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Research article

Dynamics analysis of a stochastic SIRS epidemic model with nonlinear incidence rate and transfer from infectious to susceptible

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Abstract: In this paper, we investigate the dynamics of a stochastic SIRS epidemic model with nonlinear incidence rate and transfer from infectious to susceptible. Firstly, the existence and uniqueness of global positive solution of the model with any positive initial value are proved. Next, sufficient conditions for extinction and persistence of the disease are established. It is found that a large noise intensity has the effect of suppressing the epidemic. At last, some numerical simulations are introduced to demonstrate the theoretical results. Our results generalize and improve the existing results.

Keywords: stochastic SIRS epidemic model; extinction; persistence; nonlinear incidence rate

1. Introduction

Mathematical models describing the population dynamics of infectious diseases have been playing an important role in better understanding of epidemiological patterns and disease control for a long time (see [1, 2, 3, 4]). Some infectious diseases confer temporary or permanent immunity. For some diseases, such as cholera, pertussis and influenza, temporary immunity may disappear after some time. And recovered individuals will become susceptible after losing the temporary immunity. This kind of diseases can be modeled by SIRS models (see [5, 6]). In addition, for some bacterial agent diseases such as meningitis and venereal diseases, recovery cannot generate immunity for a long time. Infected individuals may recover after some treatments and go back directly to the susceptible class on account of the presence of temporary antibodies. In [6], the authors investigated the following SIRS epidemic

model with a nonlinear incidence $\beta S f(I)$ and transfer from the infected class to the susceptible class

$$\begin{cases} \dot{S}(t) = \Lambda - \mu S(t) - \beta S(t) f(I(t)) + \gamma_1 I(t) + \delta R(t), \\ \dot{I}(t) = \beta S(t) f(I(t)) - (\mu + \gamma_1 + \gamma_2 + \alpha) I(t), \\ \dot{R}(t) = \gamma_2 I(t) - (\mu + \delta) R(t), \end{cases} \tag{1.1}$$

with the initial conditions

$$S(0) = S_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad R(0) = R_0 \geq 0, \tag{1.2}$$

where $S(t)$, $I(t)$ and $R(t)$ are respectively the number of susceptible, infectious and recovered individuals at time t. Λ is the recruitment rate of susceptible individuals, μ is the natural death rate, α is the mortality caused by the disease, γ_1 denotes the transfer rate from the infected class to the susceptible class, γ_2 is the recovery rate of the infective individuals, δ is the rate constant for loss of immunity. Suppose that Λ and μ are positive, while α, γ_1, γ_2 and δ are nonnegative. f is a real locally Lipschitz function on $\mathbb{R}_+ = [0, \infty)$ satisfying $f(0) = 0$, $f(I) > 0$, $f(I)/I$ is nonincreasing for $I > 0$ and $\lim_{I \to 0^+} \frac{f(I)}{I} = f'(0) > 0$.

Denote $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0\}$. From [6], model (1.1) always has a disease-free equilibrium $E_0 = (S_0, I_0, R_0) = (\frac{\Lambda}{\mu}, 0, 0)$. By using the next generation matrix method, [6] obtained the basic reproduction number $\mathcal{R}_0 = \frac{\Lambda \beta f'(0)}{\mu(\mu + \gamma_1 + \gamma_2 + \alpha)}$. Further, E_0 is globally asymptotically stable in D if $\mathcal{R}_0 < 1$, where $D = \{(S, I, R) \in \mathbb{R}^3_+ : S + I + R \leq \frac{\Lambda}{\mu}\}$. If $\mathcal{R}_0 > 1$, there exists a globally asymptotically stable endemic equilibrium $E^* = (S^*, I^*, R^*)$.

However, model (1.1) is just a deterministic model. In the real world, epidemic models are always affected by the environmental noise. Stochastic models may be a more appropriate way of modeling epidemics in many circumstances (see [7, 8, 9, 10]). In [11, 12, 13], the authors investigated the stochastic SIS epidemic models. In [14, 15, 16, 17, 18, 19, 20], the authors investigated the stochastic SIRS epidemic models. In this paper, we investigate the following stochastic SIRS epidemic model

$$\begin{cases} \mathrm{d}S = \Big[\Lambda - \mu S - \beta S f(I) + \gamma_1 I + \delta R\Big]\mathrm{d}t - \sigma S f(I)\mathrm{d}B(t), \\ \mathrm{d}I = \Big[\beta S f(I) - (\mu + \gamma_1 + \gamma_2 + \alpha) I\Big]\mathrm{d}t + \sigma S f(I)\mathrm{d}B(t), \\ \mathrm{d}R = \Big[\gamma_2 I - (\mu + \delta) R\Big]\mathrm{d}t, \end{cases} \tag{1.3}$$

with initial values $S_0 > 0, I_0 > 0, R_0 > 0$. Here we consider random perturbation in the environment, which is assumed to be affected by the contact rate, so that $\beta \to \beta + \sigma \dot{B}(t)$, where $B(t)$ is a standard Brownian motions defined on the compete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). σ^2 represents the intensity of $B(t)$. Other parameters are defined as system (1.1). Model (1.3) covers many existing stochastic models in [6, 9, 21, 22, 23] as special cases. For example, an epidemic model with relapse and bilinear incidence was formulated in [9]. In [21], the authors investigated the dynamics of a stochastic SIR epidemic model with saturated incidence. In [22], extinction and persistence of a stochastic SIRS epidemic model with saturated incidence rate and transfer from infectious to susceptible were considered.

In [23], the authors proposed a class of stochastic SIRS epidemic models with nonlinear incidence

$$\begin{cases} \mathrm{d}S = [\Lambda - \mu S - \beta h(S)f(I) + \delta R]\,\mathrm{d}t - \sigma h(S)f(I)\mathrm{d}B(t), \\ \mathrm{d}I = [\beta h(S)f(I) - (\mu+\gamma+\alpha)I]\,\mathrm{d}t + \sigma h(S)f(I)\mathrm{d}B(t), \\ \mathrm{d}R = [\gamma I - (\mu+\delta)R]\,\mathrm{d}t, \end{cases} \tag{1.4}$$

where $h(S)$ is continuously differentiable and monotonically increasing with respect to S, $h(0) = 0$ and for any constant $l > 0$, $M_l := \sup_{0<S\le l} \frac{h(S)}{S} < \infty$; $f(I)$ is nonnegative and twice continuously differentiable, $f(I)/I$ is monotonically decreasing with respect to I, $f(0) = 0$ and $f'(0) > 0$. Other parameters are defined as those of system (1.3). In [23], the sufficient conditions for the extinction and persistence of the disease for model (1.4) were discussed. Compared with model (1.3), model (1.4) does not include the transfer from infectious to susceptible, although the incidence rate is more general.

Denote $a \wedge b = \min\{a, b\}$ for $a, b \in \mathbb{R}$ and $\mathbb{E}(X)$ represents the exception of random variable X. Throughout this paper, we give the following assumptions.

(H_1) $f(0) = 0$, $f(I) > 0$ holds for $I > 0$ and f is a real locally Lipschitz function on $\mathbb{R}_+$;

(H_2) $f(I)/I$ is nonincreasing on $(0, +\infty)$ and $f'(0) > 0$;

(H_3) There is a constant $\vartheta > 0$, such that

$$|f(x_1)/x_1 - f(x_2)/x_2| \le \vartheta |x_1 - x_2| \ \text{ for any } x_1, x_2 \in (0, \Lambda/\mu].$$

One can easily see from (H_2) that

$$f(I) \le f'(0)I \quad \text{for } I \in \mathbb{R}_+. \tag{1.5}$$

Our general results can be applied to some specific forms of the incidence rate, for instance:

(i) linear type: $f(I) = I$ (see [9]);

(ii) saturated incidence rate: $f(I) = \frac{I}{1+aI}$ (see [21, 22]);

(iii) non-monotonic incidence rate: $f(I) = \frac{I}{1+aI^2}$ (see [24]);

(iv) incidence rate with media coverage: $f(I) = Ie^{-mI}$, where m is a positive constant (see [25]);

(v) incidence rate with media coverage: $f(I) = (1 - \frac{\widetilde{\beta} I}{\beta(m+I)})I$, where $m, \beta, \widetilde{\beta}$ are all positive constants and $\beta > \widetilde{\beta}$ (see [26]).

Our aim is to investigate how white noise affects the spread of disease, as described by model (1.3). The remaining part of this paper is organized as follows. In Section 2, a unique global positive solution of stochastic model (1.3) with any positive initial value is proved. Sufficient conditions for disease extinction and persistence in the mean are given in Sections 3 and 4, respectively. In addition, we discuss the relationship between $\mathcal{R}_0$ and $\mathcal{R}_s$ for different values of white noise intensity σ^2, and summarize a useful criterion for the extinction and persistence of the disease of model (1.3). In Section 5, some examples and numerical simulations are provided to illustrate the theoretical results. The paper ends with a conclusion that the intensity of noise plays an important role in epidemic dynamics.

2. Existence and uniqueness of global positive solution

Since $S(t)$, $I(t)$ and $R(t)$ in model (1.3) are respectively the numbers of the susceptible individuals, infected individuals and recovered individuals at time t, they should be non-negative. Therefore, we are only interested in its positive solutions. In this section, we show that model (1.3) has a unique global positive solution with positive initial value by using the Lyapunov analysis method.

Theorem 1. *Suppose that* (H_1) *and* (H_2) *hold. Model (1.3) has a unique global positive solution* $(S(t), I(t), R(t))$ *on* $[0, \infty)$ *for any* $(S_0, I_0, R_0) \in \mathbb{R}^3_+$*, that is, the solution will remain in* $\mathbb{R}^3_+$ *with probability one.*

Proof. From (H_1), it is easy to show that the coefficients of (1.3) are locally Lipschitz continuous. Thus, for any given initial value $(S_0, I_0, R_0) \in \mathbb{R}^3_+$, there is a unique maximal local solution $(S(t), I(t), R(t))$ on $[0, \tau_e)$, where τ_e is the explosion time. Now we show $\tau_e = \infty$ a.s. Let $n_0 > 0$ be sufficiently large such that S_0, I_0 and R_0 all lie within the interval $(\frac{1}{n_0}, n_0)$. For each integer $n \geq n_0$, define the stopping time

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \min\{S, I, R\} \leq \frac{1}{n} \text{ or } \max\{S, I, R\} \geq n\right\},$$

where throughout this paper, we set $\inf \emptyset = \infty$. Clearly, τ_n is increasing as $n \to \infty$. Let $\tau_\infty = \lim_{n\to\infty} \tau_n$. Thus $\tau_\infty \leq \tau_e$ a.s. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $(S(t), I(t), R(t)) \in \mathbb{R}^3_+$ a.s. for all $t \geq 0$.

