



Research article

Influence of spatial heterogeneous environment on long-term dynamics of a reaction-diffusion $SVIR$ epidemic model with relapse

Cheng-Cheng Zhu^{1,2}, Jiang Zhu^{3,*} and Xiao-Lan Liu⁴

¹ School of Science, Jiangnan University, Wuxi, Jiangsu 214122, P.R. China

² School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, P.R. China

³ School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, Jiangsu 221116, P.R. China

⁴ School of Arts and Science, Suqian College, Suqian, Jiangsu 223800, P.R. China

* **Correspondence:** Email: jzhuccy@163.com; Tel: +865163403152.

Abstract: In this paper by adding the factors of disease relapse and vaccination in the space heterogeneous environment, we establish and discuss a class of reaction-diffusion $SVIR$ model with relapse and a varying external source in spatial heterogeneous environment. By applying a different method than the Lyapunov function, we study the long-term dynamic behavior of this model by means of global exponential attractor theory and gradient flow method. The global asymptotic stability and the persistence of epidemic are proved. To test the validity of our theoretical results, we choose some specific epidemic disease with some more practical and more definitive official data to simulate the global stability and exponential attraction of the model. The simulation results showed that the factors of disease relapse, vaccination and spatial heterogeneity had a great influence on the persists uniformly of the disease.

Keywords: reaction-diffusion; global exponential attract set; gradient flow; spatial heterogeneous environment; relapse

1. Introduction

The idea of invisible living creatures as agents of disease goes back at least to the writings of Aristotle (384 BC–322 BC). In the 16th century, it developed as a theory. The AIDS epidemic, the recent SARS epidemic, MERS, Ebola virus and outbursts of diseases such as the Zika virus are events of concern and interest to many people. The mechanism of transmission of infections is now known for most diseases. Some diseases transmitted by viral agents, such as influenza, measles, rubella and chicken pox, confer immunity against reinfection, others transmitted by bacteria, such as tuberculosis,

meningitis, and gonorrhoea, confer no immunity against reinfection. While malaria, is transmitted not directly from human to human but by vectors, usually insects that are infected by humans transmit the disease to humans. There are also diseases such as West Nile virus which are transmitted back and forth between animals and vectors [1]. The first known result in mathematical epidemiology is a defense of the practice of inoculation against smallpox in 1760 by Daniel Bernoulli. The first contributions to modern mathematical epidemiology are due to P.D. En'ko between 1873 and 1894 [2], then many scholars pay more attention to the global stability problem [3–7], bifurcation problem [8–11] and the optimal control problem [12, 13] of the disease.

Recently, reaction–diffusion equations have been used by many authors in epidemiology as well as virology. Researchers have established many types of model with diffusion such as SIR model [14, 15], SIS model [16], Lyme disease model [17, 18], malaria model [19] and so on. Some authors specifically study the global stability of the spread of diseases [20–27]; others discuss the traveling wave solutions about reaction–diffusion models [28–33]. The authors of [34] applied energy estimates in their proofs to discussed cholera epidemic.

In the theory of epidemiology, it has been recognized that environmental heterogeneity is a significant factor in the spread of infectious diseases. Constrained by factors such as altitude, temperature, humidity, latitude, climate, and living factors, the diffusion of epidemics in different environments is vastly different. In 2008, Allen et al. [35] proposed a simple diffusive SIS (susceptible–infected–susceptible) model with space-dependent disease transmission rate $\beta(x)$ and recovery rate $\gamma(x)$, given by

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S - \beta(x) \frac{SI}{S+I} + \gamma(x) I, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial t} = d_I \Delta I + \beta(x) \frac{SI}{S+I} - \gamma(x) I, & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ S(x, 0) = S_0(x), I(x, 0) = I_0(x), & x \in \Omega. \end{cases} \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^m and $S + I$ is the total number of individuals at $t = 0$. They studied the existence, uniqueness, stability of the disease-free equilibrium and studied the asymptotic behavior of the unique endemic equilibrium as d_S approaches zero. More precisely, their results implied that if the spatial environment can be modified to include low-risk sites and the movement of susceptible individuals can be restricted, then it may be possible to eliminate the infectious disease. In some subsequent work, Peng [36] discussed the asymptotic profiles of the steady states for the epidemic reaction–diffusion model (1.1). The global stability of the steady states of (1.1) was also investigated by Peng and Liu [37] in 2009. Four years later, Peng and Yi [38] studied the effects of epidemic risk and population movement on the spatiotemporal transmission of infectious disease via the SIS epidemic reaction–diffusion model (1.1). In 2011, Lou and Zhao [19] introduced a reaction–diffusion malaria model with incubation period in the vector population. Kuniya and Wang [39] gave the Lyapunov functions and global stability for a spatially diffusive SIR epidemic model for only one diffusion, for systems with multiple diffusions, the author did not get the method to construct the Lyapunov function. Deng and Wu [40] discussed the global dynamics of a SIS epidemic reaction–diffusion model, which

reads as follows:

$$\begin{cases} \frac{\partial S}{\partial t} = d_S \Delta S - \beta(x) S I + \gamma(x) I, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial t} = d_I \Delta I + \beta(x) S I - \gamma(x) I, & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ S(x, 0) = S_0(x), I(x, 0) = I_0(x), & x \in \Omega. \end{cases} \quad (1.2)$$

