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Research article

Dynamic analysis of the recurrent epidemic model

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Abstract: In this work, an SIRS model with age structure is proposed for recurrent infectious disease by incorporating temporary immunity and delay. We formulate the model as an abstract non-densely defined Cauchy problem and derive the conditions for the global stability of disease free equilibrium, the local stability of endemic equilibrium, and the existence of Hopf bifurcation. Both non-periodic and periodic behaviors are possible when the disease persists in population, where time delay plays an important role. Numerical examples are provided to illustrate our theoretical results.

Keywords: age-structured model; recurrent epidemic; C_0-semigroup; asymptotical stability; Hopf bifurcation

1. Introduction

In the real world, many diseases, such as Influenza, hand-foot-mouth-disease, rotavirus, and so on, can occur the secondary infection after recovery [1–3]. We can classify such diseases as recurrent epidemics. The study of recurrent epidemics which has been attracting great attention for decades, is currently one of the hottest topics in the field of epidemiology. A great deal of significant progress has been achieved so far (for examples, see [4–12]), including derivation of the threshold for outbreak and persistence of recurrent infectious disease, and the dynamic analysis of the seasonal and non-seasonal behaviors.

At present, most of the existing deterministic models present periodic behaviors of recurrent infectious diseases by selecting seasonal parameters. Whether there are the deterministic models with non-seasonal parameters which can describe the possible periodic behaviors of recurrent infectious

disease is worth exploring. Since the most important feature of recurrent infectious diseases is that the recovered individuals' immunity are temporary, the immunity age of the recovered individuals and the time required for loss of immunity must be characterized in the model. That is, the model should be built by combining immunity age and delayed immune loss.

We assume that recovered individuals lose their immunity after a period of recovery, and then, they become susceptible again. Let $m(a)$ be the immunity loss rate function of the recovered individual with immunity age a, and $\tau > 0$ be the minimal time that a recovered individual has immunity. If the immunity age a is less than τ, then the recovered individuals remain in the recovered class, which means $m(a) = 0$. If the immunity age a is great than τ, then the recovered individuals may lose their immunity and enter into the susceptible class at the rate $m(a) = m_* > 0$. Therefore, $m(a)$ follows the following function:

$$m(a) = \begin{cases} m_*, & a \geq \tau \\ 0, & a < \tau \end{cases} \tag{1.1}$$

Apparently, $m(a) \in L^\infty_+((0, +\infty), \mathbb{R})$.

In this paper, we mainly consider the recurrent infectious diseases with temporary immunity. Therefore, we incorporate loss of immunity into epidemic model by establishing an SIRS model with age structure and delay. Denote $S(t)$ and $I(t)$ respectively as the number of susceptible and infectious individuals, and denote $R(t, a)$ as the number of recovered individuals with immunity age a at time t. The model is given as follows:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - dS(t) - \dfrac{\beta S(t)I(t)}{1 + \alpha I(t)} + \displaystyle\int_0^{+\infty} m(a)R(t, a)da, \\ \dfrac{dI(t)}{dt} = \dfrac{\beta S(t)I(t)}{1 + \alpha I(t)} - (d + \mu + \gamma)I(t), \quad t \geq 0, \\ \dfrac{\partial R(t, a)}{\partial t} + \dfrac{\partial R(t, a)}{\partial a} = -(d + m(a))R(t, a), \quad t \geq 0,\ a \geq 0, \\ R(t, 0) = \gamma I(t), \quad t \geq 0, \end{cases} \tag{1.2}$$

with the initial condition

$$S(0) = S_0 \geq 0, \quad I(0) = I_0 \geq 0, \quad R(0, a) = P_0(a) \in L^1_+(0, +\infty).$$

where Λ is the recruitment rate of susceptible individuals, β is the transmission rate, $\dfrac{1}{\alpha}$ is the half-saturation constant of infectious individuals, d is the natural death rate, μ is the mortality rate due to disease, and γ is the recovery rate. All parameters are positive and constant.

This paper is organized as follows. Some preliminary results and the well-posedness of system (1.2) are presented in Section 2. In Section 3, we prove the existence of equilibria, especially the existence and uniqueness of positive equilibrium, and linearize system (1.2) around the equilibrium E_*. In Section 4, we discuss the stability of both equilibrium, and analyze the existence of Hopf bifurcations when the stability of E_* changes. We conclude and discuss our findings in Section 5 with some numerical examples given to illustrate our theoretical results.

2. Preliminaries and Well-posedness

In this section, we will rewrite system (1.2) into an abstract equation on a suitable Banach lattice, establish the well-posedness result for the system and prove the nonnegativity and boundedness of

solutions. At first, we collect some preliminaries on linear operators and C_0-semigroup theory and some notations to be used in this paper.

Let $L : D(L) \subset X \to X$ be a linear operator on a Banach space X. Denote the resolvent set of L as $\rho(L)$. The spectrum of L is $\sigma(L) = \mathbb{C}\, n\, \rho(L)$. The point spectrum of L is the set

$$\sigma_p(L) := \{\lambda \in \mathbb{C} : N(\lambda I - L) \neq \{0\}\}.$$

Definition 2.1. *Let $L : D(L) \subseteq X \to X$ be a linear operator. If there exist real constants $M \geq 1$ and $\omega \in \mathbb{R}$, such that $(\omega, +\infty) \subseteq \rho(L)$, and*

$$\|(\lambda - L)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}, \ \text{for } n \in \mathbb{N}_+, \ \text{and all } \lambda > \omega.$$

Then the linear operator $(L, D(L))$ is called a Hille-Yosida operator.

For a Hille-Yosida operator one has the following perturbation result.

Lemma 2.1 (see [13, 14])**.** *Let $(A, D(A))$ be a Hille-Yosida operator on a Banach space X and $B \in \mathscr{L}(X)$, $\mathscr{L}(X)$ denotes the set of all bounded linear operators on X, then the sum $C = A + B$ is a Hille-Yosida operator as well.*

If $(L, D(L))$ is a Hille-Yosida operator on the Banach space X and set

$$\begin{aligned} X_0 &= (\overline{D(L)}, \|\cdot\|), \\ D(L_0) &= \{x \in D(L) : Lx \in X_0\}, \\ L_0 x &= Lx, \quad \text{for } x \in D(L_0), \end{aligned}$$

then the operator $(L_0, D(L_0))$ is called the part of L in X_0 and we have that:

Lemma 2.2 (see [13, 14])**.** *If $(L, D(L))$ is a Hille-Yosida operator, then its part $(L_0, D(L_0))$ generates a C_0-semigroup $(T_0(t))_{t\geqslant 0}$ on X_0.*

Now we set about to rewrite system (1.2) into an abstract evolution equation. Let

$$X = \mathbb{R}^3 \times L^1((0, +\infty), \mathbb{R}).$$

Define the linear operator $\mathcal{A} : D(\mathcal{A}) \subseteq X \longrightarrow X$ by

$$\mathcal{A}\begin{pmatrix} x \\ y \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} -dx \\ -(d + \mu + \gamma)y \\ -z(0) \\ -z' - (d + m(a))z \end{pmatrix},$$

with $D(\mathcal{A}) = \mathbb{R}^2 \times \{0\} \times W^{1,1}((0, +\infty), \mathbb{R})$. Then $\overline{D(\mathcal{A})} = \mathbb{R}^2 \times \{0\} \times L^1((0, +\infty), \mathbb{R})$, which shows $D(\mathcal{A})$ is not dense in X. We also introduce a nonlinear map $\mathcal{F} : \overline{D(\mathcal{A})} \longrightarrow X$ given by

$$\mathcal{F}\begin{pmatrix} x \\ y \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} \Lambda - \frac{\beta x y}{1+\alpha y} + \int_0^{+\infty} m(a)z(a)da \\ \frac{\beta x y}{1+\alpha y} \\ \gamma y \\ 0 \end{pmatrix},$$

and let

$$u(t) = (S(t), I(t), 0, R(t,\cdot))^T.$$

Then we can reformulate system (1.2) as the following abstract Cauchy problem:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}(u(t)) = \mathcal{A}u(t) + \mathcal{F}(u(t)), & t \geqslant 0, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $u_0 = (S_0, I_0, 0, R_0(a))^T$.

