-predator model (1.2)

$$\begin{cases} x_1'(t) = \alpha x_2(t) - (\gamma_1 + \Omega)x_1(t) - \eta x_1^2(t) - Ex_1(t)y(t-\tau_2), \\ x_2'(t) = \Omega x_1(t) - (\theta_1 + a)x_2(t), \\ x_3'(t) = ax_2(t) - bx_3(t), \\ y'(t) = kEx_1(t-\tau_1)y(t) - dy(t) - fy^2(t), \end{cases} \tag{1.2}$$

with three stage structure and time delay is studied in [16]. The conditions for the positive equilibrium occurring local and global Hopf bifurcation are obtained. And the properties (direction, stability, etc) of the local Hopf bifurcation are analyzed. Furthermore, a prey-predator system (1.3)

$$\begin{cases} x_1'(t) = \alpha x_2(t) - (\gamma_1 + \Omega)x_1(t) - \eta x_1^2(t) - Ex_1(t)y(t), \\ x_2'(t) = \Omega x_1(t) - (\theta_1 + a)x_2(t), \\ x_3'(t) = ax_2(t) - bx_3(t), \\ y'(t) = y(t)[kEx_1(t) - d - fy(t-\tau)], \end{cases} \tag{1.3}$$

with three stage structure and predator density dependent delay has been considered in [17, 18], by choosing time delay as a bifurcation parameter, the local and global Hopf bifurcation are investigated. The authors focus on the existence of global Hopf bifurcation in systems (1.2) and (1.3), by using the global Hopf bifurcating theorem for general functional differential equations which introduced by Wu [19]. Meanwhile, the harsh conditions for the positive equilibrium occurring local Hopf bifurcation are obtained, i.e. there are only a pair of pure imaginary roots for the characteristic equation about the positive equilibrium.

Note that, the sufficient conditions for the existence of local Hopf bifurcation of systems (1.2) and (1.3) are $C_3^1 : f\eta < KE^2$, $C_3^2 : f\eta > KE^2$, respectively, where K, E, η, f are positive. K is the rate of conversing prey into predator and E is the predation coefficient for predator population. η is the density dependent coefficient for prey populations, reflecting the competition effect between prey populations; and f is the density dependent coefficient for predator population, reflecting the competition effect between predator populations; respectively. But, the conditions C_3^1 and C_3^2 cannot hold at the same time. Then, one of them holds for any parameter values of the system exclude the special case $f\eta = KE^2$, if both digestion delay and density dependent delay considered in a new model. Therefore, there would be a natural Hopf bifurcation for the system with two different time delays τ_1

and τ_2 without any conditions for the values of the parameters. And, how does the dynamic behavior go when $\tau_1 = \tau_2 = \tau$? Does there exist a bifurcating periodic solution, stability switches or other complex dynamic behaviors, if there exist at least a pair of pure imaginary roots for the characteristic equation about the positive equilibrium?

Motivation by aforementioned observations, we consider the following prey-predator model with three stage structure and two time delays:

$$\begin{cases} x_1'(t) = \alpha x_2(t) - x_1(t)(\gamma_1 + \Omega + \eta x_1(t) + Ey(t)), \\ x_2'(t) = \Omega x_1(t) - \theta_1 x_2(t) - a x_2(t), \\ x_3'(t) = a x_2(t) - b x_3(t), \\ y'(t) = y(t)(KEx_1(t-\tau_1) - d - fy(t-\tau_2)), \end{cases} \tag{1.4}$$

where $x_1'(t), x_2'(t), x_3'(t)$ are the change of density of the prey population in the three stages of immature, mature and old, and $y'(t)$ is the change of density of the predator population at time t, respectively. All of the parameters are positive. For prey population, α is the birth rate; γ_1, θ_1, b are the death rate of the immature, mature and old stages; Ω and a are the maturity rate and ageing rate, respectively. For predator population, d is the death rate; τ_1 and τ_2 are digestion delay [16] and density dependent delay [17,18], respectively. The delays τ_1 and τ_2 in system (1.4) can be regarded as a digestion time(or conversion time) and density dependent time of the predators. For τ_1, when the predator catches the prey at time t, it needs τ_1 time to convert the energy of the prey into its own energy. For τ_2, the competition between predator populations has a time delay τ_2, as in classical delayed Logistic equation $x'(t) = rx(t)[1 - x(t-\tau)/K]$. That is to say, the change rate of the predators $y'(t)$ depends on the number of immature preys and of predators present at some previous time $x_1(t-\tau_1)$ and $y(t-\tau_2)$, respectively.

From the third equation of system (1.4), which is a linear nonhomogeneous equation about $x_3(t)$, then the asymptotic behavior of $x_3(t)$ is dependent on $x_2(t)$. Therefore, we only need to consider the following subsystem

$$\begin{cases} x_1'(t) = \alpha x_2(t) - x_1(t)(\gamma + \eta x_1(t) + Ey(t)), \\ x_2'(t) = \Omega x_1(t) - \theta x_2(t), \\ y'(t) = y(t)(KEx_1(t-\tau_1) - d - fy(t-\tau_2)), \end{cases} \tag{1.5}$$

where $\gamma = \gamma_1 + \Omega, \theta = \theta_1 + a$. And the initial conditions for system (1.5) are

$$x_i(t) = \varphi_i(t) \geq 0 (i = 1, 2), y(t) = \varphi_3(t) \geq 0, t \in [-\tau_{\max}, 0], \tau_{\max} = \max\{\tau_1, \tau_2\}.$$

The organization of this paper is as follows. We consider the stability of the equilibrium point and the existence of Hopf bifurcation, by choosing time delays as a bifurcation parameter in five different cases, firstly. And, in section 2, the conditions for the positive equilibrium occurring local Hopf bifurcation are obtained in each case. Secondly, in section 3, some numerical examples are given to support the theoretical results, which show that the delayed system considered has not only periodic oscillation, stability switches but also chaotic oscillation, even unbounded oscillation under some parameter sets of values. Finally, in section 4, delays induced Hopf bifurcation, stability switches, complicated dynamic behaviors of the system are analyzed in detail.

2. Local stability analysis and Hopf bifurcation

2.1. Local stability analysis

For system (1.5), if condition $C_1 : \alpha\Omega - \gamma\theta > 0$ holds, there're two boundary equilibrium $E_0 = (0, 0, 0), E_1(\hat{x}_1, \hat{x}_2, 0)$; and if condition $C_2 : KEx_1^* - d > 0$ holds, a unique positive equilibrium $E_2(x_1^*, x_2^*, y^*)$ exists, where

$$\hat{x}_1 = \frac{\alpha\Omega - \gamma\theta}{\eta\theta}, \hat{x}_2 = \frac{\Omega}{\theta}x_1, x_1^* = \frac{f(\alpha\Omega - \gamma\theta) + dE\theta}{(KE^2 + \eta f)\theta}, x_2^* = \frac{\Omega}{\theta}x_1^*, y^* = \frac{KEx_1^* - d}{f}.$$

Let $X(t) = (x_1(t), x_2(t), y(t))$, and $\bar{E} = (\bar{x}_1, \bar{x}_2, \bar{y})$ be any arbitrary equilibrium. The linearized equation about $\bar{E}$ is

$$X'(t) = AX(t) + B_1X(t - \tau_1) + B_2X(t - \tau_2), \tag{2.1}$$

where

$$A = \begin{pmatrix} -\gamma - 2\eta\bar{x}_1 - \bar{y}E & \alpha & -\bar{x}_1E \\ \Omega & -\theta & 0 \\ 0 & 0 & \bar{x}_1KE - d - \bar{y}f \end{pmatrix},$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ K\bar{y}E & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\bar{y}f \end{pmatrix},$$

and the characteristic equation about it is given by

$$H(\lambda, \tau_1, \tau_2) = \det(A + B_1e^{-\lambda\tau_1} + B_2e^{-\lambda\tau_2} - \lambda I) = 0. \tag{2.2}$$

Note that, $\bar{y} = 0$ for the boundary equilibrium E_0 and E_1, then the characteristic equation about E_0 and E_1 are same as in [16, 17]. Therefore, we obtain following lemma.

Lemma 2.1. *(i) If $\gamma\theta > \alpha\Omega$ then E_0 is local stable. And, if $\gamma\theta < \alpha\Omega$ then E_0 is unstable and E_1 exists.*

(ii) If $KE\hat{x}_1 < d$ then E_1 is local stable. And if $KE\hat{x}_1 > d$ then E_1 is unstable and E_2 exists.

2.2. Existence of local Hopf bifurcation

From (2.2), one obtain the characteristic equation about the positive equilibrium E_2:

$$H(\lambda, \tau_1, \tau_2) = M(\lambda) + N(\lambda)e^{-\lambda\tau_1} + P(\lambda)e^{-\lambda\tau_2} = 0, \tag{2.3}$$

where

$$\begin{aligned} &M(\lambda) = \lambda^3 + m_2\lambda^2 + m_1\lambda + m_0, \\ &N(\lambda) = n_2\lambda^2 + n_1\lambda + n_0, \\ &P(\lambda) = p_2\lambda^2 + p_1\lambda + p_0, \\ &m_2 = \gamma + Ey^* + \theta + 2\eta x_1^*, m_1 = \theta\eta x_1^*, m_0 = 0, \\ &n_2 = 0, n_1 = KE^2x_1^*y^*, n_0 = KE^2x_1^*y^*\theta, \\ &p_2 = fy^*, p_1 = fy^*(\gamma + Ey^* + \theta + 2\eta x_1^*), p_0 = fy^*\theta\eta x_1^*. \end{aligned}$$

When $\tau_1 = \tau_2 = 0$, (2.3) becomes to

$$H(\lambda, 0, 0) = \lambda^3 + h_2\lambda^2 + h_1\lambda + h_0 = 0, \tag{2.4}$$

where

$$\begin{aligned} h_2 &= \gamma + 2\eta x_1^* + Ey^* + \theta + fy^* > 0,\\ h_1 &= \theta(\eta x_1^* + fy^*) + fy^*(\gamma + 2\eta x_1^* + Ey^*) + KE^2 x_1^* y^* > 0,\\ h_0 &= \theta(f\eta + KE^2)x_1^* y^* > 0. \end{aligned}$$

By Routh-Hurwits criterion, all roots of (2.4) have negative real parts, since

$$h_2h_1 - h_0 > \theta\{[2\eta f x_1^* y^* + Ey^*(d + 2fy^*)] - (f\eta + KE^2)x_1^* y^*\} > 0.$$

Meanwhile, E_2 is local stable. We investigate the Hopf bifurcation about E_2 in following five cases.

2.2.1. The case $\tau_1 > 0, \tau_2 \equiv 0$

The equation (2.3) is

$$H(\lambda, \tau_1, 0) = M_{\tau_1}(\lambda) + N_{\tau_1}(\lambda)e^{-\lambda\tau_1} = 0, \tag{2.5}$$

where

$$M_{\tau_1}(\lambda) = M(\lambda) + P(\lambda), N_{\tau_1}(\lambda) = N(\lambda).$$

Suppose $\lambda = \mathrm{i}\omega(\omega > 0)$ is a pure imaginary root of (2.5) and separating the real and imaginary parts, one obtain

$$\begin{cases} (m_2 + p_2)\omega^2 - (m_0 + p_0) = (n_0 - n_2\omega^2)\cos\omega\tau_1 + n_1\omega\sin\omega\tau_1,\\ \omega^3 - (m_1 + p_1)\omega = n_1\omega\cos\omega\tau_1 - (n_0 - n_2\omega^2)\sin\omega\tau_1. \end{cases}$$

and

$$(n_0 - n_2\omega^2)^2 + n_1^2\omega^2 = [(m_2 + p_2)\omega^2 - (m_0 + p_0)]^2 + [\omega^3 - (m_1 + p_1)\omega]^2.$$

That is

$$F_{\tau_1}(\varpi) = \varpi^3 + f_{12}\varpi^2 + f_{11}\varpi + f_{10} = 0, \tag{2.6}$$

where

$$\begin{aligned} \varpi = \omega^2, f_{12} &= (m_2 + p_2)^2 - 2(m_1 + p_1) - n_2^2 > 0,\\ f_{11} &= (m_1 + p_1)^2 + 2n_2n_0 - n_1^2 - 2(m_2 + p_2)(m_0 + p_0),\\ f_{10} &= (m_0 + p_0)^2 - n_0^2 = \theta x_1^* y^*(m_0 + p_0 + n_0)(f\eta - KE^2). \end{aligned} \tag{2.7}$$