Now, we show that $\tau_\infty = \infty$ a.s. Otherwise, there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$. Hence, for all $n \geq n_0$

$$\mathbb{P}(\Omega_n) \geq \varepsilon, \tag{2.1}$$

where $\Omega_n = \{\omega \in \Omega : \tau_n(\omega) \leq T\}$. Moreover, for any $n \geq n_0$ and $t \leq \tau_n$,

$$\mathrm{d}[S(t) + I(t) + R(t)] \leq [\Lambda - \mu(S(t) + I(t) + R(t))]\mathrm{d}t,$$

which implies

$$S(t) + I(t) + R(t) \leq \begin{cases} \frac{\Lambda}{\mu}, & \text{if } S_0 + I_0 + R_0 \leq \frac{\Lambda}{\mu}, \\ S_0 + I_0 + R_0, & \text{if } S_0 + I_0 + R_0 > \frac{\Lambda}{\mu}. \end{cases}$$

Denote $K = \max\{\frac{\Lambda}{\mu}, S_0 + I_0 + R_0\}$. Then

$$S(t) + I(t) + R(t) \leq K \tag{2.2}$$

for any $t \leq \tau_n$. Define a C^2-function $V : \mathbb{R}^3_+ \to \mathbb{R}_+$ by

$$V(S, I, R) = (S - 1 - \ln S) + (I - 1 - \ln I) + (R - 1 - \ln R).$$

For any $n \geq n_0$, the Itô formula shows that

$$\begin{aligned} &\mathbb{E}V(S(\tau_n \wedge T), I(\tau_n \wedge T), R(\tau_n \wedge T)) \\ &= V(S_0, I_0, R_0) + \mathbb{E}\int_0^{\tau_n \wedge T} LV(S(s), I(s), R(s))\,\mathrm{d}s, \end{aligned} \tag{2.3}$$

where $LV : \mathbb{R}^3_+ \to \mathbb{R}$ is defined by

$$LV(S, I, R) = \Big(1 - \frac{1}{S}\Big)\Big(\Lambda - \mu S - \beta S f(I) + \gamma_1 I + \delta R\Big) + \Big(1 - \frac{1}{I}\Big)\Big(\beta S f(I) - (\mu + \gamma_1 + \gamma_2 + \alpha)I\Big)$$

$$
\begin{aligned}
&+\Big(1-\frac{1}{R}\Big)\Big(\gamma_2 I-(\mu+\delta)R\Big)+\frac{\sigma^2 f^2(I)}{2}+\frac{\sigma^2 S^2 f^2(I)}{2I^2}\\
=&\Lambda+\mu+(\mu+\gamma_1+\gamma_2+\alpha)+(\mu+\delta)+\beta f(I)-(\mu+\alpha)I+\frac{\sigma^2 f^2(I)}{2}+\frac{\sigma^2 S^2 f^2(I)}{2I^2}\\
&-\frac{\Lambda}{S}-\frac{\gamma_1 I}{S}-\frac{\delta R}{S}-\frac{\gamma_2 I}{R}-\frac{\beta S f(I)}{I}-\mu S-\mu R.
\end{aligned}
$$

From (1.5) and (2.2), for any $0 \le t \le \tau_n \wedge T$

$$
\begin{aligned}
LV(S,I,R) \le&\Lambda+\mu+(\mu+\gamma_1+\gamma_2+\alpha)+(\mu+\delta)+\beta f'(0)I+\frac{\sigma^2}{2}I^2+\frac{\sigma^2}{2}S^2\\
\le&\Lambda+\mu+(\mu+\gamma_1+\gamma_2+\alpha)+(\mu+\delta)+\beta f'(0)K+\sigma^2K^2=:M
\end{aligned}
$$

Thus, from (2.3), it follows that

$$
\begin{aligned}
\mathbb{E}V(S(\tau_n\wedge T),I(\tau_n\wedge T),R(\tau_n\wedge T)) &\le V(S_0,I_0,R_0)+M\mathbb{E}(\tau_n\wedge T)\\
&\le V(S_0,I_0,R_0)+MT,
\end{aligned}
$$

for any $n \ge n_0$. Note that for every $\omega \in \Omega_n$, there exists $S(\tau_n,\omega)$, $I(\tau_n,\omega)$ or $R(\tau_n,\omega)$ that equals either $1/n$ or n. Hence

$$
V(S(\tau_n,\omega),I(\tau_n,\omega),R(\tau_n,\omega)) \ge (n-1-\ln n)\wedge\left(\frac{1}{n}-1+\ln n\right).
$$

It follows from (2.1) that

$$
\begin{aligned}
V(S_0,I_0,R_0)+MT \ge&\mathbb{E}\Big[I_{\Omega_n}(\omega)V(S(\tau_n,\omega),I(\tau_n,\omega),R(\tau_n,\omega))\Big]\\
\ge&\varepsilon\left[(n-1-\ln n)\wedge\left(\frac{1}{n}-1+\ln n\right)\right],
\end{aligned}
$$

where I_{Ω_n} is the indicator function of Ω_n. Letting $n\to\infty$ leads to the contradiction

$$
\infty > V(S_0,I_0,R_0)+MT=\infty,
$$

therefore we have $\tau_\infty=\infty$ a.s. The proof is therefore complete.

Remark 1. *Theorem 1 shows that for any initial value $(S_0,I_0,R_0)\in\mathbb{R}^3_+$, model (1.3) has a unique global solution $(S(t),I(t),R(t))\in\mathbb{R}^3_+$ a.s. In addition, it follows from model (1.3) that*

$$
\begin{aligned}
\mathrm{d}[S(t)+I(t)+R(t)] =&[\Lambda-\mu(S(t)+I(t)+R(t))-\alpha I(t)]\mathrm{d}t\\
\le&[\Lambda-\mu(S(t)+I(t)+R(t))]\mathrm{d}t.
\end{aligned}
$$

Thus, $S(t)+I(t)+R(t)\le\Big(S_0+I_0+R_0-\frac{\Lambda}{\mu}\Big)e^{-\mu t}+\frac{\Lambda}{\mu}$, which implies

$$
\limsup_{t\to\infty}\Big(S(t)+I(t)+R(t)\Big)\le\frac{\Lambda}{\mu}.
$$

Moreover, if $S_0+I_0+R_0\le\frac{\Lambda}{\mu}$, then $S(t)+I(t)+R(t)\le\frac{\Lambda}{\mu}$ a.s.. Therefore, the region

$$
D=\left\{(S,I,R)\in\mathbb{R}^3_+ : S+I+R\le\frac{\Lambda}{\mu}\right\}
$$

is a positively invariant set of model (1.3).

For simplicity, we introduce the notations $X(t)=S(t)+I(t)+R(t)$, $\langle x(t)\rangle=\frac{1}{t}\int_0^t x(s)\mathrm{d}s$.

3. Extinction of disease

One of the main concerns in epidemiology is how we can regulate the disease dynamics so that the disease will be eradicated in a long term. In this section, we provide the sufficient conditions for extinction of the disease in model (1.3). Denote

$$\mathcal{R}_s = \frac{\Lambda\beta f'(0)}{\mu(\mu+\gamma_1+\gamma_2+\alpha)} - \frac{\Lambda^2\sigma^2(f'(0))^2}{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}, \quad \Phi(x) = -\frac{\sigma^2}{2}x^2 + \beta x - (\mu+\gamma_1+\gamma_2+\alpha).$$

For model (1.3), we give the following conditions.
(C_1) $\mathcal{R}_s < 1$, $\sigma^2 \le \frac{\beta\mu}{\Lambda f'(0)}$ and $\sigma^2 \le \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2}$;
(C_2) $\mathcal{R}_s < 1$, $\frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2} < \sigma^2 < \frac{\beta\mu}{\Lambda f'(0)}$;
(C_3) $\mathcal{R}_s < 1$, $\sigma^2 \ge \frac{\beta\mu}{\Lambda f'(0)}$ and $\sigma^2 > \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2}$;
(C_4) $\mathcal{R}_s \ge 1$, $\sigma^2 > \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2}$.
In addition, denote $A_1 = (\mu+\gamma_1+\gamma_2+\alpha)(\mathcal{R}_s - 1)$, $\quad A_2 = -\Big[(\mu+\gamma_1+\gamma_2+\alpha) - \frac{\beta^2}{2\sigma^2}\Big]$.

Theorem 2. *Suppose that (H_1) and (H_2) hold. Let $(S(t), I(t), R(t))$ be the solution of model (1.3) with any given initial condition $(S_0, I_0, R_0) \in \mathbb{R}^3_+$.*
(i) If (C_1) holds, then $\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le A_1 < 0$ *a.s.*
(ii) If (C_2) holds, then $\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le \min\{A_1, A_2\} < 0$ *a.s.*
(iii) If (C_3) or (C_4) holds, then $\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le A_2 < 0$ *a.s.*
Furthermore,

$$\lim_{t\to\infty} S(t) = \frac{\Lambda}{\mu}, \quad \lim_{t\to\infty} I(t) = 0, \quad \lim_{t\to\infty} R(t) = 0 \ \ a.s.$$

Proof. Applying Itô formula to the second equation of model (1.3) leads to

$$\mathrm{d}\ln I(t) = \Big[\frac{\beta S f(I)}{I} - (\mu+\gamma_1+\gamma_2+\alpha) - \frac{\sigma^2}{2}\Big(\frac{S f(I)}{I}\Big)^2\Big]\mathrm{d}t + \frac{\sigma S f(I)}{I}\mathrm{d}B(t).$$

Integrating both sides of the above equation from 0 to t, yields

$$\begin{aligned}\ln I(t) &= \int_0^t \Big[\frac{\beta S f(I)}{I} - (\mu+\gamma_1+\gamma_2+\alpha) - \frac{\sigma^2}{2}\Big(\frac{S f(I)}{I}\Big)^2\Big]\mathrm{d}s + \int_0^t \frac{\sigma S f(I)}{I}\mathrm{d}B(s) + \ln I_0 \\ &= \int_0^t \Phi\Big(\frac{S f(I)}{I}\Big)\mathrm{d}s + M_1(t) + \ln I_0, \end{aligned} \tag{3.1}$$

where $M_1(t) = \int_0^t \frac{\sigma S f(I)}{I}\mathrm{d}B(s)$.

Let us discuss three cases separately.