In general, it is difficult to find a strong solution for an abstract differential equation like (2.1). So, we solve (2.1) in integrated form

$$u(t) = u_0 + \mathcal{A}\int_0^t u(s)ds + \int_0^t \mathcal{F}(u(s))ds. \tag{2.2}$$

Set

$$X_0 = \overline{D(\mathcal{A})} = \mathbb{R}^2 \times \{0\} \times L^1((0,+\infty),\mathbb{R}),$$
$$X_{0+} = \mathbb{R}^2_+ \times \{0\} \times L^1_+((0,+\infty),\mathbb{R}).$$

We will show in Theorem 3.2 in the next section that $(\mathcal{A}, D(\mathcal{A}))$ is a Hille-Yosida operator and hence from Lemma 2.2 it generates a C_0-semigroup on the closure of its domain. As a result, we have the following well-posedness theorem for system (2.1).

Theorem 2.1. *For any $u_0 \in X_{0+}$, the system* (1.2) *represented by the integral equation* (2.2) *has a unique continuous solution with values in X_{0+}. Moreover, the map $\Phi : [0,+\infty) \times X_{0+} \mapsto X_{0+}$ defined by $\Phi(t,u_0) = u(t,u_0)$ is a continuous semi-flow, i.e. the map is continuous and satisfies that $\Phi(0,\cdot) = I$ and $\Phi(t,\Phi(s,\cdot)) = \Phi(t+s,\cdot)$.*

Due to the biological interpretation of system (1.2), only non-negative solutions are meaningful to be considered. The following result reveals that the solutions $(S(t), I(t), R(t,a))$ of system (1.2) with non-negative initial value remain non-negative and bounded ultimately.

Theorem 2.2. *All solutions of system* (1.2) *with non-negative initial value remain non-negative for all $t \geq 0$ and are ultimately bounded.*

Proof. By using the second equation of system (1.2), we have

$$I(t) = I_0 e^{\int_0^t [\frac{\beta S(\theta)}{1+\alpha I(\theta)} - (d+\mu+\gamma)]d\theta} > 0.$$

Integrating the third equation of system (1.2) along the characteristic line yields that

$$R(t,a) = \begin{cases} R(t-a,0)e^{-\int_0^a (d+m(\theta))d\theta}, & a \leq t, \\ R_0(a-t)e^{-\int_{a-t}^a (d+m(\theta))d\theta}, & a > t. \end{cases} \tag{2.3}$$

It is clear that $R(t,a)$ remains nonnegative for all nonnegative initial values.

In the following, we prove that $S(t)$ is nonnegative for $t \geq 0$. In fact, if there exists $t_1 > 0$ such that $S(t_1) = 0$, and $S(t) > 0$ for $\forall t \in (0,t_1)$, then by the first equation of system (1.2), we have $S'(t_1) = \Lambda + \int_0^{+\infty} m(a)R(t_1,a)da > 0$. It implies that $S(t) \geq 0$ for all $t \geq 0$.

Summarizing the above analysis, we know that any solution of system (1.2) with non-negative initial data remains nonnegative for all $t \geq 0$.

Next, we will show that the solutions of system (1.2) are ultimately bounded. Let $\bar{R}(t) = \int_0^{+\infty} R(t,a)da$, which represents the total number of recovered individuals at time t. Biologically, there exists a finite maximum age, so it is reasonable to assume that $\lim_{a\to+\infty} R(t,a) = 0$. Then from system (1.2), we have

$$\begin{aligned}\left(S(t)+I(t)+\bar{R}(t)\right)' &= \Lambda - dS(t) - (d+\mu+\gamma)I(t) + \int_0^{+\infty} m(a)R(t,a)da \\ &\quad + \int_0^{+\infty}\left(\frac{\partial R(t,a)}{\partial a} - (d+m(a))R(t,a)\right)da \\ &= \Lambda - dS(t) - (d+\mu)I(t) - d\bar{R}(t) \\ &\leq \Lambda - d\left(S(t)+I(t)+\bar{R}(t)\right).\end{aligned}$$

Therefore,

$$\limsup_{t\to+\infty}\left(S(t)+I(t)+\bar{R}(t)\right) \leq \frac{\Lambda}{d}.$$

It follows that the omega limit set of system (1.2) is contained in the following bounded feasible region:

$$\Gamma = \left\{(S,I,R(\cdot)) : S,I,R(\cdot) \geq 0, S + I + \int_0^{+\infty} R(t,a)da \leq \frac{\Lambda}{d}\right\}.$$

Obviously, this region is positively invariant with respect to system (1.2). It implies that the system (1.2) is ultimately bounded. □

3. Equilibria and linearized equations

In this section, we devote to linearizing the nonlinear system (1.2) around the equilibrium solutions. For this purpose, we firstly discuss the existence of equilibria of system (1.2).

It is clear that system (1.2) always has a disease free equilibrium $E_0 = (S^{(0)}, 0, 0)$ with $S^{(0)} = \frac{\Lambda}{d}$. In order to find the positive equilibrium $E_* = (S_*, I_*, R_*(a))$ of system (1.2), we have

$$\begin{cases} \Lambda - \dfrac{\beta S_* I_*}{1+\alpha I_*} - dS_* + \int_0^{+\infty} m(a)R_*(a)da = 0, \\ \dfrac{\beta S_* I_*}{1+\alpha I_*} - (d+\mu+\gamma)I_* = 0, \\ \dfrac{dR_*(a)}{da} = -(d+m(a))R_*(a), \\ R_*(0) = \gamma I_*. \end{cases} \tag{3.1}$$

Solving the third equation of (3.1), we get

$$R_*(a) = R_*(0)e^{-\int_0^a (d+m(\theta))d\theta} = \gamma I_* e^{-\int_0^a (d+m(\theta))d\theta}. \tag{3.2}$$

By using the second equation of (3.1), we have

$$S_* = \frac{d+\mu+\gamma}{\beta}(1+\alpha I_*).$$

Substituting $R_*(a)$ and S_* into the first equation of (3.1), we obtain

$$I_* = \frac{\Lambda\beta\left(1 - \frac{d(d+\mu+\gamma)}{\Lambda\beta}\right)}{(\beta + d\alpha)(d+\mu+\gamma) - \beta\gamma\int_0^{+\infty} m(a)e^{-\int_0^a (d+m(\theta))d\theta}da}.$$

Let $R_0 = \dfrac{\beta S^{(0)}}{d+\mu+\gamma} = \dfrac{\Lambda\beta}{d(d+\mu+\gamma)}$. Then, I_* can be rewritten as

$$I_* = \frac{\Lambda\beta(1 - \frac{1}{R_0})}{(\beta + d\alpha)(d+\mu+\gamma) - \beta\gamma\int_0^{+\infty} m(a)e^{-\int_0^a (d+m(\theta))d\theta}da} > 0,$$

which is positive if $R_0 > 1$. Hence, system (1.2) has a unique positive equilibrium when $R_0 > 1$.

Summarizing the above analysis, we have the following result.

Theorem 3.1. *System* (1.2) *always has a disease free equilibrium* $E_0 = (S^{(0)}, 0, 0)$. *If* $R_0 > 1$, *there also exists a unique endemic equilibrium* $E_* = (S_*, I_*, R_*(a))$ *for fixed a.*

In fact, each term in R_0 has clear epidemiological interpretation. $\dfrac{1}{d+\mu+\gamma}$ is the average infection period. β denotes the transmission rate of an infectious individual. $S^{(0)}$ is the total number of susceptible individuals. Therefore, R_0 represents average new cases generated by a typical infectious member in the entire infection period. That is, R_0 is the basic reproduction number of system (1.2).