If condition $C_3^1 : f\eta < KE^2$ holds, from (2.7) we know that (2.6) has at least one positive root. Without loss of generality, we assume that (2.6) has three different positive roots, denoted by $\omega_k = \sqrt{\varpi_k}(k = 1, 2, 3)$. And, one have

$$\cos\omega_k\tau_1 = \frac{[(m_2 + p_2)\omega_k^2 - (m_0 + p_0)](n_0 - n_2\omega_k^2) + n_1\omega_k[\omega_k^3 - (m_1 + p_1)\omega_k]}{(n_0 - n_2\omega_k^2)^2 + (n_1\omega_k)^2} \triangleq F_{\omega_k}.$$

Thus

$$\tau_{1k}^{(n)} = \frac{1}{\omega_k}\cos^{-1}[F_{\omega_k}] + \frac{2n\pi}{\omega_k}, k = 1,2,3; n = 0,1,2,\cdots, \tag{2.8}$$

and the direction of $\tau_{1k}^{(n)}$ passing through the imaginary axis [20] when $\omega = \omega_k$ is determined by

$$\text{sign}\left[\frac{\text{dRe}(\lambda(\tau))}{\text{d}\tau}\Big|_{\tau=\tau_{1k}^{(n)}}\right] = \text{sign}\left[F'_{\tau_1}(\varpi_k)\big|_{\varpi_k=\omega_k^2}\right] = \text{sign}\left(\Delta_{\tau_1}^k\right).$$

Then $\text{sign}\left(\Delta_{\tau_1}^{\text{k}}\right) \neq 0$, since $\varpi_k (k = 1,2,3)$ are three distinct positive roots of (2.6). Therefore, system (1.5) undergoes a local Hopf bifurcation at E_2 when $\tau_1 = \tau_{1k}^{(n)}$, by the Hopf bifurcation theorem for functional differential equations [21]. Furthermore, system (1.5) undergoes a local Hopf bifurcation at E_2 and $\text{sign}\left(\Delta_{\tau_1}^1\right) = 1$, if $f_{11} > 0$ and condition $C_3^1 : f\eta < KE^2$ hold. Then, (2.6) has a unique positive root ω_1, and $\tau_1 = \tau_1^{(n)} (n = 0,1,2,\cdots)$ corresponding to ω_1.

Define

$$S_{\tau_1} = \{\tau_1 | H(\lambda,\tau_1,0) = 0, \text{Re}(\lambda) < 0\}, \tau_{10} = \min\{\tau_{1k}^{(n)} | 1 \le k \le 3, n = 0,1,2,\cdots\},$$

when $\tau_1 \in S_{\tau_1}$, E_2 is local stable. Note that, if (2.6) have more than one positive roots, there would be finite stability switches when time delay τ_1 passing through the critical points $\tau_1 = \tau_{1k}^{(n)} (k = 1,2,3; n = 0,1,2,\cdots)$ and $[0,\tau_{10}) \subseteq S_{\tau_1}$. If (2.6) has only one positive root, there is no stability switches when time delay τ_1 passing through the critical points $\tau_1 = \tau_1^{(n)} (n = 1,2,\cdots)$ and $S_{\tau_1} = [0,\tau_1^{(0)})$.

Theorem 2.1 *(i) Suppose (2.6) has at least one positive roots denoted by* $\varpi_k (1 \le k \le 3)$. *There exists a nonempty set* S_{τ_1} *and* $[0,\tau_{10}) \subseteq S_{\tau_1}$, *when* $\tau_1 \in S_{\tau_1}$ *the positive equilibrium* E_2 *of system (1.5) is local stable. There is a Hopf bifurcation for system (1.5) at* E_2 *when* $\tau_1 = \tau_{1k}^{(n)} (k = 1,2,3; n = 0,1,2,\cdots)$.

(ii) Suppose (2.6) has only one positive root denoted by ϖ_1. *There exists a nonempty set* S_{τ_1} *and* $S_{\tau_1} = [0,\tau_1^{(0)})$, *when* $\tau_1 \in S_{\tau_1}$ *the positive equilibrium* E_2 *of system (1.5) is local stable and unstable when* $\tau_1 > \tau_1^{(0)}$. *There is a Hopf bifurcation for system (1.5) at* E_2 *when* $\tau_1 = \tau_1^{(n)} (n = 0,1,2,\cdots)$.

Note 2.1 *If* $f_{11} > 0$ *and condition* $C_3^1 : f\eta < KE^2$ *hold, then (2.6) have only one positive root, and this is a special case of Theorem 2.1 (ii). The local and global Hopf bifurcation in this special situation have been considered in [16]. Meanwhile, theorem 2.1 generalizes the result about local Hopf bifurcation in [16].*

2.2.2. The case $\tau_1 \equiv 0, \tau_2 > 0$

The equation (2.3) becomes to

$$H(\lambda,0,\tau_2) = M_{\tau_2}(\lambda) + N_{\tau_2}(\lambda)e^{-\lambda\tau_2} = 0, \tag{2.9}$$

where

$$M_{\tau_2}(\lambda) = M(\lambda) + N(\lambda), N_{\tau_2}(\lambda) = P(\lambda).$$

Suppose $\lambda = \text{i}\omega (\omega > 0)$ is a pure imaginary root of (2.9), similar to the case 2.2.1, one have

$$F_{\tau_2}(\varpi) = \varpi^3 + f_{22}\varpi^2 + f_{21}\varpi + f_{20} = 0, \tag{2.10}$$

where

$$\varpi = \omega^2, f_{22} = (m_2 + n_2)^2 - 2(m_1 + n_1) - p_2^2,$$
$$f_{21} = (m_1 + n_1)^2 + 2p_2p_0 - p_1^2 - 2(m_2 + n_2)(m_0 + n_0),$$
$$f_{20} = (m_0 + n_0)^2 - p_0^2 = \theta x_1^* y^*(m_0 + p_0 + n_0)(KE^2 - f\eta). \tag{2.11}$$

From (2.11) we know that (2.10) has at least one positive root, if condition $C_3^2 : f\eta > KE^2$ hold. Without loss of generality, we assume that (2.10) has three distinct positive roots, denoted by $\omega_k = \sqrt{\varpi_k}(k = 1, 2, 3)$ and we obtain

$$\cos\omega_k\tau_2 = \frac{[(m_2 + n_2)\omega_k^2 - (m_0 + n_0)](p_0 - p_2\omega_k^2) + p_1\omega_k[\omega_k^3 - (m_1 + n_1)\omega_k]}{(p_0 - p_2\omega_k^2)^2 + (p_1\omega_k)^2} \triangleq F_{\omega_k}.$$

Thus

$$\tau_{2k}^{(n)} = \frac{1}{\omega_k}\cos^{-1}[F_{\omega_k}] + \frac{2n\pi}{\omega_k}, k = 1, 2, 3; n = 0, 1, 2, \cdots, \tag{2.12}$$

and the direction of $\tau_{2k}^{(n)}$ passing through the imaginary axis [20] when $\omega = \omega_k$ is determined by

$$\text{sign}\left[\left.\frac{\text{dRe}(\lambda(\tau))}{\text{d}\tau}\right|_{\tau=\tau_{2k}^{(n)}}\right] = \text{sign}\left[F'_{\tau_2}(\varpi_k)\big|_{\varpi_k=\omega_k^2}\right] = \text{sign}\left(\Delta_{\tau_2}^k\right).$$

System (1.5) undergoes a Hopf bifurcation at E_2 when $\tau_2 = \tau_{2k}^{(n)}$ since $\text{sign}\left(\Delta_{\tau_2}^k\right) \neq 0$. Furthermore, if $f_{21} > 0, f_{22} > 0$ and condition $C_3^2 : f\eta > KE^2$ hold, then (2.10) has a unique positive root ω_1, and $\tau_2 = \tau_2^{(n)}(n = 0, 1, 2, \cdots)$ corresponding to ω_1. There is a Hopf bifurcation at E_2 since $\text{sign}\left(\Delta_{\tau_2}^1\right) = 1$.

Define

$$S_{\tau_2} = \{\tau_2 | H(\lambda, 0, \tau_2) = 0, \text{Re}(\lambda) < 0\}, \tau_{20} = \min\{\tau_{2k}^{(n)} | 1 \leq k \leq 3, n = 0, 1, 2, \cdots\}.$$

Theorem 2.2 *(i) Suppose (2.10) has at least one positive roots denoted by $\varpi_k(1 \leq k \leq 3)$. There exists a nonempty set S_{τ_2} and $[0, \tau_{20}) \subseteq S_{\tau_2}$, when $\tau_2 \in S_{\tau_2}$ the positive equilibrium E_2 of system (1.5) is local stable. There is a Hopf bifurcation for system (1.5) at E_2 when $\tau_2 = \tau_{2k}^{(n)}(k = 1, 2, 3; n = 0, 1, 2, \cdots)$.*

(ii) Suppose (2.10) has only one positive root denoted by ϖ_1. There exists a nonempty set S_{τ_2} and $S_{\tau_2} = [0, \tau_2^{(0)})$, when $\tau_2 \in S_{\tau_2}$ the positive equilibrium E_2 of (1.5) is local stable and unstable when $\tau_2 > \tau_2^{(0)}$. There is a Hopf bifurcation for system (1.5) at E_2 when $\tau_2 = \tau_2^{(n)}(n = 0, 1, 2, \cdots)$.

Note 2.2 *If $f_{21} > 0, f_{22} > 0$ and condition $C_3^2 : f\eta > KE^2$ hold, then (2.10) has only one positive root, and this is a special case of Theorem 2.2 (ii). The local and global Hopf bifurcation in this special situation have been considered in [17, 18]. Meanwhile, theorem 2.2 generalizes the result about local Hopf bifurcation in [17].*

2.2.3. The case $\tau_1 = \tau_2 = \tau > 0$

The equation (2.3) is

$$H(\lambda, \tau, \tau) = M_\tau(\lambda) + N_\tau(\lambda)e^{-\lambda\tau} = 0, \tag{2.13}$$

where

$$M_\tau(\lambda) = M(\lambda), N_\tau(\lambda) = P(\lambda) + N(\lambda).$$

Suppose $\lambda = i\omega(\omega > 0)$ is a pure imaginary root of (2.13), similar to the case 2.2.1, we have

$$F_\tau(\varpi) = \varpi^3 + f_{32}\varpi^2 + f_{31}\varpi + f_{30} = 0, \tag{2.14}$$

where

$$\varpi = \omega^2, f_{32} = m_2^2 - 2m_1 - (n_2 + p_2)^2,$$
$$f_{31} = m_1^2 + 2(p_2 + n_2)(p_0 + n_0) - (p_1 + n_1)^2 - 2m_2 m_0,$$
$$f_{30} = m_0^2 - (p_0 + n_0)^2 = -(p_0 + n_0)^2 < 0.$$

(2.14) has at least one positive root since $f_{30} < 0$. Without loss of generality, we assume that (2.14) has three different positive roots, denoted by $\omega_k = \sqrt{\varpi_k}(k = 1, 2, 3)$ and we get

$$\cos\omega_k\tau = \frac{(m_2\omega_k^2 - m_0)[p_0 + n_0 - (p_2 + n_2)\omega_k^2] + (p_1 + n_1)\omega_k(\omega_k^3 - m_1\omega_k)}{[(p_0 + n_0) - (p_2 + n_2)\omega_k^2]^2 + [(p_1 + n_1)\omega_k]^2} \triangleq F_{\omega_k}.$$

Thus

$$\tau_k^{(n)} = \frac{1}{\omega_k}\cos^{-1}[F_{\omega_k}] + \frac{2n\pi}{\omega_k}, k = 1, 2, 3; n = 0, 1, 2, \cdots, \tag{2.15}$$

and the direction of $\tau_k^{(n)}$ passing through the imaginary axis [20] when $\omega = \omega_k$ is determined by

$$\text{sign}\left[\left.\frac{\mathrm{dRe}(\lambda(\tau))}{\mathrm{d}\tau}\right|_{\tau=\tau_k^{(n)}}\right] = \text{sign}\left[\left.F_\tau'(\varpi_k)\right|_{\varpi_k=\omega_k^2}\right] = \text{sign}\left(\Delta_\tau^k\right).$$

System (1.5) undergoes a Hopf bifurcation at E_2 when $\tau = \tau_k^{(n)}$. Furthermore, if $f_{31} > 0, f_{32} > 0$ hold, then (2.14) has a unique positive root ω_1, and $\tau = \tau^{(n)}(n = 0, 1, 2, \cdots)$ corresponding to ω_1. There is a Hopf bifurcation at the positive equilibrium E_2 since $\text{sign}\left(\Delta_\tau^1\right) = 1$.

