Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/MBE

MBE, 16(5): 5991–6014.
DOI: 10.3934/mbe.2019300
Received: 01 April 2019
Accepted: 26 May 2019
Published: 28 June 2019

Research article

Dynamics of an epidemic model with advection and free boundaries

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Abstract: This paper deals with the propagation dynamics of an epidemic model, which is modeled by a partially degenerate reaction-diffusion-advection system with free boundaries and sigmoidal function. We focus on the effect of small advection on the propagation dynamics of the epidemic disease. At first, the global existence and uniqueness of solution are obtained. And then, the spreading-vanishing dichotomy and the criteria for spreading and vanishing are given. Our results imply that the small advection make the disease spread more difficult.

Keywords: epidemic model; partially degenerate; advection; free boundary; spreading and vanishing

1. Introduction

In order to describe the evolution of fecal-oral transmitted diseases in the Mediterranean regions, Capasso and Paveri-Fontana [1] proposed the following model

$$\begin{cases} u'(t) = -au + cv, \\ v'(t) = -bv + G(u), \end{cases} \tag{1.1}$$

where a, b, c are all positive constants, $u(t)$ and $v(t)$ denote the concentration of the infectious agent in the environment and the infective human population respectively. The coefficients a and b are the intrinsic decay rates of the infectious agent and the infective human population respectively, c represents the multiplication rate of the infectious agent due to the human infected population. The function $G(u)$ stands for the force of infection of the human population due to the concentration of infectious agent. We assume that $G(u)$ satisfies the two specific cases: (i) a monotone increasing function with constant concavity; (ii) a sigmoidal function of bacterial concentration tending to some finite limit, and with zero gradient at zero. These two cases contain most of the features of forces of infection in real epidemics. For some epidemic, if the density of infectious agent is small, the force of infection of the humans will be weak and may tend to zero, and the function G will satisfy case (ii). In this paper, we focus on such case, and assume that the function $G : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies:

(G1) $G \in C^2(\mathbb{R}^+)$, $G(0) = 0$, $G'(z) > 0$ for any $z > 0$ and $\lim_{z\to\infty} G(z) = 1$;

(G2) there exists $\xi > 0$ such that $G''(z) > 0$ for $z \in (0, \xi)$ and $G''(z) < 0$ for $z \in (\xi, \infty)$.

Denote

$$\theta = \frac{cG'(0)}{ab}.$$

Under two specific cases stated above, the global dynamics of the cooperative system (1.1) has been described in detail in [2]. It follows from [2, Theorem 4.3] that the global dynamics of (1.1) under conditions (G1) and (G2) can be described as follows:

(i) If $\theta < 1$ and $\frac{G(z)}{z} < \frac{ab}{c}$ for any $z > 0$, then the trivial solution is the only equilibrium for problem (1.1) and it is globally asymptotically stable in $\mathbb{R}^+ \times \mathbb{R}^+$.

(ii) If $\theta > 1$, then problem (1.1) has only one nontrivial equilibrium point (u^*, v^*) in addition to $(0, 0)$ and it is globally asymptotically stable in $\mathbb{R}^+ \times \mathbb{R}^+$.

(iii) If $\theta < 1$ and $\frac{G(z_1)}{z_1} > \frac{ab}{c}$ for some $z_1 > 0$, then problem (1.1) has three equilibrium points:

$$E_0 = (0, 0),\ E_1 = \left(K_1, \frac{aK_1}{c}\right) \text{ and } E_2 = \left(K_2, \frac{aK_2}{c}\right),$$

where $0 < K_1 < K_2$ are the positive roots of $G(z) - \frac{ab}{c} z = 0$. In this case, E_1 is a saddle point, E_0 and E_2 are stable nodes.

In 1997, Capasso and Wilson [3] further considered spatial variation and studied the problem

$$\begin{cases} u_t = d\Delta u - au + cv, & (t, x) \in (0, +\infty) \times \Omega, \\ v_t = -bv + G(u), & (t, x) \in (0, +\infty) \times \Omega, \\ u(t, x) = 0, & (t, x) \in (0, +\infty) \times \partial\Omega, \\ u(0, x) = u_0(x),\ v(0, x) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where Ω is bounded. By some numerical simulation, they speculated that the dynamical behavior of system (1.2) is similar to the ODE case. To understand the dispersal process of epidemic from outbreak to an endemic, Xu and Zhao [4] studied the bistable traveling waves of (1.2) in $x \in \mathbb{R}$.

The epidemic always spreads gradually, but the works mentioned above are hard to explain this gradual expanding process. To describe such a gradual spreading process, Du and Lin [5] introduced the free boundary condition to study the invasion of a single species. They considered the problem

$$\begin{cases} u_t - du_{xx} = u(a - bu), & t > 0,\ 0 < x < h(t), \\ u_x(t, 0) = 0,\ u(t, h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ h(0) = h_0,\ u(0, x) = u_0(x), & 0 \le x \le h_0, \end{cases} \tag{1.3}$$

and showed that (1.3) admits a unique solution which is well-defined for all $t \ge 0$ and spreading and vanishing dichotomy holds. Moreover, the criteria for spreading and vanishing are obtained: (i) for

$h_0 \geq \frac{\pi}{2}\sqrt{\frac{d}{a}}$, the species will spread; (ii) for $h_0 < \frac{\pi}{2}\sqrt{\frac{d}{a}}$ and given $u_0(x)$, there exists μ^* such that the species will spread for $\mu > \mu^*$, and the species will vanish for $0 < \mu \leq \mu^*$. Finally, they gave the spreading speed of the spreading front when spreading occurs. Since then, many problems with free boundaries and related problems have been investigated, see e.g. [6–22] and their references.

In 2016, Ahn et al. [23] considered (1.2) with monostable nonlinearity and free boundaries. They obtained the global existence and uniqueness of the solution and spreading and vanishing dichotomy. Furthermore, by introducing the so-called spatial-temporal risk index

$$R_0^F(t) = \frac{G'(0)\frac{c}{b}}{a + d\left(\frac{\pi}{h(t)-g(t)}\right)^2},$$

they proved that: (i) if $R_0 = \frac{cG'(0)}{ab} \leq 1$, the epidemic will vanish; (ii) if $R_0^F(0) \geq 1$, the epidemic will spread; (iii) if $R_0^F(0) < 1$, epidemic will vanish for the small initial densities; (iv) if $R_0^F(0) < 1 < R_0$, epidemic will spread for the large initial densities. Recently, Zhao et al. [24] determined the spreading speed of the spreading front of problem described in [23].

Inspired by the work [23], we want to study (1.2) with bistable nonlinearity and free boundaries. Meanwhile, we also want to consider the effect of the advection. In 2009, Maidana and Yang [25] studied the propagation of West Nile Virus from New York City to California. In the summer of 1999, West Nile Virus began to appear in New York City. But it was observed that the wave front traveled 187 km to the north and 1100 km to the south in the second year. Therefore, taking account of the advection movement has the greater realistic significance. Recently, there are some works considering the advection. In 2014, Gu et al. [26] was the first time to consider the long-time behavior of problem (1.3) with small advection. Then, the asymptotic spreading speeds of the free boundaries was given in [27]. For more general reaction term, Gu et al. [10] studied the long time behavior of solutions of Fisher-KPP equation with advection $\beta > 0$ and free boundaries. For single equation with advection, there are many other works. For example, [28–34] and their references. Besides, there are also several works devoted to the system with small advection, such as, [35–40] and their references.

Taking account of the effect of advection, we consider

$$\begin{cases} u_t = du_{xx} - \beta u_x - au + cv, & t > 0,\ g(t) < x < h(t), \\ v_t = -bv + G(u), & t > 0,\ g(t) < x < h(t), \\ u(t,x) = v(t,x) = 0, & t \geq 0,\ x = g(t) \text{ or } x = h(t), \\ g(0) = -h_0,\ g'(t) = -\mu u_x(t, g(t)), & t > 0, \\ h(0) = h_0,\ h'(t) = -\mu u_x(t, h(t)), & t > 0, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & -h_0 < x < h_0, \end{cases} \tag{1.4}$$

where we use the changing region $(g(t), h(t))$ to denote the infective environment of disease, where the free boundaries $x = g(t)$ and $x = h(t)$ represent the spreading fronts of epidemic. Since the diffusion coefficient of v is much smaller than that of u, we assume that the diffusion coefficient of v is zero. When u spreads into a new environment, some humans in the new environment may be infected. Hence, we can use $(g(t), h(t))$ to represent the habit of infective humans. We use $I_0 \doteq (-h_0, h_0)$ to denote the initial infective environment of epidemic. The initial functions $u_0(x)$ and $v_0(x)$ satisfy

$$u_0(x) \in \mathscr{X}_1(h_0) \doteq \left\{u_0(x) \in W_p^2(I_0):\ u_0(x) > 0 \text{ for } x \in I_0,\ u_0(x) = 0 \text{ for } x \in \mathbb{R}\backslash I_0\right\},$$
$$v_0(x) \in \mathscr{X}_2(h_0) \doteq \left\{v_0(x) \in C^2(I_0):\ v_0(x) > 0 \text{ for } x \in I_0,\ v_0(x) = 0 \text{ for } x \in \mathbb{R}\backslash I_0\right\},$$

where $p > 3$. The derivation of the stefan conditions $h'(t) = -\mu u_x(t, h(t))$ and $g'(t) = -\mu u_x(t, g(t))$ can be found in [41, 42]. In this paper, we always assume that G satisfies (G1)-(G2) and

(G3) $G(z)$ is locally Lipschitz in $z \in \mathbb{R}^+$, i.e., for any $L > 0$, there exists a constant $\rho(L) > 0$ such that

$$|G(z_1) - G(z_2)| \le \rho(L)|z_1 - z_2|,\ \forall\ z_1, z_2 \in [0, L].$$

Furthermore, we assume that $0 < \beta < \beta^*$ with

$$\beta^* = \begin{cases} \infty, & \theta < 1, \\ 2\sqrt{d\left(\frac{cG'(0)}{b} - a\right)}, & \theta > 1. \end{cases}$$

The rest of this paper is organized as follows. In Section 2, the global existence and uniqueness of solution, comparison principle and some results about the principal eigenvalue are given. Section 3 is devoted to the long time behavior of (u, v). We get a spreading and vanishing dichotomy and give the criteria for spreading and vanishing. Finally, we give some discussions in Section 4.