The authors only discussed the attractivity and global stability of the constant coefficient model, and constructed the Lyapunov function of the constant system. Wu and Zou [41] analyzed the asymptotic profiles of steady states for the model (1.2) with mass action infection mechanism. In 2017, Li et al. [42] studied the following *SIS* reaction-diffusion system with a linear external source:

$$\begin{cases} \frac{\partial S}{\partial t} - d_S \Delta S = \Lambda(x) - S - \beta(x) \frac{S I}{S+I} + \theta \gamma(x) I, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial t} - d_I \Delta I = \beta(x) \frac{S I}{S+I} - \gamma(x) I, & x \in \Omega, t > 0, \\ \frac{\partial S}{\partial \mathbf{n}} = \frac{\partial I}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ S(x, 0) = S_0(x), I(x, 0) = I_0(x), & x \in \Omega. \end{cases} \quad (1.3)$$

The authors showed the uniform bounds of solutions, established the threshold dynamics in terms of the basic reproduction number and discussed the global stability of the unique constant endemic equilibrium when spatial environment is homogeneous. In particular, the asymptotic profile of endemic equilibria is determined if the diffusion rate of the susceptible or infected population at 0 or ∞ . Since the Lyapunov function is difficult to construct, the authors did not discuss the global stability of endemic for heterogeneous models. In 2018, Cai et al. [43] investigated the dynamical outcomes of a host–parasite model incorporating both horizontal and vertical transmissions in a spatial heterogeneous environment. The work of Tong and Lei [44] concerns an *SIS* epidemic reaction–diffusion model. They also failed to construct Lyapunov functions suitable for spatial heterogeneous models. Therefore, the construction of the Lyapunov function of the spatial heterogeneous model is very difficult. In discussing the long-term dynamic behavior of disease in space, we also need to try to find other methods.

On the other hand, many diseases such as hepatitis, malaria, rabies, tuberculosis and HIV/AIDS will produce relapse after being treatment. We also know that vaccination is an important means of preventing infection and relapse of these diseases. Therefore, the effect of spatial factors on relapse and vaccination should be taken into account. Inspired by the above literature, in this manuscript, we study an *SVIR* epidemic reaction-diffusion model with relapse and a varying external source in spatially heterogeneous environment. This model has the following significant differences and difficulties compared to the above models. First, after joining the relapse, the coupling between the equations in the system is stronger. Thus, the method to prove the positivity of the solution in the prior literature can no longer be used in the coupled model. Second, because the coefficients in the model are all spatially heterogeneous and the number of equations increases, so the Lyapunov function method, which proves the global asymptotic stability of the system, is also difficult to achieve due to the increased technical difficulty of construction. For this reason, we turn to discuss the global dynamics of the system by using the attractor theory of infinite dimensional dynamical systems. It is well-known that the global attractor is an important theory to describe the long-term dynamic behavior of dissipative infinite-dimensional dynamical systems [45–54]. Finally, we conduct a large number of numerical

simulations. From the simulation we can see that spatial heterogeneity has a great influence on the long-term dynamic behavior of epidemics. Reducing the relapse rate and increasing the vaccination rate can effectively control the spread of the disease.

The organization of this paper is as follows. In Section 2, we present the model under some assumption. Then by using general comparison principle of the quasi-monotone nonlinear reaction-diffusion system and the comparison principle of parabolic equation, we prove the existence of the positive solutions. We also give the uniform boundedness and the existence of global solutions to our system. In Section 3, we prove the existence theorem of global exponential attract set for our real model and apply this result to discuss the global stability and the persistence of the infection disease. That is, by applying the property of global attractor and gradient flow method, we prove that the non-constant disease-free equilibrium is globally asymptotically stable and the endemic is persisting uniformly under some new threshold conditions. To test the validity of our theoretical results, we choose some specific epidemic disease with some more practical and more definitive official data to simulate the global stability and exponential attraction of the model. The stability and global exponential attractiveness of both constant and non-constant equilibria are simulated in Section 4. In Section 5, we give our conclusions and some discussions.