Let $S(t) = x(t) + \overline{S}$, $I(t) = y(t) + \overline{I}$, $R(t,a) = z(t,a) + \overline{R}(a)$, where $\overline{E} = (\overline{S}, \overline{I}, \overline{R}(a))$ is a steady state of system (1.2), and let $\tilde{u}(t) = (x(t), y(t), 0, z(t,a))$, $\bar{u} = (\overline{S}, \overline{I}, 0, \overline{R}(a))$. Then, system (2.1) is equivalent to the following Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{u}(t) = \mathcal{A}\tilde{u}(t) + \mathcal{F}(\tilde{u}(t) + \bar{u}) - \mathcal{F}(\bar{u}(t)), \qquad t \geqslant 0, \\ \tilde{u}(0) = u(0) - \bar{u}. \end{cases}$$

By conducting direct computations one can obtain readily the linearized system of (2.1) around $\bar{u}$ as the following form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{u}(t) = \mathcal{A}\tilde{u}(t) + D\mathcal{F}(\bar{u})(\tilde{u}(t)), \qquad t \geqslant 0, \\ \tilde{u}(0) = u(0) - \bar{u}, \end{cases} \tag{3.3}$$

in which

$$D\mathcal{F}(\bar{u})\begin{pmatrix} x(t) \\ y(t) \\ 0 \\ z(t,a) \end{pmatrix} = \begin{pmatrix} -\frac{\beta\overline{I}}{1+\alpha\overline{I}}x(t) - \frac{\beta\overline{S}}{(1+\alpha\overline{I})^2}y(t) + \int_0^{+\infty} m(a)z(t,a)da \\ \frac{\beta\overline{I}}{1+\alpha\overline{I}}x(t) + \frac{\beta\overline{S}}{(1+\alpha\overline{I})^2}y(t) \\ \gamma y(t) \\ 0 \end{pmatrix}.$$

Clearly, $D\mathcal{F}(\bar{u})$ is a compact bounded linear operator on X.

Denote $\Omega = \{\lambda \in \mathbb{C} : \ \mathrm{Re}(\lambda) > -d\}$. We claim then prove the following statement.

Theorem 3.2. *The operator* $(\mathcal{A}, D(\mathcal{A}))$ *is a Hille-Yosida operator.*

Proof. For $(\phi,\varphi,\psi,\omega)\in X$, $(\tilde{\phi},\tilde{\varphi},0,\tilde{\omega})\in D(\mathcal{A})$, $\lambda\in\Omega$, we have

$$(\lambda-\mathcal{A})^{-1}\begin{pmatrix}\phi\\ \varphi\\ \omega\\ \psi\end{pmatrix}=\begin{pmatrix}g_1\\ g_2\\ 0\\ h\end{pmatrix}\quad\Leftrightarrow\quad\begin{cases}(\lambda+d)g_1=\phi,\\ (\lambda+d+\mu+\gamma)g_2=\varphi,\\ h(0)=\omega,\\ h'+(\lambda+d+m(a))h=\psi.\end{cases}$$

It then follows that

$$\begin{cases}g_1=\frac{\phi}{\lambda+d},\\ g_2=\frac{\varphi}{\lambda+d+\mu+\gamma},\\ h=e^{-\int_0^a(\lambda+d+m(\theta))d\theta}\omega+\int_0^a e^{-\int_a^s(\lambda+d+m(\theta))d\theta}\psi(s)ds.\end{cases}\tag{3.4}$$

Integrating the last equation of (3.4) with regard to the age variable a and adding all the equations, we obtain that

$$|g_1|+|g_2|+\|h\|_{L^1}\leq\frac{1}{\lambda+d}(|\phi|+|\varphi|+|\omega|+\|\psi\|_{L^1}).$$

Thus, we have

$$\|(\lambda-\mathcal{A})^{-1}\|\leq\frac{1}{\lambda+d},\quad for\ \ all\ \ \lambda\in\Omega,$$

which shows that $(\mathcal{A},D(\mathcal{A}))$ is a Hille-Yosida operator. □

By Lemma 2.1 and Theorem 3.2, it follows that

Theorem 3.3. *The operator $\mathcal{A}+D\mathcal{F}(\bar{u})$ is a Hille-Yosida operator.*

Using Lemma 2.2, we further derive that

Theorem 3.4. *The part of $(\mathcal{A},D(\mathcal{A}))$ and $(\mathcal{A}+D\mathcal{F}(\bar{u}),D(\mathcal{A}+D\mathcal{F}(\bar{u})))$ generate C_0-semigroups $(\mathcal{S}(t))_{t\geq0}$ and $(\mathcal{T}(t))_{t\geq0}$, respectively, on space X_0.*

In order to establish the stability results for system (1.2), we will analyze the compactness of the generated C_0-semigroups. Firstly, we introduce the definition of quasi-compactness for a semigroup below.

Definition 3.1 (cf. [15])**.** *A C_0-semigroup $(\mathcal{T}(t))_{t\geq0}$ is called quasi-compact if $\mathcal{T}(t)=\mathcal{T}_1(t)+\mathcal{T}_2(t)$ with the operator families $\mathcal{T}_1(t)$ and $\mathcal{T}_2(t)$ satisfying that*

(i) $\mathcal{T}_1(t)\to0$, as $t\to+\infty$,
(ii) $\mathcal{T}_2(t)$ is eventually compact, that is, there is $t_0>0$, such that $\mathcal{T}_2(t)$ is compact for all $t>t_0$.

For a quasi-compact C_0-semigroup, one has that

Lemma 3.1 (cf. [15])**.** *Let $(\mathcal{T}(t))_{t\geq0}$ be a quasi-compact C_0-semigroup and $(B,D(B))$ its infinitesimal generator. Then $e^{\delta t}\|\mathcal{T}(t)\|\to0$, as $t\to+\infty$ for $\delta>0$ if and only if all eigenvalues of B have strictly negative real part.*

By the Hille-Yosida estimate in the proof of Theorem 3.2, we have $\|\mathcal{S}(t))\|\leq e^{-\xi t}$. Furthermore, $D\mathcal{F}(\bar{u})\mathcal{S}(t):X_0\to X$ is compact for every $t>0$. Since

$$\mathcal{T}(t)=e^{D\mathcal{F}(\bar{u})t}\mathcal{S}(t)=\mathcal{S}(t)+\sum_{k=1}^{+\infty}\frac{(D\mathcal{F}(\bar{u})t)^k}{k!}\mathcal{S}(t),$$

it is seen that $(\mathscr{T}(t))_{t\geq 0}$ is quasi-compact. Then by Lemma 3.1 we deduce that, for some $\eta > 0$, $e^{\eta t}\|\mathscr{T}(t)\| \to 0$ as $t \to +\infty$ whenever all the eigenvalues of $(\mathcal{A} + D\mathcal{F}(\bar{u}))$ have negative real part.

From the above arguments we can now make the following conclusion.

Theorem 3.5. *The solution semi-flow $\Phi(t, u_0)$ of system (1.2), defined as in Theorem 2.1, satisfies the following properties.*

(i) If all the eigenvalues of $(\mathcal{A} + D\mathcal{F}(\bar{u}))$ have strictly negative real part, then the steady state $\bar{u}$ is locally asymptotically stable.

(ii) If, however, at least one eigenvalue of $(\mathcal{A} + D\mathcal{F}(\bar{u}))$ has strictly positive part, then the steady state $\bar{u}$ is unstable.

4. The stability and Hopf bifurcation

Based on the preceding analysis, in this section, we will firstly discuss the global stability of the disease free equilibrium $E_0 = (S^{(0)}, 0, 0)$. Then, we study the stability of the endemic equilibrium $E_* = (S_*, I_*, R_*(a))$. At last, we analyze the existence of the Hopf bifurcation when E_* is unstable.