2. Preliminaries

Firstly, we prove the existence and uniqueness of the solution.

Lemma 2.1. *For any given* $(u_0, v_0) \in \mathscr{X}_1(h_0) \times \mathscr{X}_2(h_0)$ *and any* $\alpha \in (0, 1)$*, there exists a* $T > 0$ *such that problem* (1.4) *admits a unique solution*

$$(u, v, g, h) \in \left(W_p^{1,2}(\Omega_T) \cap C^{\frac{1+\alpha}{2}, 1+\alpha}(\overline{\Omega}_T)\right) \times C^1([0, T]; L^\infty([g(t), h(t)])) \times \left[C^{1+\frac{\alpha}{2}}([0, T])\right]^2, \tag{2.1}$$

moreover,

$$\|u\|_{W_p^{1,2}(\Omega_T)} + \|u\|_{C^{\frac{1+\alpha}{2}, 1+\alpha}(\overline{\Omega}_T)} + \|g\|_{C^{1+\frac{\alpha}{2}}([0,T])} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,T])} \le C, \tag{2.2}$$

where $\Omega_T = \{(t, x) \in \mathbb{R}^2 :\ 0 \le t \le T,\ g(t) \le x \le h(t)\}$*, C and T depend only on* $h_0,\ \alpha,\ \|u_0\|_{W_p^2([-h_0, h_0])}$ *and* $\|v_0\|_\infty$.

Proof. This proof can be done by the similar arguments in [43]. But there are some differences. Hence, we give the details. Let

$$y = \frac{2x - g(t) - h(t)}{h(t) - g(t)},\ w(t, y) = u\left(t, \frac{(h(t) - g(t))y + h(t) + g(t)}{2}\right),$$

and

$$z(t, y) = v\left(t, \frac{(h(t) - g(t))y + h(t) + g(t)}{2}\right).$$

Then problem (1.4) becomes

$$\begin{cases} w_t - dA^2 w_{yy} + (\beta A - B) w_y = -aw + cz, & 0 < t < T,\ -1 < y < 1, \\ w(t,-1) = w(t,1) = 0, & 0 \le t < T, \\ w(0,y) = u_0(h_0 y) \doteq w_0(y), & -1 < y < 1, \end{cases} \tag{2.3}$$

$$\begin{cases} v_t = -bv + G(u), & 0 < t < T,\ g(t) < x < h(t), \\ v(t,g(t)) = v(t,h(t)) = 0, & 0 \le t < T, \\ v(0,x) = v_0(x), & -h_0 < x < h_0, \end{cases} \tag{2.4}$$

and

$$\begin{cases} g'(t) = -\mu A w_y(t,-1), & 0 < t < T, \\ h'(t) = -\mu A w_y(t,1), & 0 < t < T, \\ g(0) = -h_0,\ h(0) = h_0, \end{cases} \tag{2.5}$$

where

$$A = A(g(t),h(t)) = \frac{2}{h(t)-g(t)} \ \text{ and } \ B = B(g(t),h(t),y) = \frac{h'(t)+g'(t)}{h(t)-g(t)} + y\frac{h'(t)-g'(t)}{h(t)-g(t)}.$$

Denote $g^* = -\frac{\mu}{h_0}u_0'(-h_0)$ and $h^* = -\frac{\mu}{h_0}u_0'(h_0)$. For $0 < T \le \frac{h_0}{2(2+g^*+h^*)}$, define

$$\begin{aligned} \triangle_T &= [0,T]\times[-1,1], \\ \mathcal{D}_{1T} &= \{w \in C(\triangle_T):\ w(0,y) = w_0(y),\ w(t,\pm 1) = 0,\ \|w - w_0\|_{C(\triangle_T)} \le 1\}, \\ \mathcal{D}_{2T} &= \{g \in C^1([0,T]):\ g(0) = -h_0,\ g'(0) = g^*,\ \|g' - g^*\|_{C([0,T])} \le 1\}, \\ \mathcal{D}_{3T} &= \{h \in C^1([0,T]):\ h(0) = h_0,\ h'(0) = h^*,\ \|h' - h^*\|_{C([0,T])} \le 1\}. \end{aligned}$$

It is easy to see that $\mathcal{D}_T \doteq \mathcal{D}_{1T}\times\mathcal{D}_{2T}\times\mathcal{D}_{3T}$ is a complete metric space with the metric

$$d\left((w_1,g_1,h_1),(w_2,g_2,h_2)\right) = \|w_1 - w_2\|_{C(\triangle_T)} + \|g_1 - g_2\|_{C^1([0,T])} + \|h_1 - h_2\|_{C^1([0,T])}.$$

For any given $(w,g,h) \in \mathcal{D}_T$, there exist some $\xi_1,\xi_2 \in (0,t)$ such that

$$|g(t)+h_0| + |h(t)-h_0| = |g'(\xi_1)|t + |h'(\xi_2)|t \le T(2+g^*+h^*) \le \frac{h_0}{2},$$

which implies that

$$2h_0 \le h(t) - g(t) \le 3h_0,\ \forall\ t \in [0,T].$$

Thus, $A(g(t),h(t))$ and $B(g(t),h(t),y)$ are well-defined. By the definition of w, we have

$$u(t,x) = w\left(t, \frac{2x - g(t) - h(t)}{h(t)-g(t)}\right). \tag{2.6}$$

Since $|w(t,y)| \le \|w_0\|_{L^\infty} + 1$ for $(t,y) \in \triangle_T$, we have

$$|u(t,x)| \le \|w_0\|_{L^\infty} + 1 \doteq M_1,\ \forall\ (t,x) \in [0,T]\times[g(t),h(t)].$$

Define

$$\widetilde{v}_0(x)=\begin{cases}v_0(x), & x\in(-h_0,h_0),\\ 0, & x\in\mathbb{R}\backslash(-h_0,h_0)\end{cases}\quad\text{and } t_x:=\begin{cases}t_x^g, & x\in[g(T),-h_0)\text{ and } x=g(t_x^g),\\ 0, & x\in[-h_0,h_0],\\ t_x^h, & x\in(h_0,h(T)]\text{ and } x=h(t_x^h).\end{cases}$$

For u defined as (2.6) and any given $x\in[g(T),h(T)]$, we consider the following ODE problem

$$\begin{cases}v_t=-bv+G(u(t,x)), & t_x<t<T,\\ v(t_x,x)=\widetilde{v}_0(x).\end{cases}\tag{2.7}$$

By the similar arguments as the step 1 in the proof of [44, Lemma 2.3], it is easy to show that (2.7) admits a unique solution $v(t,x)$ for $t\in[t_x,T_1]$, where $T_1\in\left(0,\frac{h_0}{2(2+g^*+h^*)}\right]$. Hence, problem (2.4) has a unique solution $v(t,x)\in C^1([0,T_1];L^\infty([g(t),h(t)]))$. By the continuous dependence of the solution on parameters, we can have

$$\|v_x\|_{L^\infty(\Omega_{T_1})}\le C_1.$$

Then

$$\|v_x\|_{L^\infty(\Omega_T)}\le\|v_x\|_{L^\infty(\Omega_{T_1})}\le C_1,\ \forall\, T\le T_1.$$

For this v, we can get

$$z(t,y)=v\left(t,\frac{(h(t)-g(t))y+h(t)+g(t)}{2}\right).$$

For (w,g,h) and z obtained above, we consider the following problem

$$\begin{cases}\overline{w}_t-dA^2\overline{w}_{yy}+(\beta A-B)\overline{w}_y=-aw+cz, & 0<t<T,\ -1<y<1,\\ \overline{w}(t,-1)=\overline{w}(t,1)=0, & 0\le t<T,\\ \overline{w}(0,y)=u_0(h_0y), & -1<y<1.\end{cases}\tag{2.8}$$

Applying standard L^p theory and the Sobolev imbedding theorem, we can have there exists $T_2\in(0,T_1]$ such that (2.8) admits a unique solution $\overline{w}(t,y)$ and

$$\|\overline{w}\|_{W_p^{1,2}(\triangle_{T_2})}+\|\overline{w}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\triangle_{T_2})}\le C_2,$$

where C_2 is a constant depending only on $h_0,\ \alpha$ and $\|u_0\|_{W_p^2([-h_0,h_0])}$. Then

$$\|\overline{w}\|_{W_p^{1,2}(\triangle_T)}+\|\overline{w}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\triangle_T)}\le\|\overline{w}\|_{W_p^{1,2}(\triangle_{T_2})}+\|\overline{w}\|_{C^{\frac{1+\alpha}{2},1+\alpha}(\triangle_{T_2})}\le C_2,\ \forall\, T\le T_2.\tag{2.9}$$