Case 1. Assume that $\mathcal{R}_s < 1$ and $\sigma^2 < \frac{\beta\mu}{\Lambda f'(0)}$. Then $\frac{\Lambda f'(0)}{\mu} < \frac{\beta}{\sigma^2}$ and $\Phi(\frac{\Lambda f'(0)}{\mu}) = (\mu+\gamma_1+\gamma_2+\alpha)(\mathcal{R}_s - 1) < 0$. For any $0 < \varepsilon < \frac{\beta}{\sigma^2 f'(0)} - \frac{\Lambda}{\mu}$ sufficiently small, we have $(\frac{\Lambda}{\mu}+\varepsilon)f'(0) < \frac{\beta}{\sigma^2}$ and $\Phi\Big((\frac{\Lambda}{\mu}+\varepsilon)f'(0)\Big) < 0$. From Remark 1, for any $\varepsilon > 0$, there exists a constant $T = T(\varepsilon) > 0$ such that $S(t) \le \frac{\Lambda}{\mu} + \varepsilon$ for all $t > T$. Then it follows from (1.5) that

$$\Phi\Big(\frac{S f(I)}{I}\Big) \le \Phi(f'(0)S) \le \Phi\Big(f'(0)(\frac{\Lambda}{\mu} + \varepsilon)\Big) < 0.$$

Substituting this into (3.1), we have

$$\begin{aligned}\frac{\ln I(t)}{t} =&\frac{1}{t}\int_0^T \Phi\Big(\frac{Sf(I)}{I}\Big)\mathrm{d}s + \frac{1}{t}\int_T^t \Phi\Big(\frac{Sf(I)}{I}\Big)\mathrm{d}s + \frac{M_1(t)}{t} + \frac{\ln I_0}{t}\\ \le&\frac{1}{t}\int_0^T \Phi\Big(\frac{Sf(I)}{I}\Big)\mathrm{d}s + \frac{1}{t}\int_T^t \Phi\Big(f'(0)(\frac{\Lambda}{\mu}+\varepsilon)\Big)\mathrm{d}s + \frac{M_1(t)}{t} + \frac{\ln I_0}{t}\\ =&\frac{1}{t}\int_0^T \Phi\Big(\frac{Sf(I)}{I}\Big)\mathrm{d}s + \Phi\Big(f'(0)(\frac{\Lambda}{\mu}+\varepsilon)\Big)\Big(1-\frac{T}{t}\Big) + \frac{M_1(t)}{t} + \frac{\ln I_0}{t}. \end{aligned} \tag{3.2}$$

Clearly, $M_1(t)$ is a real-valued continuous local martingale vanishing at time 0 and

$$\begin{aligned}\langle M_1, M_1\rangle_t =& \int_0^T \Big[\frac{\sigma S f(I)}{I}\Big]^2 \mathrm{d}s + \int_T^t \Big[\frac{\sigma S f(I)}{I}\Big]^2 \mathrm{d}s\\ \le& \int_0^T \Big[\frac{\sigma S f(I)}{I}\Big]^2 \mathrm{d}s + \sigma^2 (f'(0))^2 \Big(\frac{\Lambda}{\mu}+\varepsilon\Big)^2 (t-T).\end{aligned}$$

Note that

$$\limsup_{t\to\infty} \frac{\langle M_1, M_1\rangle_t}{t} \le \sigma^2 (f'(0))^2 \Big(\frac{\Lambda}{\mu}+\varepsilon\Big)^2 < \infty,$$

then, from the strong law of large numbers (see [27]), it follows that

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \frac{\sigma S f(I)}{I}\mathrm{d}B(s) = 0 \ \ a.s. \tag{3.3}$$

From (3.2) and (3.3), we have

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le \Phi\Big(f'(0)(\frac{\Lambda}{\mu}+\varepsilon)\Big) < 0 \ \ a.s.$$

By the continuity of $\Phi(t)$ and the arbitrariness of ε, we obtain

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} \le \Phi\Big(\frac{\Lambda f'(0)}{\mu}\Big) = (\mu+\gamma_1+\gamma_2+\alpha)(\mathcal{R}_s - 1) < 0 \ \ a.s.$$

Case 2. Assume that $\mathcal{R}_s < 1$ and $\sigma^2 = \frac{\beta\mu}{\Lambda f'(0)}$. Then $\frac{\Lambda f'(0)}{\mu} = \frac{\beta}{\sigma^2}$ and $\Phi\Big(\frac{Sf(I)}{I}\Big) \le \Phi\Big(\frac{\Lambda f'(0)}{\mu}\Big) < 0$. The following proof is similar to that of Case 1.

Case 3. Assume that $\sigma^2 > \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2}$. Since

$$\begin{aligned}\Phi(x) =& -\frac{\sigma^2}{2}\Big(x - \frac{\beta}{\sigma^2}\Big)^2 + \frac{\beta^2}{2\sigma^2} - (\mu+\gamma_1+\gamma_2+\alpha)\\ \le& -\Big[(\mu+\gamma_1+\gamma_2+\alpha) - \frac{\beta^2}{2\sigma^2}\Big].\end{aligned}$$

Substituting this into (3.1), we have

$$\frac{\ln I(t)}{t} \le -\Big[(\mu+\gamma_1+\gamma_2+\alpha) - \frac{\beta^2}{2\sigma^2}\Big] + \frac{M_1(t)}{t} + \frac{\ln I_0}{t}. \tag{3.4}$$

This, together with (3.3), yields

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t}\le-\Big[(\mu+\gamma_1+\gamma_2+\alpha)-\frac{\beta^2}{2\sigma^2}\Big]<0\ \ a.s..$$

From the proofs of Cases 1-3, it is easy to see that conclusions (i)-(iii) hold. Further, if one of conditions (C_1)–(C_4) holds, then

$$\lim_{t\to\infty}I(t)=0\ \ a.s. \tag{3.5}$$

That is to say, the disease $I(t)$ dies out with probability one.

Next, we show $\lim_{t\to\infty}R(t)=0$ a.s. Let $\Omega_1=\{\omega\in\Omega:\lim_{t\to\infty}I(t,\omega)=0\}$, then (3.5) implies $\mathbb{P}(\Omega_1)=1$. Hence, for any $\omega\in\Omega_1$ and any constant $\varepsilon_1>0$, there exists a constant $T_1=T_1(\omega,\varepsilon_1)>0$ such that for any $t\ge T_1$

$$I(t,\omega)\le\varepsilon_1. \tag{3.6}$$

Substituting this into the third equation of model (1.3), we obtain

$$\mathrm{d}R(t,\omega)\le[\gamma_2\varepsilon_1-(\mu+\delta)R(t,\omega)]\mathrm{d}t,\quad \omega\in\Omega_1,\ t\ge T_1.$$

Then, from the comparison theorem, it follows that

$$\limsup_{t\to\infty}R(t,\omega)\le\frac{\gamma_2\varepsilon_1}{\mu+\delta},\quad \omega\in\Omega_1.$$

Since $R(t,\omega)>0$ for all $\omega\in\Omega_1$ and $t>0$, by the arbitrariness of ε_1, we get $\lim_{t\to\infty}R(t,\omega)=0$, $\omega\in\Omega_1$. It follows from $\mathbb{P}(\Omega_1)=1$ that

$$\lim_{t\to\infty}R(t)=0\ \ a.s. \tag{3.7}$$

At last, we prove the assertion $\lim_{t\to\infty}S(t)=\frac{\Lambda}{\mu}$ a.s. It follows from model (1.3) that

$$\mathrm{d}X(t)=[\Lambda-\mu X(t)-\alpha I(t)]\mathrm{d}t,$$

which implies

$$\mathrm{d}(e^{\mu t}X(t))=e^{\mu t}[\Lambda-\alpha I(t)]\mathrm{d}t. \tag{3.8}$$

For any $t>T$, integrating both sides of (3.8) from T to t and using (3.6), we obtain that for any $\omega\in\Omega_1$

$$\begin{aligned}X(t,\omega)=&e^{-\mu(t-T)}X(T,\omega)+\int_T^t e^{-\mu(t-s)}[\Lambda-\alpha I(s)]\mathrm{d}s\\ \ge&e^{-\mu(t-T)}X(T,\omega)+\int_T^t e^{-\mu(t-s)}[\Lambda-\alpha\varepsilon_1]\mathrm{d}s\\ =&e^{-\mu(t-T)}X(T,\omega)+\frac{\Lambda-\alpha\varepsilon_1}{\mu}[1-e^{-\mu(t-T)}].\end{aligned}$$

Thus, for any $\omega \in \Omega_1$, we have $\liminf_{t\to\infty} X(t,\omega) \geq \frac{\Lambda-\alpha\varepsilon_1}{\mu}$, which, together with the arbitrariness of ε_1, yields

$$\liminf_{t\to\infty} X(t,\omega) \geq \frac{\Lambda}{\mu}, \quad \text{for all } \omega \in \Omega_1.$$

From $\mathbb{P}(\Omega_1) = 1$, we get $\liminf_{t\to\infty} X(t) \geq \frac{\Lambda}{\mu}$ a.s. On the other hand, from Remark 1, it follows that $\limsup_{t\to\infty} X(t) \leq \frac{\Lambda}{\mu}$. Thus, we have

$$\lim_{t\to\infty} X(t) = \lim_{t\to\infty}[S(t) + I(t) + R(t)] = \frac{\Lambda}{\mu} \ a.s.,$$

which, together with (3.5) and (3.7), yields

$$\lim_{t\to\infty} S(t) = \frac{\Lambda}{\mu} \ a.s.$$

The proof is therefore complete.