2. Positive and uniform boundedness of solutions to a reaction-diffusion $SVIR$ epidemic model with relapse in heterogeneous environment

In this section, we consider a reaction-diffusion $SVIR$ model with relapse and spatially heterogeneous environment. Our model is divided into four compartments, namely the susceptible individuals (S), vaccinated individuals (V), infected individuals (I) and temporary restorers (R). The transfer diagram and parameters description as shown below:

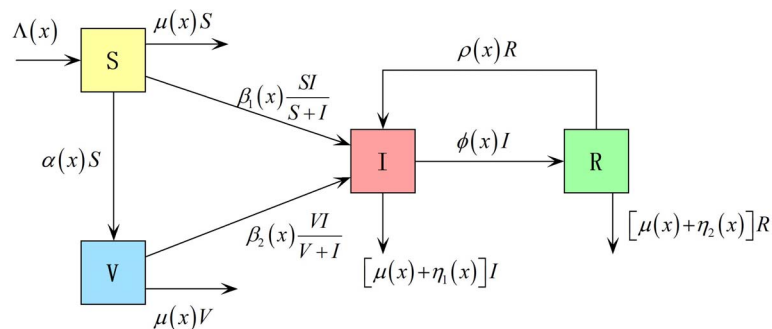


Figure 1. Transfer diagram for the $SVIR$ model with relapse and spatially heterogeneous environment.

Table 1. State variables and parameters of *S VIR* model.

Parameter	Description
$S(x, t)$	Density of susceptible individuals at location x and time t .
$V(x, t)$	Density of vaccinated individuals at location x and time t .
$I(x, t)$	Density of infected individuals at location x and time t .
$R(x, t)$	Density of temporary restorers at location x and time t .
$\Lambda(x)$	Total recruitment scale into this homogeneous social mixing community at location x .
$\alpha(x)$	Vaccination rate at location x .
$\beta_i(x), i = 1, 2$	Effective transmission rate (average number of effective interactions per susceptible, vaccination individuals and infected individuals) at location x .
$\rho(x)$	Relapse rate at location x .
$\phi(x)$	Per-capita recovery (treatment) rate at location x .
$\mu(x)$	Natural mortality rate at location x .
$\eta_i(x), i = 1, 2$	Disease-related death rate at location x .

From Figure 1, the following system with the initial-boundary-value conditions is constructed by:

$$\begin{cases}
 \frac{\partial S}{\partial t} = d_S \Delta S + \Lambda(x) - \beta_1(x) \frac{SI}{S+I} - [\mu(x) + \alpha(x)] S, & x \in \Omega, t > 0, \\
 \frac{\partial V}{\partial t} = d_V \Delta V + \alpha(x) S - \beta_2(x) \frac{VI}{V+I} - \mu(x) V, & x \in \Omega, t > 0, \\
 \frac{\partial I}{\partial t} = d_I \Delta I + \beta_1(x) \frac{SI}{S+I} + \beta_2(x) \frac{VI}{V+I} + \rho(x) R - [\mu(x) + \eta_1(x) + \phi(x)] I, & x \in \Omega, t > 0, \\
 \frac{\partial R}{\partial t} = d_R \Delta R + \phi(x) I - [\rho(x) + \mu(x) + \eta_2(x)] R, & x \in \Omega, t > 0, \\
 \frac{\partial S}{\partial n} = \frac{\partial V}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial R}{\partial n} = 0, & x \in \partial\Omega, t > 0, \\
 S(x, 0) = S_0(x), V(x, 0) = V_0(x), I(x, 0) = I_0(x), R(x, 0) = R_0(x), & x \in \Omega.
 \end{cases} \quad (2.1)$$

Here, Ω is a bounded domain in \mathbb{R}^m ($m \geq 1$) with smooth boundary $\partial\Omega$ (when $m > 1$), $d_S, d_V, d_I, d_R > 0$ are diffusion coefficients and $\alpha(x), \beta_1(x), \beta_2(x), \rho(x), \phi(x), \mu(x), \eta_1(x), \eta_2(x)$ and $\Lambda(x)$ are positive Hölder continuous functions on accounting for the rates of vaccination, disease transmission, relapse, recovery, natural death, disease-related death and total recruitment scale respectively. Neumann boundary conditions $\frac{\partial S}{\partial n} = \frac{\partial V}{\partial n} = \frac{\partial I}{\partial n} = \frac{\partial R}{\partial n} = 0$ denotes that the change rate on the boundary of the region Ω is equal to 0. It is straightforward to verify that $\frac{SI}{S+I} \left(\frac{VI}{V+I} \right)$ is a Lipschitz continuous function of S (V) and I in the open first quadrant. Therefore, we can extend it to the entire first quadrant by defining it to be zero whenever $S = 0$ ($V = 0$) or $I = 0$. Throughout the paper, we assume that the initial value $S_0(x), V_0(x), I_0(x)$ and $R_0(x)$ are nonnegative continuous functions on $\bar{\Omega}$, and the number of infected individuals is positive, i.e., $\int_{\Omega} I_0(x) dx > 0$. Specific parameters described in Table 1.

