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Research article

Positive steady states of a ratio-dependent predator-prey system with cross-diffusion

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Abstract: In this paper, we study a ratio-dependent predator-prey system with diffusion and cross-diffusion under the homogeneous Neumann boundary condition. By applying the maximum principle and Harnack's inequality, we present a priori estimates of the positive steady state of the system. The existence and non-existence of non-constant positive steady states are established. Our findings show that under certain hypotheses, non-constant positive steady states can exist due to the emergence of cross-diffusion, which reveals that cross-diffusion can induce stationary patterns but the random diffusion fails.

Keywords: cross-diffusion; priori estimates; non-constant positive steady state; maximum principle; Leray-Schauder degree theory

In Memory of Geoffrey J. Butler and Herbert I. Freedman

1. Introduction

The dynamic relationship between prey and predator has been one of the most interesting topics in the qualitative theory of population dynamics [1, 2]. For example, Bazykin [2] considered a prey-predator system with the bounded functional response:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u - ku^2 - \dfrac{uv}{1+mu}, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = -av - bv^2 + \dfrac{uv}{1+mu}, \end{cases} \tag{1.1}$$

where u and v represent the densities of the prey and predator, respectively. Here, a, b, k and m are positive constants. The predator consumes the prey based on the prey-dependent functional response $u/(1 + mu)$, and the term $-bv^2$ represents the self-limitation for the predator. The above model takes account of not only the interspecies interactions between the prey species and predators, but also the density dependence of predator species.

Recently, there is much evidence from biology and physiology to show that in many situations, especially when predators have to search, share or compete for food, the modeling and analysis of predator-prey systems for each specific case should be further developed according to the ratio-dependent theory [3, 4, 5], which can be roughly stated as that the per capita predator growth rate should depend on the ratio of prey to predators, but not just prey numbers, so it should be the ratio-dependent function response [6, 7, 8, 9], i.e., a function of u/v. On the basis of this consideration, substituting $\frac{u/v}{1+mu/v}$ for $\frac{u}{1+mu}$ in the above-mentioned model (1.1), we arrive at the following system:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = u - ku^2 - \dfrac{uv}{mu+v}, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = -av - bv^2 + \dfrac{uv}{mu+v}. \end{cases} \tag{1.2}$$

If $ma < 1$, system (1.2) has a unique positive equilibrium $\widetilde{\mathbf{w}} = (\tilde{u}, \tilde{v})^{\mathsf{T}}$, where

$$\tilde{u} = \frac{m(a+b\tilde{v})}{k}, \quad \tilde{v} = \frac{m(1-ma)}{k+m^2b}. \tag{1.3}$$

Taking into account the inhomogeneous distribution of predator and prey populations in different spatial locations, in this study we will incorporate diffusion and cross-diffusion to system (1.2) and consider a more general system:

$$\begin{cases} u_t = d_1\Delta u + d_{12}\Delta v + u - ku^2 - \dfrac{uv}{mu+v}, & x \in \Omega,\ t > 0, \\ v_t = d_{21}\Delta u + d_2\Delta v - av - bv^2 + \dfrac{uv}{mu+v}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.4}$$

where Ω is a fixed bounded domain with the sufficient smooth boundary $\partial\Omega$ in $\mathbb{R}^N$, ν is the outward unit normal vector of the boundary $\partial\Omega$, and Δ is the Laplace operator. The positive constants d_1 and d_2 are the random diffusion coefficients of prey and predator, respectively, which describe the natural dispersive forces of random movement of individuals. Here, d_{12} and d_{21} are the cross-diffusion coefficients that express population fluxes of preys and predators due to the presence of other species [10, 11, 12]. The non-flux boundary condition indicates that the system is self-contained with no external energy exchange. For more related studies on dynamics of population models with diffusion and cross-diffusion, we refer the reader to [12, 13, 14, 15, 16, 17] and references therein.

One of our motivations of this study lies in a fact that, in the past few decades there have been continuous interests in the existence of positive steady states of diffusive predator-prey systems, and the majority of works have devoted to discovering the effect of diffusion on positive steady states [18, 19, 20, 21, 22, 23, 24]. However, little attention has been paid to the case that both diffusion and cross-diffusion are present in population systems [25, 26, 27, 28]. Therefore, the main purpose of this paper is to make an attempt on the existence and non-existence of non-constant positive steady states of system (1.4). The corresponding steady-state equations of system (1.4) are:

$$\begin{cases} -d_1\Delta u - d_{12}\Delta v = u - ku^2 - \dfrac{uv}{mu+v}, & x \in \Omega, \\ -d_{21}\Delta u - d_2\Delta v = -av - bv^2 + \dfrac{uv}{mu+v}, & x \in \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

This paper is organized as follows. In Section 2, we discuss a priori estimate of the positive steady state for system (1.4), that is, a priori upper and lower bounds for positive solutions of the elliptic system (1.5). In Section 3, we investigate the existence and non-existence of non-constant positive solutions of system (1.5). In particular, the impact of diffusion and cross-diffusion on the existence of non-constant positive steady states is explored.

2. A priori estimates

In this section, to establish a priori estimates of upper and lower bounds of positive solutions for system (1.5) in a straightforward manner, we need the following two lemmas.

Lemma 2.1. *[22, Maximun Principle] Assume that $g \in C(\overline{\Omega} \times \mathbb{R}^1)$.*
(i) *If $\omega \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies $\Delta\omega(x) + g(x, \omega(x)) \geq 0,\ x \in \Omega;\ \frac{\partial\omega}{\partial\nu} \leq 0,\ x \in \partial\Omega$; and $\omega(x_0) = \max_{\overline{\Omega}} \omega$, then $g(x_0, \omega(x_0)) \geq 0$.*
(ii) *If $\omega \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies $\Delta\omega(x) + g(x, \omega(x)) \leq 0,\ x \in \Omega;\ \frac{\partial\omega}{\partial\nu} \geq 0,\ x \in \partial\Omega$; and $\omega(x_0) = \min_{\overline{\Omega}} \omega$, then $g(x_0, \omega(x_0)) \leq 0$.*

Lemma 2.2. *[27, Harnack's Inequality] Suppose that $c(x) \in C(\overline{\Omega})$. Let $\omega(x) \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a positive solution to*

$$\Delta\omega(x) + c(x)\omega(x) = 0,\ x \in \Omega;\ \frac{\partial\omega}{\partial\nu} = 0,\ x \in \partial\Omega.$$

Then there exists a positive constant $C_* = C_*(\|c(x)\|_\infty, \Omega)$ such that

$$\max_{\overline{\Omega}} \omega \leq C_* \min_{\overline{\Omega}} \omega.$$

In the following, the positive constants C_1, C_2, C_1^*, C_2^*, c_1 and c_2 will rely upon the domain Ω. However, while Ω is fixed, we will not mention this dependence explicitly. Also, for convenience, we shall write Λ instead of the set of constants (a, b, m, k) in the sequel.