4.1. Stability of disease-free equilibrium

Theorem 4.1. *If $R_0 < 1$, then disease free equilibrium $E_0 = (S^{(0)}, 0, 0)$ of system (1.2) is globally asymptotically stable. While if $R_0 > 1$, E_0 is unstable.*

Proof. Let $x(t) = S(t) - S^{(0)}$, $y(t) = I(t)$, $z(t, a) = R(t, a)$. Linearizing system (3.3) at E_0 turns out to be the following system:

$$\begin{cases} x'(t) = -dx(t) - \beta S^{(0)} y(t) + \int_0^{+\infty} m(a) z(t, a) da, \\ y'(t) = \beta S^{(0)} y(t) - (d + \mu + \gamma) y(t), \\ \dfrac{\partial z(t, a)}{\partial t} + \dfrac{\partial z(t, a)}{\partial a} = -(d + m(a)) z(t, a), \\ z(t, 0) = \gamma y(t). \end{cases} \tag{4.1}$$

Solving the second and the third equations of (4.1), we obtain

$$\begin{aligned} z(t, a) &= \begin{cases} R_0(a - t) e^{-\int_{a-t}^{a} (d + m(\theta)) d\theta}, & a > t, \\ \gamma y(t - a) e^{-\int_0^t (d + m(\theta)) d\theta}, & a < t, \end{cases} \\ y(t) &= y(0) e^{\left(\beta S^{(0)} - (d + \mu + \gamma)\right) t}. \end{aligned} \tag{4.2}$$

The second equation of (4.2) implies that $\lim_{t\to+\infty} y(t) = 0$ when $R_0 < 1$. Furthermore, we have $\lim_{t\to+\infty} z(t, a) = 0$. Substituting $\lim_{t\to+\infty} y(t) = 0$ and $\lim_{t\to+\infty} z(t, a) = 0$ into the first equation of (4.1) and solving for $x(t)$, we obtain that $\lim_{t\to+\infty} x(t) = 0$ as well. It implies that E_0 is locally asymptotically stable when $R_0 < 1$. In addition, when $R_0 > 1$, we have $y(t) \to \infty$ as $t \to +\infty$, which implies that E_0 is unstable when $R_0 > 1$.

In the following, we construct the Lyapunov function to investigate the global stability of the disease free equilibrium E_0 of system (1.2). Define a Lyapunov function

$$V(t) = S(t) - S^{(0)} - S^{(0)} \ln \frac{S(t)}{S^{(0)}} + I(t) + \theta \int_0^{+\infty} \alpha(a) R(t, a) da,$$

where $\theta > 0$ is constant, which will be determined later, and $\alpha(a) = \int_a^{+\infty} m(\xi)e^{-\int_a^{\xi}(d+m(\eta))d\eta}d\xi$ is a differential function of a on $[0, \infty)$, then the derivative of $V(t)$ along the solution of system is given by

$$\frac{dV(t)}{dt} = (1 - \frac{S^{(0)}}{S})\frac{dS}{dt} + \frac{dI}{dt} + \theta\int_0^{+\infty}\alpha(a)\frac{\partial}{\partial t}R(t,a)da.$$

By using of $\Lambda = dS^{(0)}$, we obtain

$$\begin{aligned}\frac{dV}{dt} &= -d\tfrac{(S-S^{(0)})^2}{S} - (1-\tfrac{S^{(0)}}{S})\tfrac{\beta SI}{1+\alpha I} + (1-\tfrac{S^{(0)}}{S})\textstyle\int_0^{+\infty} m(a)R(t,a)da + \tfrac{\beta SI}{1+\alpha I} - (d+\mu+\gamma)I\\ &\quad - \theta\textstyle\int_0^{+\infty}\alpha(a)[\tfrac{\partial}{\partial a}R(t,a) + (d+m(a))R(t,a)]da\\ &= -d\tfrac{(S-S^{(0)})^2}{S} + \tfrac{\beta S^{(0)}I}{1+\alpha I} + \tfrac{S-S^{(0)}}{S}\textstyle\int_0^{+\infty} m(a)R(t,a)da - (d+\mu+\gamma)I - \theta\int_0^{+\infty}\alpha(a)\tfrac{\partial}{\partial a}R(t,a)da\\ &\quad - \theta\textstyle\int_0^{+\infty}\alpha(a)(d+m(a))R(t,a)da.\end{aligned}$$

Because of

$$\begin{aligned}\int_0^{+\infty}\alpha(a)\frac{\partial}{\partial a}R(t,a)da &= \alpha(a)R(t,a)\big|_{a=0}^{+\infty} - \int_0^{+\infty}\alpha'(a)R(t,a)da\\ &= -\alpha(0)R(t,0) + \alpha(a)R(t,a)\,|_{a=+\infty} - \int_0^{+\infty}[(d+m(a))\alpha(a) - m(a)]R(t,a)da,\end{aligned}$$

we have

$$\begin{aligned}\frac{dV}{dt} &= -d\frac{(S-S^{(0)})^2}{S} + \frac{\beta S^{(0)}I}{1+\alpha I} + \frac{S-\frac{\Lambda}{d}}{S}\int_0^{+\infty} m(a)R(t,a)da - (d+\mu+\gamma)I\\ &\quad + \theta\alpha(0)\gamma I - \theta\alpha(a)R(t,a)\,|_{a=+\infty} - \theta\int_0^{+\infty} m(a)R(t,a)da\\ &\le -d\frac{(S-S^{(0)})^2}{S} + \frac{S-\frac{\Lambda}{d}}{S}\int_0^{+\infty} m(a)R(t,a)da - \theta\alpha(a)R(t,a)\,|_{a=+\infty}\\ &\quad - \theta\int_0^{+\infty} m(a)R(t,a)da + \beta S^{(0)}I - (d+\mu+\gamma)I + \theta\alpha(0)\gamma I\\ &= -d\frac{(S-S^{(0)})^2}{S} + \frac{S-\frac{\Lambda}{d}}{S}\int_0^{+\infty} m(a)R(t,a)da - \theta\alpha(a)R(t,a)\,|_{a=+\infty}\\ &\quad - \theta\int_0^{+\infty} m(a)R(t,a)da - ((d+\mu+\gamma)(1-R_0) - \theta\alpha(0)\gamma)I.\end{aligned}$$

It is clear that $S - \frac{\Lambda}{d} \le 0$. If $R_0 < 1$, assuming that there exists $\varepsilon > 0$ such that $\theta\alpha(0) = \frac{(d+\mu+\gamma)(1-R_0)}{\gamma} - \varepsilon > 0$, we obtain

$$\begin{aligned}\frac{dV}{dt} &\le -d\frac{(S-S^{(0)})^2}{S} + \frac{S-\frac{\Lambda}{d}}{S}\int_0^{+\infty} m(a)R(t,a)da\\ &\quad - \theta\alpha(a)R(t,a)\,|_{a=+\infty} - \theta\int_0^{+\infty} m(a)R(t,a)da - \varepsilon\gamma I \le 0.\end{aligned}$$

Therefore, $R_0 < 1$ ensures that the positive-definite function $V(t)$ has negative derivative $\frac{dV(t)}{dt}$. Moreover, the strict equality $\frac{dV(t)}{dt} = 0$ holds if and only if $S(t) = \frac{\Lambda}{d}$, $I(t) = 0$ and $R(t,a) = 0$. Thus, the

singleton E_0 is the largest invariant subset of $\frac{dV(t)}{dt} = 0$. By the LaSalle's invariant principle, the disease free equilibrium E_0 is globally attractive. Therefore, E_0 is globally asymptotically stable when $R_0 < 1$. □