Remark 2. *Theorem 2 shows that the disease of system* (1.3) *dies out with probability one if* $\mathcal{R}_s < 1$ *and* $\sigma^2 < \frac{\beta\mu}{\Lambda f'(0)}$. *Moreover, we note* $\mathcal{R}_s$ *in Theorem 2 is smaller than the basic reproduction number* $\mathcal{R}_0$ *of system* (1.1)*, and hence environmental noise may lead the disease to extinction. If* $\sigma = 0$*, then system* (1.3) *can be reduced to system* (1.1)*. Further, Theorem 2 is consistent with the result in [6].*

Remark 3. *If we let* $f(I) = \frac{I}{1+aI}$, $\gamma_1 = 0$, $\gamma_2 = \gamma$, $\delta = 0$, *system* (1.3) *can be transformed into system* (1.3) *in [21]. From Theorem 2, it follows that for any initial condition* $(S_0, I_0, R_0) \in \mathbb{R}^3_+$, *if one of conditions* (C_1)–(C_4) *holds then the disease of system* (1.3) *will extinct with probability one. However, Theorem 3.1 in [21] shows that for any initial condition* $(S_0, I_0, R_0) \in \Gamma$, *if* $\sigma^2 > \frac{\beta^2}{2(\mu+\gamma+\alpha)}$, *then the disease of system* (1.3) *will tend to zero exponentially with probability one. Obviously, Theorem 2 generalizes and improves the corresponding result in [21].*

Remark 4. *Let* $f(I) = \frac{I}{1+aI}$, *system* (1.3) *can be changed into* (1.2) *in [22]. Theorem 2 shows that the disease of system dies out with probability one. Particularly, if* $\mathcal{R}_s \geq 1$ *and* $\frac{\beta\mu}{\Lambda} \cdot \frac{\mathcal{R}_0}{2} < \sigma^2 < \frac{\beta\mu}{\Lambda}$, *Theorem 2 in this paper shows the extinction of the disease; however, Theorem 3.2 in [22] can not show the extinction of the disease. Furthermore, we obtain* $\lim_{t\to\infty} S(t) = \frac{\Lambda}{\mu}$ *and* $\lim_{t\to\infty} R(t) = 0$ *a.s. But, only the disease extinction is found in Theorem 3.2 in [22]. Therefore, Theorem 2 in this paper improves and generalizes Theorem 3.2 in [22].*

Remark 5. *Let* $h(S) = S$, $\gamma_1 = 0$, $\gamma_2 = \gamma$. *From Theorem 2, extinction of model (1.3) requires that* $f(I)$ *is a real locally Lipschitz function on* $\mathbb{R}_+$. *However, from Corollary 1 in [23], extinction of model (1.3) requires that* $f(I)$ *is twice continuously differentiable on* $\mathbb{R}_+$. *Obviously, the conditions of Theorem 2 are weaker than those of Corollary 1 in [23].*

4. Persistence of disease

When considering epidemic models, we are interested in when the disease will prevail in the population. In the deterministic models, the problem can be solved by proving that the endemic equilibrium

of the corresponding model is globally asymptotically stable. But for model (1.3), there is no endemic equilibrium. In this section, we will establish sufficient conditions to ensure that the disease in model (1.3) is permanent in the time mean. Now, we give the definition of persistence in the mean as follows.

Definition 1. *Model (1.3) is said to be persistent in the mean if* $\liminf_{t\to\infty}\langle I(t)\rangle > 0$ *a.s.*

For the convenience, we give the following useful lemma.

Lemma 1. (See [28]). *Suppose* $x \in C(\Omega\times[0,+\infty),\mathbb{R}_+)$ *and* $F \in C(\Omega\times[0,+\infty),(-\infty,+\infty))$. *If there are three positive constants* λ, λ_0 *and* T *such that*

$$\ln x(t) \geq \lambda t - \lambda_0\int_0^t x(s)\mathrm{d}s + F(t), \quad a.s., \text{ for all } t \geq T,$$

where $\lim\limits_{t\to\infty}\frac{F(t)}{t} = 0$ *a.s., then* $\liminf_{t\to\infty}\frac{1}{t}\int_0^t x(s)\mathrm{d}s \geq \frac{\lambda}{\lambda_0}$ *a.s.*

Theorem 3. *Suppose that* $(H_1)-(H_3)$ *hold. Let* $(S(t),I(t),R(t))$ *be the solution of model (1.3) with any given initial condition* $(S_0,I_0,R_0)\in\mathbb{R}^3_+$. *If* $\mathcal{R}_s > 1$, *then*

$$\text{(i)}\ \liminf_{t\to\infty}\langle S(t)\rangle \geq \frac{\Lambda\mu}{\mu^2+\Lambda\beta f'(0)} > 0 \ \ a.s.;$$

$$\text{(ii)}\ \liminf_{t\to\infty}\langle I(t)\rangle \geq \frac{\mu^2\Phi\Big(\frac{\Lambda f'(0)}{\mu}\Big)}{\beta\Lambda(f'(0))^2\Big(1-\frac{\Lambda\sigma^2 f'(0)}{2\mu\beta}\Big)(\mu\vartheta+\beta)} > 0 \ \ a.s.;$$

$$\text{(iii)}\ \liminf_{t\to\infty}\langle R(t)\rangle \geq \frac{\gamma_2\mu^2\Phi\Big(\frac{\Lambda f'(0)}{\mu}\Big)}{(\mu+\delta)\beta\Lambda(f'(0))^2\Big(1-\frac{\Lambda\sigma^2 f'(0)}{2\mu\beta}\Big)\Big(\mu\vartheta+\beta\Big)} > 0 \ \ a.s.$$

Proof. From Remark 1, it follows that, for any $0<\varepsilon<\frac{2\beta}{\sigma^2 f'(0)}\Big(1-\frac{\Lambda\sigma^2 f'(0)}{2\mu\beta}\Big)$, there exists a constant $T=T(\varepsilon)>0$ such that $X(t)\leq\frac{\Lambda}{\mu}+\varepsilon$ for all $t\geq T$. Note that $0<f(I)\leq f'(0)I$, then from the first equation of model (1.3), we can derive that

$$\begin{aligned}
S(t)-S_0 &\geq \int_0^T\Big[\Lambda-\mu S(s)-\beta S(s)f(I(s))\Big]\mathrm{d}s + \int_T^t\Big[\Lambda-\mu S(s)-\beta f'(0)\Big(\frac{\Lambda}{\mu}+\varepsilon\Big)S(s)\Big]\mathrm{d}s\\
&\quad -\int_0^t\sigma S(s)f(I(s))\mathrm{d}B(s)\\
&= \int_0^T\Big[\Lambda-\mu S(s)-\beta S(s)f(I(s))\Big]\mathrm{d}s + \int_T^t\Big[\Lambda-\Big(\mu+\beta f'(0)\Big(\frac{\Lambda}{\mu}+\varepsilon\Big)\Big)S(s)\Big]\mathrm{d}s\\
&\quad -\int_0^t\sigma S(s)f(I(s))\mathrm{d}B(s),
\end{aligned}$$

which implies

$$\begin{aligned}
&\Big(\mu+\beta f'(0)\Big(\frac{\Lambda}{\mu}+\varepsilon\Big)\Big)\frac{1}{t}\int_0^t S(s)\mathrm{d}s\\
&\quad\geq \Lambda-\frac{\Lambda T}{t}-\frac{S(t)-S_0}{t}-\frac{M_2(t)}{t}+\frac{1}{t}\int_0^T\Big[\Lambda-\mu S(s)-\beta S(s)f(I(s))\Big]\mathrm{d}s, \qquad (4.1)
\end{aligned}$$

where $M_2(t) = \int_0^t \sigma S(s) f(I(s)) \mathrm{d}B(s)$. Clearly, $M_2(t)$ is a real-valued continuous local martingale vanishing at time 0 and

$$\begin{aligned}\langle M_2, M_2\rangle_t &= \int_0^T \Big[\sigma S(s) f(I(s))\Big]^2 \mathrm{d}s + \int_T^t \Big[\sigma S(s) f(I(s))\Big]^2 \mathrm{d}s \\ &\le \int_0^T \Big[\sigma S(s) f(I(s))\Big]^2 \mathrm{d}s + \sigma^2 (f'(0))^2 \Big(\frac{\Lambda}{\mu} + \varepsilon\Big)^4 (t - T).\end{aligned}$$

Note that $\limsup_{t\to\infty} \frac{\langle M_2, M_2\rangle_t}{t} \le \sigma^2 (f'(0))^2 \Big(\frac{\Lambda}{\mu} + \varepsilon\Big)^4 < \infty$. Then, from the strong law of large numbers (see [27]), it follows that

$$\lim_{t\to\infty} \frac{M_2(t)}{t} = \lim_{t\to\infty} \frac{1}{t} \int_0^t \sigma S(s) f(I(s)) \mathrm{d}B(s) = 0 \ \ a.s. \tag{4.2}$$

This, together with (4.1), yields

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t S(s) \mathrm{d}s \ge \frac{\Lambda}{\mu + \beta f'(0)\Big(\frac{\Lambda}{\mu} + \varepsilon\Big)} \ \ a.s.,$$

From the arbitrariness of ε, it follows that

$$\liminf_{t\to\infty} \frac{1}{t} \int_0^t S(s) \mathrm{d}s \ge \frac{\Lambda\mu}{\mu^2 + \Lambda\beta f'(0)} > 0 \ \ a.s.,$$

In the following, we will prove that (ii) holds. Note that $S(t) \le \frac{\Lambda}{\mu} + \varepsilon$ for all $t \ge T$ and $0 < f(I) \le f'(0)I$, then, for any $t \ge T$, $0 < \frac{Sf(I)}{I} \le f'(0)S \le f'(0)\Big(\frac{\Lambda}{\mu} + \varepsilon\Big)$, which implies $0 < \frac{Sf(I)}{f'(0)(\frac{\Lambda}{\mu}+\varepsilon)I} \le 1$. Then, we have

$$\Phi\Big(\frac{Sf(I)}{I}\Big) - \Phi\Big(f'(0)(\frac{\Lambda}{\mu} + \varepsilon)\Big) \ge -\beta f'(0)\Big[1 - \frac{\sigma^2 f'(0)}{2\beta}\Big(\frac{\Lambda}{\mu} + \varepsilon\Big)\Big]\Big(\frac{\Lambda}{\mu} + \varepsilon - \frac{Sf(I)}{f'(0)I}\Big).$$

That is,

$$\Phi\Big(\frac{Sf(I)}{I}\Big) \ge \Phi\Big(f'(0)(\frac{\Lambda}{\mu} + \varepsilon)\Big) - \beta f'(0)\Big[1 - \frac{\sigma^2 f'(0)}{2\beta}\Big(\frac{\Lambda}{\mu} + \varepsilon\Big)\Big]\Big(\frac{\Lambda}{\mu} + \varepsilon - \frac{Sf(I)}{f'(0)I}\Big).$$

Substituting this into (3.1) yields

$$\begin{aligned}\ln I(t) \ge &\int_0^T \Phi\Big(\frac{Sf(I)}{I}\Big)\mathrm{d}s + \Phi\Big(f'(0)(\frac{\Lambda}{\mu} + \varepsilon)\Big)(t - T) + M_1(t) + \ln I_0 \\ &- \beta f'(0)\Big[1 - \frac{\sigma^2 f'(0)}{2\beta}\Big(\frac{\Lambda}{\mu} + \varepsilon\Big)\Big] \int_T^t \Big(\frac{\Lambda}{\mu} + \varepsilon - \frac{Sf(I)}{f'(0)I}\Big)\mathrm{d}s.\end{aligned} \tag{4.3}$$