Define

$$S_\tau = \{\tau | H(\lambda, \tau, \tau) = 0, \text{Re}(\lambda) < 0\}, \tau_0 = \min\{\tau_k^{(n)} | 1 \le k \le 3, n = 0, 1, 2, \cdots\}.$$

Theorem 2.3 *(i) Suppose (2.14) has at least one positive roots denoted by $\varpi_k(1 \le k \le 3)$. There exists a nonempty set S_τ and $[0, \tau_0) \subseteq S_\tau$, when $\tau \in S_\tau$ the positive equilibrium E_2 of system (1.5) is local stable. There is a Hopf bifurcation for system (1.5) at E_2 when $\tau = \tau_k^{(n)}(k = 1, 2, 3; n = 0, 1, 2, \cdots)$.*

(ii) Suppose (2.14) has only one positive root denoted by ϖ_1. There exists a nonempty set S_τ and $S_\tau = [0, \tau^{(0)})$, when $\tau \in S_\tau$ the positive equilibrium E_2 of system (1.5) is local stable and unstable when $\tau > \tau^{(0)}$. There is a Hopf bifurcation for system (1.5) at E_2 when $\tau = \tau^{(n)}(n = 0, 1, 2, \cdots)$.

Note 2.3 *If $f_{32} > 0, f_{31} > 0$ hold, then (2.14) has only one positive root, and this is a special case of Theorem 2.3 (ii).*

2.2.4. The case $\tau_1 > 0$ and fixed $\tau_2 \in S_{\tau_2}$

The characteristic equation about E_2 becomes to

$$H(\lambda, \tau_1, \tau_2) = \left(M(\lambda) + P(\lambda)e^{-\lambda\tau_2}\right) + N(\lambda)e^{-\lambda\tau_1} = 0, \tag{2.16}$$

Suppose $\lambda = \mathrm{i}\omega(\omega > 0)$ is a pure imaginary root of (2.16), similar to the case 2.2.1, one have

$$\begin{cases} A_1 + B_1 \cos\omega\tau_2 - C_1 \sin\omega\tau_2 = -E_1 \cos\omega\tau_1 + F_1 \sin\omega\tau_1, \\ D_1 - B_1 \sin\omega\tau_2 - C_1 \cos\omega\tau_2 = E_1 \sin\omega\tau_1 + F_1 \cos\omega\tau_1, \end{cases}$$

where

$$A_1 = m_2\omega^2 - m_0, B_1 = p_2\omega^2 - p_0, C_1 = p_1, D_1 = \omega^3 - m_1\omega, E_1 = n_2\omega^2 - n_0, F_1 = n_1\omega.$$

And

$$F_{\tau_1(\tau_2)}(\omega) = \omega^6 + f_{45}\omega^5 + f_{44}\omega^4 + f_{43}\omega^3 + f_{42}\omega^2 + f_{41}\omega + f_{40} = 0, \tag{2.17}$$

where

$$\begin{aligned} f_{45} &= -2p_2 \sin\omega\tau_2, \\ f_{44} &= m_2^2 - 2m_1 - n_2^2 + p_2^2 + 2(m_2p_2 - p_1)\cos\omega\tau_2, \\ f_{43} &= 2(p_0 + m_1p_2 - m_2p_1)\sin\omega\tau_2, \\ f_{42} &= m_1^2 - 2m_2m_0 + 2n_2n_0 - n_1^2 + p_1^2 - 2p_2p_0 + 2(p_1m_1 - p_0m_2 - m_0p_2)\cos\omega\tau_2, \\ f_{41} &= 2(m_0p_1 - p_0m_1)\sin\omega\tau_2, \\ f_{40} &= p_0^2 + m_0^2 + 2p_0m_0\cos\omega\tau_2 - n_0^2. \end{aligned}$$

Assumed that condition $C_3^1 : f\eta < KE^2$ holds, then

$$F_{\tau_1(\tau_2)}(0) = f_0 = (m_0 + p_0)^2 - n_0^2 = \theta x_1^* y^*(m_0 + p_0 + n_0)(f\eta - KE^2) < 0, \tag{2.18}$$

and $F_{\tau_1(\tau_2)}(+\infty) = +\infty$. Therefore, (2.17) has at least one positive root. Without loss of generality, we assume that (2.17) has $N_1(N_1 \in \mathbb{N}^+)$ different positive roots, denoted by $\omega_k = \sqrt{\varpi_k}(k = 1, 2, \cdots, N_1)$ and we have

$$\cos\omega_k\tau_1 = \frac{F_1D_1 - E_1A_1 - (F_1C_1 + E_1B_1)\cos\omega_k\tau_2 + (E_1C_1 - F_1B_1)\sin\omega_k\tau_2}{E_1^2 + F_1^2} \triangleq F_{\omega_k}.$$

Thus

$$\tau_{1k}^{(n)}(\tau_2) = \frac{1}{\omega_k}\cos^{-1}[F_{\omega_k}] + \frac{2n\pi}{\omega_k}, k = 1, 2, \cdots, N_1; n = 0, 1, 2, \cdots, \tag{2.19}$$

and the direction of $\tau_{1k}^{(n)}(\tau_2)$ passing through the imaginary axis [20] when $\omega = \omega_k$ is determined by

$$\mathrm{sign}\left[\frac{\mathrm{dRe}(\lambda(\tau))}{\mathrm{d}\tau}\bigg|_{\tau=\tau_{1k}^{(n)}}\right] = \mathrm{sign}\left[F'_{\tau_1(\tau_2)}(\varpi_k)\big|_{\varpi_k=\omega_k^2}\right] = \mathrm{sign}\left(\Delta_{\tau_1(\tau_2)}^k\right).$$

Then $\mathrm{sign}\left(\Delta_{\tau_1(\tau_2)}^k\right) \neq 0$, since $\omega_k(k = 1, 2, \cdots, N_1)$ are N_1 distinct positive roots of (2.17). And, system (1.5) undergos a Hopf bifurcation at E_2 when $\tau_1 = \tau_{1k}^{(n)}(\tau_2)$.

Define

$$S_{\tau_1(\tau_2)} = \{\tau_1 | H(\lambda, \tau_1, \tau_2) = 0, \mathrm{Re}(\lambda) < 0, \tau_2 \in S_{\tau_2}\},$$

$$\tau_{10}(\tau_2) = \min\{\tau_{1k}^{(n)}(\tau_2) | 1 \leq k \leq N_1, n = 0, 1, 2, \cdots\},$$

when $\tau_1 \in S_{\tau_1(\tau_2)}$ the positive equilibrium E_2 is local stable. Note that, if (2.17) has more than one positive root, there would be finite stability switches when time delay τ_1 passing through the critical points

$$\tau_1 = \tau_{1k}^{(n)}(\tau_2)(k = 1, 2, \cdots, N_1; n = 0, 1, 2, \cdots)$$

and $[0, \tau_{10}(\tau_2)) \subseteq S_{\tau_1(\tau_2)}$. If $f_{4i} > 0 (i = 1, 2, \cdots, 5)$ and condition $C_3^1 : f\eta < KE^2$ hold, (2.17) has only one positive root, there is no stability switches when time delay τ_1 passing through the critical points $\tau_1 = \tau_1^{(n)}(\tau_2)(n = 1, 2, \cdots)$ and $S_{\tau_1(\tau_2)} = [0, \tau_1^{(0)}(\tau_2))$.

Theorem 2.4 *(i) Suppose (2.17) has at least one positive roots denoted by $\omega_k (1 \leq k \leq N_1)$. There exists a nonempty set $S_{\tau_1(\tau_2)}$ and $[0, \tau_{10}(\tau_2)) \subseteq S_{\tau_1(\tau_2)}$, when $\tau_1 \in S_{\tau_1(\tau_2)}$ the positive equilibrium E_2 of (1.5) is local stable, system (1.5) can undergoes a Hopf bifurcation at the positive equilibrium E_2 when*

$$\tau_1 = \tau_{1k}^{(n)}(\tau_2)(k = 1, 2, \cdots, N_1; n = 0, 1, 2, \cdots).$$

(ii) Suppose (2.17) has only one positive root denoted by ω_1. There exists a nonempty set $S_{\tau_1(\tau_2)}$ and $S_{\tau_1(\tau_2)} = [0, \tau_1^{(0)}(\tau_2))$, when $\tau_1(\tau_2) \in S_{\tau_1(\tau_2)}$ the positive equilibrium E_2 of (1.5) is local stable and unstable when $\tau_1 > \tau_1^{(0)}(\tau_2)$, system (1.5) can undergoes a Hopf bifurcation at the positive equilibrium E_2 when $\tau_1 = \tau_1^{(n)}(\tau_2)(n = 0, 1, 2, \cdots)$.

Note 2.4 *If $f_{4i} > 0 (i = 1, 2, \cdots, 5)$ and condition $C_3^1 : f\eta < KE^2$ hold, then (2.17) has only one positive root, and this is a special case of Theorem 2.4 (ii).*

2.2.5. The case $\tau_2 > 0$ and fixed $\tau_1 \in S_{\tau_1}$

The characteristic equation about E_2 is given by

$$H(\lambda, \tau_1, \tau_2) = \left(M(\lambda) + N(\lambda)e^{-\lambda\tau_1}\right) + P(\lambda)e^{-\lambda\tau_2} = 0, \tag{2.20}$$

Suppose $\lambda = \mathrm{i}\omega(\omega > 0)$ is a pure imaginary root of (2.20), similar to the case 2.2.1, we have

$$\begin{cases} A_2 + B_2 \cos\omega_0\tau_1 - C_2 \sin\omega_0\tau_1 = -E_2 \cos\omega_0\tau_2 + F_2 \sin\omega_0\tau_2, \\ D_2 - B_2 \sin\omega_0\tau_1 - C_2 \cos\omega_0\tau_1 = E_2 \sin\omega_0\tau_2 + F_2 \cos\omega_0\tau_2, \end{cases}$$

where

$$A_2 = m_2\omega_0^2 - m_0, B_2 = n_2\omega_0^2 - n_0, C_2 = n_1, D_2 = \omega_0^3 - m_1\omega_0, E_2 = p_2\omega_0^2 - p_0, F_2 = p_1\omega_0.$$

And

$$F_{\tau_2(\tau_1)}(\omega) = \omega^6 + f_{55}\omega^5 + f_{54}\omega^4 + f_{53}\omega^3 + f_{52}\omega^2 + f_{51}\omega + f_{50} = 0, \tag{2.21}$$

where

$$\begin{aligned}
f_{55} &= -2n_2 \sin\omega\tau_1, \\
f_{54} &= m_2^2 - 2m_1 - p_2^2 + n_2^2 + 2(m_2n_2 - p_1)\cos\omega\tau_1, \\
f_{53} &= 2(n_0 + m_1n_2 - m_2n_1)\sin\omega\tau_1, \\
f_{52} &= m_1^2 - 2m_2m_0 + 2p_2p_0 - p_1^2 + n_1^2 - 2n_2n_0 + 2(n_1m_1 - n_0m_2 - m_0n_2)\cos\omega\tau_1, \\
f_{51} &= 2(m_0n_1 - n_0m_1)\sin\omega\tau_1, \\
f_{50} &= n_0^2 + m_0^2 + 2n_0m_0\cos\omega\tau_1 - p_0^2,
\end{aligned}$$

Assumed that condition $C_3^2 : f\eta > KE^2$ hold, then

$$F_{\tau_2(\tau_1)}(0) = f_0 = (m_0 + n_0)^2 - p_0^2 = \theta x_1^* y^*(m_0 + p_0 + n_0)(KE^2 - f\eta) < 0, \tag{2.22}$$

and $F_{\tau_2(\tau_1)}(+\infty) = +\infty$, therefore, (2.21) has at least one positive root. Without loss of generality, we assume that (2.21) has $N_2(N_2 \in \mathbb{N}^+)$ distinct positive roots, denoted by $\omega_k = \sqrt{\varpi_k}(k = 1, 2, \cdots, N_2)$ and we have

$$\cos\omega_k\tau_2 = \frac{F_2D_2 - E_2A_2 - (F_2C_2 + E_2B_2)\cos\omega_k\tau_1 + (E_2C_2 - F_2B_2)\sin\omega_k\tau_1}{E_2^2 + F_2^2} \triangleq F_{\omega_k}.$$

Thus

$$\tau_{2k}^{(n)}(\tau_1) = \frac{1}{\omega_k}\cos^{-1}\left[F_{\omega_k}\right] + \frac{2n\pi}{\omega_k}, k = 1, 2, \cdots, N_2; n = 0, 1, 2, \cdots, \tag{2.23}$$

and the direction of $\tau_{2k}^{(n)}(\tau_1)$ passing through the imaginary axis [20] when $\omega = \omega_k$ is determined by

$$\text{sign}\left[\frac{\text{dRe}(\lambda(\tau))}{\text{d}\tau}\bigg|_{\tau=\tau_{2k}^{(n)}}\right] = \text{sign}\left[F'_{\tau_2(\tau_1)}(\varpi_k)\big|_{\varpi_k=\omega_k^2}\right] = \text{sign}\left(\Delta_{\tau_2(\tau_1)}^k\right).$$

Then $\text{sign}\left(\Delta_{\tau_2(\tau_1)}^k\right) \neq 0$, since $\omega_k(k = 1, 2, \cdots, N_2)$ are N_2 distinct positive roots of (2.21). System (1.5) undergoes a Hopf bifurcation at E_2 when $\tau_2 = \tau_{2k}^{(n)}(\tau_1)$.