Theorem 2.1. *[Upper Bounds] Suppose that $\frac{d_{12}}{d_1} \leq D$ and $\frac{d_{21}}{d_2} \leq D$ for an arbitrary fixed number D. Then there exist positive constants $C_i = C_i(D, \Lambda)$, $i = 1, 2$, such that the positive solution (u, v) of* *system* (1.5) *satisfies*

$$\max_{\overline{\Omega}} u \le C_1 \ \ and \ \ \max_{\overline{\Omega}} v \le C_2. \tag{2.1}$$

Proof. Set $\phi = d_1 u + d_{12} v$ and $\psi = d_{21} u + d_2 v$. Then the problem (1.5) becomes

$$\begin{cases} -\Delta\phi = \frac{1-ku-\frac{v}{mu+v}}{d_1+d_{12}\frac{v}{u}}\phi, & x \in \Omega, \\ -\Delta\psi = \frac{-a-bv+\frac{u}{mu+v}}{d_{21}\frac{u}{v}+d_2}\psi, & x \in \Omega, \\ \frac{\partial\phi}{\partial\nu} = \frac{\partial\psi}{\partial\nu} = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

Let $x_1 \in \overline{\Omega}$ be a point satisfying $\phi(x_1) = \max_{\overline{\Omega}} \phi$. Applying Lemma 2.1 to the first equation of (2.2), we can obtain $u(x_1) \le \frac{1}{k}$ and $v(x_1) \le \frac{m}{k}(1 - ku(x_1)) \le \frac{m}{k}$. Hence, we obtain

$$\begin{aligned} \max_{\overline{\Omega}} u &\le \frac{1}{d_1} \max_{\overline{\Omega}} \phi \\ &= \frac{1}{d_1}(d_1 u(x_1) + d_{12} v(x_1)) \\ &\le \frac{1}{k} + \frac{d_{12}}{d_1} \cdot \frac{m}{k} \\ &\triangleq C_1. \end{aligned} \tag{2.3}$$

Analogously, let $x_2 \in \overline{\Omega}$ be a point satisfying $\psi(x_2) = \max_{\overline{\Omega}} \psi$. Using Lemma 2.1 to the second equation of (2.2), we have $v(x_2) \le \frac{1-ma}{bm}$. Then we have

$$\begin{aligned} \max_{\overline{\Omega}} v &\le \frac{1}{d_2} \max_{\overline{\Omega}} \psi \\ &= \frac{1}{d_2}(d_{21} u(x_2) + d_2 v(x_2)) \\ &\le \frac{d_{21} C_1}{d_2} + \frac{1 - ma}{bm} \\ &\triangleq C_2. \end{aligned} \tag{2.4}$$

□

Before presenting the lower bound, we present the following lemma.

Lemma 2.3. *Let $ma < 1$, $d_{1,n}$, $d_{2,n}$, $d_{12,n}$, $d_{21,n}$ be positive constants, $n = 1, 2, ...,$ and (u_n, v_n) be the positive solution of system* (1.5) *with $d_1 = d_{1,n}$, $d_{12} = d_{12,n}$, $d_{21} = d_{21,n}$ and $d_2 = d_{2,n}$. Suppose that $(d_{1,n}, d_{2,n}, d_{12,n}, d_{21,n}) \to (d_1, d_2, d_{12}, d_{21})$, and $(u_n, v_n) \to (u^*, v^*)$ uniformly on $\overline{\Omega}$. If u^* and v^* are positive constants, then (u^*, v^*) satisfies*

$$1 - ku^* - \frac{v^*}{mu^* + v^*} = 0, \ \ -a - bv^* + \frac{u^*}{mu^* + v^*} = 0.$$

That is, $(u^*, v^*) = (\tilde{u}, \tilde{v})$, is the unique positive constant solution of system (1.5).

Proof. It is clear that $\int_\Omega u_n\left(1-ku_n-\frac{v_n}{mu_n+v_n}\right)dx = 0$ holds for all n. If $1-ku^*-\frac{v^*}{mu^*+v^*} > 0$, then $1-ku_n-\frac{v_n}{mu_n+v_n} > 0$ as n is getting large. However, u_n is positive, so this is impossible. Similarly, $1-ku^*-\frac{v^*}{mu^*+v^*} < 0$ is impossible either. Hence, $1-ku^*-\frac{v^*}{mu^*+v^*} = 0$. Using the same argument leads to $-a-bv^*+\frac{u^*}{mu^*+v^*} = 0$. Consequently, $(u^*,v^*) = (\tilde{u},\tilde{v})$. □

Theorem 2.2. *(Lower Bounds). Assume that d and D are fixed positive constants. Then there exist positive constants $c_i = c_i(d,D,\Lambda)$, $i=1,2$, such that when $d_1,d_2 \ge d$ and $\frac{d_{12}}{d_1}, \frac{d_{21}}{d_2} \le D$, the positive solution (u,v) of system* (1.4) *satisfies*

$$\min_{\overline{\Omega}} u \ge c_1 \ \text{and} \ \min_{\overline{\Omega}} v \ge c_2. \tag{2.5}$$

Proof. By a straightforward calculation, we have

$$\begin{aligned}
\left\|\frac{u\left(1-ku-\frac{v}{mu+v}\right)}{d_1u+d_{12}v}\right\|_\infty &\le \frac{1}{d_1}\left(1+k\max_{\overline{\Omega}} u+\max_{\overline{\Omega}}\left\{\frac{1}{m\frac{u}{v}+1}\right\}\right)\\
&= \frac{1}{d_1}\left(1+k\max_{\overline{\Omega}} u+\frac{1}{\min_{\overline{\Omega}}(m\frac{u}{v}+1)}\right)\\
&\le \frac{1}{d_1}\left(2+k\max_{\overline{\Omega}} u\right)\\
&\le \frac{1}{d_1}\left(2+kC_1\right),
\end{aligned}$$

and

$$\begin{aligned}
\left\|\frac{v\left(-a-bv+\frac{u}{mu+v}\right)}{d_{21}u+d_2v}\right\|_\infty &\le \frac{1}{d_2}\left(a+b\max_{\overline{\Omega}} v+\max_{\overline{\Omega}}\left\{\frac{1}{m+\frac{v}{u}}\right\}\right)\\
&= \frac{1}{d_2}\left(a+b\max_{\overline{\Omega}} v+\frac{1}{\min_{\overline{\Omega}}(m+\frac{v}{u})}\right)\\
&\le \frac{1}{d_2}\left(a+b\max_{\overline{\Omega}} v+\frac{1}{m}\right)\\
&\le \frac{1}{d_2}\left(a+\frac{1}{m}+bC_2\right).
\end{aligned}$$