In addition, from the first equation of model (1.3), it follows that

$$S(t) - S(T) = \int_T^t [\Lambda - \mu S(s) - \beta S(s) f(I(s)) + \gamma_1 I(s) + \delta R(s)]\mathrm{d}s - \int_T^t \sigma S(s) f(I(s)) \mathrm{d}B(s)$$

$$\geq \int_T^t [\Lambda - \mu S(s) - \beta S(s) f(I(s))]\mathrm{d}s - \int_T^t \sigma S(s) f(I(s))\mathrm{d}B(s)$$
$$= \int_T^t \Big[\mu\Big(\frac{\Lambda}{\mu} + \varepsilon - \frac{S(s)f(I(s))}{f'(0)I(s)}\Big) - \mu S(s)\Big(1 - \frac{f(I(s))}{f'(0)I(s)}\Big) - \mu\varepsilon - \beta S(s) f(I(s))\Big]\mathrm{d}s - M_2(t). \tag{4.4}$$

By the Assumption (H_3), we know that there exists a constant $\vartheta > 0$, such that $1 - \frac{f(I)}{I} \leq f'(0)\vartheta I$. Note that $f(I) \leq f'(0)I$ and $S(t) \leq \frac{\Lambda}{\mu} + \varepsilon$ for any $t > T$, we have

$$S(t) - S(T) \geq \int_T^t \Big[\mu\Big(\frac{\Lambda}{\mu} + \varepsilon - \frac{S(s)f(I(s))}{f'(0)I(s)}\Big) - f'(0)S(s)(\mu\vartheta + \beta)I(s) - \mu\varepsilon\Big]\mathrm{d}s - M_2(t)$$
$$\geq \int_T^t \Big[\mu\Big(\frac{\Lambda}{\mu} + \varepsilon - \frac{S(s)f(I(s))}{f'(0)I(s)}\Big) - f'(0)\Big(\frac{\Lambda}{\mu} + \varepsilon\Big)(\mu\vartheta + \beta)I(s) - \mu\varepsilon\Big]\mathrm{d}s - M_2(t),$$

which implies

$$\mu \int_T^t \Big(\frac{\Lambda}{\mu} + \varepsilon - \frac{S(s)f(I(s))}{f'(0)I(s)}\Big)\mathrm{d}s \leq f'(0)\Big(\frac{\Lambda}{\mu} + \varepsilon\Big)(\mu\vartheta + \beta)\int_T^t I(s)\mathrm{d}s + \mu\varepsilon(t - T) + S(t) - S(T) + M_2(t). \tag{4.5}$$

Substituting (4.5) into (4.3), we obtain

$$\ln I(t) \geq \Phi\Big(f'(0)(\frac{\Lambda}{\mu} + \varepsilon)\Big)t - \beta f'(0)\Big[1 - \frac{\sigma^2 f'(0)}{2\beta}\Big(\frac{\Lambda}{\mu} + \varepsilon\Big)\Big]\varepsilon t$$
$$- \frac{\beta(f'(0))^2}{\mu}\Big[1 - \frac{\sigma^2 f'(0)}{2\beta}\Big(\frac{\Lambda}{\mu} + \varepsilon\Big)\Big]\Big(\frac{\Lambda}{\mu} + \varepsilon\Big)(\mu\vartheta + \beta)\int_0^t I(s)\mathrm{d}s + F_1(t),$$

where

$$F_1(t) = \int_0^T \Phi\Big(\frac{Sf(I)}{I}\Big)\mathrm{d}s - \Phi\Big(f'(0)(\frac{\Lambda}{\mu} + \varepsilon)\Big)T + M_1(t) + \ln I_0$$
$$- \frac{\beta f'(0)}{\mu}\Big[1 - \frac{\sigma^2 f'(0)}{2\beta}\Big(\frac{\Lambda}{\mu} + \varepsilon\Big)\Big]\Big[-\mu\varepsilon T + M_2(t) + S(t) - S(T)\Big].$$

From (3.3) and (4.2), it follows that $\lim_{t\to\infty} \frac{F_1(t)}{t} = 0$ a.s. Note that $\mathcal{R}_s > 1$. For any $\varepsilon > 0$ sufficiently small, we have

$$\Phi\Big(f'(0)(\frac{\Lambda}{\mu} + \varepsilon)\Big) = (\mu + \gamma_1 + \gamma_2 + \alpha)(\mathcal{R}_s - 1) + \Big(\beta - \frac{\Lambda\sigma^2 f'(0)}{\mu}\Big)f'(0)\varepsilon - \frac{\sigma^2(f'(0))^2}{2}\varepsilon^2 > 0,$$

and $1 - \frac{\sigma^2 f'(0)}{2\beta}\Big(\frac{\Lambda}{\mu} + \varepsilon\Big) > 0$. Using Lemma 1, it follows from the arbitrariness of ε that

$$\liminf_{t\to\infty} \langle I(t)\rangle \geq \frac{\mu^2\Phi\Big(\frac{\Lambda f'(0)}{\mu}\Big)}{\beta\Lambda(f'(0))^2\Big(1 - \frac{\Lambda\sigma^2 f'(0)}{2\mu\beta}\Big)(\mu\vartheta + \beta)} > 0 \ \ a.s. \tag{4.6}$$

The assertion (ii) is proved.

Now, we will give the proof of (iii). It follows from the third equation of model (1.3) that

$$\frac{R(t)-R_0}{t}=\gamma_2\langle I(t)\rangle-(\mu+\delta)\langle R(t)\rangle, \tag{4.7}$$

which, together with (4.6), yields

$$\liminf_{t\to\infty}\langle R(t)\rangle\geq\frac{\gamma_2\mu^2\Phi\Big(\frac{\Lambda f'(0)}{\mu}\Big)}{(\mu+\delta)\beta\Lambda(f'(0))^2\Big(1-\frac{\Lambda\sigma^2 f'(0)}{2\mu\beta}\Big)\Big(\mu\vartheta+\beta\Big)}>0\ \ a.s.$$

The proof is therefore complete.

Remark 6. *From Theorem 3, one can see that if $\mathcal{R}_s>1$, the disease persists and prevails. Clearly, if we let $\sigma=0$, then Theorem 3 is coincident with the result in [6].*

Remark 7. *Theorems 2 and 3 imply that $\mathcal{R}_s$ is the threshold of system* (1.3) *if $\sigma^2<\frac{\beta\mu}{\Lambda f'(0)}$. Namely, the disease dies out with probability one if $\mathcal{R}_s<1$, whereas if $\mathcal{R}_s>1$, the infection may become almost surely persistent in the time mean.*

Remark 8. *By constructing $V(I)=\ln I$ and using Lemma 1, we obtain that, for any $(S_0,I_0,R_0)\in\mathbb{R}^3_+$, if $\mathcal{R}_s>1$, then $\liminf_{t\to\infty}\langle S(t)\rangle>0$, $\liminf_{t\to\infty}\langle I(t)\rangle>0$, $\liminf_{t\to\infty}\langle R(t)\rangle>0$ a.s. However, Theorem 3.5 in [22] only obtained that, for any $(S_0,I_0,R_0)\in\Gamma$, $\liminf_{t\to\infty}\langle I(t)\rangle>0$ by constructing $V(I)=\ln I+aI$, which can be seen in the proof.*

Remark 9. *Let $h(S)=S$, $\gamma_1=0$ and $\gamma_2=\gamma$, then system* (1.3) *is consistent with system* (1.4). *The methods in proofs of Theorem 3 in our paper and [23] are different. Further, the results in [23] and our paper are different. From Theorem 3 in our paper, we have*

$$\liminf_{t\to\infty}\langle I(t)\rangle\geq\frac{(\mu+\gamma+\alpha)(\mathcal{R}_s-1)}{\frac{\beta\Lambda}{\mu^2}(f'(0))^2\Big(1-\frac{\Lambda\sigma^2 f'(0)}{2\mu\beta}\Big)(\mu\vartheta+\beta)}=:A_1>0\ \ a.s.$$

From Theorem 3 in [23], we have $\liminf_{t\to\infty}\langle I(t)\rangle\geq\frac{(\mu+\gamma+\alpha)(\mathcal{R}_s-1)}{D^*}=:A_2>0$ a.s., where $D^*=f'(0)\left[\beta\Big(\frac{\mu+\alpha}{\mu}+\frac{\gamma}{\mu+\delta}\Big)+(\mu+\gamma+\alpha)\max_{0\leq\zeta\leq\frac{\Lambda}{\mu}}\frac{f(\zeta)-\zeta f'(\zeta)}{f^2(\zeta)}\right]$.

For general function f, it is difficult to compare A_1 with A_2. However, when f is a specific function, we can compare A_1 with A_2. For example, let $f(I)=(1-\frac{\widetilde{\beta}I}{m+I})I$, where $m>0$ and $0<\widetilde{\beta}<1$. If $0<m<\frac{\mu\widetilde{\beta}(\mu+\delta)(\mu+\gamma+\alpha)}{\beta\delta\gamma}\Big(\frac{1}{(1-\widetilde{\beta})^2}-1\Big)$, then $A_3:=\frac{2\mu\beta}{\Lambda}\Big[1-\frac{D^*\mu^2 m}{\beta\Lambda(\mu\widetilde{\beta}+\beta)}\Big]<\frac{2\mu\beta}{\Lambda}\Big[1-\frac{\mu(\mu+\gamma+\alpha)}{\beta\Lambda}\Big]=:A_4$. It is easy to show that $A_3<\sigma^2<A_4$ yields $A_1=\frac{(\mu+\gamma+\alpha)(\mathcal{R}_s-1)}{\frac{\beta\Lambda}{\mu^2}\Big(1-\frac{\Lambda\sigma^2}{2\mu\beta}\Big)(\frac{\mu\widetilde{\beta}}{m}+\beta)}>\frac{(\mu+\gamma+\alpha)(\mathcal{R}_s-1)}{D^*}=A_2$. In addition, condition $A_3<\sigma^2<A_4$ ensures that the conditions of Theorem 3 in our paper and [23] hold. In this case, our results are better than those in [23].

In the following, we will summarize a useful criterion for the extinction and persistence of the disease of model (1.3). First, we discuss the relationship between $\mathcal{R}_0$ and $\mathcal{R}_s$ for different values of white noise intensity σ^2 in the following three cases.

Case of $\mathcal{R}_0 \leq 1$. From $\mathcal{R}_s = \mathcal{R}_0 - \frac{\Lambda^2\sigma^2(f'(0))^2}{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}$, it follows that $\mathcal{R}_s < 1$ for any $\sigma > 0$. Clearly, $\frac{\beta\mu}{\Lambda f'(0)} > \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2}$. Now, we differ three cases

$$\begin{cases} \text{if } \sigma^2 \geq \frac{\beta\mu}{\Lambda f'(0)} > \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2}, \text{ then } (C_3) \text{ holds;} \\ \text{if } \frac{\beta\mu}{\Lambda f'(0)} > \sigma^2 > \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2}, \text{ then } (C_2) \text{ holds;} \\ \text{if } \frac{\beta\mu}{\Lambda f'(0)} > \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2} \geq \sigma^2, \text{ then } (C_1) \text{ holds.} \end{cases}$$

Hence, if $\mathcal{R}_0 \leq 1$, then for any $\sigma > 0$, the disease dies out with probability one.