Define

$$\overline{g}(t)=-h_0-\int_0^t\mu A(g(\tau),h(\tau))\overline{w}_y(\tau,-1)d\tau,$$
$$\overline{h}(t)=h_0-\int_0^t\mu A(g(\tau),h(\tau))\overline{w}_y(\tau,1)d\tau,$$

then we have $\overline{g}(0) = -h_0,\ \overline{h}(0) = h_0$,

$$\overline{g}'(t) = -\mu A(g(t), h(t))\overline{w}_y(t, -1),\ \overline{h}'(t) = -\mu A(g(t), h(t))\overline{w}_y(t, 1),$$

and hence

$$\|\overline{g}'\|_{C^{\frac{\alpha}{2}}([0,T])},\ \|\overline{h}'\|_{C^{\frac{\alpha}{2}}([0,T])} \le \mu h_0^{-1} C_2 \doteq C_3. \tag{2.10}$$

Now, we can define the mapping $\mathcal{F} : \mathcal{D}_T \to C(\triangle_T) \times C^1([0, T]) \times C^1([0, T])$ by

$$\mathcal{F}(w, g, h) = (\overline{w}, \overline{g}, \overline{h}).$$

Obviously, $\mathcal{D}_T$ is a bounded and closed convex set of $C(\triangle_T) \times C^1([0, T]) \times C^1([0, T])$, $\mathcal{F}$ is continuous in $\mathcal{D}_T$, and (w, g, h) is a fixed point of $\mathcal{F}$ if and only if (w, v, g, h) solve (2.3), (2.4) and (2.5). By (2.9) and (2.10), we have $\mathcal{F}$ is compact and

$$\|\overline{w} - w_0\|_{C(\triangle_T)} \le C_2 T^{\frac{1+\alpha}{2}},\ \|\overline{g}' - g^*\|_{C([0,T])} \le C_3 T^{\frac{\alpha}{2}},\ \|\overline{h}' - h^*\|_{C([0,T])} \le C_3 T^{\frac{\alpha}{2}}.$$

Therefore if we take $T \le \min\left\{T_2,\ C_2^{-\frac{2}{1+\alpha}},\ C_3^{-\frac{2}{\alpha}}\right\} \doteq T_3$, then $\mathcal{F}$ maps $\mathcal{D}_T$ into itself. It now follows from the Schauder fixed point theorem that $\mathcal{F}$ has a fixed point (w, g, h) in $\mathcal{D}_T$. Moreover, we have (w, v, g, h) solve (2.3), (2.4) and (2.5),

$$\|w\|_{W_p^{1,2}(\triangle_T)} + \|w\|_{C^{\frac{1+\alpha}{2}, 1+\alpha}(\triangle_T)} \le C_2,\ \|v_x\|_{L^\infty(\Omega_T)} \le C_1,\ \forall\ T \le T_3.$$

Define as before,

$$u(t, x) = w\left(t, \frac{2x - g(t) - h(t)}{h(t) - g(t)}\right).$$

Then (u, v, g, h) solve (1.4), and satisfies (2.1) and (2.2).

In the following, we prove the uniqueness of (u, v, g, h). Let (u_i, v_i, g_i, h_i) $(i = 1, 2)$ be the two solutions of problem (1.4) for $T \in (0, T_3]$ sufficiently small. Let

$$w_i(t, y) = u_i\left(t, \frac{(h_i(t) - g_i(t))y + h_i(t) + g_i(t)}{2}\right).$$

Then it is easy to see that (w_i, v_i, g_i, h_i) solve (2.3), (2.4) and (2.5). Denoting

$$A_i = A(g_i(t), h_i(t)),\ B_i = B(g_i(t), h_i(t), y),\ W = w_1 - w_2,\ Z = z_1 - z_2,\ G = g_1 - g_2,\ H = h_1 - h_2,$$

we can have

$$\begin{cases} W_t - dA_1^2 W_{yy} + (\beta A_1 - B_1)W_y = -aW + cZ & \\ \qquad + (dA_1^2 - dA_2^2)w_{2yy} + [-(\beta A_1 - B_1) + (\beta A_2 - B_2)]w_{2y}, & 0 < t < T,\ -1 < y < 1, \\ W(t, -1) = W(t, 1) = 0, & 0 \le t < T, \\ W(0, y) = 0, & -1 < y < 1, \end{cases}$$

and

$$\begin{cases} G' = -\mu A_1 W_y(t, -1) + \mu(A_2 - A_1)w_{2y}(t, -1), & 0 < t < T, \\ H' = -\mu A_1 W_y(t, 1) + \mu(A_2 - A_1)w_{2y}(t, 1), & 0 < t < T, \\ G(0) = 0,\ H(0) = 0. \end{cases} \tag{2.11}$$

Using the L^p estimates for parabolic equations and Sobolev imbedding theorem, we obtain

$$\|W\|_{W_p^{1,2}(\triangle_T)} \le C_4 \left(\|Z\|_{C(\triangle_T)} + \|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])} \right), \tag{2.12}$$

where C_4 depends on $C_2,\ C_3$ and the functions A and B. Next we should estimate $\|z_1 - z_2\|_{C(\triangle_T)}$. For convenience, we define

$$\begin{aligned} &H_m(t) \doteq \min\{h_1(t), h_2(t)\},\ H_M(t) \doteq \max\{h_1(t), h_2(t)\},\\ &G_m(t) \doteq \min\{g_1(t), g_2(t)\},\ G_M(t) \doteq \max\{g_1(t), g_2(t)\},\\ &\Omega_T^{G_m,H_M} \doteq [0,T] \times [G_m(t), H_M(t)]. \end{aligned}$$

By direct calculations, we have

$$\begin{aligned} &\|z_1(t,y) - z_2(t,y)\|_{C(\triangle_T)}\\ = &\left\| v_1\left(t, \frac{(h_1(t)-g_1(t))y + h_1(t) + g_1(t)}{2}\right) - v_2\left(t, \frac{(h_2(t)-g_2(t))y + h_2(t) + g_2(t)}{2}\right) \right\|_{C(\triangle_T)}\\ \le &\left\| v_1\left(t, \frac{(h_1(t)-g_1(t))y + h_1(t) + g_1(t)}{2}\right) - v_2\left(t, \frac{(h_1(t)-g_1(t))y + h_1(t) + g_1(t)}{2}\right) \right\|_{C(\triangle_T)}\\ &+ \left\| v_2\left(t, \frac{(h_1(t)-g_1(t))y + h_1(t) + g_1(t)}{2}\right) - v_2\left(t, \frac{(h_2(t)-g_2(t))y + h_2(t) + g_2(t)}{2}\right) \right\|_{C(\triangle_T)}\\ \le &\ \|v_1(t,x) - v_2(t,x)\|_{C(\Omega_T^{G_m,H_M})} + \|v_{2x}\|_{L^\infty(\Omega_T^{G_m,H_M})} \left(\|G\|_{C([0,T])} + \|H\|_{C([0,T])} \right). \end{aligned} \tag{2.13}$$

Now we estimate $|(v_1 - v_2)(t^*, x^*)|$ for any fixed $(t^*, x^*) \in \Omega_T^{G_m,H_M}$. It will be divided into the following three cases.

Case 1. $x^* \in [-h_0, h_0]$.

Since (2.4) is equivalent to the following integral equation:

$$v(t,x) = e^{-bt}\left[v_0(x) + \int_0^t e^{bs} G(u)(s,x)ds \right],$$

we have

$$v_1(t,x) - v_2(t,x) = e^{-bt}\left[\int_0^t e^{bs} \left(G(u_1) - G(u_2) \right)(s,x)ds \right].$$

Then,

$$|v_1(t^*, x^*) - v_2(t^*, x^*)| \le \frac{\rho(M_1)}{b} \|u_1 - u_2\|_{C(\Omega_T^{G_m,H_M})}. \tag{2.14}$$

Case 2. $x^* \in (h_0, H_m(t^*))$.

In this case, there exist $t_1^*,\ t_2^* \in (0, t^*)$ such that $h_1(t_1^*) = h_2(t_2^*) = x^*$. Without loss of generality, we may assume that $0 \le t_1^* \le t_2^*$. Then,

$$v_1(t^*, x^*) - v_2(t^*, x^*) = e^{-bt^*}\left[v_1(t_2^*, x^*)e^{bt_2^*} + \int_{t_2^*}^{t^*} e^{bs} \left(G(u_1) - G(u_2) \right)(s, x^*)ds \right].$$

Thus,

$$|v_1(t^*, x^*) - v_2(t^*, x^*)| \le |v_1(t_2^*, x^*)| + \frac{\rho(M_1)}{b}\|u_1 - u_2\|_{C(\Omega_T^{G_m,H_M})}.$$

By (G1) and (G2), we can have that there exists γ such that $G(z) \le \gamma z$ for $z \ge 0$. Now we estimate $v_1(t_2^*, x^*)$. Direct calculations give that

$$v_1(t_2^*, x^*) = e^{-bt_2^*}\int_{t_1^*}^{t_2^*} e^{bs}G(u_1)(s, x^*)ds \le \frac{\gamma}{b}\max_{t\in[t_1^*,t_2^*]}|u_1(t, x^*)| = \frac{\gamma}{b}\max_{t\in[t_1^*,t_2^*]}|(u_1 - u_2)(t, x^*)|.$$

Hence,

$$|v_1(t^*, x^*) - v_2(t^*, x^*)| \le \frac{\gamma + \rho(M_1)}{b}\|u_1 - u_2\|_{C(\Omega_T^{G_m,H_M})}. \tag{2.15}$$

Case 3. $x^* \in [H_m(t^*), H_M(t^*)]$.