Then, Lemma 2.2 indicates that there exist positive constants $C_1^* = C_1^*(d,\Lambda)$ and $C_2^* = C_2^*(d,\Lambda)$ such that $\max_{\overline{\Omega}}\phi \le C_1^*\min_{\overline{\Omega}}\phi$ and $\max_{\overline{\Omega}}\psi \le C_2^*\min_{\overline{\Omega}}\psi$. Thus we obtain

$$\frac{\max_{\overline{\Omega}} u}{\min_{\overline{\Omega}} u} \le \frac{\max_{\overline{\Omega}}\phi}{\min_{\overline{\Omega}}\left(d_1+d_{12}\frac{v}{u}\right)}\Bigg/\frac{\min_{\overline{\Omega}}\phi}{\max_{\overline{\Omega}}\left(d_1+d_{12}\frac{v}{u}\right)} = \frac{\max_{\overline{\Omega}}\phi}{\min_{\overline{\Omega}}\phi}\cdot\frac{\max_{\overline{\Omega}}\left(d_1+d_{12}\frac{v}{u}\right)}{\min_{\overline{\Omega}}\left(d_1+d_{12}\frac{v}{u}\right)} \le C_1^*, \tag{2.6}$$

and

$$\frac{\max_{\overline{\Omega}} v}{\min_{\overline{\Omega}} v} \le \frac{\max_{\overline{\Omega}} \psi}{\min_{\overline{\Omega}}\left(d_{21}\frac{u}{v}+d_2\right)} \Bigg/ \frac{\min_{\overline{\Omega}} \psi}{\max_{\overline{\Omega}}\left(d_{21}\frac{u}{v}+d_2\right)} = \frac{\max_{\overline{\Omega}} \psi}{\min_{\overline{\Omega}} \psi} \cdot \frac{\max_{\overline{\Omega}}\left(d_{21}\frac{u}{v}+d_2\right)}{\min_{\overline{\Omega}}\left(d_{21}\frac{u}{v}+d_2\right)} \le C_2^*. \tag{2.7}$$

Now, we estimate the positive lower bounds of u and v. Suppose that (2.5) is not true. Then there exists a sequence $\{d_{1,n}, d_{2,n}, d_{12,n}, d_{21,n}\}_{n=1}^{\infty}$ with $(d_{1,n}, d_{2,n}, d_{12,n}, d_{21,n}) \in [d,\infty)\times[d,\infty)\times(0,\infty)\times(0,\infty)$ such that the positive solution (u_n, v_n) of system (1.5) satisfies

$$\begin{cases} -\Delta(d_{1,n}u_n + d_{12,n}v_n) = u_n - ku_n^2 - \dfrac{u_nv_n}{mu_n+v_n}, & x\in\Omega,\\ -\Delta(d_{21,n}u_n + d_{2,n}v_n) = -av_n - bv_n^2 + \dfrac{u_nv_n}{mu_n+v_n}, & x\in\Omega,\\ \partial_\nu u_n = \partial_\nu v_n = 0, & x\in\partial\Omega, \end{cases} \tag{2.8}$$

and

$$\min_{\overline{\Omega}} u_n \to 0 \text{ or } \min_{\overline{\Omega}} v_n \to 0, \text{ as } n\to\infty. \tag{2.9}$$

For (2.8), it follows from the standard regularity theorem of elliptic equations [29] that there exists a subsequence of $\{(u_n, v_n)\}_{n=1}^{\infty}$, still denoted by $\{(u_n, v_n)\}_{n=1}^{\infty}$, and two nonnegative functions $u, v \in C^2(\overline{\Omega})$ satisfying $(u_n, v_n) \to (u, v)$ as $n\to\infty$. Assume that $(d_{1,n}, d_{2,n}, d_{12,n}, d_{21,n}) \to (d_1, d_2, d_{12}, d_{21}) \in [d,\infty)\times[d,\infty)\times(0,\infty)\times(0,\infty)$. In view of inequalities (2.6) and (2.7), it is easy to see that either $\max_{\overline{\Omega}} u_n \to 0$ or $\max_{\overline{\Omega}} v_n \to 0$, as $n\to\infty$.

So, there are three possible cases.

If $\max_{\overline{\Omega}} u_n \to 0$ and $\max_{\overline{\Omega}} v_n \to 0$, then $(u_n, v_n) \to (0,0)$ uniformly on $\overline{\Omega}$, which is a contradiction to Lemma 2.3.

If $\max_{\overline{\Omega}} u_n \to 0$ and $\max_{\overline{\Omega}} v_n \to \overline{v}$, where $\overline{v}$ is a positive constant. By integrating the second equation of system (2.8) in Ω, it gives

$$\int_\Omega v_n\,(a+bv_n)\,dx = 0.$$

Then we get $v_n \to -\frac{a}{b}$, which yields a contradiction to $\overline{v} > 0$. Hence, $\min_{\overline{\Omega}} u > 0$.

Similarly, if $\max_{\overline{\Omega}} u_n \to \overline{u}$ and $\max_{\overline{\Omega}} v_n \to 0$, where $\overline{u}$ is a positive constant. By integrating the first equation of system (2.8) in Ω, it gives

$$\int_\Omega u_n\,(1-ku_n)\,dx = 0.$$

Then $\overline{u} = \frac{1}{k}$. It implies that $(u_n, v_n) \to \left(\frac{1}{k}, 0\right)$ uniformly on $\overline{\Omega}$. Apparently, this leads to a contradiction to Lemma 2.3. Consequently, $\min_{\overline{\Omega}} v > 0$. □

3. Non-constant positive steady states

In this section, we study the non-existence and existence of non-constant positive steady states of system (1.5).