Case of $1 < \mathcal{R}_0 \leq 2$. It follows that

$$\frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1) \leq \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2} \leq \frac{\beta\mu}{\Lambda f'(0)}.$$

On the other hand, $\mathcal{R}_s < 1$ is equivalent to $\sigma^2 > \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1)$. Now, we differ four cases

$$\begin{cases} \text{if } \sigma^2 < \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1), \text{ then condition in Theorem 3 holds;} \\ \text{if } \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1) < \sigma^2 < \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2} \leq \frac{\beta\mu}{\Lambda f'(0)}, \text{ then } (C_1) \text{ holds;} \\ \text{if } \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1) \leq \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2} < \sigma^2 < \frac{\beta\mu}{\Lambda f'(0)}, \text{ then } (C_2) \text{ holds;} \\ \text{if } \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1) \leq \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2} \leq \frac{\beta\mu}{\Lambda f'(0)} < \sigma^2, \text{ then } (C_3) \text{ holds.} \end{cases}$$

Thus, for $1 < \mathcal{R}_0 \leq 2$, if $\sigma^2 > \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1)$ (i.e. $\mathcal{R}_s < 1$), then the disease dies out with probability one, whereas if $\sigma^2 < \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1)$ (i.e. $\mathcal{R}_s > 1$), then the infection may become almost surely persistent in the time mean. Therefore, if $1 < \mathcal{R}_0 \leq 2$, then $\mathcal{R}_s$ is the threshold parameter.

Case of $\mathcal{R}_0 > 2$. On the one hand, it follows that

$$\frac{\beta\mu}{\Lambda f'(0)} < \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1) < \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2}.$$

On the other hand, $\mathcal{R}_s < 1$ is equivalent to $\sigma^2 > \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1)$. Now, we differ four cases

$$\begin{cases} \text{if } \sigma^2 \leq \frac{\beta\mu}{\Lambda f'(0)} < \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1) < \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2}, \text{ then } \mathcal{R}_s > 1; \\ \text{if } \frac{\beta\mu}{\Lambda f'(0)} < \sigma^2 < \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1) < \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2}, \text{ then } \mathcal{R}_s > 1; \\ \text{if } \frac{\beta\mu}{\Lambda f'(0)} < \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1) \leq \sigma^2 \leq \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2}, \text{ then conditions} \\ \quad \text{in Theorems 2 and 3 are not satisfied;} \\ \text{if } \frac{\beta\mu}{\Lambda f'(0)} < \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1) < \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2} < \sigma^2, \text{ then } (C_3) \text{ holds.} \end{cases}$$

Thus, if $\sigma^2 < \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1)$ (i.e. $\mathcal{R}_s > 1$), then the infection may become almost surely persistent in the mean, whereas if $\sigma^2 > \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2}$ (i.e. (C_3) in Theorem 2 holds), then the disease dies out.

We summarize the above analysis as follows.

Corollary 1. *Let Assumptions $(H_1) - (H_3)$ hold and let $(S(t), I(t), R(t))$ be the solution of model (1.3) with any given initial condition $(S_0, I_0, R_0) \in \mathbb{R}^3_+$. The solution $(S(t), I(t), R(t))$ has the property:*

(i) *if* $\mathcal{R}_0 \leq 1$, *then the disease dies out with probability one for any* $\sigma > 0$;
(ii) *if* $1 < \mathcal{R}_0 \leq 2$, *then* $\mathcal{R}_s$ *is the threshold parameter, i.e., the disease dies out with probability one if* $\mathcal{R}_s < 1$, *whereas if* $\mathcal{R}_s > 1$, *the infection may become almost surely persistent in the time mean;*
(iii) *if* $\mathcal{R}_0 > 2$, *then the disease dies out with probability one if* $\sigma^2 > \frac{\beta\mu}{\Lambda f'(0)} \cdot \frac{\mathcal{R}_0}{2}$, *while if* $\sigma^2 < \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2(f'(0))^2}(\mathcal{R}_0 - 1)$, *the infection may become almost surely persistent in the time mean.*

5. Numerical simulations

In this section, we analyze the stochastic behaviors of model (1.3) by means of the numerical simulations in order to make readers understand our results more better. Throughout the following numerical simulations, we choose $f(I) = I$, $f(I) = \frac{I}{1+aI^2}$ and $f(I) = Ie^{-mI}$, respectively. Thus, $f(I) \leq I$. The numerical simulations of epidemic dynamics are carried out for the academic tests with the arbitrary values of the parameters, which do not correspond to some epidemic and exhibit only the theoretical properties of numerical solutions of considered models.

Example 1. *Let* $f(I) = I$. *Numerical experiments were made by using the following parameters:*

$$\Lambda = 0.8,\ \mu = 0.4,\ \gamma_1 = 0.4,\ \gamma_2 = 0.1,\ \alpha = 0.3,\ \delta = 0.5,\ (S_0, I_0, R_0) = (1, 1, 0).$$

(i) In order to demonstrate the conclusion (i) in Corollary 1, we take $\beta = 0.3$ and $\sigma^2 = 0.2$. By a simple computation, $\mathcal{R}_0 = 0.5 < 1$. From the numerical simulations given in Figure 1, the disease $I(t)$ in deterministic model (1.1) and stochastic model (1.3) will die out with probability one (Figure 1).

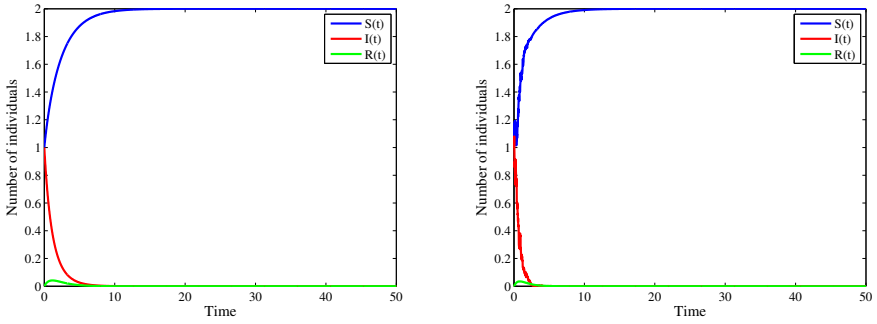


Figure 1. (a) The trajectories of deterministic SIRS model (1.1). (b) The trajectories of stochastic SIRS model (1.3) with $\sigma^2 = 0.2$.

(ii) In order to demonstrate the conclusion (ii) in Corollary 1, we take $\beta = 0.8$. By a simple computation, we obtain $1 < \mathcal{R}_0 \approx 1.33 < 2$. From the numerical simulations given in Figure 2(a), it is shown that disease $I(t)$ of deterministic model (1.1) is permanent in the population.

1° Assume that $\sigma^2 = 0.4$. By a simple computation, we obtain $\mathcal{R}_s \approx 0.66 < 1$. It can be seen from Figure 2(b) that disease $I(t)$ in model (1.3) will die out with probability one.

2° Assume that $\sigma^2 = 0.02$. By computing, $\mathcal{R}_s \approx 1.297 > 1$. From the numerical simulations given in Figure 3, it is shown that disease $I(t)$ of model (1.3) is permanent in the mean with probability one.

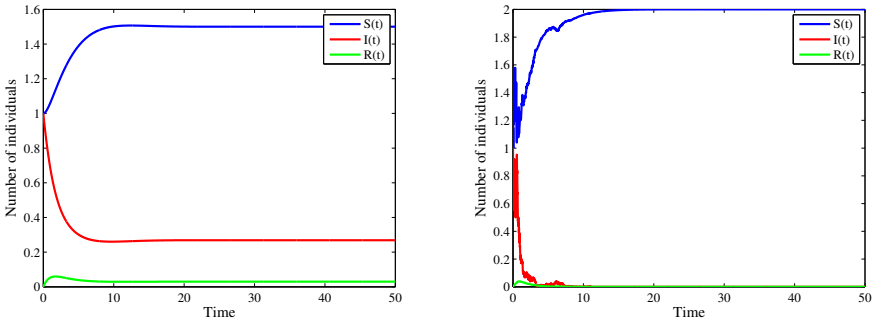


Figure 2. (a) The trajectories of deterministic SIRS model (1.1). (b) The trajectories of stochastic SIRS model (1.3) with $\sigma^2 = 0.4$.

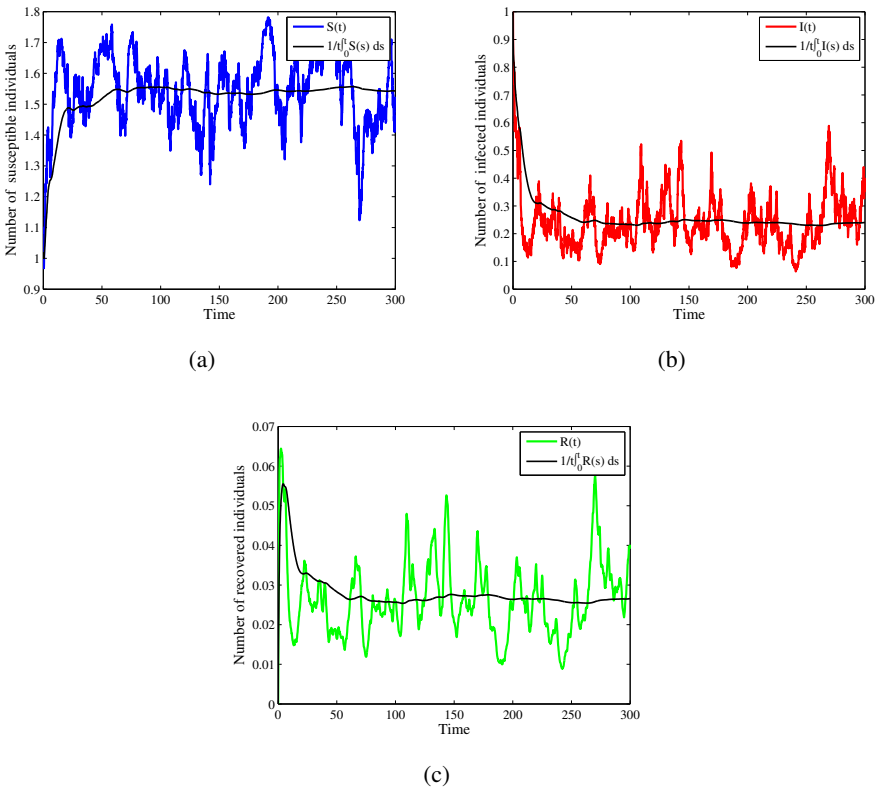


Figure 3. The trajectories of stochastic SIRS model (1.3) with $\sigma^2 = 0.02$. (a) the trajectories of susceptible individuals, (b) the trajectories of infectious individuals, (c) the trajectories of recovered individuals.