4.2. Hopf bifurcation around endemic equilibrium E_*

Next, we are interested in the stability of endemic equilibrium $E_* = (S_*, I_*, R_*(a))$ and the existence of Hopf bifurcations around E_* when its stability changes. In order to show the local stability of E_*, we linearize system (1.2) around E_* and put it as follows:

$$\begin{cases} x'(t) = -\left(d + \frac{\beta I_*}{1+\alpha I_*}\right) x(t) - \frac{\beta S_*}{(1+\alpha I_*)^2} y(t) + \int_0^{+\infty} m(a) z(t,a) da, \\ y'(t) = \frac{\beta I_*}{1+\alpha I_*} x(t) + \frac{\beta S_*}{(1+\alpha I_*)^2} y(t) - (d+\mu+\gamma) y(t), \\ \dfrac{\partial z(t,a)}{\partial t} + \dfrac{\partial z(t,a)}{\partial a} = -(d + m(a)) z(t,a), \\ z(t,0) = \gamma y(t). \end{cases}$$

To analyze the asymptotic behavior of E_*, we look for solutions of the form $x(t) = x_0 e^{\lambda t}$, $y(t) = y_0 e^{\lambda t}$, $z(t,a) = z_0(a) e^{\lambda t}$. Thus, we have the following eigenvalue problem:

$$\begin{cases} \lambda x_0 = -(d + \frac{\beta I_*}{1+\alpha I_*}) x_0 - \frac{\beta S_*}{(1+\alpha I_*)^2} y_0 + \int_0^{+\infty} m(a) z_0(a) da, \\ \lambda y_0 = \frac{\beta I_*}{1+\alpha I_*} x_0 + \frac{\beta S_*}{(1+\alpha I_*)^2} y_0 - (d+\mu+\gamma) y_0, \\ \dfrac{dz_0(a)}{da} = -(\lambda + d + m(a)) z_0(a), \\ z_0(0) = \gamma y_0. \end{cases} \tag{4.3}$$

Solving the second and the third equation in (4.3), we obtain

$$z_0(a) = \gamma y_0 e^{-\int_0^a (\lambda + d + m(\theta)) d\theta}, \quad x_0 = \frac{\lambda + d + \mu + \gamma - \frac{\beta S_*}{(1+\alpha I_*)^2}}{\frac{\beta I_*}{1+\alpha I_*}} y_0. \tag{4.4}$$

Substituting $z_0(a)$ and x_0 into the first equation of (4.3), we get the following characteristic equation:

$$\Delta_1(\lambda, \tau) = \frac{b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 + a_0 e^{-\lambda\tau}}{\lambda + d + m_*} = \frac{f(\lambda,\tau)}{g(\lambda)} = 0,$$

where

$$\begin{aligned} b_3 &= \frac{1+\alpha I_*}{\beta I_*}, \\ b_2 &= 1 + \frac{2d + m_* + \alpha I_*(3d + \mu + \gamma + m_*)}{\beta I_*}, \\ b_1 &= \left(1 + \frac{\alpha(2d+m_*)}{\beta}\right)(d+\mu+\gamma) + (d+m_*)\left(d + \frac{\beta I_*}{1+\alpha I_*}\right)\frac{1+\alpha I_*}{\beta I_*}, \\ b_0 &= (d+m_*)(1 + \frac{\alpha d}{\beta})(d+\mu+\gamma), \\ a_0 &= -\gamma m_* e^{-d\tau}. \end{aligned}$$

It is easy to see that

$$\{\lambda \in \Omega : \det(\Delta_1(\lambda,\tau)) = 0\} = \{\lambda \in \Omega : f(\lambda,\tau) = 0\}.$$

In addition, if $\tau = 0$, then

$$f(\lambda, 0) = \tilde{b}_3\lambda^3 + \tilde{b}_2\lambda^2 + \tilde{b}_1\lambda + \tilde{b}_0 + \tilde{a}_0 = 0, \tag{4.5}$$

where $\tilde{b}_i = b_i|_{\tau=0}$, $i = 0,\ 1,\ 2,\ 3,\ \ \tilde{a}_0 = a_0|_{\tau=0}$. It is easy to check that $\tilde{b}_i > 0$ for $i = 0,\ 1,\ 2,\ 3$, and $\tilde{a}_0 + \tilde{b}_0 > 0$. Therefore, by using of the Routh-Hurwitz criterion, we know if

$$\tilde{b}_1\tilde{b}_2 > \tilde{b}_3(\tilde{a}_0 + \tilde{b}_0), \tag{4.6}$$

then all roots of (4.5) have negative real parts, which implies that the endemic equilibrium E_* is locally asymptotically stable. That is, the following result holds:

Theorem 4.2. *If $R_0 > 1$, $\tau = 0$, and $\tilde{b}_1\tilde{b}_2 > \tilde{b}_3(\tilde{a}_0 + \tilde{b}_0)$, then the endemic equilibrium E_* of system* (1.2) *is locally asymptotically stable.*

In fact, the roots of $f(\lambda, \tau) = 0$ depend on τ continuously, and the roots may pass through the imaginary axis and enter the right side as τ increasing. In the following, we discuss the case where $\tau > 0$.

Let $\lambda = i\omega$ ($\omega > 0$) be purely imaginary roots of $f(\lambda, \tau) = 0$. By submitting $\lambda = i\omega$ into $f(\lambda, \tau) = 0$ and separating the real and imaginary parts, we have

$$\begin{cases} -b_3\omega^3 + b_1\omega = a_0 \sin\omega\tau, \\ -b_2\omega^2 + b_0 = -a_0\cos\omega\tau, \end{cases} \tag{4.7}$$

which yields that

$$p_3\omega^6 + p_2\omega^4 + p_1\omega^2 + p_0 = 0, \tag{4.8}$$

where $p_3 = b_3^2$, $p_2 = b_2^2 - 2b_1b_3$, $p_1 = b_1^2 - 2b_0b_2$, $p_0 = b_0^2 - a_0^2$. It is clear that $p_3 > 0$, and $p_0 > 0$.

Put $\Theta = \omega^2$, (4.8) turns out to be

$$Q(\Theta) = p_3\Theta^3 + p_2\Theta^2 + p_1\Theta + p_0 = 0. \tag{4.9}$$

Let $F(\Theta) = 3p_3\Theta^2 + 2p_2\Theta + p_1$. When $p_2^2 - 3p_1p_3 < 0$, we know $F(\Theta) = 0$ has no real roots. When $p_2^2 - 3p_1p_3 \geq 0$, we know $F(\Theta) = 0$ has two real roots, which are $\Theta_1 = \dfrac{-p_2 - \sqrt{p_2^2 - 3p_1p_3}}{3p_3}$, and $\Theta_2 = \dfrac{-p_2 + \sqrt{p_2^2 - 3p_1p_3}}{3p_3}$, respectively. The following lemma gives the results on the positive root of the equation $Q(\Theta) = 0$.

Lemma 4.1.

(i) If $p_2^2 - 3p_1p_3 < 0$, then $Q(\Theta) = 0$ has no positive root;

(ii) If $p_2^2 - 3p_1p_3 \geq 0$ and $\Theta_2 \leq 0$, then $Q(\Theta) = 0$ has no positive root;

(iii) If $p_2^2 - 3p_1p_3 \geq 0$, $\Theta_2 > 0$, and $Q(\Theta_2) > 0$, then $Q(\Theta) = 0$ has no positive root;

(iv) If $p_2^2 - 3p_1p_3 \geq 0$, then $Q(\Theta) = 0$ has positive roots if and only if $\Theta_2 > 0$ and $Q(\Theta_2) \leq 0$.