3.1. Non-existence of non-constant positive steady states

To consider the non-existence of non-constant solutions of system (1.5) in this subsection, for convenience, we denote

$$f(u,v) = u - ku^2 - \frac{uv}{mu+v}, \quad g(u,v) = -av - bv^2 + \frac{uv}{mu+v}.$$

Theorem 3.1. *Let d_2^* be a fixed positive constant and satisfy $\mu_1 d_2^* > \frac{1}{m}$, where μ_1 is the least positive eigenvalue of $-\Delta$ on Ω under the Neumann boundary condition. Then there exists a positive constant $D_1 = D_1(\Lambda, d_2^*)$ such that when $d_1 \geq D_1$ and $d_2 \geq d_2^*$, system* (1.5) *with $d_{12} = d_{21} = 0$ has no non-constant positive solution.*

Proof. Suppose that $\mathbf{w} = (u,v)^{\mathsf{T}}$ is a positive solution of system (1.5). Let $\tilde{\varphi} = \frac{1}{|\Omega|}\int_\Omega \varphi dx$ for any $\varphi \in L^1(\Omega)$. Multiplying the first two equations of (1.5) by $u - \tilde{u}$ and $v - \tilde{v}$, respectively, and then integrating on Ω, by integration by parts we obtain

$$\begin{aligned}
& \int_\Omega d_1|\nabla u|^2 dx + \int_\Omega d_2 |\nabla v|^2 dx \\
= & \int_\Omega f(u,v)(u-\tilde{u})dx + \int_\Omega g(u,v)(v-\tilde{v})dx \\
= & \int_\Omega [f(u,v) - f(\tilde{u},\tilde{v})](u-\tilde{u})dx + \int_\Omega [g(u,v)-g(\tilde{u},\tilde{v})](v-\tilde{v})dx \\
= & \int_\Omega \Big\{1 - k(u+\tilde{u}) - \frac{v\tilde{v}}{(mu+v)(m\tilde{u}+\tilde{v})}\Big\}(u-\tilde{u})^2 \\
& + \int_\Omega \Big\{ - \frac{mu\tilde{u}}{(mu+v)(m\tilde{u}+\tilde{v})} + \frac{v\tilde{v}}{(mu+v)(m\tilde{u}+\tilde{v})}\Big\}(u-\tilde{u})(v-\tilde{v})dx \\
& + \int_\Omega \Big\{ - a - b(v+\tilde{v}) + \frac{mu\tilde{u}}{(mu+v)(m\tilde{u}+\tilde{v})}\Big\}(v-\tilde{v})^2 dx \\
\leq & \int_\Omega \Big\{(u-\tilde{u})^2 + \frac{mu\tilde{u}+v\tilde{v}}{(mu+v)(m\tilde{u}+\tilde{v})}|u-\tilde{u}||v-\tilde{v}| + \frac{1}{m}(v-\tilde{v})^2\Big\}dx.
\end{aligned} \tag{3.1}$$

In view of Theorems 2.1 and 2.2, using Young's inequality to (3.1) we find

$$\begin{aligned}
& \int_\Omega d_1|\nabla u|^2 dx + \int_\Omega d_2|\nabla v|^2 dx \\
\leq & \int_\Omega \Big\{(u-\tilde{u})^2 + 2L|u-\tilde{u}||v-\tilde{v}| + \frac{1}{m}(v-\tilde{v})^2\Big\}dx \\
\leq & \int_\Omega \Big\{\Big(1+\frac{L}{\varepsilon}\Big)(u-\tilde{u})^2 + \Big(\frac{1}{m}+\varepsilon L\Big)(v-\tilde{v})^2\Big\}dx
\end{aligned} \tag{3.2}$$

for some positive constant L, where ε is the arbitrary small positive constant arising from Young's inequality.

Using Poincaré's inequality, we see that $\mu_1 \int_\Omega (u - \tilde{u})^2 dx \le \int_\Omega |\nabla u|^2 dx$ and $\mu_1 \int_\Omega (v - \tilde{v})^2 dx \le \int_\Omega |\nabla v|^2 dx$, where μ_1 is the least positive eigenvalue of $-\Delta$ on Ω under the Neumann boundary condition. It follows from inequality (3.2) that

$$\mu_1 \int_\Omega [d_1(u - \tilde{u})^2 + d_2(v - \tilde{v})^2] dx \le \int_\Omega \left\{ \left(1 + \frac{L}{\varepsilon}\right)(u - \tilde{u})^2 + \left(\frac{1}{m} + \varepsilon L\right)(v - \tilde{v})^2 \right\} dx.$$

Choosing a sufficiently small $\varepsilon_0 > 0$ such that $d_2^* \mu_1 \ge \frac{1}{m} + \varepsilon_0 L$, and taking $D_1 \triangleq \frac{1}{\mu_1}(1 + \frac{L}{\varepsilon_0})$, we arrive at the desired result $(u, v) = (\tilde{u}, \tilde{v})$. □

3.2. Existence of non-constant positive solutions

From the discussion in the preceding subsection, we know that when the cross-diffusion terms are absent, there might be no non-constant positive solutions for system (1.5). In this subsection, we shall discuss the existence of non-constant positive solutions of system (1.5) with respect to cross-diffusion coefficients d_{21} and d_{12} as the other parameters d_1 and d_2 are fixed by means of the Leray-Schauder degree theory.

To facilitate the discussion, we rewrite system (1.5) as

$$\begin{cases} -\Delta \mathbf{\Phi}(\mathbf{w}) = \mathbf{F}(\mathbf{w}), & x \in \Omega, \\ \frac{\partial \mathbf{w}}{\partial \nu} = \mathbf{0}, & x \in \partial\Omega, \end{cases} \tag{3.3}$$

where $\mathbf{w} = (u, v)^\mathsf{T}$, $\mathbf{\Phi}(\mathbf{w}) = (\phi, \psi)^\mathsf{T}$, and $\mathbf{F}(\mathbf{w}) = (f(u, v),\ g(u, v))^\mathsf{T}$.

Let $\mathbf{X} = \left\{ \mathbf{w} \in \left[C^1(\overline{\Omega})\right]^2 \middle| \frac{\partial \mathbf{w}}{\partial \nu} = \mathbf{0} \text{ on } \partial\Omega \right\}$, and define

$$\mathbf{X}^+ = \{\mathbf{w} \in \mathbf{X} | \mathbf{w} > 0 \ \ on \ \ \overline{\Omega}\},$$
$$\mathcal{B}(c) = \{\mathbf{w} \in \mathbf{X} | c^{-1} < u, v < c \ \ on \ \ \overline{\Omega}\},$$

where c is a positive constant that is guaranteed to exist by Theorems 2.1 and 2.2.