(iii) In order to demonstrate the conclusion (iii) in Corollary 1, we take $\beta = 1.5$. By a simple computation, $\mathcal{R}_0 = 2.5 > 2$. From Figure 4(a), it follows that $I(t)$ of (1.1) is permanent in the population.

1° Assume that $\sigma^2 = 0.03$ ($\sigma^2 = 0.3$). By a simple computation, $\sigma^2 < \frac{2\mu^2(\mu+\gamma_1+\gamma_2+\alpha)}{\Lambda^2}(\mathcal{R}_0 - 1) = 0.9$. It can be seen from Figure 4(b) (4(c)) that disease $I(t)$ in model (1.3) is permanent in the time mean.

2° Assume that $\sigma^2 = 1$. By computing, $\sigma^2 > \frac{\beta\mu}{\Lambda} \cdot \frac{\mathcal{R}_0}{2} = 0.9375$. From the numerical simulations given in Figure 4(d), it is shown that disease $I(t)$ of model (1.3) will die out with probability one.

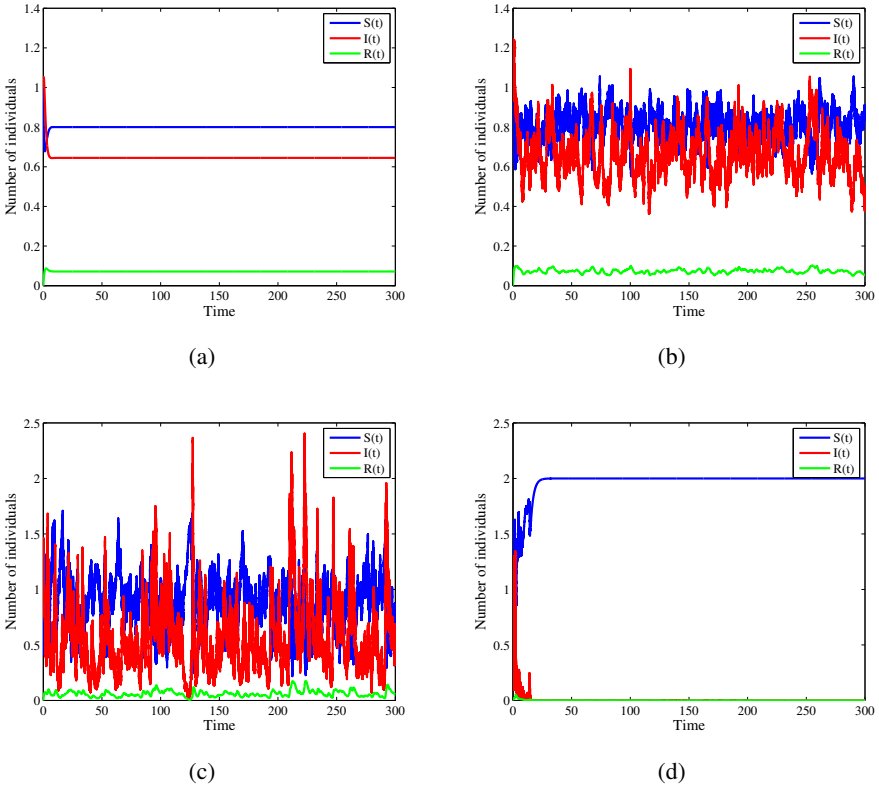


Figure 4. The trajectories of stochastic SIRS model (1.3). (a) $\sigma^2 = 0$, (b) $\sigma^2 = 0.03$, (c) $\sigma^2 = 0.3$, (d) $\sigma^2 = 1$.

Example 2. *Let* $f(I) = \frac{I}{1+aI^2}$ *and* $a = 1$. *Other parameters are defined as Example 1.*

(i)From the numerical simulations given in Figure 5, the disease $I(t)$ in deterministic model (1.1) and stochastic model (1.3) will die out with probability one.

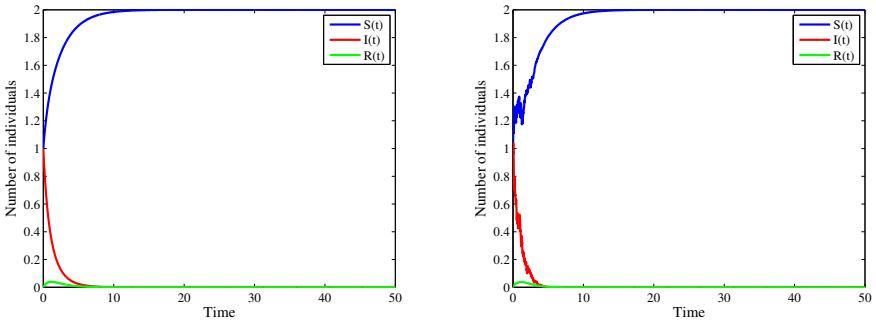
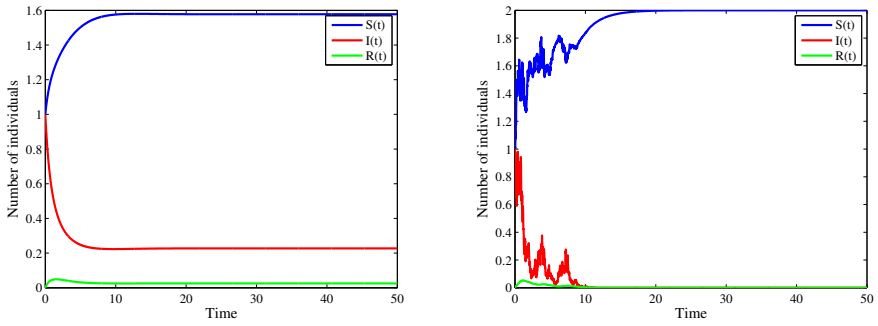


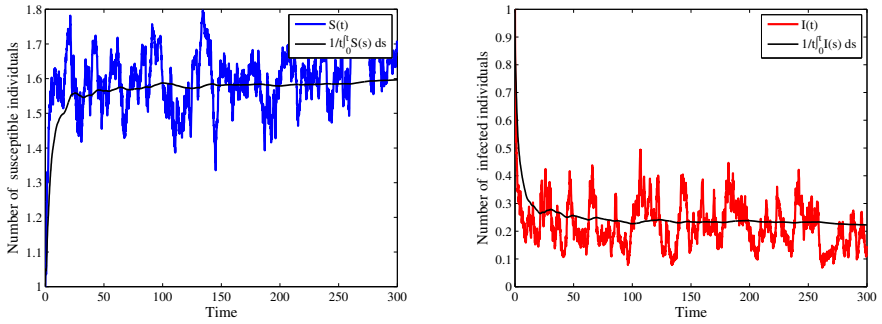
Figure 5. (a) The trajectories of deterministic SIRS model (1.1). (b) The trajectories of stochastic SIRS model (1.3) with $\sigma^2 = 0.2$.

(ii)From Figure 6(a), we obtain that disease $I(t)$ of (1.1) is permanent in the population.
1° If $\sigma^2 = 0.4$. From Figure 6(b), disease $I(t)$ in model (1.3) will be extinct with probability one.
2° If $\sigma^2 = 0.02$. From Figure 7, disease $I(t)$ of model (1.3) is permanent in the mean.

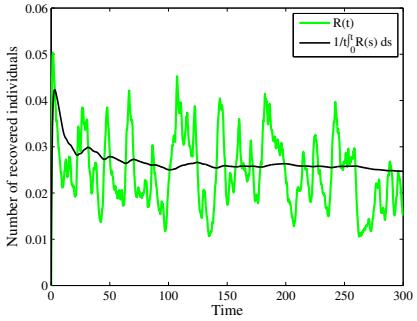


(a) (b)

Figure 6. (a) The trajectories of deterministic SIRS model (1.1). (b) The trajectories of stochastic SIRS model (1.3) with $\sigma^2 = 0.4$.



(a) (b)



(c)

Figure 7. The trajectories of SIRS model (1.3) with $\sigma^2 = 0.02$. (a) the trajectories of susceptible individuals, (b) the trajectories of infectious individuals, (c) the trajectories of recovered individuals.

(iii) From the numerical simulations given in Figure 8(a), it is shown that disease $I(t)$ of determin-

istic model (1.1) is permanent in the population.

1° If $\sigma^2 = 0.03$ ($\sigma^2 = 0.3$). From Figure 8(b) (8(c)), disease $I(t)$ in (1.3) is permanent in mean.

2° If $\sigma^2 = 1$. From Figure 8(d), we obtain that disease $I(t)$ of model (1.3) will be extinct.

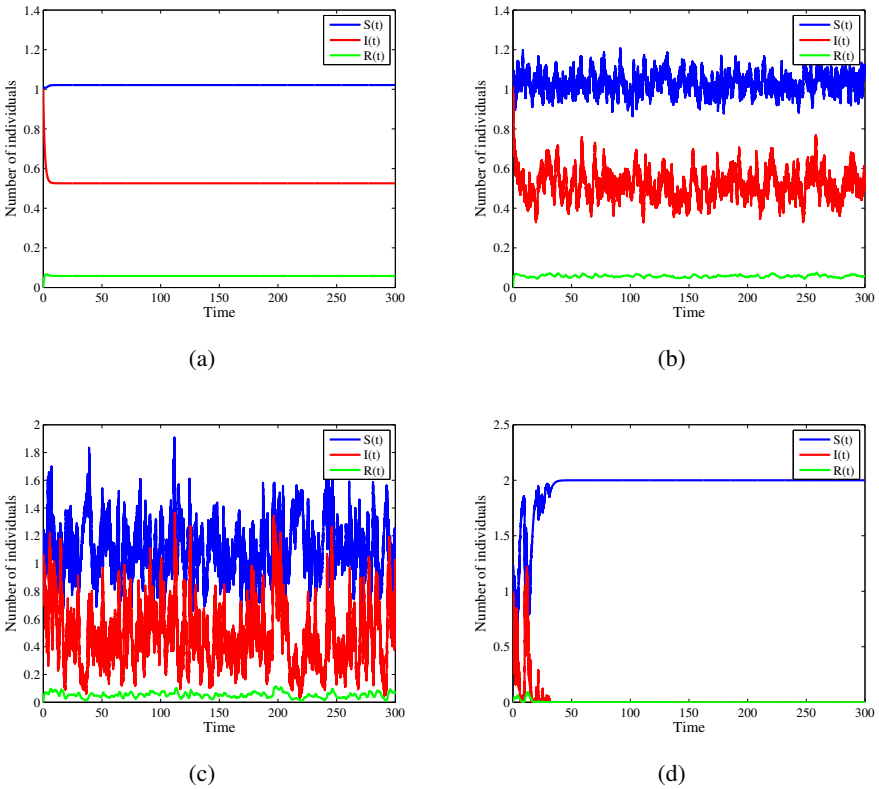


Figure 8. The trajectories of stochastic SIRS model (1.3). (a) $\sigma^2 = 0$, (b) $\sigma^2 = 0.03$, (c) $\sigma^2 = 0.3$, (d) $\sigma^2 = 1$.