With the development of science and technology and the improvement of medical standards, vaccines play an important role in the prevention and control of disease transmission. So we considered a group of people who were vaccinated with preventive vaccines in the model (2.1). In the model, I denotes the infected person, and R denotes the person who is temporarily recovered by treatment after the infection but may also relapse. These two groups of people carry the disease-causing virus, so we need to consider their disease-induced rates. S denotes a susceptible population, and V denotes an

uninfected person who has been vaccinated. They are not ill before they are exposed to pathogenic bacteria or infected people, so we only need to consider the natural mortality of these two types of people. There are many diseases in real life that have such properties, such as hepatitis B, smallpox, etc.,

In the following, we discuss the positivity and boundedness of the solution of system (2.1). In many articles, the positivity, boundedness, and existence of solutions of the model have been discussed, such as models that lose immunity in [55–57]. The authors in [55] construct a model of ordinary differential equations, which is naturally less difficult for the proof of positive, boundedness, and existence of solutions. For example, the positive problem of the solution can be easily obtained through contradiction analysis even without using the comparison principle. The model in our manuscript is diffused, and the difficulty of analysis is naturally higher than that of the ODE model. Models in [56] and [57] are diffusion models, and the model of [57] assumes that all diffusion coefficients are equal at the time of construction. In [56], when discussing the existence of the solution of the model (Theorem 2.1 in [56]), the condition that the diffusion coefficients are equal is added. We all know that it is very easy to add the equations in the system when the diffusion coefficients are equal, and then researchers can discuss positivity, boundedness and existence of the solution for this equation about the rate of change of the total population. However, the model diffusion coefficients in our manuscripts are different, we refuse to add any conditions to the diffusion coefficient from beginning to end. Therefore, from the perspective of the diffusion system, our model is more general. These are places where our models and discussion methods differ from other existing results. Firstly, we give an comparison theorem for semilinear parabolic differential equation that we need to use later.

Denote

$$\begin{aligned} Q_T &= \Omega \times (0, T], \quad S_T = \partial\Omega \times (0, T], \\ \mathcal{L}_i &\equiv - \sum_{j,k=1}^n a_{jk}^{(i)}(x) \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^n b_j^{(i)}(x) \frac{\partial}{\partial x_j}, \\ B_i u_i &= u_i \text{ or } B_i u_i = \frac{\partial u_i}{\partial \mathbf{n}} + b_i(x) u_i, \end{aligned}$$

where \mathbf{n} is the unit outside normal vector of $\partial\Omega$ and $b_i(x) \geq 0$. The following lemma is similar to the Theorem 4.2.9 in Chinese books [58], but for the convenience of the reader, we give its conclusions and a brief proof.

Lemma 2.1. *Let $u, v \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$, satisfy*

$$\begin{cases} \mathcal{L}_i u_i - f_i(x, t, u) \geq \mathcal{L}_i v_i - f_i(x, t, v), & (x, t) \in Q_T, \\ B_i u_i \geq B_i v_i, & (x, t) \in S_T, \\ u_i(x, 0) \geq v_i(x, 0), & i = 1, 2, \dots, m, \end{cases}$$

and

$$\frac{\partial f_i}{\partial u_j} \geq 0, \quad i \neq j, \quad i, j = 1, 2, \dots, m.$$

Then

$$(1) \quad u(x, t) \geq v(x, t) \text{ in } Q_T.$$

(2) If there exist $(x_0, t_0) \in Q_T$ and $\mu : 0 \leq \mu \leq m$, such that $(u - v)_\mu(x_0, t_0) = 0$, then for any $(x, t) \in \overline{Q_{t_0}}$, there has $(u - v)_\mu(x, t) = 0$.

Proof. Let $w = u - v$, then by using differential mean value theorem for multivariate functions

$$\begin{aligned} & \mathcal{L}_i u_i - \mathcal{L}_i v_i - [f_i(x, t, u) - f_i(x, t, v)] \\ &= \mathcal{L}_i w_i - [f_i(x, t, u_1, u_2, u_3, \dots, u_{m-1}, u_m) - f_i(x, t, v_1, u_2, u_3, \dots, u_{m-1}, u_m) \\ & \quad + f_i(x, t, v_1, u_2, u_3, \dots, u_{m-1}, u_m) - f_i(x, t, v_1, v_2, u_3, \dots, u_{m-1}, u_m) \\ & \quad + \dots + f_i(x, t, v_1, v_2, v_3, \dots, v_{m-1}, u_m) - f_i(x, t, v_1, v_2, v_3, \dots, v_{m-1}, v_m)] \\ &= \mathcal{L}_i w_i - \sum_{j=1}^m \frac{\partial f_i}{\partial u_j} (x, t, v_1, \dots, v_{j-1}, v_j + \xi_j (u_j - v_j), v_{j+1}, \dots, u_m) (u_j - v_j) \\ &=: \mathcal{L}_i w_i + \sum_{j=1}^m h_{ij} w_j \geq 0, \quad i = 1, 2, \dots, m. \end{aligned}$$

In view of $\frac{\partial f_i}{\partial u_j} \geq 0, i \neq j, i, j = 1, 2, \dots, m$, therefore, $h_{ij} = -\frac{\partial f_i}{\partial u_j} \leq 0$. If $x \in \overline{\Omega}$, we have $w(x, 0) \geq 0$ and $B_i w_i|_{S_T} \geq 0, i = 1, 2, \dots, m$. Then by the strong maximum principle in the form of equation derivatives, we can prove that $w(x, t) \geq 0$, that is $u(x, t) \geq v(x, t)$ in Q_T . Finally, if there exist $(x_0, t_0) \in Q_T$ and $\mu : 0 \leq \mu \leq m$, such that $w_\mu(x_0, t_0) = 0$, then for any $(x, t) \in \overline{Q_{t_0}}$, there has $w_\mu(x, t) = 0$, that is $(u - v)_\mu(x, t) = 0$. \square