Without loss of generality, we assume that $h_2(t^*) < h_1(t^*)$. In this case, there exists t_1^* such that $h_1(t_1^*) = x^*$. Then $v_1(t_1^*, x^*) = 0$, $u_2(t, x^*) = v_2(t, x^*) = 0$ for $t \in [t_1^*, t^*]$. Hence, $V(t^*, x^*) = v_1(t^*, x^*)$ and

$$v_1(t^*, x^*) = e^{-bt^*}\int_{t_1^*}^{t^*} e^{bs}G(u_1)(s, x)ds \le \frac{\gamma}{b}\max_{t\in[t_1^*,t^*]}|u_1(t, x^*)| = \frac{\gamma}{b}\max_{t\in[t_1^*,t^*]}|(u_1 - u_2)(t, x^*)|.$$

Hence,

$$|v_1(t^*, x^*) - v_2(t^*, x^*)| \le \frac{\gamma}{b}\|u_1 - u_2\|_{C(\Omega_T^{G_m,H_M})}. \tag{2.16}$$

By (2.14), (2.15) and (2.16), we have

$$\|v_1 - v_2\|_{C(\Omega_T^{G_m,H_M})} \le C_5\|u_1 - u_2\|_{C(\Omega_T^{G_m,H_M})}, \tag{2.17}$$

where C_5 depends on b, ρ, M_1 and γ. Now we estimate $\|u_1(t, x) - u_2(t, x)\|_{C(\Omega_T^{G_m,H_M})}$.

$$\begin{aligned}
&\|u_1(t, x) - u_2(t, x)\|_{C(\Omega_T^{G_m,H_M})}\\
={}& \left\|w_1\left(t, \frac{2x - g_1(t) - h_1(t)}{h_1(t) - g_1(t)}\right) - w_2\left(t, \frac{2x - g_2(t) - h_2(t)}{h_2(t) - g_2(t)}\right)\right\|_{C(\Omega_T^{G_m,H_M})}\\
\le{}& \left\|w_1\left(t, \frac{2x - g_1(t) - h_1(t)}{h_1(t) - g_1(t)}\right) - w_2\left(t, \frac{2x - g_1(t) - h_1(t)}{h_1(t) - g_1(t)}\right)\right\|_{C(\Omega_T^{G_m,H_M})}\\
&+ \left\|w_2\left(t, \frac{2x - g_1(t) - h_1(t)}{h_1(t) - g_1(t)}\right) - w_2\left(t, \frac{2x - g_2(t) - h_2(t)}{h_2(t) - g_2(t)}\right)\right\|_{C(\Omega_T^{G_m,H_M})}\\
\le{}& \|w_1(t, y) - w_2(t, y)\|_{C(\triangle_T)} + C_6\left(\|G\|_{C([0,T])} + \|H\|_{C([0,T])}\right),
\end{aligned} \tag{2.18}$$

where C_6 only depends on h_0 and $\|w_{2x}\|_{C(\triangle_{T_3})}$. By $\overline{W}(0, y) = 0$ and Sobolev imbedding theorem, we have

$$\|W(t, y)\|_{C(\triangle_T)} \le [W]_{C^{\frac{\alpha}{2},0}(\triangle_T)}T^{\frac{\alpha}{2}} \le C_7T^{\frac{\alpha}{2}}[W]_{C^{\frac{\alpha}{2},\alpha}(\triangle_T)} \le C_8T^{\frac{\alpha}{2}}\|W\|_{W_p^{1,2}(\triangle_T)}, \tag{2.19}$$

where C_7 and C_8 do not depend on T. By (2.12), (2.13), (2.17), (2.18) and (2.19), we can get

$$\|W\|_{W_p^{1,2}(\triangle_T)} \le C_9T^{\frac{\alpha}{2}}\|W\|_{W_p^{1,2}(\triangle_T)} + C_{10}\left(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}\right),$$

where C_9 depends on $C_4,\ C_5$ and C_8; C_{10} depends on $C_1,\ C_5$ and C_6. If $T \in \min\left\{T_3, (2C_9)^{-\frac{2}{\alpha}}\right\} \doteq T_4$,

$$\|W\|_{W_p^{1,2}(\triangle_T)} \leq 2C_{10}\left(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}\right). \tag{2.20}$$

In the following, we estimate $\|G\|_{C^1([0,T])}$ and $\|H\|_{C^1([0,T])}$. Since $G(0) = G'(0) = 0$, we have

$$\begin{aligned}\|G\|_{C^1([0,T])} &= \max_{t\in[0,T]} G(t) + \max_{t\in[0,T]} G'(t) \leq \max_{\xi\in[0,T]} G'(\xi)T + \max_{t\in[0,T]} G'(t)\\ &\leq (1+T)\max_{t\in[0,T]} \frac{G'(t)-G'(0)}{(t-0)^{\frac{\alpha}{2}}}T^{\frac{\alpha}{2}} = T^{\frac{\alpha}{2}}(1+T)[G']_{C^{\frac{\alpha}{2}}([0,T])}.\end{aligned}$$

By (2.11), we have

$$[G']_{C^{\frac{\alpha}{2}}([0,T])} = C_{11}\left[[W_y(t,-1)]_{C^{\frac{\alpha}{2},0}([0,T])} + (\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])})[w_{2y}(t,-1)]_{C^{\frac{\alpha}{2}}([0,T])}\right],$$

where C_{11} depends on $\mu,\ A$ and h_0. It follows from the proof of [45, Theorem 1.1] that we have

$$[W_y(t,y)]_{C^{\frac{\alpha}{2},0}(\triangle_T)} \leq C_{12}[W_y(t,y)]_{C^{\frac{\alpha}{2},\alpha}(\triangle_T)} \leq C_{13}\|W\|_{W_p^{1,2}(\triangle_T)},$$

where C_{12} and C_{13} do not depend on T. Therefore, we have

$$\|G\|_{C^1([0,T])} \leq C_{14}T^{\frac{\alpha}{2}}(1+T)(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}), \tag{2.21}$$

where C_{14} depends on $C_2,\ C_{10},\ C_{11}$ and C_{13}. Similarly, there exists C_{15} such that

$$\|H\|_{C^1([0,T])} \leq C_{15}T^{\frac{\alpha}{2}}(1+T)(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}). \tag{2.22}$$

It follows from (2.21) and (2.22) that

$$\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])} = C_{16}T^{\frac{\alpha}{2}}(1+T)(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])}) \leq \frac{1}{2}(\|G\|_{C^1([0,T])} + \|H\|_{C^1([0,T])})$$

if $T \leq \min\left\{T_4,\ 1,\ (4C_{16})^{-\frac{2}{\alpha}}\right\} \doteq T_5$, where $C_{16} = C_{14} + C_{15}$. Hence, $G = H = 0$ for $T \leq T_5$. It follows from (2.20) that $W = 0$. This implies that $u_1 \equiv u_2$. By (2.17), we have $v_1 \equiv v_2$. The uniqueness is obtained. □

Then it follows from the arguments in [23] that we can get the following estimates.

Lemma 2.2. *Let (u, v, g, h) be a solution of problem (1.4) defined for $t \in (0, T_0]$, where $T_0 \in (0, +\infty)$. Then there exist $M_1,\ M_2$ and M_3 independent of T_0 such that*
(i) $0 < u(t,x) \leq M_1,\ 0 < v(t,x) \leq M_2$ *for* $t \in (0, T_0]$ *and* $x \in [g(t), h(t)]$.
(ii) $0 < -g'(t),\ h'(t) \leq M_3$ *for* $t \in (0, T_0]$.

Just like the proof of [37, Theorem 3.2], we can obtain the global existence and uniqueness.

Theorem 2.3. *The solution exists and is unique for all $t > 0$.*

Then, we exhibit the following comparison principle, which can be proven by the similar argument in [23, Lemma 2.5].

Theorem 2.4. *Assume that*

$$\overline{g},\ \overline{h} \in C^1([0,+\infty)),\ \overline{u}(t,x),\ \overline{v}(t,x) \in C(\overline{D}) \cap C^{1,2}(D),$$
$$\overline{u}(0,x) \in \mathscr{X}_1(h_0),\ \overline{v}(0,x) \in \mathscr{X}_2(h_0)$$

with

$$D := \left\{(t,x) \in \mathbb{R}^2:\ 0 < t < \infty,\ \overline{g}(t) < x < \overline{h}(t)\right\},$$

and $(\overline{u}, \overline{v}, \overline{g}, \overline{h})$ *satisfies*

$$\begin{cases} \overline{u}_t \geq d\overline{u}_{xx} - \beta\overline{u}_x - a\overline{u} + c\overline{v}, & t > 0,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{v}_t \geq -b\overline{v} + G(\overline{u}), & t > 0,\ \overline{g}(t) < x < \overline{h}(t), \\ \overline{u}(t,\overline{g}(t)) = \overline{u}(t,\overline{h}(t)) = 0, & t \geq 0, \\ \overline{v}(t,\overline{g}(t)) = \overline{v}(t,\overline{h}(t)) = 0, & t \geq 0, \\ \overline{g}(0) \leq -h_0,\ \overline{g}'(t) \leq -\mu\overline{u}_x(t,\overline{g}(t)), & t > 0, \\ \overline{h}(0) \geq h_0,\ \overline{h}'(t) \geq -\mu\overline{u}_x(t,\overline{h}(t)), & t > 0, \\ \overline{u}(0,x) \geq u_0(x),\ \overline{v}(0,x) \geq v_0(x), & -h_0 < x < h_0. \end{cases}$$

Then the solution (u,v,g,h) *of the free boundary problem* (1.4) *satisfies*

$$h(t) \leq \overline{h}(t),\ g(t) \geq \overline{g}(t),\ \forall\ t \geq 0,$$

$$u(t,x) \leq \overline{u}(t,x),\ v(t,x) \leq \overline{v}(t,x),\ \forall\ t \geq 0,\ g(t) \leq x \leq h(t).$$

Remark 2.5. The pair $(\overline{u}, \overline{v}, \overline{g}, \overline{h})$ in Theorem 2.4 is usually called an upper solution of problem (1.4). Similarly, we can define a lower solution by reversing all the inequalities in the suitable places.