If $Q(\Theta) = 0$ does not have a positive root, then the stability of E_* will not change as τ increasing. Therefore, the following result holds:

Theorem 4.3. *Assume that $R_0 > 1$, $\tau > 0$, and $\tilde{b}_1\tilde{b}_2 > \tilde{b}_3(\tilde{a}_0 + \tilde{b}_0)$.*

(i) If $p_2^2 - 3p_1p_3 < 0$, then the endemic equilibrium E_* of system (1.2) *is locally asymptotically stable;*

(ii) If $p_2^2 - 3p_1p_3 \geq 0$ and $\Theta_2 \leq 0$, then the endemic equilibrium E_* of system (1.2) *is locally asymptotically stable;*

(iii) If $p_2^2 - 3p_1p_3 \geq 0$, $\Theta_2 > 0$, and $Q(\Theta_2) > 0$, then the endemic equilibrium E_* of system (1.2) *is locally asymptotically stable.*

While if $Q(\Theta) = 0$ has positive root, then the stability of E_* may change when τ passes through some specific values. Let Θ_* be the positive real root of (4.9). Then $\omega_* = \sqrt{\Theta_*}$ is the only one positive real root of (4.8), so $f(\lambda, \tau) = 0$ with $\tau = \tau_k$, $k = 0, 1, 2, \cdots$, has a pair of purely imaginary roots $\pm i\omega_*$, where

$$\tau_k = \begin{cases} \frac{1}{\omega_*}\left(\arccos\frac{b_2\omega_*^2 - b_0}{a_0} + 2k\pi\right), & c \geq 0, \\ \frac{1}{\omega_*}\left(-\arccos\frac{b_2\omega_*^2 - b_0}{a_0} + 2(k+1)\pi\right), & c < 0, \end{cases} \tag{4.10}$$

for $k = 0,\ 1,\ 2,\ \cdots$, and $c = \dfrac{-b_3\omega_*^3 + b_1\omega_*}{a_0}$.

Differentiating both sides of $f(\lambda, \tau) = 0$ with respect to τ yields

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(3b_3\lambda^2 + 2b_2\lambda + b_1)e^{\lambda\tau}}{\lambda a_0} - \frac{\tau}{\lambda}. \tag{4.11}$$

From (4.7) and the fact that $sign\left\{Re\left[\left(\frac{d\lambda}{d\tau}\right)^{-1}\Big|_{\lambda=i\omega_*}\right]\right\} = sign\left\{\frac{dRe(\lambda)}{d\tau}\Big|_{\tau=\tau_k}\right\}$, we have

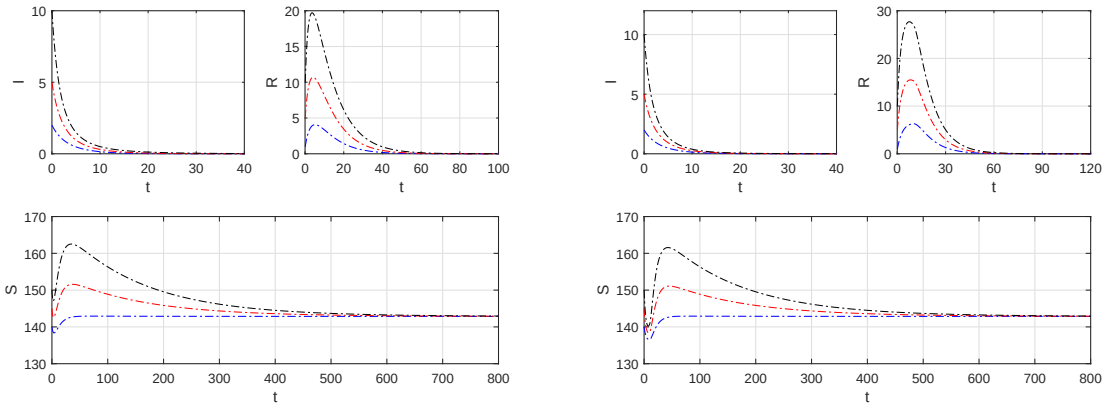
$$\begin{aligned} sign\left\{\frac{dRe(\lambda)}{d\tau}\Big|_{\tau=\tau_k}\right\} &= sign\left\{Re\left[\left(\frac{(3b_3\lambda^2+2b_2\lambda+b_1)e^{\lambda\tau}}{\lambda a_0} - \frac{\tau}{\lambda}\right)\Big|_{\lambda=i\omega_*}\right]\right\} \\ &= sign\left\{\frac{3b_3^2\omega_*^6+2(b_2^2-2b_1b_2)\omega_*^4+(b_1^2-2b_0b_2)\omega_*^2}{(b_3\omega_*^4-b_1\omega_*^2)^2+(b_0\omega_*-b_2\omega_*^3)^2}\right\} \\ &= sign\left\{\frac{Q'(\omega_*^2)}{(b_3\omega_*^3-b_1\omega_*)^2+(b_0-b_2\omega_*^2)^2}\right\} \\ &= sign\left\{Q'(\omega_*^2)\right\} \\ &= sign\left\{Q'(\Theta_*)\right\}. \end{aligned}$$

The transversality condition holds and a Hopf bifurcation occurs at $\tau = \tau_k$, $k = 0, 1, 2, \cdots$ when $Q'(\Theta_*) \neq 0$. According to the Hopf bifurcation theorem for functional differential equations [17], we have the following result.

Theorem 4.4. *Assume $R_0 > 1$.*

(i) If $p_2^2 - 3p_1p_3 \geq 0$, $\Theta_2 > 0$, and $Q(\Theta_2) \leq 0$, then the endemic equilibrium E_* of system (1.2) *is asymptotically stable for all $\tau \in [0, \tau_0)$ under condition* (4.6)*;*

(ii) If $p_2^2 - 3p_1p_3 \geq 0$, $\Theta_2 > 0$, $Q(\Theta_2) \leq 0$, and $Q'(\Theta_*) \neq 0$, then system (1.2) *undergoes Hopf bifurcation at E_* when $\tau = \tau_k$, $k = 0, 1, 2, \cdots$.*



(a) $\tau = 0$ (b) $\tau = 10$

Figure 1. The disease free equilibrium E_0 of system (1.2) is globally asymptotically stable for any $\tau \geq 0$ when $R_0 < 1$.

5. Numerical simulations and conclusion

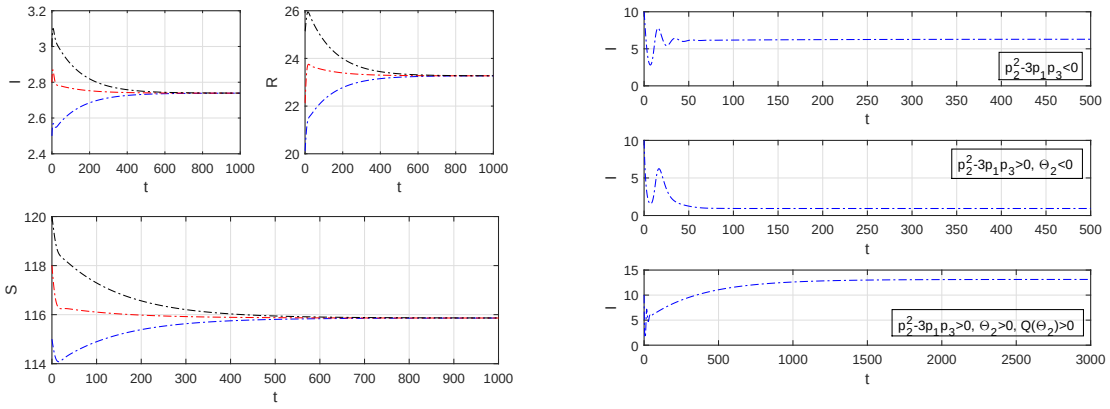
In this paper, we have proposed and analyzed an SIRS model with age structure for recurrent infectious disease by incorporating temporary immunity. A delayed differential equation system can be derived from this model and the delay corresponds to the time that recovery individuals lose their immunity. The purpose of this article is to explore the conditions switching between periodic and non-periodic behavior of recurrent infectious diseases which have the temporary immunity.