Assume that

$$d_1 d_2 - d_{12} d_{21} \neq 0. \tag{3.4}$$

Since $\mathbf{\Phi}_\mathbf{w}(\mathbf{w}) = \begin{pmatrix} d_1 & d_{12} \\ d_{21} & d_2 \end{pmatrix}$, the determinant $\det \mathbf{\Phi}_\mathbf{w}(\mathbf{w})$ is nonzero for all non-negative $\mathbf{w}$, $\mathbf{\Phi}_\mathbf{w}^{-1}(\mathbf{w})$ exists and $\det \mathbf{\Phi}_\mathbf{w}^{-1}(\mathbf{w})$ is of the same sign as $\det \mathbf{\Phi}_\mathbf{w}(\mathbf{w})$. Thus, $\mathbf{w}$ is a positive solution of system (3.3) if and only if

$$\mathbf{G}(\mathbf{w}) \triangleq \mathbf{w} - (\mathbf{I} - \Delta)^{-1}\{\mathbf{\Phi}_\mathbf{w}^{-1}(\mathbf{w})[\mathbf{F}(\mathbf{w}) + \nabla\mathbf{w}\mathbf{\Phi}_{\mathbf{ww}}(\mathbf{w})\nabla\mathbf{w}] + \mathbf{w}\} = 0, \ \ \mathbf{w} \in \mathbf{X}^+, \tag{3.5}$$

where $(\mathbf{I} - \Delta)^{-1}$ is the inverse of $\mathbf{I} - \Delta$ in $\mathbf{X}$ under the homogeneous Neumann boundary condition. Since $\mathbf{G}(\cdot)$ is a compact perturbation of the identity operator, for any $\mathcal{B} = \mathcal{B}(c)$, the Leray-Schauder degree of $\deg(\mathbf{G}(\cdot), \mathbf{0}, \mathcal{B})$ is well-defined if $\mathbf{G}(\mathbf{w}) \neq 0$ on $\partial\mathcal{B}$.

Note that

$$D_{\mathbf{w}}\mathbf{G}(\widetilde{\mathbf{w}}) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\mathbf{\Phi}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\mathbf{F}_{\mathbf{w}}(\widetilde{\mathbf{w}}) + \mathbf{I}\},$$

where

$$\mathbf{\Phi}_{\mathbf{w}}(\widetilde{\mathbf{w}}) = \begin{pmatrix} d_1 & d_{12} \\ d_{21} & d_2 \end{pmatrix}, \quad \mathbf{F}_{\mathbf{w}}(\widetilde{\mathbf{w}}) = \begin{pmatrix} -\frac{m^2\tilde{u}^2}{(m\tilde{u}+\tilde{v})^2} & -\frac{m\tilde{u}^2}{(m\tilde{u}+\tilde{v})^2} \\ \frac{\tilde{v}^2}{(m\tilde{u}+\tilde{v})^2} & -\frac{\tilde{u}\tilde{v}}{(m\tilde{u}+\tilde{v})^2} - b\tilde{v} \end{pmatrix}.$$

We recall that if $D_{\mathbf{w}}\mathbf{G}(\widetilde{\mathbf{w}})$ is invertible, the index of $\mathbf{G}$ at $\widetilde{\mathbf{w}}$ is defined by

$$\text{index}(\mathbf{G}(\cdot), \widetilde{\mathbf{w}}) = (-1)^{\gamma},$$

where γ is the total number of eigenvalues of $D_{\mathbf{w}}\mathbf{G}(\widetilde{\mathbf{w}})$ with negative real parts (counting multiplicities), then the degree $\deg(\mathbf{G}(\cdot), \mathbf{0}, \mathcal{B})$ is equal to the sum of the indices over all isolated solutions when $\mathbf{G} = \mathbf{0}$ in $\mathcal{B}(c)$, provided that $\mathbf{G} \neq \mathbf{0}$ on $\partial\mathcal{B}$.

Let $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω under the homogeneous Neumann boundary condition and $E(\mu_i)$ be the eigenspaces with respect to μ_i. Let $\{\phi_{ij}; j = 1, 2, \cdots, \dim E(\mu_i)\}$ be a set of orthonormal basis of $E(\mu_i)$ and $\mathbf{X}_{ij} = \{\mathbf{c}\phi_{ij} | \mathbf{c} \in \mathbb{R}^2\}$.

Denote

$$\mathbf{X} = \left\{\mathbf{w} \in \left[C^1(\overline{\Omega})\right]^2 \left| \frac{\partial \mathbf{w}}{\partial \nu} = \mathbf{0} \text{ on } \partial\Omega \right.\right\} \text{ and } \mathbf{X}_{ij} = \{\mathbf{c}\phi_{ij} \mid \mathbf{c} \in \mathbb{R}^2\}.$$

Then

$$\mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i \quad \textit{and} \quad \mathbf{X}_i = \bigoplus_{j=1}^{\dim E(\mu_i)} \mathbf{X}_{ij}.$$

We refer to the decomposition above in the following discussions of the eigenvalues of $D_{\mathbf{w}}\mathbf{G}(\widetilde{\mathbf{w}})$. We know that $\mathbf{X}_{ij}$ is invariant under $D_{\mathbf{w}}\mathbf{G}(\widetilde{\mathbf{w}})$ for each $i \in \mathbb{N}$ and each $j \in [1, \dim E(\mu_i)] \cap \mathbb{N}$, i.e., $D_{\mathbf{w}}\mathbf{G}(\widetilde{\mathbf{w}})\mathbf{w} \in \mathbf{X}_{ij}$ for any $\mathbf{w} \in \mathbf{X}_{ij}$. Thus, λ is an eigenvalue of $D_{\mathbf{w}}\mathbf{G}(\widetilde{\mathbf{w}})$ on $\mathbf{X}_{ij}$ if and only if it is an eigenvalue of the matrix

$$\mathbf{I} - \frac{1}{1 + \mu_i}\left[\mathbf{\Phi}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\mathbf{F}_{\mathbf{w}}(\widetilde{\mathbf{w}}) + \mathbf{I}\right] = \frac{1}{1 + \mu_i}\left[\mu_i\mathbf{I} - \mathbf{\Phi}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\mathbf{F}_{\mathbf{w}}(\widetilde{\mathbf{w}})\right].$$

Hence, $D_{\mathbf{w}}\mathbf{G}(\widetilde{\mathbf{w}})$ is invertible if and only if the matrix

$$\mathbf{I} - \frac{1}{1 + \mu_i}\left[\mathbf{\Phi}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\mathbf{F}_{\mathbf{w}}(\widetilde{\mathbf{w}}) + \mathbf{I}\right]$$

is non-singular for any $i \geq 0$.

Denote

$$H(\mu) = H(\widetilde{\mathbf{w}}; \mu) \triangleq \det\left\{\mu\mathbf{I} - \mathbf{\Phi}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\mathbf{F}_{\mathbf{w}}(\widetilde{\mathbf{w}})\right\}. \tag{3.6}$$

We observe that if $H(\mu_i) \neq 0$, then for each $1 \leq j \leq \dim E(\mu_i)$, the number of negative eigenvalues of $D_{\mathbf{w}}\mathbf{G}(\widetilde{\mathbf{w}})$ on $\mathbf{X}_{ij}$ is odd if and only if $H(\mu_i) < 0$.