Define

$$S_{\tau_2(\tau_1)} = \{\tau_2 | H(\lambda, \tau_1, \tau_2) = 0, \text{Re}(\lambda) < 0, \tau_1 \in S_{\tau_1}\},$$

$$\tau_{20}(\tau_1) = \min\{\tau_{2k}^{(n)}(\tau_1) | 1 \leq k \leq N_2, n = 0, 1, 2, \cdots\},$$

when $\tau_2 \in S_{\tau_2(\tau_1)}$ the positive equilibrium E_2 is local stable. Note that, if (2.21) has more than one positive root, there would be finite stability switches when time delay τ_2 passing through the critical points

$$\tau_2 = \tau_{2k}^{(n)}(\tau_1)(k = 1, 2, \cdots, N_2; n = 0, 1, 2, \cdots)$$

and $[0, \tau_{20}(\tau_1)) \subseteq S_{\tau_2(\tau_1)}$. If $f_{5i} > 0(i = 1, 2, \cdots, 5)$ and condition $C_3^2 : f\eta > KE^2$ hold, (2.21) has only one positive root, there is no stability switches when time delay τ_1 passing through the critical points $\tau_2 = \tau_2^{(n)}(\tau_1)(n = 1, 2, \cdots)$ and $S_{\tau_2(\tau_1)} = [0, \tau_2^{(0)}(\tau_1))$.

Theorem 2.5 *(i) Suppose (2.21) has at least one positive roots denoted by $\omega_k(1 \leq k \leq N_2)$. There exists a nonempty set $S_{\tau_2(\tau_1)}$ and $[0, \tau_{20}(\tau_1)) \subseteq S_{\tau_2(\tau_1)}$, when $\tau_2 \in S_{\tau_2(\tau_1)}$ the positive equilibrium E_2 of system (1.5) is local stable. There is a Hopf bifurcation at E_2 when*

$$\tau_2 = \tau_{2k}^{(n)}(\tau_1)(k = 1, 2, \cdots, N_2; n = 0, 1, 2, \cdots).$$

(ii) Suppose (2.21) has only one positive root denoted by ω_1. There exists a nonempty set $S_{\tau_2(\tau_1)}$ and $S_{\tau_2(\tau_1)} = [0, \tau_2^{(0)}(\tau_1))$, when $\tau_2(\tau_1) \in S_{\tau_2(\tau_1)}$ the positive equilibrium E_2 of (1.5) is local stable and unstable when $\tau_2 > \tau_2^{(0)}(\tau_1)$. There is a Hopf bifurcation at E_2 when $\tau_2 = \tau_2^{(n)}(\tau_1)(n = 0, 1, 2, \cdots)$.

Note 2.5 *If $f_{5i} > 0(i = 1, 2, \cdots, 5)$ and condition $C_3^2 : f\eta > KE^2$ hold, then (2.21) has only one positive root, and this is a special case of Theorem 2.5 (ii).*

3. Numerical simulations

3.1. Example 1

We consider following system

$$\begin{cases} x_1'(t) = 2.5x_2(t) - x_1(t)(1.05 + 0.2x_1(t) + 1.25y(t)), \\ x_2'(t) = 0.9x_1(t) - 0.7x_2(t), \\ y'(t) = y(t)(0.75x_1(t-\tau_1) - 0.1 - 1.8y(t-\tau_2)), \end{cases} \tag{3.1}$$

where $\alpha = 2.5, \gamma_1 = 0.15, \Omega = 0.9, \eta = 0.2, E = 1.25, \theta_1 = 0.2, a = 0.5, K = 0.6, d = 0.1, f = 1.8, X(0) = (4.0, 5.0, 1.3)$.

In case 2.2.1, $\tau_1 > 0, \tau_2 \equiv 0$, from (2.6) we have $f_{12} = 24.6366, f_{11} = 84.7024, f_{10} = -5.3829$, the unique positive root $\omega = 0.2498$ and

$$\tau_1^{(n)} = 8.4802 + 0.5n\pi, n = 0, 1, 2, \cdots,$$

$\text{sign}\left(\Delta^1_{\tau_1}\right) = 1$. According to Theorem 2.2.1 (ii),

$$\tau_{10} = 8.4802, S_{\tau_1} = [0, 8.4802).$$

The positive equilibrium point E_2 is local stable when $\tau_1 = 8.3 < \tau_{10}$, and unstable when $\tau_1 = 8.6 > \tau_{10}$ (Figure 1). And increasing time delay τ_1, the prey and predator populations can coexist with stable limit cycles when $\tau_2 = 0$ and $\tau_1 = 8.5, 8.6, 8.7, 8.8, 8.9, 9, 9.3, 10, 11, 12, 13, 15, 20, 25, 40, 80$, respectively (Figure 2). Then, there is a global Hopf bifurcation when time delay τ_1 far away from the first bifurcating critical point τ_{10} [16], and the amplitudes of period oscillation are increasing with time delay τ_1 increased. By the fast-slow oscillations, too large time delay τ_1 would make the population to be die out, since the populations are very close to zero when time delay τ_1 increase to some critical value (Figure 3).

In case 2.2.2, $\tau_2 > 0, \tau_1 \equiv 0$, from (2.10) we have $f_{22} = 7.5640, f_{21} = -103.9972, f_{20} = 5.3829$, and there are two positive roots $\omega_1 = 7.0595, \omega_2 = 0.0520$,

$$\tau_{21}^{(n)} = 0.68 + 0.2833n\pi, \tau_{22}^{(n)} = 17.4608 + 38.4615n\pi, n = 0, 1, 2, \cdots,$$

$\text{sign}\left(\Delta^1_{\tau_2}\right) = 1, \text{sign}\left(\Delta^2_{\tau_2}\right) = -1$. Note that $\tau_{21}^{(0)} < \tau_{21}^{(1)} < \tau_{22}^{(0)}$, there is no stability switches for τ_2 passing through the critical points $\tau_{21}^{(n)}$ and $\tau_{22}^{(n)}$. According to Theorem 2.2.2 (i),

$$\tau_{20} = 0.68, S_{\tau_2} = [0, 0.68).$$

The positive equilibrium point E_2 is local stable when $\tau_2 = 0.66 < \tau_{20}$, and unstable when $\tau_2 = 0.70 > \tau_{20}$ (Figure 4). And increasing time delay τ_2, the prey and predator populations can coexist with stable limit cycles when $\tau_1 = 0$ and $\tau_2 = 0.7, 0.8, 0.9, 1.0, 1.1, 1.2$, respectively (Figure 5), and the amplitudes of period oscillation are increased. And, time delay τ_2 would make the population to be die out, because the populations are very close to zero and then tend to unbounded solutions as time delay $\tau_2 = 1.23$ (Figure 6).

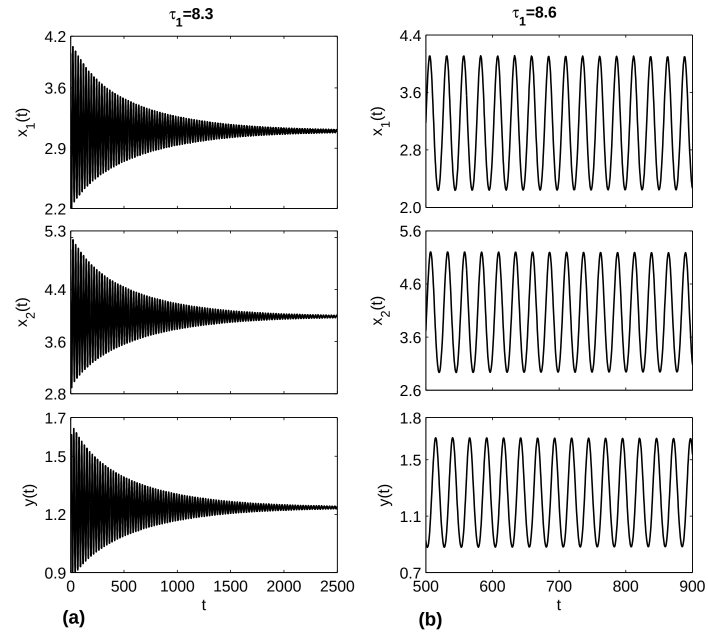

Figure 1. The time-series plot of the system (3.1). (a) E_2 is local asymptotically stable for $\tau_1 = 8.3 < \tau_{10}$, (b) A local Hopf bifurcation for $\tau_1 = 8.6 > \tau_{10}$ near positive equilibrium point E_2.

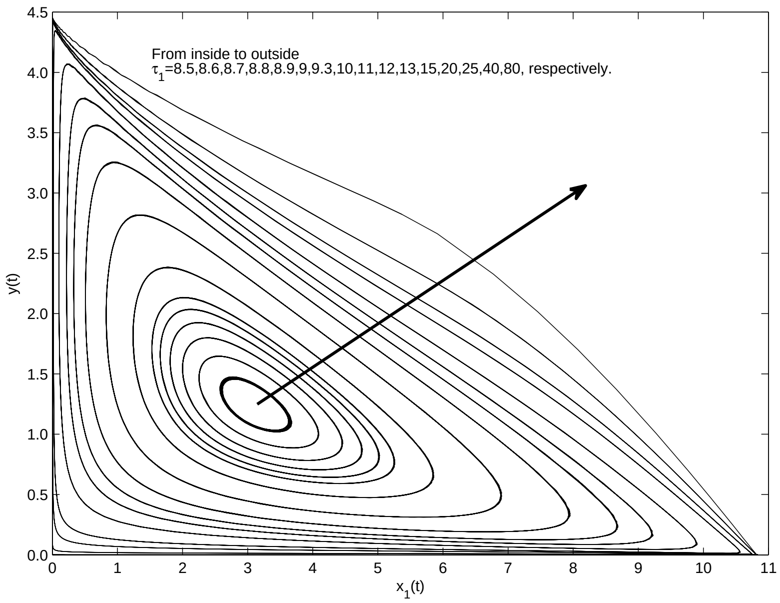

Figure 2. Prey and predator populations coexist with stable limit cycles for system (3.1) when $\tau_2 = 0$ and $\tau_1 = 8.5, 8.6, 8.7, 8.8, 8.9, 9, 9.3, 10, 11, 12, 13, 15, 20, 25, 40, 80$, respectively.

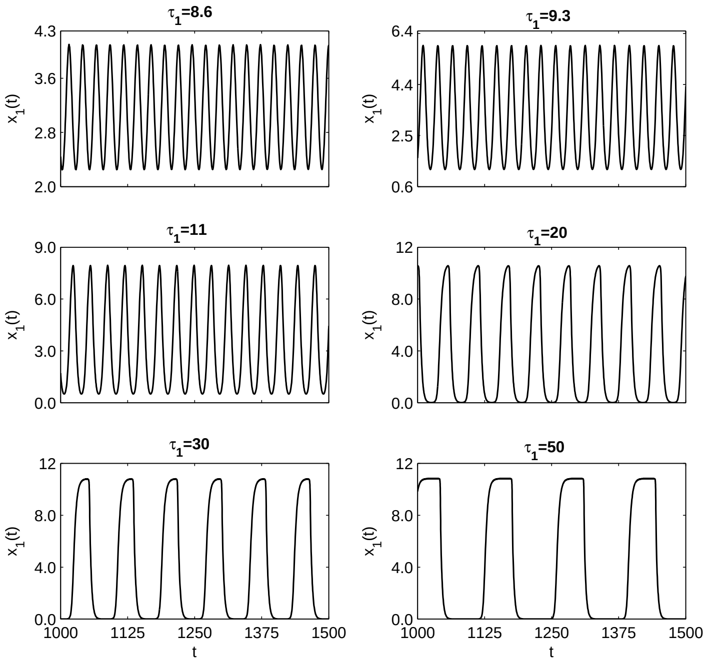

Figure 3. The time-series plot of the system (3.1) when $\tau_2 = 0$ and $\tau_1 = 8.6, 9.3, 11, 20, 30, 50$, respectively.