The above lemma can also be derived indirectly from [59, Theorem 7.3.4]. Let $\mathbf{X} := \mathbf{L}^2(\Omega)$ be a Banach space, L_1, L_2, L_3 and L_4 be a linear operator on $D(L) = (D(L_1), D(L_2), D(L_3), D(L_4)) \subset \mathbf{X}$ defined by

$$\begin{aligned} L_1 S(x) &:= d_S \Delta S(x), \quad L_2 V(x) := d_V \Delta V(x), \\ L_3 I(x) &:= d_I \Delta I(x), \quad L_4 R(x) := d_R \Delta R(x), \\ D(L_1) &:= \left\{ S \in \mathbf{H}^2(\Omega); \frac{\partial S}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \right\}, \\ D(L_2) &:= \left\{ V \in \mathbf{H}^2(\Omega); \frac{\partial V}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \right\}, \\ D(L_3) &:= \left\{ I \in \mathbf{H}^2(\Omega); \frac{\partial I}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \right\}, \\ D(L_4) &:= \left\{ R \in \mathbf{H}^2(\Omega); \frac{\partial R}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \right\}. \end{aligned}$$

Then, we can see that L_1, L_2, L_3, L_4 are respectively the infinitesimal generator of strongly continuous semigroup $\{e^{tL_1}\}_{t \geq 0}, \{e^{tL_2}\}_{t \geq 0}, \{e^{tL_3}\}_{t \geq 0}, \{e^{tL_4}\}_{t \geq 0}$ in $\mathbf{H}^2(\Omega)$. Let $(\mathbf{H}^2(\Omega))^4 = \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega) \times \mathbf{H}^2(\Omega)$ and define $L : (\mathbf{H}^2(\Omega))^4 \rightarrow (\mathbf{H}^2(\Omega))^4, \forall u := (S, V, I, R) \in (\mathbf{H}^2(\Omega))^4$, then

$$L(S, V, I, R)(x) := \begin{pmatrix} L_1 S(x) \\ L_2 V(x) \\ L_3 I(x) \\ L_4 R(x) \end{pmatrix}, \quad D(L) := D(L_1) \times D(L_2) \times D(L_3) \times D(L_4)$$

is also the infinitesimal generator of a strongly continuous semigroup $\{e^{tL}\}_{t \geq 0}$ in $\mathbf{Y} := (\mathbf{H}^2(\Omega))^4$, where

$$e^{tL} = \begin{pmatrix} e^{tL_1} \\ e^{tL_2} \\ e^{tL_3} \\ e^{tL_4} \end{pmatrix} \text{ and } e^{tL}u = \begin{pmatrix} e^{tL_1}S \\ e^{tL_2}V \\ e^{tL_3}I \\ e^{tL_4}R \end{pmatrix}.$$

Note that \mathbf{Y} is a Banach space equipped with norm

$$\|(S, V, I, R)^T\|_{\mathbf{Y}} := \max \{ \|S\|_{\mathbf{H}^2(\Omega)}, \|V\|_{\mathbf{H}^2(\Omega)}, \|I\|_{\mathbf{H}^2(\Omega)}, \|R\|_{\mathbf{H}^2(\Omega)} \}.$$

Let G be a nonlinear operator on \mathbf{Y} defined by

$$G(S, V, I, R)(x) := (g_1(S, V, I, R)(x), g_2(S, V, I, R)(x), g_3(S, V, I, R)(x), g_4(S, V, I, R)(x))^T,$$

where

$$\begin{aligned} g_1(S, V, I, R)(x) &= \Lambda(x) - \beta_1(x) \frac{SI}{S+I} - [\mu(x) + \alpha(x)]S, \\ g_2(S, V, I, R)(x) &= \alpha(x)S - \beta_2(x) \frac{VI}{V+I} - \mu(x)V, \\ g_3(S, V, I, R)(x) &= \beta_1(x) \frac{SI}{S+I} + \beta_2(x) \frac{VI}{V+I} + \rho(x)R - [\mu(x) + \eta_1(x) + \phi(x)]I, \\ g_4(S, V, I, R)(x) &= \phi(x)I - [\rho(x) + \mu(x) + \eta_2(x)]R. \end{aligned}$$

Let $F = (f_1, f_2, f_3, f_4)$, where

$$f_i(S, V, I, R) = \begin{cases} g_i(S, V, I, R)(x), & S, V, I, R > 0, \\ 0, & \text{others,} \end{cases} \quad i = 1, 2, 3, 4.$$

Using the operators L and F defined by above, we can rewrite the problem (2.1) into the following abstract form in \mathbf{Y} .