We summarize our result as follows.

Theorem 3.2. *Assume that the matrix $\mu_i \mathbf{I} - \mathbf{\Phi}_{\mathrm{w}}^{-1}(\widetilde{\mathbf{w}})\mathbf{F}_{\mathrm{w}}(\widetilde{\mathbf{w}})$ is non-singular for an arbitrary $i \geq 0$. Then there holds*

$$\mathrm{index}(\mathbf{G}(\cdot), \widetilde{\mathbf{w}}) = (-1)^{\sigma},$$

where $\sigma = \sum_{i \geq 0,\, H(\mu_i) < 0} \dim E(\mu_i)$.

According to the above Theorem, we need to consider the sign of $H(\mu_i)$ in order to calculate the index of $(\mathbf{G}(\cdot), \widetilde{\mathbf{w}})$. In addition, by (3.6) we have

$$H(\mu) = \det\left\{\mathbf{\Phi}_{\mathrm{w}}^{-1}(\widetilde{\mathbf{w}})\right\} \det\left\{\mu \mathbf{\Phi}_{\mathrm{w}}(\widetilde{\mathbf{w}}) - \mathbf{F}_{\mathrm{w}}(\widetilde{\mathbf{w}})\right\}.$$

Hence, we need to consider the signs of $\det\{\mathbf{\Phi}_{\mathrm{w}}^{-1}(\widetilde{\mathbf{w}})\}$ and $\det\{\mu \mathbf{\Phi}_{\mathrm{w}}(\widetilde{\mathbf{w}}) - \mathbf{F}_{\mathrm{w}}(\widetilde{\mathbf{w}})\}$, respectively.

By a direct calculation we get

$$\begin{aligned} \det\left\{\mathbf{\Phi}_{\mathrm{w}}^{-1}(\widetilde{\mathbf{w}})\right\} &= \frac{1}{A}, \\ \det\left\{\mu \mathbf{\Phi}_{\mathrm{w}}(\widetilde{\mathbf{w}}) - \mathbf{F}_{\mathrm{w}}(\widetilde{\mathbf{w}})\right\} &= A\mu^2 - B\mu + C \triangleq \mathcal{A}(\mu), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} A &= d_1 d_2 - d_{12} d_{21}, \\ B &= -\frac{m^2 \tilde{u}^2}{(m\tilde{u} + \tilde{v})^2} d_2 - \left(\frac{\tilde{u}\tilde{v}}{(m\tilde{u} + \tilde{v})^2} + b\tilde{v}\right) d_1 - \frac{\tilde{v}^2}{(m\tilde{u} + \tilde{v})^2} d_{12} + \frac{m\tilde{u}^2}{(m\tilde{u} + \tilde{v})^2} d_{21}, \\ C &= \det\left\{\mathbf{F}_{\mathrm{w}}(\widetilde{\mathbf{w}})\right\} = \frac{ma\tilde{u}\tilde{v}}{(m\tilde{u} + \tilde{v})^2} + kb\tilde{u}\tilde{v} > 0. \end{aligned}$$

Let $\tilde{\mu}_1$ and $\tilde{\mu}_2$ be the two roots of $\mathcal{A}(\mu) = 0$ with $Re\{\tilde{\mu}_1\} \leq Re\{\tilde{\mu}_2\}$. Then $\tilde{\mu}_1 \tilde{\mu}_2 = \frac{C}{A}$, which is of the same sign as A.

Next, we discuss the dependence of $\mathcal{A}(\mu)$ on d_{12} and d_{21}, respectively. Due to the following limits:

$$\begin{aligned} &\lim_{d_{12} \to \infty} \frac{A}{d_{12}} = -d_{21} < 0, \quad \lim_{d_{21} \to \infty} \frac{A}{d_{21}} = -d_{12} < 0, \\ &\lim_{d_{12} \to \infty} \frac{B}{d_{12}} = -\frac{\tilde{v}^2}{(m\tilde{u} + \tilde{v})^2} < 0, \quad \lim_{d_{21} \to \infty} \frac{B}{d_{21}} = \frac{m\tilde{u}^2}{(m\tilde{u} + \tilde{v})^2} > 0, \end{aligned}$$

it follows from (3.7) that

$$\begin{aligned} \lim_{d_{12} \to \infty} \frac{\mathcal{A}(\mu)}{d_{12}} &= \mu \left[-d_{21}\mu + \frac{\tilde{v}^2}{(m\tilde{u} + \tilde{v})^2}\right], \\ \lim_{d_{21} \to \infty} \frac{\mathcal{A}(\mu)}{d_{21}} &= \mu \left[-d_{12}\mu - \frac{m\tilde{u}^2}{(m\tilde{u} + \tilde{v})^2}\right]. \end{aligned} \tag{3.8}$$

Note that the above two limits hold only when d_{12} (or d_{21}) is chosen to be large enough. This is certainly possible if $A = d_1 d_2 - d_{12} d_{21} < 0$ for the fixed d_1 and d_2. In this case, we have

$$\det\left\{\mathbf{\Phi}_{\mathrm{w}}^{-1}(\widetilde{\mathbf{w}})\right\} = \frac{1}{A} < 0 \ \text{ and } \ \tilde{\mu}_1 \tilde{\mu}_2 = \frac{C}{A} < 0.$$

Based on the above analyses, in summary we obtain the following results.

Theorem 3.3. *Assume $A < 0$. Then there exists a positive constant d_{12}^* such that when $d_{12} \geq d_{12}^*$, both of the two roots $\tilde{\mu}_1(d_{12})$ and $\tilde{\mu}_2(d_{12})$ of $\mathcal{A}(\mu) = 0$ are real and satisfy*

$$\begin{aligned} &\lim_{d_{12}\to\infty} \tilde{\mu}_1(d_{12}) = 0, \\ &\lim_{d_{12}\to\infty} \tilde{\mu}_2(d_{12}) = \frac{\tilde{v}^2}{d_{21}(m\tilde{u}+\tilde{v})^2} \triangleq \bar{\mu} > 0, \end{aligned} \tag{3.9}$$

where

$$\begin{cases} \tilde{\mu}_1(d_{12}) < 0 < \tilde{\mu}_2(d_{12}), \\ \mathcal{A}(\mu; d_{12}) < 0 \text{ when } \mu \in (-\infty, \tilde{\mu}_1(d_{12})) \cup (\tilde{\mu}_2(d_{12}), \infty), \\ \mathcal{A}(\mu; d_{12}) > 0 \text{ when } \mu \in (\tilde{\mu}_1(d_{12}), \tilde{\mu}_2(d_{12})). \end{cases} \tag{3.10}$$