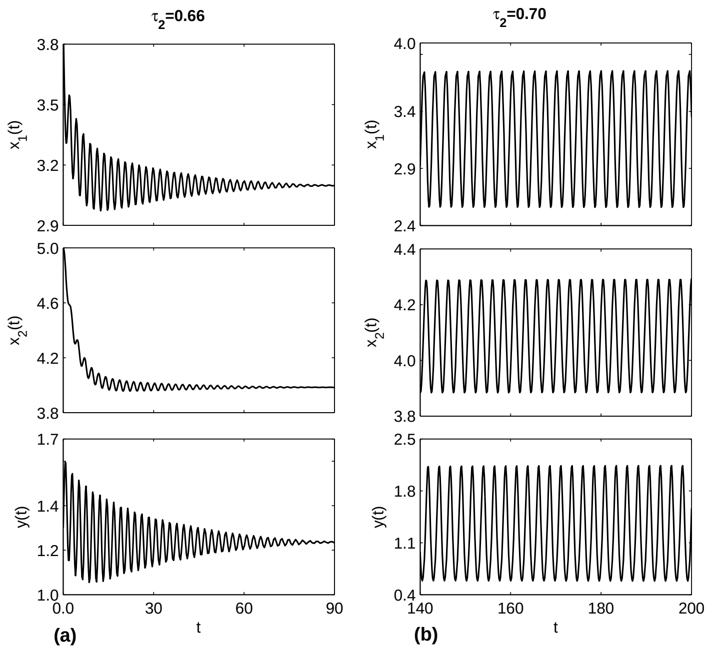

Figure 4. The time-series plot of the system (3.1). (a) E_2 is local asymptotically stable for $\tau_2 = 0.66 < \tau_{10}$, (b) A local Hopf bifurcation for $\tau_2 = 0.70 > \tau_{10}$ near positive equilibrium point E_2.

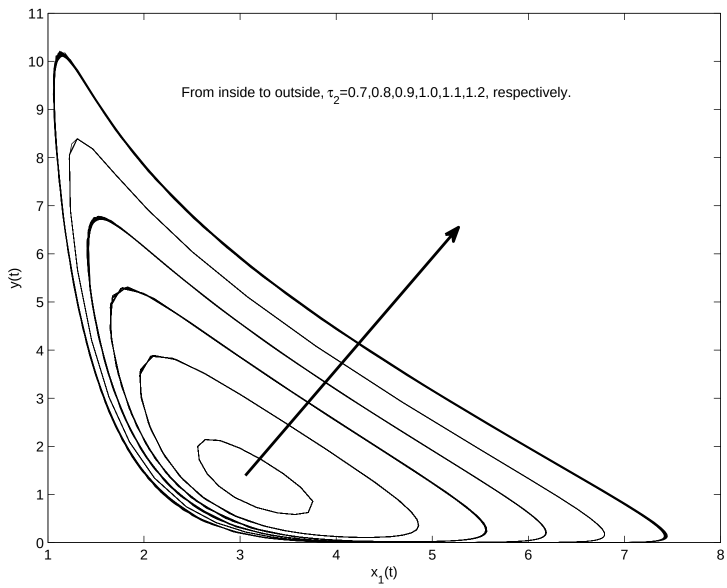

Figure 5. Prey and predator populations coexist with stable limit cycles for system (3.1) when $\tau_1 = 0$ and $\tau_2 = 0.7, 0.8, 0.9, 1.0, 1.1, 1.2$, respectively.

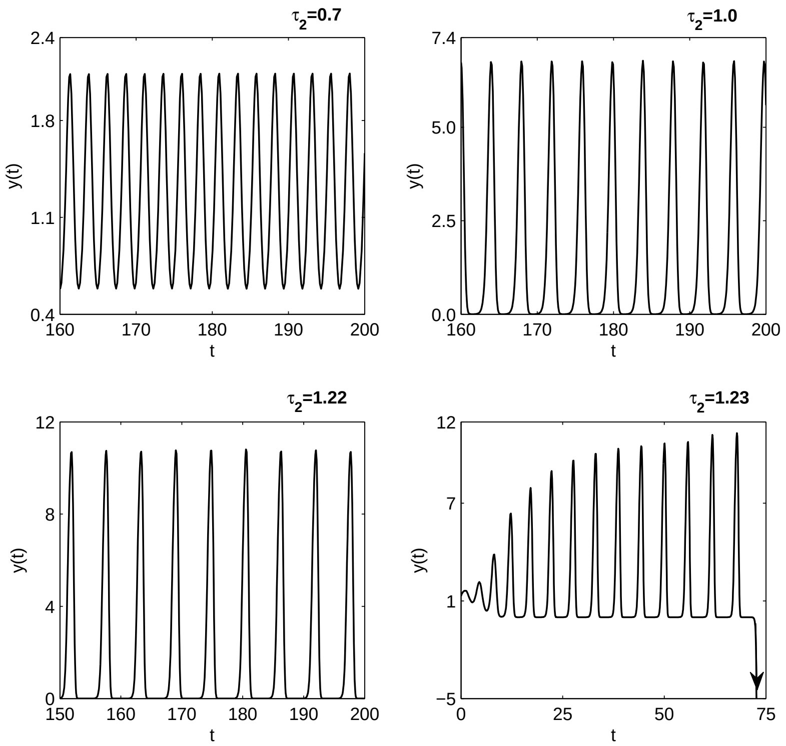

Figure 6. The time-series plot of the system (3.1) when $\tau_1 = 0$ and $\tau_2 = 0.7, 1.0, 1.22, 1.23$, respectively.

In case 2.2.3, $\tau_1 = \tau_2 = \tau$, $f_{32} = 14.7433$, $f_{31} = -171.3174$, $f_{30} = -12.0940$, and the unique positive root $\omega = 2.7753$,

$$\tau^{(n)} = 0.5015 + 0.7206n\pi, n = 0, 1, 2, \cdots,$$

$\text{sign}\left(\Delta_\tau^1\right) = 1$. Then $\tau_0 = 0.5017$, According to Theorem 2.2.3 (i),

$$\tau_0 = 0.5017, S_\tau = [0, 0.5017).$$

The positive equilibrium point E_2 is local stable when $\tau = 0.48 < \tau_0$, and unstable when $\tau = 0.52 > \tau_0$ (Figure 7). And increasing time delay τ, the prey and predator populations can coexist with stable limit cycles when $\tau = 0.503, 0.505, 0.508, 0.513, 0.518, 0.523, 0.526, 0.53$, respectively (Figure 8), and the amplitudes of period oscillation are increased. And, time delay τ would make the population to be die out, because the populations are very close to zero and then tend to unbounded solution as time delay $\tau = 0.536$ (Figure 9).

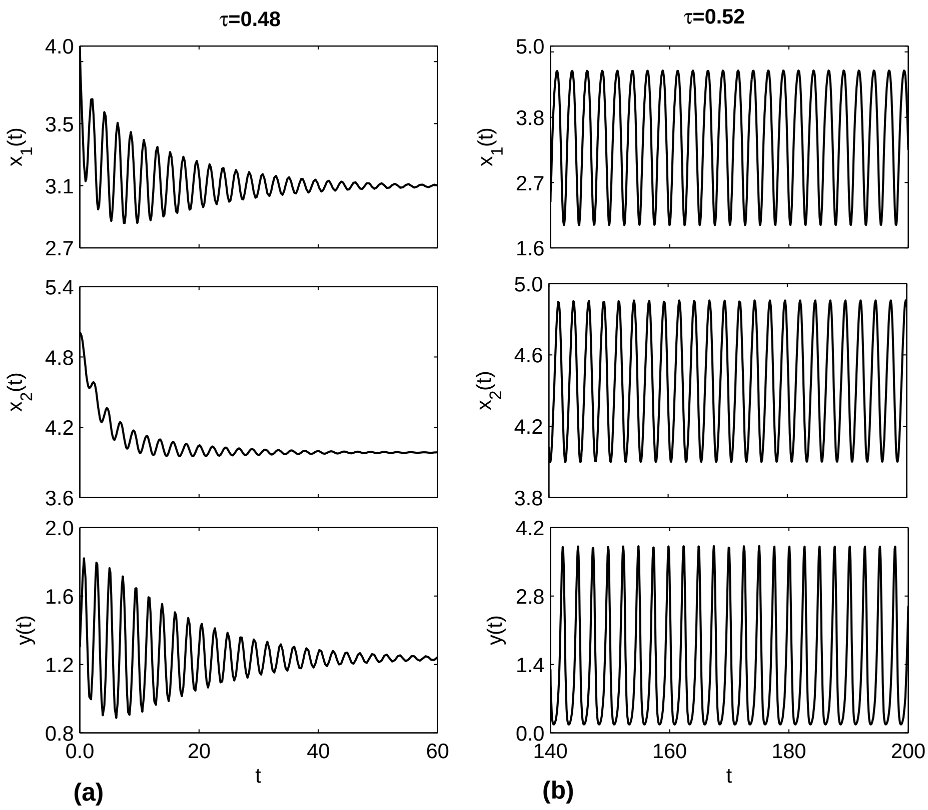

Figure 7. The time-series plot of the system (3.1). (a) E_2 is local asymptotically stable for $\tau = 0.48 < \tau_0$, (b) A local Hopf bifurcation for $\tau = 0.52 > \tau_0$ near positive equilibrium point E_2.

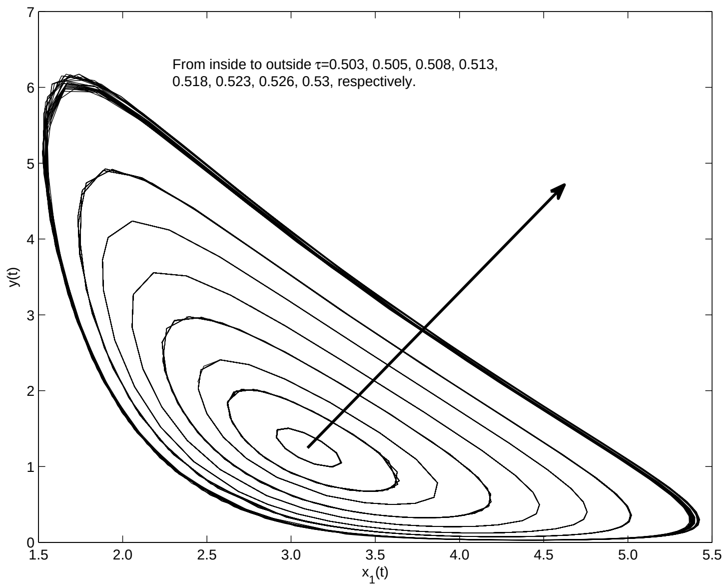

Figure 8. Prey and predator populations coexist with stable limit cycles for system (3.1) when $\tau = 0.503, 0.505, 0.508, 0.513, 0.518, 0.523, 0.526, 0.53$, respectively.

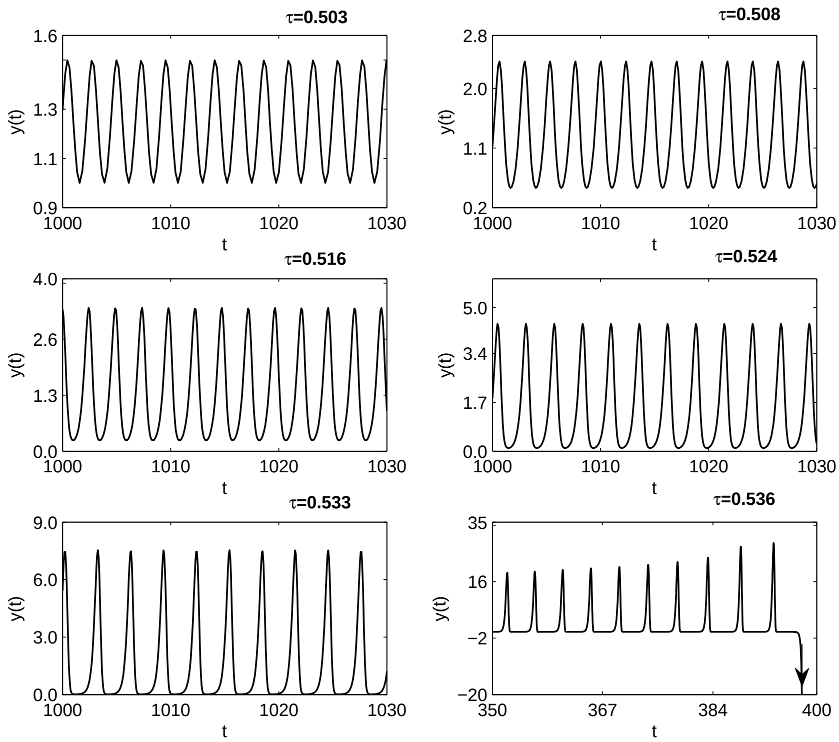

Figure 9. The time-series plot of the system (3.1) when $\tau = 0.503, 0.508, 0.516,$ $0.524, 0.533, 0.536$, respectively.