In the following part, we consider the following eigenvalue problem

$$\begin{cases} -\lambda\phi = d\phi_{xx} - \beta\phi_x - a\phi + \frac{cG'(0)}{b}\phi, & -l < x < l, \\ \phi(-l) = \phi(l) = 0. \end{cases} \tag{2.23}$$

Denote by $\lambda_0(l)$ the principal eigenvalue of problem (2.23) with some fixed l.

Lemma 2.6. $\lambda_0(l)$ *has the following form:*

$$\lambda_0(l) = \frac{\beta^2}{4d} + \frac{d\pi^2}{4l^2} - \left(\frac{cG'(0)}{b} - a\right).$$

Proof. We choose β to be small and determine it later. By a simple calculation, we can achieve the characteristic equation

$$d\mu^2 - \beta\mu + \lambda - a + \frac{cG'(0)}{b} = 0, \tag{2.24}$$

and let μ_i $(i = 1,2)$ be the roots of (2.24). Then the solution of (2.23) is

$$\phi(x) = c_1 e^{\mu_1 x} + c_2 e^{\mu_2 x},$$

where c_1 and c_2 will be determined later. Since $\phi(-l) = \phi(l) = 0$, we can derive that

$$\Delta = \beta^2 - 4d\left(\lambda - a + \frac{cG'(0)}{b}\right) < 0.$$

In fact, if $\Delta = \beta^2 - 4d\left(\lambda - a + \frac{cG'(0)}{b}\right) \geq 0$, we have $\phi \equiv 0$, which is a contradiction. Hence, (2.24) has two complex roots:

$$\mu_1 = \frac{\beta + i\sqrt{4d\left(\lambda - a + \frac{cG'(0)}{b}\right) - \beta^2}}{2d}, \ \mu_2 = \frac{\beta - i\sqrt{4d\left(\lambda - a + \frac{cG'(0)}{b}\right) - \beta^2}}{2d}.$$

Then

$$\begin{aligned}\phi(x) = {} & c_1 e^{\frac{\beta}{2d}x}\left[\cos\frac{\sqrt{4d\left(\lambda - a + \frac{cG'(0)}{b}\right) - \beta^2}}{2d}x + i\sin\frac{\sqrt{4d\left(\lambda - a + \frac{cG'(0)}{b}\right) - \beta^2}}{2d}x\right] \\ & + c_2 e^{\frac{\beta}{2d}x}\left[\cos\frac{\sqrt{4d\left(\lambda - a + \frac{cG'(0)}{b}\right) - \beta^2}}{2d}x - i\sin\frac{\sqrt{4d\left(\lambda - a + \frac{cG'(0)}{b}\right) - \beta^2}}{2d}x\right].\end{aligned}$$

By $\phi(-l) = \phi(l) = 0$, we have $c_1 = c_2$ and

$$\frac{\sqrt{4d\left(\lambda - a + \frac{cG'(0)}{b}\right) - \beta^2}}{2d}l = \frac{\pi}{2} + k\pi, \ \forall\, k \in \mathbb{N}.$$

When $k = 0$, λ attain its minimum, we have

$$\lambda_0(l) = \frac{\beta^2}{4d} + \frac{d\pi^2}{4l^2} - \left(\frac{cG'(0)}{b} - a\right),$$

and the corresponding eigenfunction $\phi(x) = e^{\frac{\beta}{2d}x}\cos\left(\frac{\pi}{2l}x\right)$. □

Then we have the following properties about $\lambda_0(l)$.

Lemma 2.7. *The following assertions hold:*

(i) $\lambda_0(l)$ *is continuous and strictly decreasing in l,*

$$\lim_{l\to 0}\lambda_0(l) = \infty, \ \lim_{l\to\infty}\lambda_0(l) = \frac{\beta^2}{4d} - \left(\frac{cG'(0)}{b} - a\right).$$

(ii) *If* $\frac{cG'(0)}{ab} > 1$ *and* $0 < \beta < 2\sqrt{d\left(\frac{cG'(0)}{b} - a\right)}$, *then there exists*

$$l^* = 2d\pi \Big/ \sqrt{4d\left(\frac{cG'(0)}{b} - a\right) - \beta^2}$$

such that $\lambda_0(l^*) = 0$. *Furthermore,* $\lambda_0(l) > 0$ *for* $0 < l < l^*$, *and* $\lambda_0(l) < 0$ *for* $l > l^*$.

(iii) *If* $\frac{cG'(0)}{ab} \leq 1$, *then* $\lambda_0(l) > \frac{\beta^2}{4d} - \left(\frac{cG'(0)}{b} - a\right) > 0$.

Proof. By the expression of $\lambda_0(l)$ in Lemma 2.6, the proof of lemma is obvious. We omit it here. □

3. Spreading and vanishing

Firstly, we give the definitions of spreading and vanishing of the disease:

Definition 3.1. *We say that **vanishing** happens if*

$$h_\infty - g_\infty < \infty \text{ and } \lim_{t\to\infty}(\|u(t,\cdot)\|_{C([g(t),h(t)])} + \|v(t,\cdot)\|_{C([g(t),h(t)])}) = 0,$$

*and **spreading** happens if*

$$h_\infty - g_\infty = \infty \text{ and } \limsup_{t\to\infty}(\|u(t,\cdot)\|_{C([g(t),h(t)])} + \|v(t,\cdot)\|_{C([g(t),h(t)])}) > 0.$$

Then, we give the following lemmas.

Lemma 3.2. *Let* (u, v, g, h) *be the solution of* (1.4)*. If* $h_\infty - g_\infty < \infty$*, then there exists a constant* $C > 0$ *such that*

$$\|u(t,\cdot)\|_{C^1([g(t),h(t)])} \le C, \ \forall\, t > 1. \tag{3.1}$$

Moreover,

$$\lim_{t\to\infty} g'(t) = \lim_{t\to\infty} h'(t) = 0. \tag{3.2}$$

Proof. We can use the method in [46, Theorem 2.1] to get (3.1). Then the proof of (3.2) can be done as [16, Theorem 4.1]. □

Lemma 3.3. *Let* d, μ *and* h_0 *be positive constants,* $w \in C^{\frac{1+\alpha}{2},1+\alpha}([0,\infty)\times[g(t),h(t)])$ *and* $g,\ h \in C^{1+\frac{\alpha}{2}}([0,\infty))$ *for some* $\alpha > 0$*. We further assume that* $w_0(x) \in \mathscr{X}_1(h_0)$*. If* (w, g, h) *satisfies*

$$\begin{cases} w_t \ge dw_{xx} - \beta w_x - aw, & t > 0,\ g(t) < x < h(t), \\ w(t,x) = 0, & t \ge 0,\ x \le g(t), \\ w(t,x) = 0, & t \ge 0,\ x \ge h(t), \\ g(0) = -h_0,\ g'(t) \le -\mu w_x(t, g(t)), & t > 0, \\ h(0) = h_0,\ h'(t) \ge -\mu w_x(t, h(t)), & t > 0, \\ w(0,x) = w_0(x) \ge, \not\equiv 0, & -h_0 < x < h_0, \end{cases} \tag{3.3}$$

and

$$\lim_{t\to\infty} g(t) = g_\infty > -\infty,\ \lim_{t\to\infty} g'(t) = 0,\ \lim_{t\to\infty} h(t) = h_\infty < \infty,\ \lim_{t\to\infty} h'(t) = 0,$$
$$\|w(t,\cdot)\|_{C^1([g(t),h(t)])} \le M,\ \forall\, t > 1$$

for some constant $M > 0$*. Then*

$$\lim_{t\to\infty} \max_{g(t)\le x\le h(t)} w(t,x) = 0.$$

Proof. It can be proved by the similar arguments in [16, Theorem 4.2]. □

By above Lemmas 3.2 and 3.3, we can derive the following result.

Theorem 3.4. *If $h_\infty - g_\infty < \infty$, then*

$$\lim_{t\to\infty}(\|u(t,\cdot)\|_{C([g(t),h(t)])} + \|v(t,\cdot)\|_{C([g(t),h(t)])}) = 0.$$

Proof. Firstly, we can use the method in the proof of [46, Theorem 2.1] to get

$$\|u\|_{C^{\frac{1+\alpha}{2},1+\alpha}([0,\infty)\times[g(t),h(t)])} + \|g\|_{C^{1+\frac{\alpha}{2}}([0,\infty))} + \|h\|_{C^{1+\frac{\alpha}{2}}([0,\infty))} \le C.$$

Recall that u satisfies (3.3). By Lemmas 3.2 and 3.3, we can get $\lim_{t\to\infty}\|u(t,\cdot)\|_{C([g(t),h(t)])} = 0$.