Example 3. *Let* $f(I) = Ie^{-mI}$ *and* $m = 1$. *Other parameters are defined as Example 1.*

(i)From Figure 9, the disease $I(t)$ in model (1.1) and model (1.3) will die out with probability one.

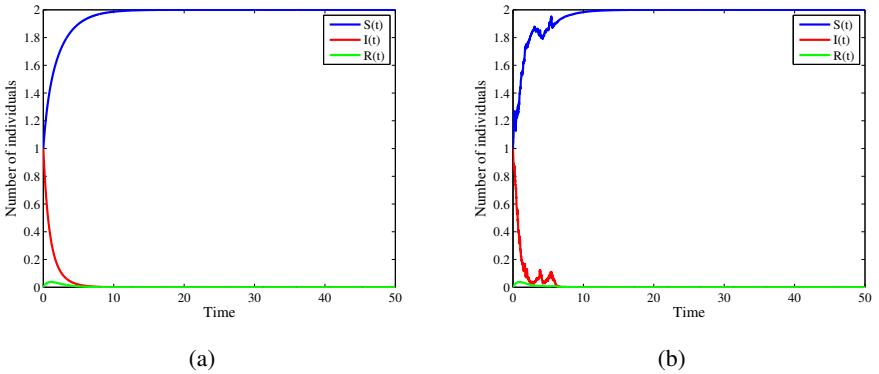
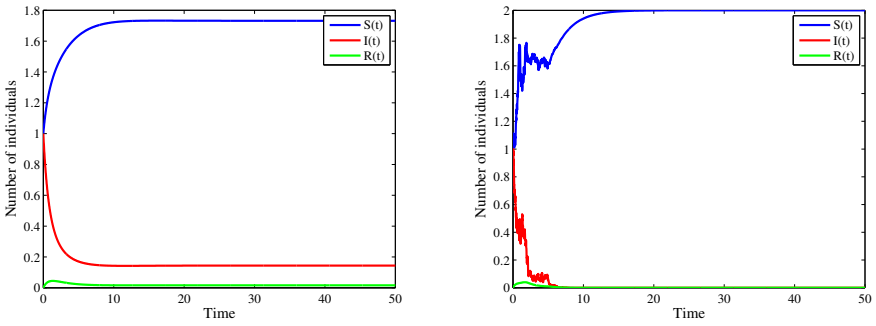


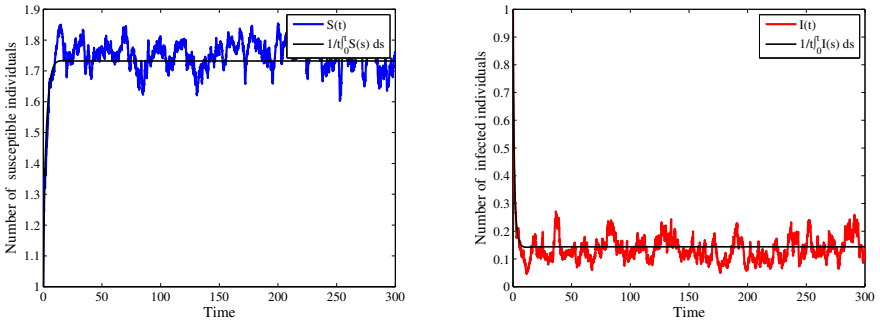
Figure 9. (a) The trajectories of deterministic SIRS model (1.1). (b) The trajectories of stochastic SIRS model (1.3) with $\sigma^2 = 0.2$.

(ii)From Figure 10(a), we obtain that disease $I(t)$ of (1.1) is permanent in the population.
1° If $\sigma^2 = 0.4$. From Figure 10(b), disease $I(t)$ in model (1.3) will be extinct with probability one.
2° If $\sigma^2 = 0.02$. From Figure 11, disease $I(t)$ of model (1.3) is permanent in the mean.

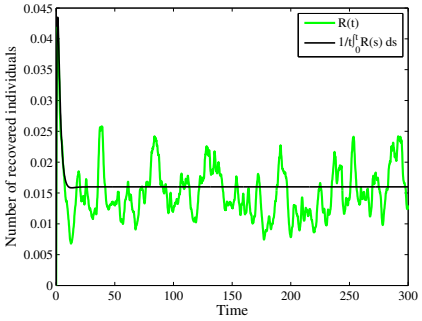


(a) (b)

Figure 10. (a) The trajectories of deterministic SIRS model (1.1). (b) The trajectories of stochastic SIRS model (1.3) with $\sigma^2 = 0.4$.



(a) (b)



(c)

Figure 11. The trajectories of stochastic SIRS model (1.3) with $\sigma^2 = 0.02$. (a) the trajectories of susceptible individuals, (b) the trajectories of infectious individuals, (c) the trajectories of recovered individuals.

(iii) From the numerical simulations given in Figure 12(a), it is shown that disease $I(t)$ of determin-

istic model (1.1) is permanent in the population.

1° If $\sigma^2 = 0.03$ ($\sigma^2 = 0.3$). From Figure 12(b) (12(c)), we get that disease $I(t)$ in model (1.3) is permanent in the mean.

2° If $\sigma^2 = 1$. From Figure 12(d), we obtain that disease $I(t)$ of model (1.3) will be extinct with probability one.

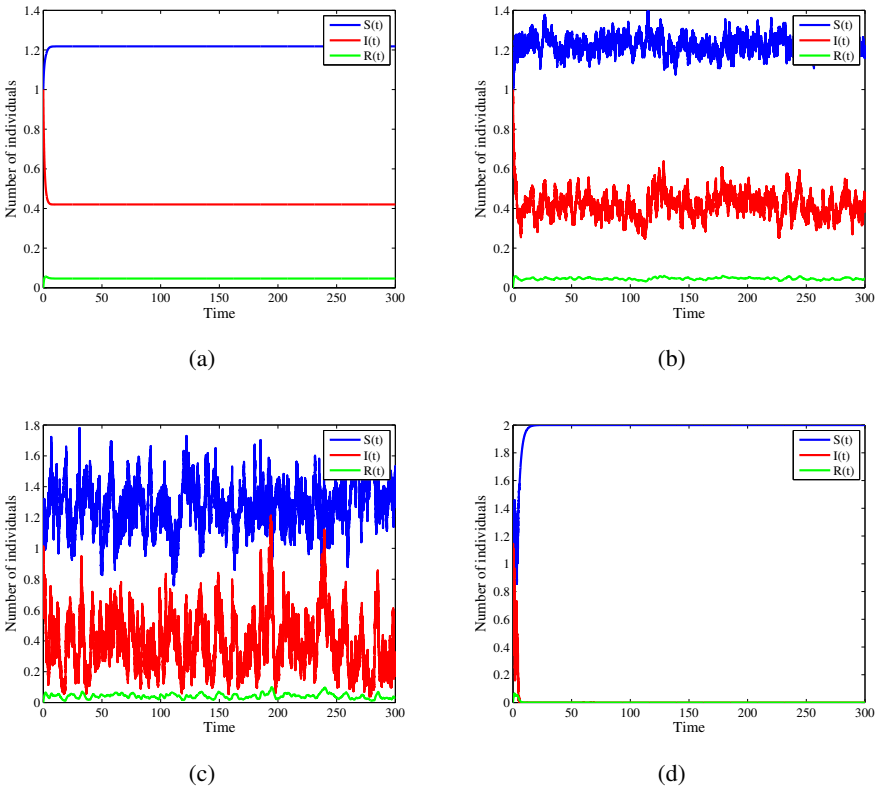


Figure 12. The trajectories of stochastic SIRS model (1.3). (a) $\sigma^2 = 0$, (b) $\sigma^2 = 0.03$, (c) $\sigma^2 = 0.3$, (d) $\sigma^2 = 1$.

From Examples 1, 2 and 3, we can see that different incidence rates of diseases have different effects on the extinction and persistence in the mean of diseases. In addition, the intensity of noise plays an important role in epidemic dynamics. Comparing Figures 3 and 11 (or Figures 7 and 11), we can conclude that massive media coverages are needed to prevent the disease to spread widely in the population. This is consistent with the results in [26].

Comparing Figures 5(a) and 5(b), we conclude that if $\mathcal{R}_0 < 1$, then $I(t)$ becomes extinct regardless of the intensity of noise. From Figures 6(a), 6(b) and 7, if $1 < \mathcal{R}_0 < 2$, then great intensity of the noise can make diseases extinction. From Figure 8(d), if $\mathcal{R}_0 > 2$, the disease becomes extinct as the intensity of the noise is large.

6. Conclusions and discussions

This paper is concerned with the persistence and extinction of a stochastic SIRS epidemic model with nonlinear incidence rate and transfer from infectious to susceptible. To begin with, we consider the

global existence and uniqueness of the positive solution to model (1.3) with any positive initial value. Next, sufficient criteria for the persistence and extinction of the disease are established. Then, we discuss the relationship between $\mathcal{R}_0$ and $\mathcal{R}_s$ for different values of white noise intensity σ^2. In addition, we find that in case of neglecting the impact of environmental noises, the deterministic threshold $\mathcal{R}_0$ may exist and the threshold parameter will be overestimated. Furthermore, we discover that a large noise has the effect of suppressing the epidemic. So these results show that noises have important effects on the persistence and extinction of the disease. In addition, we can see that different incidence rates of diseases have different effects on epidemic dynamics.

Although there are important discoveries revealed by these studies, there are also limitations for the model. Theorem 2 shows that the particular expression for the nonlinear transmission makes no influence on the extinction of the disease. We leave these as our future work.

Acknowledgments

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Conflict of interest

The authors declare there is no conflict of interest.

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