We plot the stable and unstable regions with $\tau_1 \times \tau_2 = [0, 10] \times [0, 1.4]$ (Figure 10) by using the publicly available Matlab package Trace-DDE [22], which by the pseudospectral method for the computation of characteristic roots of delay differential equations introduced in [23, 24]. From Figure 10, we see that, if one fixed τ_2 about 0.55, there would be stability switches when τ_1 increasing from 0 to 10. Let $\tau_2 = 0.52 \in S_{\tau_2}$, in case 2.2.4, from the Figure 11 we see that $F_{\tau_1(\tau_2)} = 0$ have three positive roots

$$\omega_1 = 2.896366, \omega_2 = 2.473462, \omega_3 = 0.288596,$$

and

$$\tau_{11}^{(n)} = 0.338236 + 0.690520n\pi, \tau_{12}^{(n)} = 0.786103 + 0.808583n\pi,$$

$$\tau_{13}^{(n)} = 7.871032 + 6.930103n\pi, (n = 0, 1, 2, \cdots), \text{sign}\left(\Delta^1_{\tau_1(\tau_2)}\right) = 1, \text{sign}\left(\Delta^2_{\tau_1(\tau_2)}\right) = -1, \text{sign}\left(\Delta^3_{\tau_1(\tau_2)}\right) = 1.$$

Note that,

$$\tau_{11}^{(0)} = 0.338236, \tau_{12}^{(0)} = 0.786103, \tau_{11}^{(1)} = 2.507569, \tau_{12}^{(1)} = 3.326342, \tau_{11}^{(2)} = 4.676903,$$

$$\tau_{12}^{(2)} = 5.866582, \tau_{11}^{(3)} = 6.846236, \tau_{13}^{(0)} = 7.8710317, \tau_{12}^{(3)} = 8.406821, \tau_{11}^{(4)} = 9.015570.$$

then

$$\tau_{11}^{(0)} < \tau_{12}^{(0)} < \tau_{11}^{(1)} < \tau_{12}^{(1)} < \tau_{11}^{(2)} < \tau_{12}^{(2)} < \tau_{11}^{(3)} < \tau_{13}^{(0)} < \tau_{12}^{(3)} < \tau_{11}^{(4)},$$

and

$$S_{\tau_1(\tau_2)} = \left[0, \tau_{11}^{(0)}\right) \bigcup \left(\tau_{12}^{(0)}, \tau_{11}^{(1)}\right) \bigcup \left(\tau_{12}^{(1)}, \tau_{11}^{(2)}\right) \bigcup \left(\tau_{12}^{(2)}, \tau_{11}^{(3)}\right).$$

When $\tau_2 = 0.52, \tau_1 \in S_{\tau_1(\tau_2)}$, the positive equilibrium point E_2 is local stable, where $S_{\tau_1(\tau_2)}$ composed of four an increasing intervals. There are four times stability switches when time delay τ_1 crossing $S_{\tau_1(\tau_2)}$. And continuously increasing time delay τ_1, the prey and predator populations coexist with period oscillation, quasi-period oscillation, even chaotic oscillation when $\tau_1 = 8, 9, 11, 14, 15, 19, 37$, and tend to unbounded oscillation for $\tau_1 = 38$ (Figure 12).

Let $\tau_1 = 3.1 \in \left(\tau_{11}^{(1)}, \tau_{12}^{(1)}\right)$ (unstable region). We investigate the effect time delay τ_2 on system (3.1). The bifurcation diagrams of time delay τ_2 over $[0.52, 0.68]$ show that system (3.1) has rich dynamics (Figure 13), including (1) periodic oscillating, (2) period-doubling bifurcations, and (3) chaos, and the solution tend to unbounded oscillation for $\tau_2 = 0.69$ at time $t = 210$ (Figure 14).

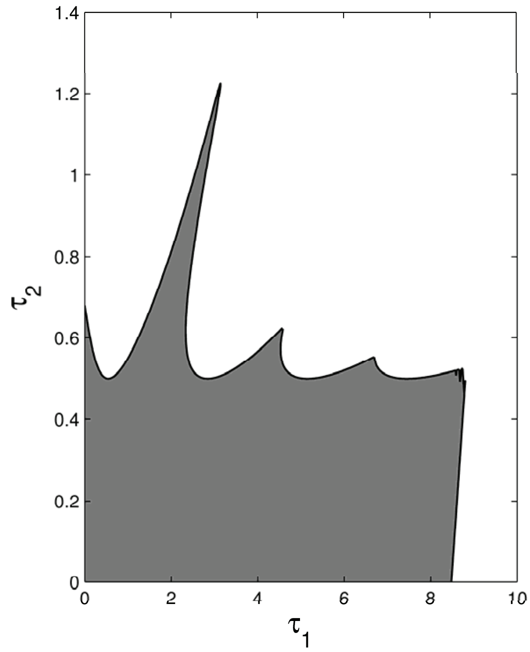

Figure 10. The stable regions (gray) and unstable regions (white) of the positive equilibrium point E_2 of system (3.1) with $\tau_1 \times \tau_2 = [0, 10] \times [0, 1.4]$.

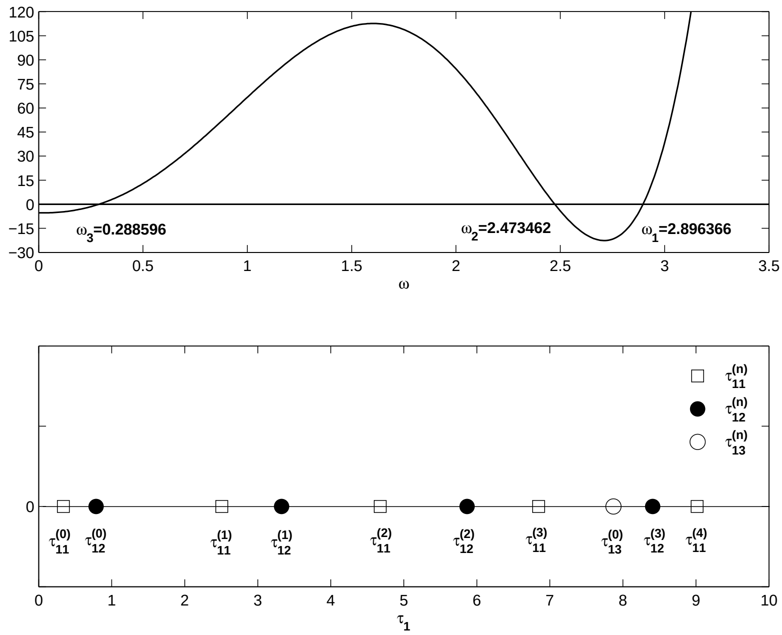

Figure 11. The graphic of function $F_{\tau_1(\tau_2)}(\omega) = 0$ (top) and the critical time delay series $\tau_{11}^{(n)}, \tau_{12}^{(n)}, \tau_{13}^{(n)}$ (bottom) when $\tau_2 = 0.52$ for system (3.1).

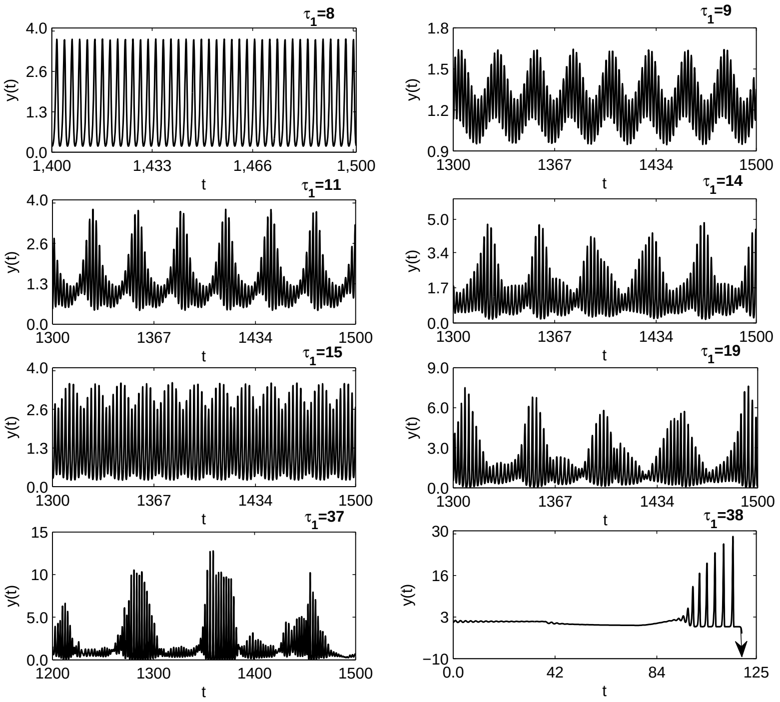

Figure 12. The time-series plot of the system (3.1) when $\tau_2 = 0.52$ and $\tau_1 = 8, 9, 11, 14, 15, 19, 37, 38$, respectively.

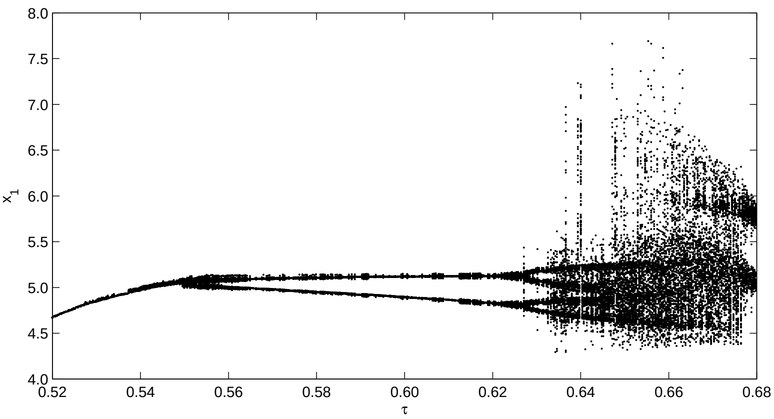

Figure 13. The bifurcation diagrams of system (3.1) when time delay $\tau_1 = 3.1$ and time delay τ_2 over $[0.52, 0.68]$.

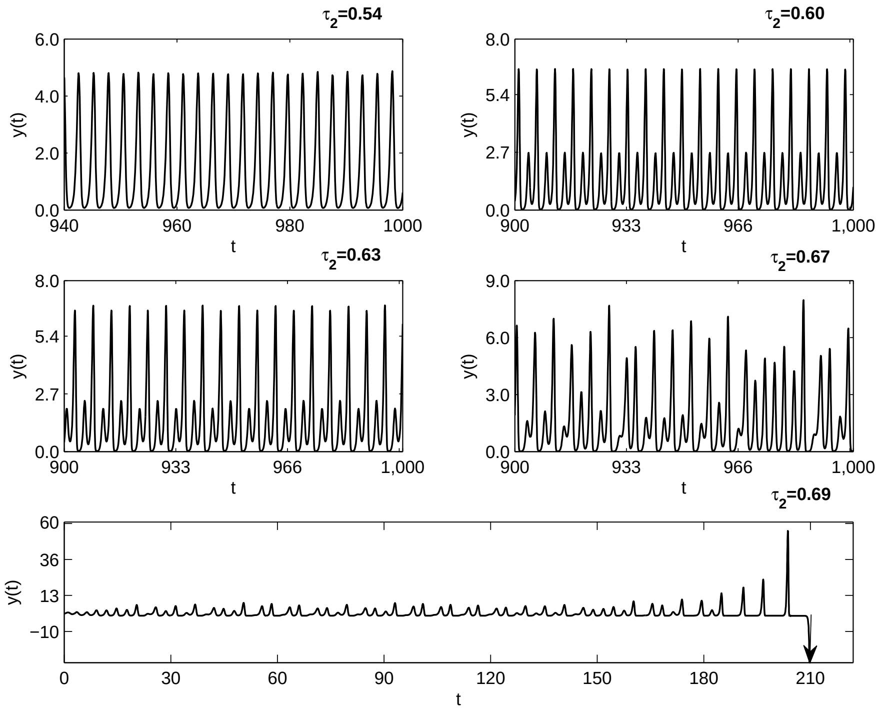

Figure 14. The time-series plot of the system (3.1) when $\tau_1 = 3.1$ and $\tau_2 = 0.54, 0.60, 0.63, 0.67, 0.69$, respectively.