On the dynamic behavior analysis of system (1.2), we showed the well-posedness, gave the basic reproduction number R_0, and proved that $R_0 = 1$ is the threshold that determines whether the epidemic persists or not by studying the stability of both disease free equilibrium E_0 and the endemic equilibrium E_*. The disease free equilibrium E_0 is globally asymptotically stable if $R_0 < 1$ and is unstable if $R_0 > 1$. Moreover, the disease persists in the later case, in the sense that infected individuals survive above a certain number for any initial infection numbers. We also proved the existence of Hopf bifurcation around the endemic equilibrium E_* when E_* is unstable.

In order to display our conclusions more intuitively, we will use Matlab to demonstrate the nonlinear dynamics behavior of system (1.2). We denote the numbers of recovery individuals at time t as $R(t) = \int_0^{+\infty} R(t, a)da$. Numerically, we set the maximum immunity age as 100.

Firstly, we illustrate that the disease free equilibrium E_0 is globally asymptotically stable when $R_0 < 1$. The parameter values are chosen as $\Lambda = 1$, $d = 0.007$, $\mu = 0.0025$, $\alpha = 0.1$, $\gamma = 0.9$, $m_* = 0.1$, and $\beta = 0.005$. Accordingly, we obtain $R_0 < 1$. Since R_0 is independent of τ, R_0 does not change with time delay τ. When $\tau = 0$, the solutions of system (1.2) with three different initial values all approach to E_0 as t trends to infinity(see Figure 1(a)). When $\tau = 10$, something similar happens(see Figure 1(b)). It means if we control the basic regeneration number R_0 to be less than unity, the disease will die out in population. In this case, we don't need to worry about the recurrence of the epidemic disease.

Secondly, we illustrate that the stability of the endemic equilibrium E_* when $R_0 > 1$. Here, we only change the values of γ and β, and keep all other parameter values same as theses in Figure 1. In



(a) $\tau = 0$ (b) $\tau = 10$

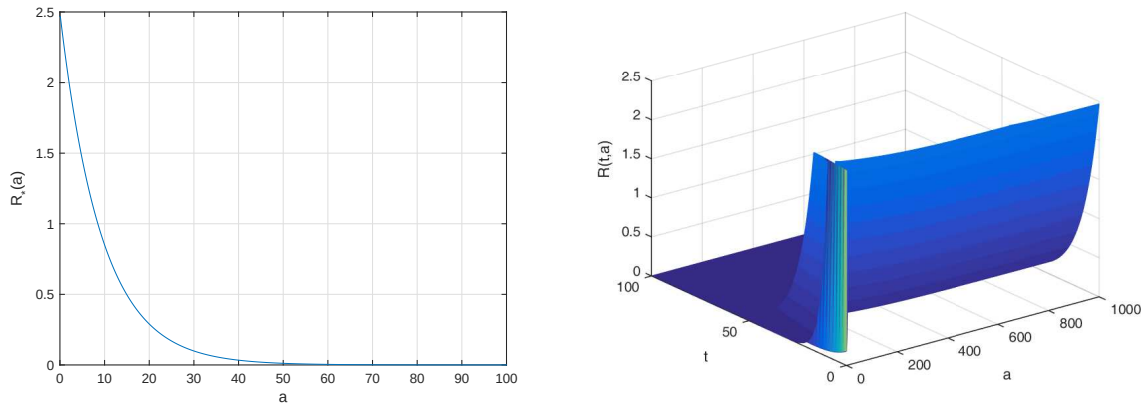
Figure 2. The endemic equilibrium E_* of system (1.2) is asymptotically stable when $R_0 > 1$ and $Q(\Theta) = 0$ does not have positive roots.

the case where $\tau = 0$, we take $\gamma = 0.9$ and $\beta = 0.01$, then, $R_0 > 1$, and $\tilde{b}_1\tilde{b}_2 > \tilde{b}_3(\tilde{a}_0 + \tilde{b}_0)$. Figure 2(a) shows that the solutions of system (1.2) with three different initial values all approach to E_* as t trends to infinity. In the case where $\tau = 10 > 0$, we take three different pairs of value for β and γ, which are $(\beta, \gamma) = (0.05, 0.9)$, $(\beta, \gamma) = (0.03, 0.1)$, and $(\beta, \gamma) = (0.03, 1)$, respectively. Accordingly, we have (i) $p_2^2 - 3p_1p_3 < 0$, (ii) $p_2^2 - 3p_1p_3 > 0$ and $\Theta_2 < 0$, (iii) $p_2^2 - 3p_1p_3 > 0$, $\Theta_2 > 0$, and $Q(\Theta_2) > 0$. Theorem 4.3 indicates that the endemic equilibrium E_* of system (1.2) is asymptotically stable under these conditions, which is consistent with the results shown in Figure 2(b). That is, under these conditions of Theorem 4.2 and Theorem 4.3, no matter how the delay τ changes, the system (1.2) doesn't have the periodic behavior even if the disease persists in the population. Accordingly, the distribution of recovery individuals with respect to immunity age at the endemic equilibrium E_*, $R_*(a)$ is shown in Figure 3(a), which corresponding to the second solution line in Figure 2(a), and the distributions with both immunity age and time, $R(t, a)$ is shown in Figure 3(b).

Thirdly, we demonstrate the case when $R_0 > 1$ and $Q(\Theta) = 0$ has the positive root. According to Theorem 4.4, the endemic equilibrium E_* is asymptotically stable for $\tau \in [0, \tau_0)$, and periodic solutions occur as the stability of E_* changes, which means that Hopf bifurcation appears. Setting $\Lambda = 1$, $d = 0.0006$, $\mu = 0.00003$, $\gamma = 3$, $\alpha = 0.000001$, $\beta = 0.02$, and $m_* = 0.0085$, the conditions $R_0 > 1$, $p_2^2 - 3p_1p_3 > 0$, $\Theta_2 > 0$ and $Q(\Theta_2) < 0$ are satisfied. The critical time value is $\tau_0 = 10$.

When time delay is small, say $\tau = 0, 2, 4, 6, 8, 10$, the solutions shown in Figure 4 all approaches to the endemic equilibrium E_* through damped oscillations. While if we choose $\tau = 12, 14, 16, 18, 20$, Figure 5 displays periodic solutions with different periods and amplitudes. That is, with increasing of time delay, the stable endemic equilibrium is replaced by stable periodic orbits. Accordingly, when $\tau = 12$, we displays the distributions $R(t, a)$ with both immunity age and time in Figure 6.

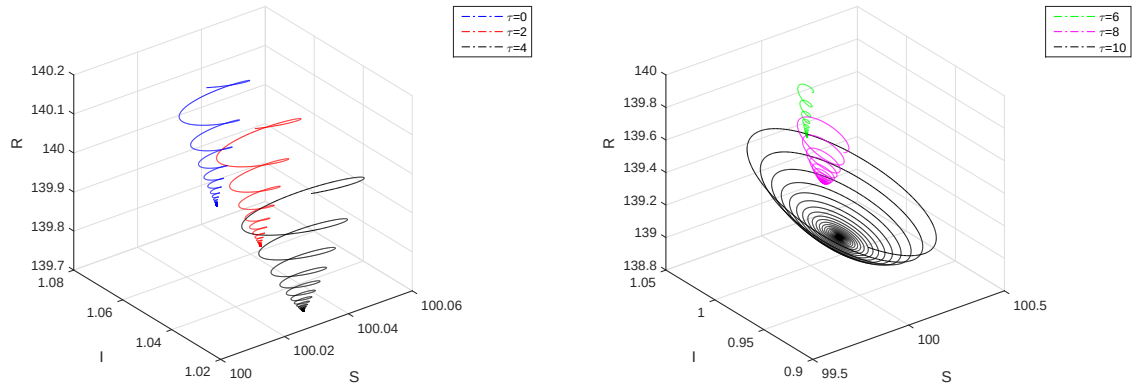
In fact, when Hopf bifurcation exists, it is global continuation. That is, Hopf bifurcation always exits for any $\tau \in [\tau_k, \tau_{k+1}]$. We show this numerically by the following Figure 7, where $\tau = 12.2, 12.4, 12.6, 12.8, 13$, and other parameter values are the same as those of Figure 5. The system presents periodic behaviors for all these chosen time delays.