Noting that $v(t,x)$ satisfies

$$v_t = -bv + G(u),\ t > 0,\ g(t) < x < h(t)$$

and $G(u) \to 0$ uniformly for $x \in [g(t), h(t)]$ as $t \to \infty$, we have $\lim_{t\to\infty}\|v(t,\cdot)\|_{C([g(t),h(t)])} = 0$. □

Lemma 3.5. *If $\frac{G(z)}{z} < \frac{ab}{c}$ for any $z > 0$, then $h_\infty - g_\infty < \infty$.*

Proof. Direct calculations yield

$$\begin{aligned}
&\frac{d}{dt}\int_{g(t)}^{h(t)}\Big(u(t,x) + \frac{c}{b}v(t,x)\Big)dx\\
=&\int_{g(t)}^{h(t)}\Big(u_t + \frac{c}{b}v_t\Big)dx\\
=&\int_{g(t)}^{h(t)}\Big(du_{xx} - \beta u_x - au + \frac{c}{b}G(u)\Big)dx\\
=&-\frac{d}{\mu}(h'(t) - g'(t)) + \int_{g(t)}^{h(t)}\Big(-au + \frac{c}{b}G(u)\Big)dx.
\end{aligned}$$

Integrating from 0 to t gives

$$\begin{aligned}
&\int_{g(t)}^{h(t)}\Big(u(t,x) + \frac{c}{b}v(t,x)\Big)dx\\
=&\int_{-h_0}^{h_0}\Big(u_0(x) + \frac{c}{b}v_0(x)\Big)dx - \frac{d}{\mu}(h(t) - g(t))\\
&+\frac{d}{\mu}2h_0 + \int_0^t\int_{g(s)}^{h(s)}\Big(-au + \frac{c}{b}G(u)\Big)dxds.
\end{aligned}$$

Since $u \ge 0$, $v \ge 0$ and $G(u) \le \frac{ab}{c}u$ for $u \ge 0$, we have

$$h(t) - g(t) \le \frac{\mu}{d}\int_{-h_0}^{h_0}\Big(u_0(x) + \frac{c}{b}v_0(x)\Big)dx + 2h_0 < \infty.$$

Letting $t \to \infty$, we have $h_\infty - g_\infty < \infty$. □

Lemma 3.6. *Assume that $\frac{G(z_1)}{z_1} > \frac{ab}{c}$ for some $z_1 > 0$. If $\lambda_0(h_0) > 0$ holds, then vanishing will happen provided that u_0 and v_0 are sufficiently small.*

Proof. We prove this result by constructing the appropriate upper solution. Let ϕ be the corresponding eigenfunction of $\lambda_0(h_0)$. Since $\lambda_0(h_0) > 0$, we can choose some small δ such that

$$-\delta - \frac{\beta h_0 \delta^2}{2d(2+\delta)} + \frac{3}{4}\lambda_0 \frac{1}{(1+\delta)^2} > 0.$$

Set

$$\begin{aligned}
&\sigma(t) = h_0(1+\delta - \frac{\delta}{2}e^{-\delta t}),\ t \geq 0,\\
&\overline{u}(t,x) = \varepsilon e^{-\delta t}\phi\left(\frac{xh_0}{\sigma(t)}\right)e^{\frac{\beta}{2d}\left(1-\frac{h_0}{\sigma(t)}\right)x},\ t\geq 0,\ -\sigma(t)\leq x\leq \sigma(t),\\
&\overline{v}(t,x) = \left(\frac{G'(0)}{b} + \frac{\lambda_0}{4c}\right)\frac{h_0^2}{\sigma^2}\overline{u},\ t\geq 0,\ -\sigma(t)\leq x\leq\sigma(t).
\end{aligned}$$

Direct computations yield

$$\begin{aligned}
&\overline{u}_t - d\overline{u}_{xx} + \beta\overline{u}_x + a\overline{u} - c\overline{v}\\
=&\,\overline{u}\left(-\delta - \frac{\phi'}{\phi}\frac{xh_0\sigma'}{\sigma^2} + \frac{\beta h_0 x}{2d}\frac{\sigma'}{\sigma^2}\right)\\
&- d\varepsilon e^{-\delta t}e^{\frac{\beta}{2d}(1-\frac{h_0}{\sigma})x}\left[\phi''\left(\frac{h_0}{\sigma}\right)^2 + 2\phi'\frac{h_0}{\sigma}\frac{\beta}{2d}\left(1-\frac{h_0}{\sigma}\right) + \phi\left(\frac{\beta}{2d}\right)^2\left(1-\frac{h_0}{\sigma}\right)^2\right]\\
&+ \beta\varepsilon e^{-\delta t}e^{\frac{\beta}{2d}(1-\frac{h_0}{\sigma})x}\left[\phi'\frac{h_0}{\sigma} + \phi\frac{\beta}{2d}\left(1-\frac{h_0}{\sigma}\right)\right] + a\overline{u} - c\left(\frac{G'(0)}{b}+\frac{\lambda_0}{4c}\right)\frac{h_0^2}{\sigma^2}\overline{u}\\
=&\,\overline{u}\left(-\delta - \frac{\phi'}{\phi}\frac{xh_0\sigma'}{\sigma^2} + \frac{\beta h_0 x}{2d}\frac{\sigma'}{\sigma^2}\right)\\
&+ \varepsilon e^{-\delta t}e^{\frac{\beta}{2d}(1-\frac{h_0}{\sigma(t)})x}\left[\frac{h_0^2}{\sigma^2}(-d\phi''+\beta\phi') + \phi\frac{\beta^2}{4d}\left(1-\frac{h_0^2}{\sigma^2}\right)\right] + a\overline{u} - c\left(\frac{G'(0)}{b}+\frac{\lambda_0}{4c}\right)\frac{h_0^2}{\sigma^2}\overline{u}\\
\geq&\,\overline{u}\left(-\delta - \frac{\beta h_0}{2d}\frac{\sigma'}{\sigma} + \frac{3}{4}\lambda_0\frac{h_0^2}{\sigma^2}\right) + \left(1-\frac{h_0^2}{\sigma^2}\right)\left(\frac{\beta^2}{4d}\overline{u} + a\overline{u}\right)\\
>&\,\overline{u}\left[-\delta - \frac{\beta h_0\delta^2}{2d(2+\delta)} + \frac{3}{4}\lambda_0\frac{1}{(1+\delta)^2}\right] > 0,
\end{aligned}$$

and

$$\begin{aligned}
&\overline{v}_t + b\overline{v} - G(\overline{u})\\
=&\,-\left(\frac{G'(0)}{b}+\frac{\lambda_0}{4c}\right)\frac{2h_0^2\sigma'}{\sigma^3}\overline{u} + \left(\frac{G'(0)}{b}+\frac{\lambda_0}{4c}\right)\frac{h_0^2}{\sigma^2}(\overline{u}_t + b\overline{u}) - G'(\xi)\overline{u}\\
\geq&\,-\left(\frac{G'(0)}{b}+\frac{\lambda_0}{4c}\right)\frac{2h_0^2}{\sigma^2}\frac{\delta^2}{2+\delta}\overline{u} + \left(\frac{G'(0)}{b}+\frac{\lambda_0}{4c}\right)\frac{h_0^2}{\sigma^2}\left[-\delta - \frac{\beta h_0\delta^2}{2d(2+\delta)} + b\right]\overline{u} - G'(\xi)\overline{u}\\
=&\,\overline{u}\left\{\left(\frac{G'(0)}{b}+\frac{\lambda_0}{4c}\right)\frac{h_0^2}{\sigma^2}\left[-\delta - \frac{\beta h_0\delta^2}{2d(2+\delta)}\right] + G'(0)\frac{h_0^2}{\sigma^2}\left[1-\frac{2\delta^2}{b(2+\delta)}\right]\right.\\
&\left.-G'(\xi) + \frac{\lambda_0 h_0^2}{4c\sigma^2}\left(b - \frac{2\delta^2}{2+\delta}\right)\right\} \doteq B
\end{aligned}$$

for all $t > 0$ and $-\sigma(t) < x < \sigma(t)$, where $\xi \in (0, \overline{u})$. Let

$$\varepsilon = \frac{\delta^2 h_0(1+\frac{\delta}{2})}{2\mu} \min\left\{-\frac{1}{\phi'(h_0)} e^{-\frac{\beta}{2d}\delta h_0}, \frac{1}{\phi'(-h_0)} e^{\frac{\beta}{4d}\delta h_0}\right\}.$$

Since $\overline{u} \le \varepsilon e^{\frac{\beta}{2d}h_0\delta}$, we can choose δ to be sufficiently small such that $B > 0$. Noting that

$$\sigma'(t) = h_0\frac{\delta^2}{2}e^{-\delta t},\ \overline{u}_x(t,\sigma(t)) = \varepsilon e^{-\delta t}\phi'(h_0)\frac{h_0}{\sigma}e^{\frac{\beta}{2d}(\sigma(t)-h_0)},$$
$$\overline{u}_x(t,-\sigma(t)) = \varepsilon e^{-\delta t}\phi'(-h_0)\frac{h_0}{\sigma}e^{\frac{\beta}{2d}(h_0-\sigma(t))},$$

then we have

$$\begin{cases} \overline{u}_t \ge d\overline{u}_{xx} - \beta\overline{u}_x - a\overline{u} + c\overline{v}, & t>0,\ -\sigma(t)<x<\sigma(t),\\ \overline{v}_t \ge -b\overline{v} + G(\overline{u}), & t>0,\ -\sigma(t)<x<\sigma(t),\\ \overline{u}(t,-\sigma(t)) = \overline{u}(t,\sigma(t)) = 0, & t\ge 0,\\ \overline{v}(t,-\sigma(t)) = \overline{v}(t,\sigma(t)) = 0, & t\ge 0,\\ -\sigma(0) \le -h_0,\ -\sigma'(t) \le -\mu\overline{u}_x(t,-\sigma(t)), & t>0,\\ \sigma(0)\ge h_0,\ \sigma'(t) \ge -\mu\overline{u}_x(t,\sigma(t)), & t>0. \end{cases}$$

If u_0 and v_0 are sufficiently small such that

$$u_0(x) \le \varepsilon\phi\left(\frac{x}{1+\delta/2}\right)e^{\frac{\beta\delta x}{2d(2+\delta)}},\ \forall\ x\in[-h_0(1+\delta/2), h_0(1+\delta/2)]$$

and

$$v_0(x) \le \left(\frac{G'(0)}{b} + \frac{\lambda_0}{4c}\right)\frac{1}{(1+\delta/2)^2}\varepsilon\phi\left(\frac{x}{1+\delta/2}\right)e^{\frac{\beta\delta x}{2d(2+\delta)}},\ \forall\ x\in[-h_0(1+\delta/2), h_0(1+\delta/2)],$$

then

$$u_0(x) \le \overline{u}(0,x),\ v_0(x) \le \overline{v}(0,x),\ \forall\ x\in(-h_0,h_0).$$

Applying Theorem 2.4 gives that $h(t) \le \sigma(t)$ and $g(t) \ge -\sigma(t)$. Hence, $h_\infty - g_\infty \le 2h_0(1+\delta) < \infty$. By Theorem 3.4, we have $\lim_{t\to\infty}(\|u(t,\cdot)\|_{C([g(t),h(t)])} + \|v(t,\cdot)\|_{C([g(t),h(t)])}) = 0$. □

By Lemma 3.6, we can derive the following corollary directly.