Furthermore, increasing τ_2 from 0.52 to 1.4, then $\tau_{11}^{(n)}$ decreased and $\tau_{12}^{(n)}$ increased, and the stability switches disappear one by one when $\tau_{12}^{(n)} > \tau_{11}^{(n+1)}$ for $n = 2, 1, 0$ (Figure 15). We plot the stable and unstable regions (Figures 16 and 17) by choose $f = 0.6, 0.8, 1.2, 1.5, 1.9, 2.2, 2.4, 2.9$ respectively, and remained other parameters in example 1. By increasing the values of parameter f, the stable and unstable regions showing that $\tau_{11}^{(0)}$ increased and $\tau_{21}^{(0)}$ decreased, and the stable regions changed more and more complexity, which is a connect region from the view of topology. If we increasing the values of parameter f and choose τ_2 less than and closed to the first critical point $\tau_{21}^{(0)}$, then there would be more and more stability switches by increasing time delay τ_1 from 0 to 15.

3.2. Example 2

We consider following system

$$\begin{cases} x_1'(t) = 2.6x_2(t) - x_1(t)(1.1 + 0.3x_1(t) + 1.1y(t)), \\ x_2'(t) = 0.9x_1(t) - 0.8x_2(t), \\ y'(t) = y(t)(0.88x_1(t-\tau_1) - 0.15 - fy(t-\tau_2)), \end{cases} \tag{3.2}$$

where $\alpha = 2.6, \gamma_1 = 0.2, \Omega = 0.9, \eta = 0.3, E = 1.1, \theta_1 = 0.15, a = 0.65, k = 0.8, d = 0.15, X(0) = (4.0, 5.0, 1.3)$. We consider the case 2.2.2 with different value of parameter f.

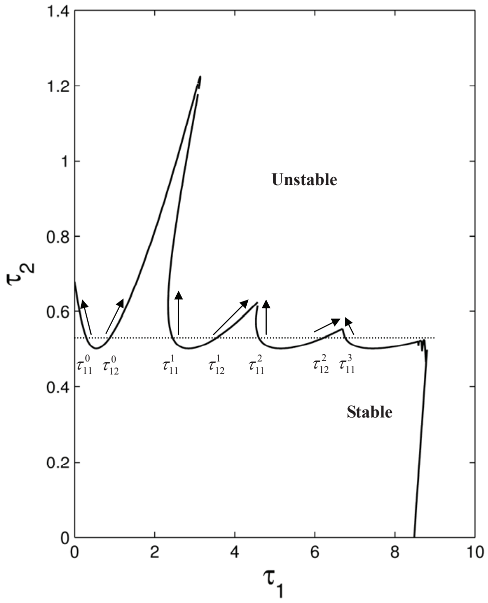

Figure 15. Location about the critical time delay series $\tau_{11}^{(n)}, \tau_{12}^{(n)}, \tau_{13}^{(n)}$ of the system (3.1) in the stable-unstable regions when increased time delay τ_2.

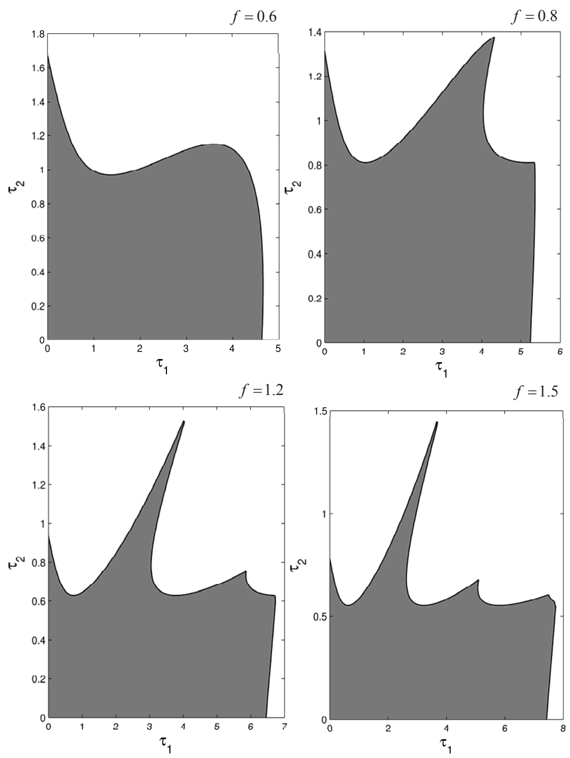

Figure 16. The stable regions (gray) and unstable regions (white) of the positive equilibrium point E_2 of system (3.1) with parameter $f = 0.6, 0.8, 1.2, 1.5$, respectively.

Let $f = 0.095$, from (2.10) we have $\omega_1 = 0.3760, \omega_2 = 0.2963$ and

$$\tau_{21}^{(n)} = 6.7472 + 5.3191n\pi, \tau_{22}^{(n)} = 12.1964 + 6.7499n\pi, \text{sign}\left(\Delta_{\tau_2}^1\right) = 1, \text{sign}\left(\Delta_{\tau_2}^2\right) = -1,$$

$$\tau_{21}^{(0)} = 6.7472, \tau_{22}^{(0)} = 12.1964, \tau_{21}^{(1)} = 23.4585, \tau_{22}^{(1)} = 33.4014,$$
$$\tau_{21}^{(2)} = 40.1697, \tau_{22}^{(2)} = 54.6066, \tau_{21}^{(3)} = 56.8809, \tau_{21}^{(4)} = 73.5921, \tau_{22}^{(3)} = 75.8117,$$

then

$$\tau_{21}^{(0)} < \tau_{22}^{(0)} < \tau_{21}^{(1)} < \tau_{22}^{(1)} < \tau_{21}^{(2)} < \tau_{22}^{(2)} < \tau_{21}^{(3)} < \tau_{21}^{(4)} < \tau_{22}^{(3)},$$

and

$$S_{\tau_2} = \left[0, \tau_{21}^{(0)}\right) \bigcup \left(\tau_{22}^{(0)}, \tau_{21}^{(1)}\right) \bigcup \left(\tau_{22}^{(1)}, \tau_{21}^{(2)}\right) \bigcup \left(\tau_{22}^{(2)}, \tau_{21}^{(3)}\right).$$

Let $f = 0.11$, from (2.10) we have $\omega_1 = 0.4037, \omega_2 = 0.2957$ and

$$\tau_{21}^{(n)} = 6.0146 + 4.9542n\pi, \tau_{22}^{(n)} = 12.5688 + 6.7636n\pi, \text{sign}\left(\Delta_{\tau_2}^1\right) = 1, \text{sign}\left(\Delta_{\tau_2}^2\right) = -1,$$

$$\tau_{21}^{(0)} = 6.0146, \tau_{22}^{(0)} = 12.5688, \tau_{21}^{(1)} = 21.5800,$$
$$\tau_{22}^{(1)} = 33.8150, \tau_{21}^{(2)} = 37.1453, \tau_{21}^{(3)} = 52.7107, \tau_{22}^{(2)} = 55.0613,$$

then

$$\tau_{21}^{(0)} < \tau_{22}^{(0)} < \tau_{21}^{(1)} < \tau_{22}^{(1)} < \tau_{21}^{(2)} < \tau_{21}^{(3)} < \tau_{22}^{(2)},$$

and

$$S_{\tau_2} = \left[0, \tau_{21}^{(0)}\right) \bigcup \left(\tau_{22}^{(0)}, \tau_{21}^{(1)}\right) \bigcup \left(\tau_{22}^{(1)}, \tau_{21}^{(2)}\right).$$

Let $f = 0.16$, from (2.10) we have $\omega_1 = 0.4874, \omega_2 = 0.2972$ and

$$\tau_{21}^{(n)} = 4.6008 + 4.1034n\pi, \tau_{22}^{(n)} = 13.0647 + 6.7295n\pi, \text{sign}\left(\Delta_{\tau_2}^1\right) = 1, \text{sign}\left(\Delta_{\tau_2}^2\right) = -1,$$

$$\tau_{21}^{(0)} = 4.6008, \tau_{22}^{(0)} = 13.0647, \tau_{21}^{(1)} = 17.4916, \tau_{21}^{(2)} = 30.3824, \tau_{22}^{(1)} = 34.2025,$$

then

$$\tau_{21}^{(0)} < \tau_{22}^{(0)} < \tau_{21}^{(1)} < \tau_{21}^{(2)} < \tau_{22}^{(1)}, S_{\tau_2} = \left[0, \tau_{21}^{(0)}\right) \bigcup \left(\tau_{22}^{(0)}, \tau_{21}^{(1)}\right).$$

Let $f = 0.35$, from (2.10) we have $\omega_1 = 0.7658, \omega_2 = 0.2976$ and

$$\tau_{21}^{(n)} = 2.6501 + 2.6116n\pi, \tau_{22}^{(n)} = 13.5430 + 6.7204n\pi, \text{sign}\left(\Delta_{\tau_2}^1\right) = 1, \text{sign}\left(\Delta_{\tau_2}^2\right) = -1,$$

$$\tau_{21}^{(0)} = 2.6501, \tau_{21}^{(1)} = 10.8544, \tau_{21}^{(1)} = 13.5430,$$

then

$$\tau_{21}^{(0)} < \tau_{21}^{(1)} < \tau_{22}^{(0)}, S_{\tau_2} = \left[0, \tau_{21}^{(0)}\right).$$

From above numerical analysis, we see that, the times of stability switches are decreased from four to one by increasing the values of parameter f from 0.095 to 0.35; and the first critical point $\tau_{21}^{(0)}$ also decreased (Figure 18). From Figure 19, we see the stable regions changed more and more simple by increasing the values of parameter f, and the stable regions from four parts to three parts, and to two parts, finally to one connect region.

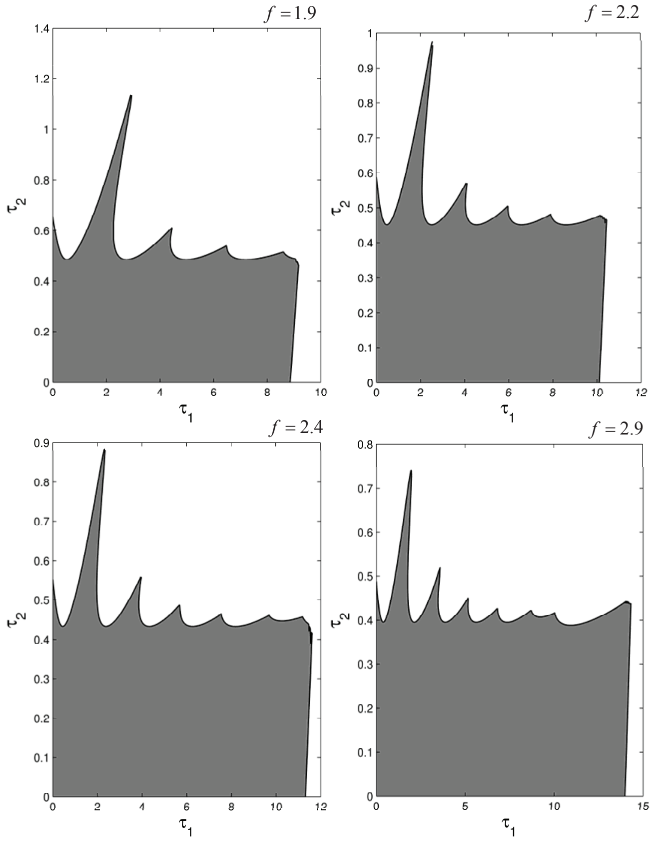

Figure 17. The stable regions (gray) and unstable regions (white) of the positive equilibrium point E_2 of system (3.1) with parameter $f = 1.9, 2.2, 2.4, 2.9$, respectively.

4. Conclusion

We have considered a prey-predator system with three stage structure and two delays, and analyzed the stability of the equilibrium point, obtained the conditions for the positive equilibrium E_2 occurring Hopf bifurcation by analyzing the characteristic equation in five cases. From the numerical examples and analysis, we know that the time delays would make the system subject to period oscillation, quasi-period oscillation, chaotic oscillation, finite stability switches, even unbounded oscillation and extinct. That is to say, time delays are important factors to affect the dynamic behaviors of the system.