(a) The distributions $R_*(a)$ at E_*

(b) The distributions $R(t, a)$

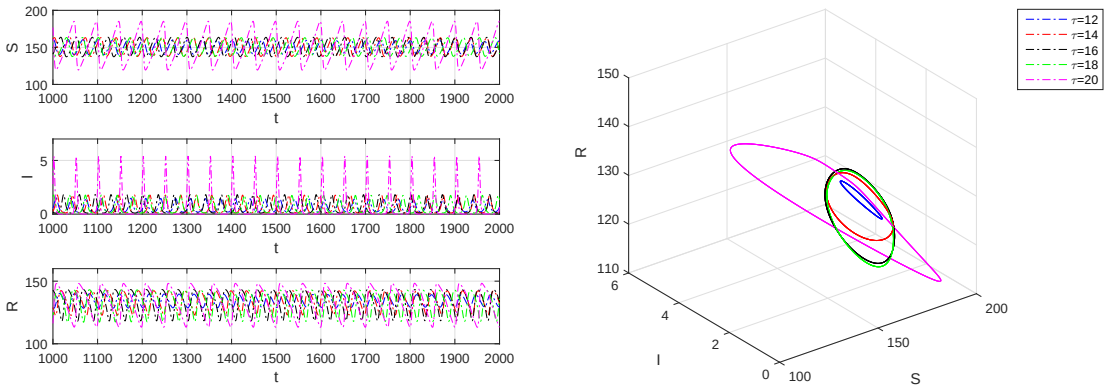
Figure 3. The distributions of recovery individuals when E_* is asymptotically stable under $R_0 > 1$ and $\tau = 0$.



(a) $\tau = 0, 2, 4$

(b) $\tau = 6, 8, 10$

Figure 4. The endemic equilibrium E_* of system (1.2) is asymptotically stable for $\tau \in [0, 10]$ when $R_0 > 1$, and $Q(\Theta) = 0$ has the positive root.



(a) The periodic solution (b) Phase plane

Figure 5. Periodic solutions occur via Hopf bifurcation when E_* loses its stability.

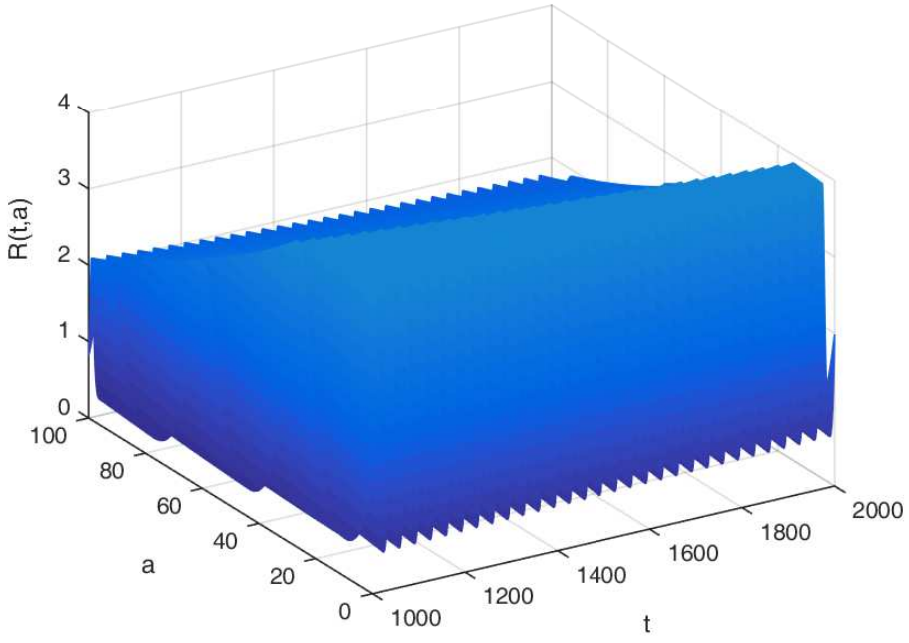


Figure 6. Periodic solutions of system (1.2) when E_* loses its stability.

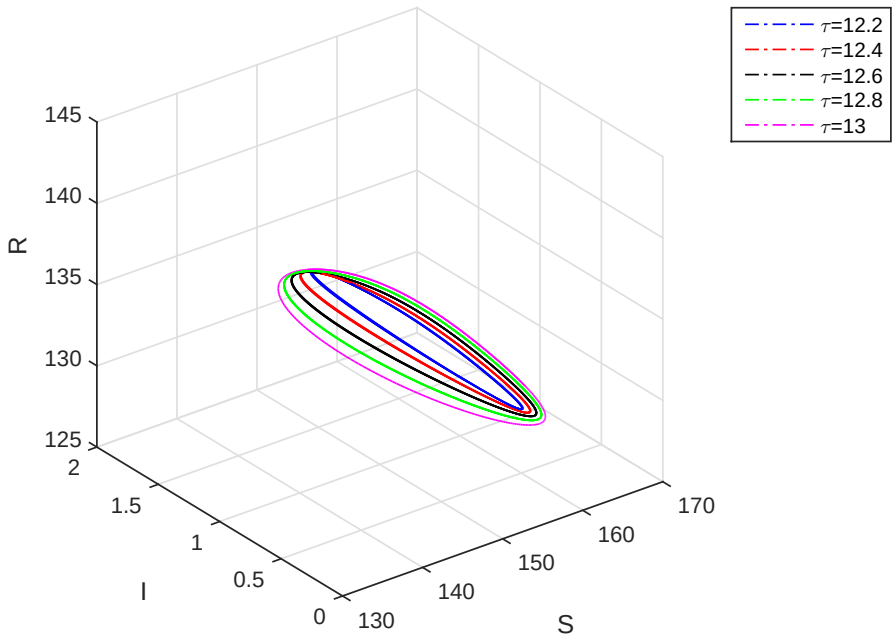


Figure 7. Hopf bifurcation around endemic equilibrium E_* exits for any $\tau \in [t_k, t_{k+1}]$.

Based on our analysis, we know that both non-periodic and periodic behaviors are possible when the disease persists in population. These findings are consistent with the results in [7, 11]. It means that the dynamical behaviors of recurrent epidemics is dependent on the parameters of system (1.2). In particular, immune age and time delay is an important effects on the transmission of recurrent epidemics. Once $R_0 > 1$, we can control how disease spreads through controlling the conditions theorem 4.3 and theorem 4.4. If the parameters of system (1.2) satisfy these conditions of theorem 4.3, or the conditions of theorem 4.4(i), the recurrent infectious disease is going to be a steady state as t goes to infinity. While if the parameters of system (1.2) satisfy these conditions of theorem 4.4(ii), the recurrent infectious disease will persist in the population in the form of periodic oscillations.

Although the simple delay differential equations model may show the switching between periodic and non-periodic behavior, it cannot express the immunity age of the recovery individual. In fact, the immunity age of the individual is important since the immunity age of the individual can be used to describe whether the recovered individual has the immunity or not. The model (1.2) may more accurately describe how the recurrent infectious disease spreads. In addition, our results also implies that model (1.2) may be closer to the transmission mechanism of the recurrent infectious disease. We suspect that the seasonality of recurrent epidemics is also related to age structure and the delay.

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Conflict of interest

The authors have declared that no competing interests exist.

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