Theorem 3.4. *Assume $A < 0$. Then there exists a positive constant d_{21}^* such that when $d_{21} \geq d_{21}^*$, both of the two roots $\tilde{\mu}_1(d_{21})$ and $\tilde{\mu}_2(d_{21})$ of $\mathcal{A}(\mu) = 0$ are real and satisfy*

$$\begin{aligned} &\lim_{d_{21}\to\infty} \tilde{\mu}_1(d_{21}) = -\frac{m\tilde{u}^2}{d_{12}(m\tilde{u}+\tilde{v})^2} \triangleq \bar{\bar{\mu}} < 0, \\ &\lim_{d_{21}\to\infty} \tilde{\mu}_2(d_{21}) = 0, \end{aligned} \tag{3.11}$$

where

$$\begin{cases} \tilde{\mu}_1(d_{21}) < 0 < \tilde{\mu}_2(d_{21}), \\ \mathcal{A}(\mu; d_{21}) < 0 \ \text{ when } \ \mu \in (-\infty, \tilde{\mu}_1(d_{21})) \cup (\tilde{\mu}_2(d_{21}), \infty), \\ \mathcal{A}(\mu; d_{21}) > 0 \ \text{ when } \ \mu \in (\tilde{\mu}_1(d_{21}), \tilde{\mu}_2(d_{21})). \end{cases} \tag{3.12}$$

The following theorem is regarding the existence of non-constant positive solutions to system (3.3) with respect to the cross-diffusion coefficient d_{12}, while all of other parameters are fixed.

Theorem 3.5. *Suppose that $A < 0$, and the parameters Λ, d_1, d_2 and d_{21} are fixed. Let $\bar{\mu}$ be given by* (3.9). *If $\bar{\mu} \in (\mu_n, \mu_{n+1})$ for some $n \geq 1$, and the sum $\sigma_n = \sum_{i=1}^{n} \dim E(\mu_i)$ is odd, then there exists a positive constant d_{12}^* such that when $d_{12} \geq d_{12}^*$, system* (3.3) *has at least one non-constant positive solution.*

Proof. According to Theorem 3.3, there exists a positive constant d_{12}^* such that, if $d_{12} \geq d_{12}^*$, then (3.10) holds and

$$\tilde{\mu}_1(d_{12}) < 0 = \mu_0 < \tilde{\mu}_2(d_{12}), \quad \tilde{\mu}_2(d_{12}) \in (\mu_n, \mu_{n+1}). \tag{3.13}$$

It suffices to prove that for all $d_{12} \geq d_{12}^*$, system (3.3) has at least one non-constant positive solution. By the way of contradiction, we assume that this is not true for some d_{12} ($\geq d_{12}^*$). By applying the homotopy invariance of the topological degree, we can see a contradiction explicitly.

For any fixed d_{12} that satisfies $d_{12} \geq d_{12}^*$, we take $\hat{d}_2 \geq d_2^*$ and $\hat{d}_1 \geq D_1$, where $\mu_1 d_2^* > \frac{1}{m}$ and $D_1 = D_1(\Lambda, d_2^*)$ are given by Theorem 3.1. For $t \in [0, 1]$, we define

$$\mathbf{\Phi}(t; \mathbf{w}) = \begin{pmatrix} [td_1 + (1-t)\hat{d}_1]u + td_{12}v \\ td_{21}u + [td_2 + (1-t)\hat{d}_2]v \end{pmatrix}.$$

Consider the system

$$\begin{cases} -\Delta \mathbf{\Phi}(t;\mathbf{w}) = \mathbf{F}(\mathbf{w}), & x \in \Omega, \\ \frac{\partial \mathbf{w}}{\partial \nu} = \mathbf{0}, & x \in \partial\Omega. \end{cases} \tag{3.14}$$

Then $\mathbf{w}$ is a non-constant positive solution of system (3.3) if and only if it is a positive solution of system (3.14) when $t = 1$. It is obvious that $\widetilde{\mathbf{w}}$ is the unique positive constant solution of system (3.14) for any $t \in [0, 1]$. From (3.5), we know that for any $t \in [0, 1]$, $\mathbf{w}$ is a positive solution of system (3.3) if and only if

$$\begin{aligned} \mathbf{G}(t;\mathbf{w}) &\triangleq \mathbf{w} - (\mathbf{I} - \Delta)^{-1}\{\mathbf{\Phi}_{\mathbf{w}}^{-1}(t;\mathbf{w})[\mathbf{F}(\mathbf{w}) + \nabla\mathbf{w}\mathbf{\Phi}_{\mathbf{ww}}(t;\mathbf{w})\nabla\mathbf{w}] + \mathbf{w}\} \\ &= 0, \text{ for } \mathbf{w} \in \mathbf{X}^+. \end{aligned}$$

Analogous to (3.6), we set

$$H(t,\mu) = \det\left\{\mathbf{\Phi}_{\mathbf{w}}^{-1}(t,\widetilde{\mathbf{w}})\right\}\det\{\mu\mathbf{\Phi}_{\mathbf{w}}(t,\widetilde{\mathbf{w}}) - \mathbf{F}_{\mathbf{w}}(\widetilde{\mathbf{w}})\}.$$

It is easy to see that $\mathbf{G}(1;\mathbf{w}) = \mathbf{G}(\mathbf{w})$ and

$$D_{\mathbf{w}}\mathbf{G}(t;\widetilde{\mathbf{w}}) = \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\mathbf{\Phi}_{\mathbf{w}}^{-1}(t;\widetilde{\mathbf{w}})\mathbf{F}_{\mathbf{w}}(\widetilde{\mathbf{w}}) + \mathbf{I}\}.$$

In particular, we obtain

$$\begin{aligned} D_{\mathbf{w}}\mathbf{G}(0;\widetilde{\mathbf{w}}) &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\hat{\mathbf{\Phi}}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\mathbf{F}_{\mathbf{w}}(\widetilde{\mathbf{w}}) + \mathbf{I}\}, \\ D_{\mathbf{w}}\mathbf{G}(1;\widetilde{\mathbf{w}}) &= \mathbf{I} - (\mathbf{I} - \Delta)^{-1}\{\mathbf{\Phi}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\mathbf{F}_{\mathbf{w}}(\widetilde{\mathbf{w}}) + \mathbf{I}\} = D_{\mathbf{w}}\mathbf{G}(\widetilde{\mathbf{w}}), \end{aligned}$$

where

$$\hat{\mathbf{\Phi}}_{\mathbf{w}}(\widetilde{\mathbf{w}}) = \begin{pmatrix} \hat{d}_1 & 0 \\ 0 & \hat{d}_2 \end{pmatrix} \text{ and } \mathbf{\Phi}_{\mathbf{w}}(\widetilde{\mathbf{w}}) = \begin{pmatrix} d_1 & d_{12} \\ d_{21} & d_2 \end{pmatrix}.$$