4.1. Delays induced Hopf bifurcation

From the analysis in section 2, we know that $f_{30} < 0$ in (2.14) for case 2.2.3, then (2.14) has at least one positive root, and there is a natural Hopf bifurcation for system (1.5) without any conditions according to theorem 2.2.3 (i). If condition $C_3^1 : f\eta < KE^2$ holds then $f_{10} < 0$ in (2.6) for case 2.2.1, and that (2.6) has at least one positive root, and there is a Hopf bifurcation for system (1.5) according to theorem 2.2.1 (i). Similarly, if condition $C_3^2 : f\eta > KE^2$ holds then $f_{20} < 0$ in (2.10) for case 2.2.2, and that (2.10) has at least one positive root, there is a Hopf bifurcation for system (1.5) according to theorem 2.2.2 (i). Note that conditions $C_3^1 : f\eta < KE^2$ and $C_3^2 : f\eta > KE^2$ cannot hold at the

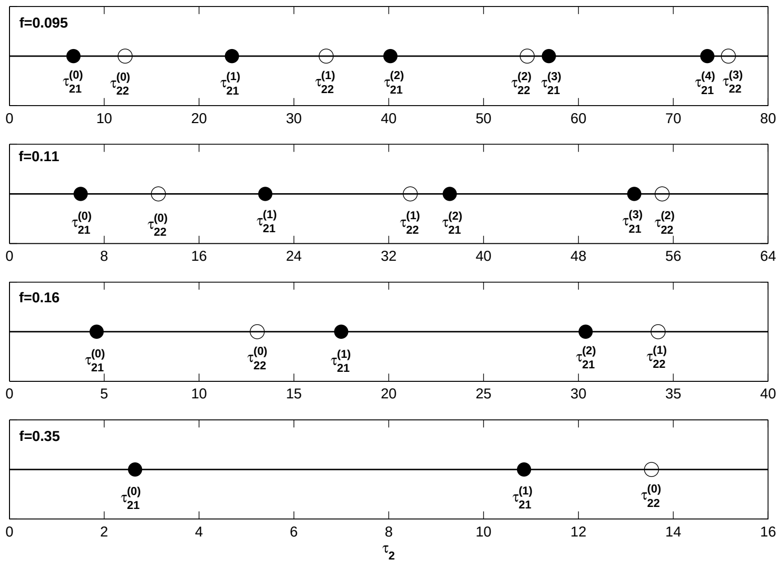

Figure 18. The location of the critical time delay points $\tau_{21}^{(n)}$ and $\tau_{22}^{(n)}$ of the system (3.2) with $f = 0.095, 0.11, 0.16, 0.35$, respectively.

same time, but one of them can hold for any parameter values of the system exclude the special case $f\eta = KE^2$. Therefore, there is a Hopf bifurcation for system (1.5) with only one time delay τ_1 or τ_2. And then, one of S_{τ_1} and S_{τ_2} is nonempty set. So, there is a natural Hopf bifurcation for system (1.5), and large time delays would make the positive point E_2 eventually unstable. These are harmful delays for system (1.5).

4.2. Delays induced stability switches

From the analysis in section 2, we know that there would be finite stability switches for system (1.5) when the equation has more than one positive roots $\omega_k (k > 1)$. From example 1 in case 2.2.2, only time delay τ_2, there are two positive roots and two critical delay sequences $\tau_{21}^{(n)}$ and $\tau_{22}^{(n)}$. But, there is no stability switches since $\tau_{21}^{(0)} < \tau_{21}^{(1)} < \tau_{22}^{(0)}$. From example 1 in case 2.2.4 fixed $\tau_2 = 0.52 \in S_{\tau_2}$, there are three positive roots and three critical delay sequences $\tau_{11}^{(n)}, \tau_{12}^{(n)}$ and $\tau_{13}^{(0)}$. Note that

$$\tau_{11}^{(0)} < \tau_{12}^{(0)} < \tau_{11}^{(1)} < \tau_{12}^{(1)} < \tau_{11}^{(2)} < \tau_{12}^{(2)} < \tau_{11}^{(3)} < \tau_{13}^{(0)} < \tau_{12}^{(3)} < \tau_{11}^{(4)},$$

and

$$S_{\tau_1(\tau_2)} = \left[0, \tau_{11}^{(0)}\right) \bigcup \left(\tau_{12}^{(0)}, \tau_{11}^{(1)}\right) \bigcup \left(\tau_{12}^{(1)}, \tau_{11}^{(2)}\right) \bigcup \left(\tau_{12}^{(2)}, \tau_{11}^{(3)}\right),$$

there are four times stability switches when time delay τ_1 increasing from 0 to infinity. From the stable and unstable regions in example 1 (Figures 16 and 17), we see that, there is no stability switches on τ_1-axis or τ_2-axis, but there are several times stability switches on τ_2-axis in example 2 for some suitable parameter values (Figure 19). And, the stability regions in examples 1 and 2 are two different types in view of topology. The former is a connected region varying the parameter f from 0.6 to 2.9, and the latter from four parts to three parts, to two parts and to one connected region varying the parameter f from 0.095 to 0.35. That is to say, parameter f would change the stability switches times for some suitable parameter values of the system.

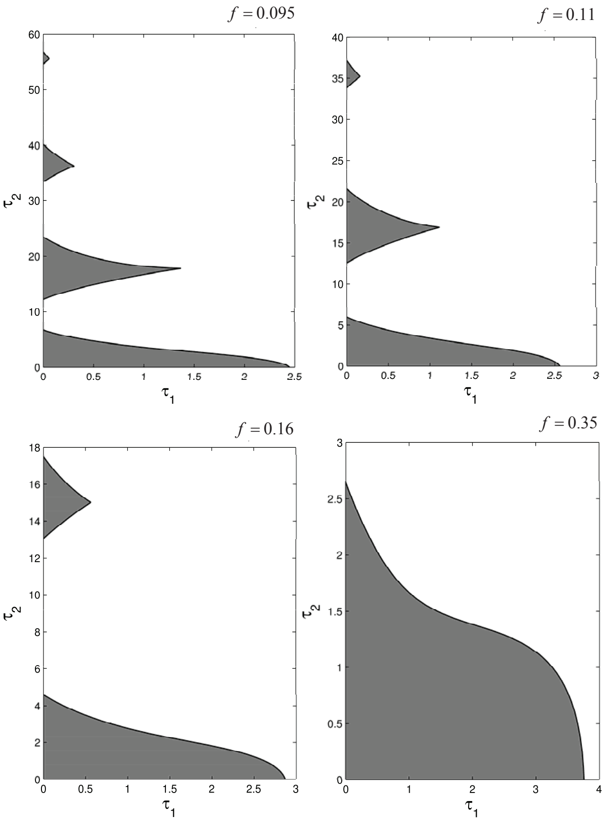

Figure 19. The stable regions (gray) and unstable regions (white) of the positive equilibrium point E_2 of the system (3.2) with $f = 0.095, 0.11, 0.16, 0.35$, respectively.

4.3. Delays induced complicated dynamic behaviors

The numerical simulations show that delayed system (1.5) has complicated dynamic behaviors (Figures 12, 13 and 14) when we change the time delays and far away from the first bifurcating critical time delay point, including periodic oscillating, quasi-periodic oscillating, period-doubling bifurcations, chaos, and those behaviors undiscovered if the system (1.5) has only one time delay [16–18]. That is to say, time delays are important factor to affect the complex dynamic behaviors of the system, since the positive equilibrium point E_2 of the system (1.5) is global asymptotically stable in the absence of time delays [16]. When time delay far away from the first critical point and increased, large time delays would make system (1.5) extinct (unbounded oscillation) undergoing a series of fast-slow oscillations or chaotic oscillations which make the prey and predator populations very closed to zero, and destroyed the permanence of it. And these are not found in [16–18]. All of the analysis show that the time delays would destroy the stability of the system, and induced complicated dynamic behaviors, even make the system die out.

All in all, time delays induced Hopf bifurcation, stability switches, and complicated dynamic behaviors for system (1.5), and make the system (1.5) subject to period oscillations and finite times stability switches via local Hopf bifurcation, and quasi-period oscillations, period-doubling bifurcations, chaotic oscillations and unbounded oscillations. Harmful time delays destroy the stability of the system, even make the system die out. How to control the bifurcation, unbounded oscillations and even chaos, arising from the multiple time delays system? The impulsive control strategies and the time-varying control strategies would be considered [25, 26], which could both improve the stability of the system and control periodic and chaotic oscillations effectively. We will continue to study these problems in the future.

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Conflict of interest

The author declares that there is no conflict of interest.

References

1. W. Aiello and H. Freedman, A time-delay model of single-species growth with stage structure, *Math. Biosci.*, **101** (1990), 139–153.
2. W. Aiello, H. Freedman and J. Wu, Analysis of a model representing stage-structured populations growth with stage-dependent time delay, *SIAM J. Appl. Math.*, **3** (1992), 855–869.
3. W. Wang and L. Chen, A predator-prey system with stage-structure for predator, *Comp. Math. Appl.*, **33** (1997), 83–91.
4. S. Liu, L. Chen and R. Agarwal, Recent progress on stage-structured population dynamics, *Math. Comput. Model.*, **36** (2002), 1319–1360.
5. S. Gao, Models for single species with three life history stages and cannibalism, *J. Biomath.*, **20** (2005), 385–391.
6. S. Yang and B. Shi, Periodic solution for a three-stage-structured predator-prey system with time delay, *J. Math. Anal. Appl.*, **341** (2008), 287–294.
7. H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Springer Science+Business Media, LLC, 201l.
8. E. Beretta and D. Breda, Discrete or distributed delay? Effects on stability of population growth, *Math. Biosci. Eng.*, **13** (2016), 19–41.
9. Z. Shen and J. Wei, Hopf bifurcation analysis in a diffusive predator-prey system with delay and surplus killing effect, *Math. Biosci. Eng.*, **15** (2018), 693–715.
10. S. Li and Z. Xiong, Bifurcation analysis of a predator-prey system with sex-structure and sexual favoritism, *Adv. Differ. Equ.*, **219** (2013), 1–24.
11. Z. Ma and S. Wang, A delay-induced predator Cprey model with Holling type functional response and habitat complexity, *Nonl. Dyna.*, **93** (2018), 1519–1544.

12. S. Kundu and S. Maitra Dynamical behaviour of a delayed three species predator Cprey model with cooperation among the prey species, *Nonl. Dyna.*, **92** (2018), 627–643.
13. L. Li and J. Shen, Bifurcations and Dynamics of a Predator CPrey Model with Double Allee Effects and Time Delays, *Int. J. Bifurc. Chaos*, **28** (2018), 1–14. (No. 1850135)
14. T. Caraballo, R. Colucci and L. Guerrini, On a predator prey model with nonlinear harvesting and distributed delay, *Comm. on Pure Appl. Anal.*, **17** (2018), 2703–2727.
15. X. Xu, Y. Wang and Y. Wang, Local bifurcation of a Ronsenzwing-MacArthur predator prey model with two prey-taxis, *Math. Biosci. Eng.*, **16** (2019), 1786-1797.
16. S. Li, Y. Xue and W. Liu, Hopf bifurcation and global periodic solutions for a three-stage-structured prey-predator system with delays, *Int. J. Info. Syst. Scie.*, **8** (2012), 142–156.
17. S. Li and X. Xue, Hopf bifurcation in a three-stage-structured prey-predator system with predator density dependent, *Comm. Comp. Info. Scie.*, **288** (2012), 740–747.
18. S. Li and W. Liu, Global hopf bifurcation in a delayed three-stage-structured prey-predator system, *Proceedings-5th Int. Conf. Info. Comp. Scie.*, (2012), 206–209.
19. J. Wu, Symmetric functional differential equations and neural networks with memory, *Trans. Am. Math. Soc.*, **350** (1998), 4799–4838.
20. Z. Wang, A very simple criterion for characterizing the crossing direction of time-delay systems with delay-dependent parameters, *Int. J. Bifu. Chaos*, **22** (2012), 1–7.
21. J. Hale, *Theory of Functional Differential Equations*, Springer, New York, 1977.
22. D. Breda, S. Maset and R. Vermiglio, TRACE-DDE: a tool for robust analysis and characteristic equations for delay differential equations, *Lect. Notes Cont. Info. Scie.*, **388** (2009), 145–155.
23. D. Breda, S. Maset and R. Vermiglio, Pseudospectral differencing methods for characteristic roots of delay differential equations, *SIAM J. Sci. Comput.*, **27** (2005), 482–495.
24. D. Breda, S. Maset and R. Vermiglio, An adaptive algorithm for efficient computation of level curves of surfaces, *Numer. Algorithms*, **52** (2009), 605–628.
25. Y. Zhao, X. Yu and L. Wang, Bifurcation and control in an inertial two-neuron system with time delays, *Int. J. Bifurc. Chaos*, **22** (2012), 1–15.
26. S. Li, W. Liu and X. Xue, Hopf bifurcation, chaos and impulsive control in a sex-structured prey-predator system with time delay, *J. Biomath.*, **30** (2015), 443–452.