Corollary 3.7. *Assume that* $\frac{G(z_1)}{z_1} > \frac{ab}{c}$ *for some* $z_1 > 0$*, then the following statements holds:*

(i) *If* $\frac{cG'(0)}{ab} < 1$*, then vanishing will happen for* u_0 *and* v_0 *sufficiently small.*

(ii) *If* $\frac{cG'(0)}{ab} > 1$ *and* $h_0 < l^*$*, then vanishing will happen for* u_0 *and* v_0 *sufficiently small.*

Lemma 3.8. *Assume that* $\frac{G(z_1)}{z_1} > \frac{ab}{c}$ *for some* $z_1 > 0$ *and* $\frac{cG'(0)}{ab} > 1$*. If* $h_0 > l^*$*, then spreading will happen.*

Proof. Let ϕ be the corresponding eigenfunction of $\lambda_0(h_0)$. Since $\frac{cG'(0)}{ab} > 1$ and $h_0 > l^*$, we have $\lambda_0(h_0) < 0$. Then we construct a suitable lower solution. Since

$$\frac{cG'(0)}{b} + \frac{\lambda_0}{4} = \frac{\beta^2}{4d} + \frac{d\pi^2}{4l^2} + a - \frac{3\lambda_0}{4} > 0,$$

we can define

$$\underline{u}(t,x) = \epsilon\phi(x),\ t \geq 0,\ -h_0 \leq x \leq h_0,$$
$$\underline{v}(t,x) = \left(\frac{G'(0)}{b} + \frac{\lambda_0}{4c}\right)\epsilon\phi(x),\ t \geq 0,\ -h_0 \leq x \leq h_0.$$

Direct computations yield

$$\underline{u}_t - d\underline{u}_{xx} + \beta\underline{u}_x + a\underline{u} - c\underline{v} = \epsilon\left(-d\phi_{xx} + \beta\phi_x + a\phi - \frac{cG'(0)}{b}\phi - \frac{\lambda_0}{4}\phi\right) = \frac{3}{4}\lambda_0\epsilon\phi < 0,$$

and

$$\underline{v}_t + b\underline{v} - G(\underline{u}) = \epsilon\phi\left(G'(0) - G'(\xi) + \frac{b\lambda_0}{4c}\right)$$

for all $t > 0$ and $-h_0 < x < h_0$, where $\xi \in (0, \underline{u})$. We can choose ϵ small enough such that

$$G'(0) - G'(\xi) + \frac{b\lambda_0}{4c} \leq 0,\ \epsilon\phi(x) \leq u_0(x),\ \left(\frac{G'(0)}{b} + \frac{\lambda_0}{4c}\right)\epsilon\phi(x) \leq v_0(x).$$

Then

$$\begin{cases} \underline{u}_t \leq d\underline{u}_{xx} - \beta\underline{u}_x - a\underline{u} + c\underline{v}, & t > 0,\ -h_0 < x < h_0, \\ \underline{v}_t \leq -b\underline{v} + G(\underline{u}), & t > 0,\ -h_0 < x < h_0, \\ \underline{u}(t,-h_0) = \underline{u}(t,h_0) = 0, & t \geq 0, \\ \underline{v}(t,-h_0) = \underline{v}(t,h_0) = 0, & t \geq 0, \\ 0 \geq -\mu\underline{u}_x(t,-h_0),\ 0 \leq -\mu\underline{u}_x(t,h_0), & t > 0, \\ \underline{u}(0,x) \leq u(0,x),\ \underline{v}(0,x) \leq v(0,x), & -h_0 < x < h_0. \end{cases}$$

It follows from Remark 2.5 that $u(t,x) \geq \underline{u}(t,x)$ in $[0,\infty) \times [-h_0, h_0]$. Hence,

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C([g(t),h(t)])} \geq \epsilon\phi(x) > 0.$$

By Theorem 3.4, we have $h_\infty - g_\infty = \infty$. □

Lemma 3.9. *Assume that $\frac{G(z_1)}{z_1} > \frac{ab}{c}$ for some $z_1 > 0$ and $\frac{cG'(0)}{ab} > 1$. If $h_0 < l^*$, then $h_\infty - g_\infty = \infty$ provided that u_0 and v_0 are sufficiently large.*

Proof. We first note that there exists $\sqrt{T^*} > l^*$ such that $\lambda_0(\sqrt{T^*}) < 0$.

Inspired by the argument of [8, proposition 5.3], we consider

$$\begin{cases} -d\varphi'' - \left(\frac{1}{2} + \sqrt{T^* + 1}\right)\varphi' = \widetilde{\lambda}_0\varphi, & 0 < x < 1, \\ \varphi'(0) = \varphi(1) = 0. \end{cases} \tag{3.4}$$

It is well-known that the first eigenvalue $\widetilde{\lambda}_0$ of (3.4) is simple and the corresponding eigenfunction φ can be chosen positive in $[0,1)$ and $\|\varphi\|_{L^\infty(-1,1)} = 1$. Moreover, one can easily see that $\widetilde{\lambda}_0 > 0$ and $\varphi'(x) < 0$ in $(0,1]$. We extend φ to $[-1,1]$ as an even function. Then clearly

$$\begin{cases} -d\varphi'' - \left(\frac{1}{2} + \sqrt{T^*+1}\right)\mathrm{sgn}(x)\varphi' = \widetilde{\lambda}_0\varphi, & -1 < x < 1, \\ \varphi(-1) = \varphi(1) = 0. \end{cases}$$

Now we construct a suitable lower solution to (1.4). Define

$$\eta(t) = \sqrt{t+\varrho},\ 0 \le t \le T^*,$$
$$\underline{u}(t,x) = \begin{cases} \frac{m}{(t+\varrho)^k}\varphi\left(\frac{x}{\sqrt{t+\varrho}}\right), & 0 \le t \le T^*,\ -\eta(t) < x < \eta(t), \\ 0, & 0 \le t \le T^*,\ |x| \ge \eta(t), \end{cases}$$

where the constants $\varrho,\ m,\ k$ are chosen as follows:

$$0 < \varrho \le \min\left\{1, h_0^2\right\},\ k \ge \widetilde{\lambda}_0 + a(T^*+1),\ m \ge \frac{(T^*+1)^k}{2\mu\min\{\varphi'(-1), -\varphi'(1)\}}.$$

Let

$$t_x := \begin{cases} t_x^1, & x \in [-\eta(T^*), -\sqrt{\varrho}) \text{ and } x = -\eta(t_x^1), \\ 0, & x \in [-\sqrt{\varrho}, \sqrt{\varrho}], \\ t_x^2, & x \in (\sqrt{\varrho}, \eta(T^*)] \text{ and } x = \eta(t_x^2) \end{cases}$$

and

$$\underline{v}_0(x) = \begin{cases} \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\cos\left(\frac{\pi}{\sqrt{\varrho}}x\right), & -\sqrt{\varrho} \le x \le \sqrt{\varrho}, \\ 0, & |x| > \sqrt{\varrho}, \end{cases}$$

where we choose ε small enough such that

$$\underline{v}_0(x) \le v_0(x),\ \forall\ x \in (-\sqrt{\varrho}, \sqrt{\varrho}).$$

Then we define

$$\underline{v}(t,x) = e^{-bt}\left(\int_{t_x}^{t} e^{b\tau}G(\underline{u}(\tau,x))d\tau + \underline{v}_0(x)\right),\ t_x \le t \le T^*,\ -\eta(t) \le x \le \eta(t).$$

Direct computations yield

$$\begin{aligned} &\underline{u}_t - d\underline{u}_{xx} + \beta\underline{u}_x + a\underline{u} - c\underline{v} \\ \le\ & -\frac{m}{(t+\varrho)^{k+1}}\left[k\varphi + \frac{x}{2\sqrt{t+\varrho}}\varphi' + d\varphi'' - \sqrt{t+\varrho}\varphi' - a(t+\varrho)\varphi\right] \\ \le\ & -\frac{m}{(t+\varrho)^{k+1}}\left[k\varphi + \left(\frac{1}{2} + \sqrt{T^*+1}\right)\mathrm{sgn}(x)\varphi' + d\varphi'' - a(T^*+1)\varphi\right] \\ \le\ & -\frac{m}{(t+\varrho)^{k+1}}\left[d\varphi'' + \left(\frac{1}{2} + \sqrt{T^*+1}\right)\mathrm{sgn}(x)\varphi' + \widetilde{\lambda}_0\varphi\right] = 0, \end{aligned}$$

and

$$\underline{v}_t + b\underline{v} - G(\underline{u}) = 0,\ 0 < t \le T^*,\ -\eta(t) < x < \eta(t).$$

For $x \in [-\sqrt{\varrho},\ \sqrt{\varrho}]$, we have $t_x = 0$. Then

$$\underline{v}(0, x) = \underline{v}_0(x) \le v_0(x),\ \forall\ x \in [-\sqrt{\varrho},\ \sqrt{\varrho}].$$