$$\frac{d}{dt}u(t) = Lu(t) + F(u(t)), \quad u(t) := \begin{pmatrix} S(\cdot, t) \\ V(\cdot, t) \\ I(\cdot, t) \\ R(\cdot, t) \end{pmatrix}, \quad u(0) := \begin{pmatrix} S_0(\cdot) \\ V_0(\cdot) \\ I_0(\cdot) \\ R_0(\cdot) \end{pmatrix}. \quad (2.2)$$

Because L is the generator of a C_0 -semigroup of contractions on \mathbf{Y} and F is locally Lipschitz with respect to u , apply [60, Corollary 11.1.2] and [60, Corollary 11.3.1], we know that there exists at least one saturated solution $u : [0, T) \rightarrow \mathbf{Y}$, which is a classical solution of system (2.2), such that

$$u(t) = e^{tL}u_0 + \int_0^t e^{(t-s)L}F(u(s)) ds. \quad (2.3)$$

Moreover, if $S(x, 0), V(x, 0), I(x, 0), R(x, 0) \in C^2(\Omega)$, then $u(x, t) \in C^{2,1}(\Omega \times (0, T])$.

In the following, we discuss the positivity of the solution of system (2.2) and the existence of the solution of system (2.1).

Theorem 2.2. *Assume that $(S(x, t), V(x, t), I(x, t), R(x, t))$ be a solution of system (2.2), and $S(x, t), V(x, t), I(x, t), R(x, t) \in C(\bar{\Omega} \times (0, T]) \cap C^{2,1}(\Omega \times (0, T])$ where T is the maximal existing time. If $S(x, 0) > 0, V(x, 0) > 0, I(x, 0) > 0, R(x, 0) > 0$ for $x \in \Omega$, then the solution $S(x, t) > 0, V(x, t) > 0, I(x, t) > 0, R(x, t) > 0, (x, t) \in \Omega \times (0, T]$ and $u(t) = (S(x, t), V(x, t), I(x, t), R(x, t))$ is a solution of system (2.1).*

Proof. We denote that

$$\begin{aligned}\widetilde{f}_1(S, V, I, R) &= \begin{cases} -\beta_1(x) \frac{SI}{S+I} - [\mu(x) + \alpha(x)] S, & S, V, I, R > 0, \\ 0, & \text{others,} \end{cases} \\ \widetilde{f}_2(S, V, I, R) &= \begin{cases} -\beta_2(x) \frac{VI}{V+I} - \mu(x) V, & S, V, I, R > 0, \\ 0, & \text{others.} \end{cases}\end{aligned}$$

It is obviously that $\widetilde{f}_1(S, V, I, R) \leq f_1(S, V, I, R)$ and 0 is a solution of equation

$$\begin{cases} \frac{\partial S}{\partial t} - d_S \Delta S = \widetilde{f}_1(S, V, I, R), \\ \frac{\partial S}{\partial \mathbf{n}} = 0, \\ S(x, 0) = 0. \end{cases}$$

If S is the solution of the equation

$$\begin{cases} \frac{\partial S}{\partial t} - d_S \Delta S = f_1(S, V, I, R), \\ \frac{\partial S}{\partial \mathbf{n}} = 0, \\ S(x, 0) = S_0(x) > 0. \end{cases}$$

then by using [61, Theorem 2.2.1], we get $S(x, t) > 0$. Similarly, it is obviously that $\widetilde{f}_2(S, V, I, R) \leq f_2(S, V, I, R)$ and 0 is a solution of equation

$$\begin{cases} \frac{\partial V}{\partial t} - d_V \Delta V = \widetilde{f}_2(S, V, I, R), \\ \frac{\partial V}{\partial \mathbf{n}} = 0, \\ V(x, 0) = 0. \end{cases}$$

If V is the solution of the equation

$$\begin{cases} \frac{\partial V}{\partial t} - d_V \Delta V = f_2(S, V, I, R), \\ \frac{\partial V}{\partial \mathbf{n}} = 0, \\ V(x, 0) = V_0(x) > 0. \end{cases}$$

then by using [61, Theorem 2.2.1], we get $V(x, t) > 0$. At this time, S and V are just the solutions of the first and the second equations system (2.2). Then for the third and the fourth equations of system (2.2), we know that

$$\begin{aligned}\frac{\partial f_3}{\partial R} &= \begin{cases} \rho(x), & S, V, I, R > 0, \\ 0, & \text{others,} \end{cases} \\ \frac{\partial f_4}{\partial I} &= \begin{cases} \phi(x), & S, V, I, R > 0, \\ 0, & \text{others,} \end{cases}\end{aligned}$$

therefore, $\frac{\partial f_3}{\partial R} \geq 0$, $\frac{\partial f_4}{\partial I} \geq 0$. By the comparison principle Lemma 2.1, we can obtain $I(x, t) > 0$ and $R(x, t) > 0$. Because under the condition of $S, V, I, R > 0$, we have $F = G$, therefore, $u(t)$ is also the solution of (2.1). \square

From now on, for any given continuous function f on $\overline{\Omega}$, we denote

$$f^* = \max_{x \in \Omega} f(x) \text{ and } f_* = \min_{x \in \Omega} f(x).$$

In the following, we need to give the Gronwall type differential inequality. Moreover, we do not make assumptions about the positivity of all the functions that appear in the lemma. This lemma will play an important role in the proof of the following lemmas and theorems.