More specifically, when $t = 1$, from (3.6) and (3.7) we have

$$\begin{aligned} H(1,\mu) &= H(\mu) \\ &= \det\{\mathbf{\Phi}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\}\det\{\mu\mathbf{\Phi}_{\mathbf{w}}(\widetilde{\mathbf{w}}) - \mathbf{F}_{\mathbf{w}}(\widetilde{\mathbf{w}})\} \\ &= \det\{\mathbf{\Phi}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\}\mathcal{A}(\mu), \end{aligned} \tag{3.15}$$

where

$$\det\{\mathbf{\Phi}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\} = (\det\{\mathbf{\Phi}_{\mathbf{w}}(\widetilde{\mathbf{w}})\})^{-1} = \frac{1}{A} < 0,$$

and $\mathcal{A}(\mu)$ is defined by (3.7).

According to (3.10), (3.13) and (3.15), we deduce that

$$\begin{cases} H(1,\mu_0) = H(0) < 0, \\ H(1,\mu_i) < 0 & \text{when } 1 \le i \le n, \\ H(1,\mu_{i+1}) > 0 & \text{when } i \ge n+1. \end{cases}$$

Therefore, zero is not an eigenvalue of the matrix $\mu_i\mathbf{I} - \mathbf{\Phi}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\mathbf{F}_{\mathbf{w}}(\widetilde{\mathbf{w}})$ for any $i \geq 0$, and

$$\sum_{i\geq 0, H(1,\mu_i)<0} \dim E(\mu_i) = \sum_{i=1}^{n} \dim E(\mu_i) = \sigma_n$$

is odd. It follows from Theorem 3.2 that

$$\text{index}(\mathbf{\Phi}(1;\cdot), \widetilde{\mathbf{w}}) = (-1)^{\gamma} = (-1)^{\sigma_n} = -1. \tag{3.16}$$

When $t = 0$, we have

$$\begin{aligned} H(0,\mu) &= \det\{\hat{\mathbf{\Phi}}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\}\det\{\mu\hat{\mathbf{\Phi}}_{\mathbf{w}}(\widetilde{\mathbf{w}}) - \mathbf{F}_{\mathbf{w}}(\widetilde{\mathbf{w}})\} \\ &= \det\{\hat{\mathbf{\Phi}}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\}\hat{\mathcal{A}}(\mu), \end{aligned} \tag{3.17}$$

where

$$\det\{\hat{\mathbf{\Phi}}_{\mathbf{w}}^{-1}(\widetilde{\mathbf{w}})\} = \left(\hat{d}_1\hat{d}_2\right)^{-1} > 0,$$

and we use $\hat{\mathcal{A}}(\mu)$ to represent $\mathcal{A}(\mu)$ given in (3.7) under the restriction of $d_{12} = d_{21} = 0$. In this case, by Theorem 3.1 it implies that $\mathbf{G}(0;\mathbf{w})$=0 only has the positive constant solution $\widetilde{\mathbf{w}}$ in $\mathbf{X}^+$. By a direct calculation we have

$$\hat{\mathcal{A}}(\mu) = \hat{d}_1\hat{d}_2\mu^2 + \left[\left(\frac{\tilde{u}\tilde{v}}{(m\tilde{u}+\tilde{v})^2} + b\tilde{v}\right)\hat{d}_1 + \frac{m^2\tilde{u}^2}{(m\tilde{u}+\tilde{v})^2}\hat{d}_2\right]\mu + \frac{ma\tilde{u}\tilde{v}}{(m\tilde{u}+\tilde{v})^2} + kb\tilde{u}\tilde{v},$$

and so $H(0,\mu_i) > 0$ for all $i \geq 0$.

Discussing in the same manner, we can prove that

$$\text{index}(\mathbf{\Phi}(0;\cdot), \widetilde{\mathbf{w}}) = (-1)^0 = 1. \tag{3.18}$$

According to Theorems 2.1 and 2.2, there exists a positive constant c such that the positive solution of system (3.14) satisfies $c^{-1} < u$ and $v < c$ for any $t \in [0,1]$. Thus for any $t \in [0,1]$, there holds $\mathbf{\Phi}(t;\mathbf{w}) \neq 0$ on $\partial\mathcal{B}(M)$. By the homotopy invariance of the topological degree, we have

$$\deg(\mathbf{\Phi}(1;\cdot), 0, \mathcal{B}(M)) = \deg(\mathbf{\Phi}(0;\cdot), 0, \mathcal{B}(M)). \tag{3.19}$$

Note that both $\mathbf{\Phi}(1;\mathbf{w}) = 0$ and $\mathbf{\Phi}(0;\mathbf{w}) = 0$ have the unique positive solution $\widetilde{\mathbf{w}}$ in $\mathcal{B}(M)$. From (3.16) and (3.18) we get

$$\begin{cases} \deg(\mathbf{\Phi}(0;\cdot), 0, \mathcal{B}(M)) = \text{index}(\mathbf{\Phi}(0;\cdot), \widetilde{\mathbf{w}}) = 1, \\ \deg(\mathbf{\Phi}(1;\cdot), 0, \mathcal{B}(M)) = \text{index}(\mathbf{\Phi}(1;\cdot), \widetilde{\mathbf{w}}) = -1, \end{cases} \tag{3.20}$$

which yields a contradiction with (3.19). □

Processing in an analogous way as we just did in the proof of Theorem 3.5, we can obtain the following result regarding the existence of non-constant positive steady states for system (3.3) with respect to the cross-diffusion coefficient d_{21}. So, we omit the proof.

Theorem 3.6. *Suppose that $A < 0$, and the parameters Λ, d_1, d_2 and d_{12} are fixed. Let $\bar{\bar{\mu}}$ be given by* (3.11). *If $\bar{\bar{\mu}} \in (\mu_n, \mu_{n+1})$ for some $n \geq 1$, and the sum $\sigma_n = \sum_{i=1}^{n} \dim E(\mu_i)$ is odd, then there exists a positive constant d_{21}^* such that when $d_{21} \geq d_{21}^*$, system* (3.3) *has at least one non-constant positive solution.*

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Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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