Moreover,

$$\eta'(t) + \mu\underline{u}_x(t, \eta(t)) = \frac{1}{2\sqrt{t+\varrho}} + \frac{\mu m}{(t+\varrho)^{k+\frac{1}{2}}}\varphi'(1) \le 0,\ \forall\ t \in (0, T^*),$$
$$\eta'(t) - \mu\underline{u}_x(t, -\eta(t)) = \frac{1}{2\sqrt{t+\varrho}} - \frac{\mu m}{(t+\varrho)^{k+\frac{1}{2}}}\varphi'(-1) \le 0,\ \forall\ t \in (0, T^*).$$

If u_0 is sufficiently large such that $\underline{u}(0, x) = \frac{m}{\varrho^k}\varphi\left(\frac{x}{\sqrt{\varrho}}\right) \le u_0(x)$ for $x \in [-\sqrt{\varrho},\ \sqrt{\varrho}]$, then we have

$$\begin{cases} \underline{u}_t \le d\underline{u}_{xx} - \beta\underline{u}_x - a\underline{u} + c\underline{v}, & 0 < t \le T^*,\ -\eta(t) < x < \eta(t), \\ \underline{v}_t \le -b\underline{v} + G(\underline{u}), & 0 < t \le T^*,\ -\eta(t) < x < \eta(t), \\ \underline{u}(t, x) = \underline{v}(t, x) = 0, & 0 \le t \le T^*,\ x \le -\eta(t), \\ \underline{u}(t, x) = \underline{v}(t, x) = 0, & 0 \le t \le T^*,\ x \ge \eta(t), \\ -\eta'(t) \ge -\mu\underline{u}_x(t, -\eta(t)), & 0 < t \le T^*, \\ \eta'(t) \le -\mu\underline{u}_x(t, \eta(t)), & 0 < t \le T^*, \\ \underline{u}(0, x) \le u_0(x),\ \underline{v}(0, x) \le v_0(x), & -\eta(0) < x < \eta(0). \end{cases}$$

Noting that $\eta(0) = \sqrt{\varrho} \le h_0$, we can use Remark 2.5 to conclude that $h(t) \ge \eta(t)$ and $g(t) \le -\eta(t)$ in $[0, T^*]$. Specially, we obtain $h(T^*) \ge \eta(T^*) = \sqrt{T^* + \varrho} > \sqrt{T^*}$ and $g(T^*) < -\sqrt{T^*}$. Then

$$(-l^*, l^*) \subseteq (-\sqrt{T^*},\ \sqrt{T^*}) \subseteq (g(t), h(t)),\ \forall\ t \ge T^*.$$

Hence, we have $h_\infty - g_\infty = +\infty$ by Lemma 3.8. □

Next, we present the sharp criteria on initial value, which separates spreading and vanishing.

Theorem 3.10. *For some $\gamma > 0$ and ω_1 and ω_2 in $\mathscr{X}(h_0)$, let (u, v, g, h) be a solution of* (1.4) *with $(u_0, v_0) = \gamma(\omega_1, \omega_2)$, then the following statements holds:*

(i) *Assume that $\frac{cG'(0)}{ab} < 1$. If $\frac{G(z)}{z} < \frac{ab}{c}$ for any $z > 0$, then vanishing will happen. If $\frac{G(z_1)}{z_1} > \frac{ab}{c}$ for some $z_1 > 0$, then vanishing will happen for u_0 and v_0 sufficiently small.*

(ii) *Assume that $\frac{cG'(0)}{ab} > 1$ and $0 < \beta < 2\sqrt{d\left(\frac{cG'(0)}{b} - a\right)}$. If $\frac{G(z)}{z} < \frac{ab}{c}$ for any $z > 0$, then vanishing will happen. If $\frac{G(z_1)}{z_1} > \frac{ab}{c}$ for some $z_1 > 0$, then the following will hold:*

(a) *If $h_0 > l^*$, then spreading will happen;* (b) *If $h_0 < l^*$, then there exists $\gamma^* \in (0, \infty)$ such that spreading occurs for $\gamma > \gamma^*$, and vanishing happens for $0 < \gamma \le \gamma^*$.*

Proof. This theorem follows from Lemma 3.5, Corollary 3.7, Lemmas 3.8 and 3.9. The conclusion (b) can be proven by the same arguments in [23, Theorem 4.3]. □

Finally, we give the asymptotic behavior of (1.4) when spreading happens.

Theorem 3.11. *Assume that* $\frac{cG'(0)}{ab} > 1$, $0 < \beta < 2\sqrt{d\left(\frac{cG'(0)}{b} - a\right)}$ *and* $\frac{G(z_1)}{z_1} > \frac{ab}{c}$ *for some* $z_1 > 0$*. If* $h_\infty - g_\infty = \infty$*, then*

$$(\underline{u}^*(x), \underline{v}^*(x)) \le \liminf_{t\to\infty}(u(t,x), v(t,x)) \le \limsup_{t\to\infty}(u(t,x), v(t,x)) \le (u^*, v^*)$$

for $x \in \mathbb{R}$*, where* $(\underline{u}^*(x), \underline{v}^*(x))$ *will be given in the proof.*

Proof. We denote by $(u(t), v(t))$ the solution of (1.1) with

$$u(0) = \|u_0\|_{L^\infty([-h_0,h_0])} \text{ and } v(0) = \|v_0\|_{L^\infty([-h_0,h_0])}.$$

Applying the comparison principle gives

$$(u(t,x), v(t,x)) \le (u(t), v(t)) \text{ for } t > 0 \text{ and } g(t) \le x \le h(t).$$

Since $\frac{cG'(0)}{ab} > 1$, $\lim_{t\to\infty}(u(t), v(t)) = (u^*, v^*)$. Hence,

$$\limsup_{t\to\infty}(u(t,x), v(t,x)) \le (u^*, v^*) \text{ uniformly for } x \in \mathbb{R}.$$

By Lemma 2.7, we can find some $L > l^*$ such that $\lambda_0(L) < 0$, where $\lambda_0(L)$ is the principal eigenvalue of problem (2.23) with $l = L$ and $\phi(x)$ is the corresponding eigenfunction. For such L, it follows from $h_\infty - g_\infty = \infty$ that there exists T_L such that

$$[-L, L] \subset [g(t), h(t)], \forall\ t \ge T_L.$$

Let $(\underline{u}(t,x), \underline{v}(t,x)) = \delta\left(\phi(x), \left(\frac{G'(0)}{b} + \frac{\lambda_0}{4c}\right)\phi(x)\right)$, then we can choose small δ such that

$$\begin{cases} \underline{u}_t - d\underline{u}_{xx} + \beta\underline{u}_x + a\underline{u} - c\underline{v} \le 0, & t > T_L,\ -L < x < L, \\ \underline{v}_t + b\underline{v} - G(\underline{u}) \le 0, & t > T_L,\ -L < x < L, \\ \underline{u}(t,x) = \underline{v}(t,x) = 0, & t \ge T_L,\ x = -L \text{ or } x = L, \\ \underline{u}(T_L, x) \le u(T_L, x),\ \underline{v}(T_L, x) \le v(T_L, x), & -L < x < -L. \end{cases}$$

Applying the comparison principle gives that

$$(u(t,x), v(t,x)) \ge \delta\left(\phi(x), \left(\frac{G'(0)}{b} + \frac{\lambda_0}{4c}\right)\phi(x)\right),\ t \ge T_L,\ -L \le x \le L.$$

We extend $\delta\left(\phi(x), \left(\frac{G'(0)}{b} + \frac{\lambda_0}{4c}\right)\phi(x)\right)$ to $(\underline{u}^*(x), \underline{v}^*(x))$ by defining

$$(\underline{u}^*(x), \underline{v}^*(x)) = \begin{cases} \delta\left(\phi(x), \left(\frac{G'(0)}{b} + \frac{\lambda_0}{4c}\right)\phi(x)\right), & -L \le x \le L, \\ 0, & x < -L \text{ or } x > L. \end{cases}$$

Then we have $\liminf_{t\to\infty}(u(t,x), v(t,x)) \ge (\underline{u}^*(x), \underline{v}^*(x))$ for $x \in \mathbb{R}$. □

4. Discussion

In this paper, we have dealt with a partially degenerate epidemic model with free boundaries and small advection. At first, we obtain the global existence and uniqueness of the solution. Then the effect of small advection is considered. We have proved that the results is similar to that in [20, 23] under the condition $0 < \beta < \beta^*$. But we should explain that, for the case that $\frac{cG'(0)}{ab} > 1$ and $\beta \geq 2\sqrt{d\left(\frac{cG'(0)}{b} - a\right)}$, the criteria for spreading and vanishing is hard to get by using the results of eigenvalue problem to construct the suitable upper and lower solution. We will study it in the future. When spreading occurs, the precise long-time behavior also needs a further consideration.

In order to study the spreading of disease, the asymptotic spreading speed of the spreading fronts is one of the most important subjects. To estimate the precise asymptotic spreading speed, we need to study the corresponding semi-wave problem or some other new technique. This may be not an easy task and deserves further study. We will consider it in another paper.

Due to the advection term, we find that the spreading barrier l^* becomes larger if we increase the size of β for $\beta \in (0, \beta^*)$. This means that if $\beta \in (0, \beta^*)$, the more lager the size of advection is, the more difficult the disease will spread. This result may provide us a suggestion in controlling and preventing the disease. It may be an effective measure to make the infectious agents move along a certain direction by artificial means.

Acknowledgments

We are very grateful to the anonymous referee for careful reading and helpful comments which led to improvements of our original manuscript. The first author was supported by FRFCU (lzujbky-2017-it55) and the second author was partially supported by NSF of China (11731005, 11671180).

Conflict of interest

The authors declare there is no conflict of interest.

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