Lemma 2.3. $u(t), c(t), h(t)$ are three real functions defined on $[a, T]$, $u(t)$ is absolutely continuous on $[a, T]$, $c(t)$ is integrable on $[a, T]$. If the following inequality holds,

$$\frac{d}{dt}u(t) + c(t)u(t) \leq h(t), \text{ a.e. } t \in [a, T], \quad (2.4)$$

then

$$u(t) \leq e^{-\int_a^t c(\tau)d\tau}u(a) + \int_a^t e^{-\int_s^t c(\tau)d\tau}h(s)ds.$$

Theorem 2.4. Assume that $(S(x, t), V(x, t), I(x, t), R(x, t))$ be a solution of system (2.1), and $S(x, t), V(x, t), I(x, t), R(x, t) \in \mathbf{C}(\bar{\Omega} \times (0, T]) \cap \mathbf{C}^{2,1}(\Omega \times (0, T])$ where T is the maximal existing time. If $S(x, 0) > 0, V(x, 0) > 0, I(x, 0) > 0, R(x, 0) > 0$ for $x \in \Omega$, then the solution $(S(x, t), V(x, t), I(x, t), R(x, t))$ uniformly bounded on $\bar{\Omega}$.

Proof. We consider the following total population at time t . Define

$$U(t) = \int_{\Omega} [S(x, t) + V(x, t) + I(x, t) + R(x, t)] dx.$$

From system (2.1), it is easy to see that

$$\begin{aligned} \frac{dU(t)}{dt} &= \int_{\Omega} \left[\frac{\partial}{\partial t}S(x, t) + \frac{\partial}{\partial t}V(x, t) + \frac{\partial}{\partial t}I(x, t) + \frac{\partial}{\partial t}R(x, t) \right] dx \\ &= \int_{\Omega} \{d_S \Delta S + d_V \Delta V + d_I \Delta I + d_R \Delta R + \Lambda(x) - \mu(x)S \\ &\quad - \mu(x)V - [\mu(x) + \eta_1(x)]I - [\mu(x) + \eta_2(x)]R\} dx \\ &\leq d_S \int_{\Omega} \Delta S dx + d_V \int_{\Omega} \Delta V dx + d_I \int_{\Omega} \Delta I dx + d_R \int_{\Omega} \Delta R dx \\ &\quad + \int_{\Omega} \{\Lambda^* - \mu_* [S + V + I + R]\} dx \\ &= d_S \int_{\partial\Omega} \frac{\partial S}{\partial \mathbf{n}} ds + d_V \int_{\partial\Omega} \frac{\partial V}{\partial \mathbf{n}} ds + d_I \int_{\partial\Omega} \frac{\partial I}{\partial \mathbf{n}} ds + d_R \int_{\partial\Omega} \frac{\partial R}{\partial \mathbf{n}} ds \\ &\quad + \Lambda^* |\Omega| - \mu_* U(t) \\ &= \Lambda^* |\Omega| - \mu_* U(t). \end{aligned}$$

By Lemma 2.3, we can obtain that

$$U(t) \leq U(0)e^{-\mu_* t} + \frac{\Lambda^* |\Omega|}{\mu_*} (1 - e^{-\mu_* t}).$$

So $U(t) \leq \max \left\{ U(0), \frac{\Lambda^* |\Omega|}{\mu_*} \right\}$, where

$$\begin{aligned} U(0) &= \int_{\Omega} [S(x, 0) + V(x, 0) + I(x, 0) + R(x, 0)] dx \\ &\leq \int_{\Omega} \|S(x, 0) + V(x, 0) + I(x, 0) + R(x, 0)\|_{\infty} dx \end{aligned}$$

$$= \|S(x, 0) + V(x, 0) + I(x, 0) + R(x, 0)\|_{\infty} |\Omega|.$$

This shows that $U(t) = \int_{\Omega} (S + V + I + R) dx$ is bounded. By Theorem 2.2, we obtain that

$$\begin{aligned} \|S + V + I + R\|_{L^1(\Omega)} &= \int_{\Omega} |S + V + I + R|(x, t) dx \\ &= \int_{\Omega} (S + V + I + R)(x, t) dx \\ &\leq \max \left\{ \|S(x, 0) + V(x, 0) + I(x, 0) + R(x, 0)\|_{\infty} |\Omega|, \frac{\Lambda^* |\Omega|}{\mu_*} \right\}. \end{aligned}$$

We denote that $K = \max \left\{ \|S(x, 0) + V(x, 0) + I(x, 0) + R(x, 0)\|_{\infty} |\Omega|, \frac{\Lambda^* |\Omega|}{\mu_*} \right\}$, then we know

$$\int_{\Omega} (S + V + I + R) dx \leq K.$$

In view of [62, Lemma 2.1], there exists a positive constant K^* depending on K such that

$$\|S + V + I + R\|_{L^{\infty}(\Omega)} \leq K^*.$$

Thus, we can obtain that $S(x, t), V(x, t), I(x, t), R(x, t)$ are uniformly bounded on $\overline{\Omega}$. This implies the boundedness of the solution of system (2.1). \square

By [60, Corollary 11.3.2] we can deduce that the saturated solution $u(t) = (S(x, t), V(x, t), I(x, t), R(x, t))$ is either global or $\lim_{t \rightarrow +\infty} \|u(t)\|_Y = +\infty$. From Theorem 2.4 we know that $u(t)$ is bounded, hence, $u(t)$ is a global solution.

3. Threshold dynamics of the reaction-diffusion *SVIR* epidemic model with relapse in heterogeneous environment

In this section, we will analyze the qualitative behavior of system (2.1). It is clearly seen that system (2.1) admits a disease-free equilibrium $E^0(x) = (S^0(x), V^0(x), 0, 0)$, then we prove that the existence of principal eigenvalues of system (2.1). Linearizing the second and the third equations of (2.1) at disease-free equilibrium, we get

$$\begin{cases} \frac{\partial I}{\partial t} = d_I \Delta I + [\beta_1(x) + \beta_2(x)] I + \rho(x) R - [\mu(x) + \eta_1(x) + \phi(x)] I, & x \in \Omega, t > 0, \\ \frac{\partial R}{\partial t} = d_R \Delta R + \phi(x) I - [\rho(x) + \mu(x) + \eta_2(x)] R, & x \in \Omega, t > 0, \\ \frac{\partial I}{\partial \mathbf{n}} = \frac{\partial R}{\partial \mathbf{n}} = 0, & x \in \partial \Omega, t > 0. \end{cases} \quad (3.1)$$

Let $I = e^{\lambda t} \varphi(x)$, $R = e^{\lambda t} \psi(x)$, equations (3.1) can be rewritten as

$$\begin{cases} d_I \Delta \varphi(x) + [\beta_1(x) + \beta_2(x) - (\mu(x) + \eta_1(x) + \phi(x))] \varphi(x) + \rho(x) \psi(x) = \lambda \varphi(x), & x \in \Omega, \\ d_R \Delta \psi(x) + \phi(x) \varphi(x) - [\rho(x) + \mu(x) + \eta_2(x)] \psi(x) = \lambda \psi(x), & x \in \Omega, \\ \frac{\partial \varphi}{\partial \mathbf{n}} = \frac{\partial \psi}{\partial \mathbf{n}} = 0, & x \in \partial \Omega. \end{cases} \quad (3.2)$$

Denote $\Phi(x) = (\varphi(x), \psi(x))^T$, $D = \begin{bmatrix} d_I & 0 \\ 0 & d_R \end{bmatrix}$ and

$$M(x) = (m_{ij}(x)) = \begin{bmatrix} \beta_1(x) + \beta_2(x) - (\mu(x) + \eta_1(x) + \phi(x)) & \rho(x) \\ \phi(x) & -[\rho(x) + \mu(x) + \eta_2(x)] \end{bmatrix},$$

where $m_{ij}(x) \geq 0$, $i \neq j$, $x \in \bar{\Omega}$. Therefore, equations (3.2) can be rewritten as

$$\begin{cases} \lambda \Phi(x) = D \Delta \Phi(x) + M(x) \Phi(x), & x \in \Omega, \\ \frac{\partial \Phi}{\partial n} = 0, & x \in \partial \Omega \end{cases} \quad (3.3)$$

By [59, Theorem 7.6.1], we can obtain that there exists a real eigenvalue λ^* of Eq (3.3) and a corresponding eigenvector $\Phi^*(x)$ satisfying $\Phi^*(x) \gg 0$ for all $x \in \Omega$ in the case of Neumann boundary conditions.

3.1. Long-term dynamics of the reaction-diffusion SVIR epidemic model

Before discussing the global dynamics of the reaction-diffusion SVIR model with relapse and spatial heterogeneous in detail, we need to recall some concept and existing results of exponential attractor and measure of noncompactness.

Let \mathbf{X} be a Banach space with the decomposition

$$\mathbf{X} = \mathbf{X}_1 \oplus \mathbf{X}_2; \dim \mathbf{X}_1 < \infty$$

and denote orthogonal projector by $P : \mathbf{X} \rightarrow \mathbf{X}_1$ and $(I - P) : \mathbf{X} \rightarrow \mathbf{X}_2$. In addition, let $\{Q(t)\}_{t \geq 0}$ be a continuous semigroup on \mathbf{X} . The following Condition (C*) introduced in [53] is an important condition for verifying the global exponential attraction.

Condition (C*): For any bounded set $\mathbf{B} \subset \mathbf{X}$ there exist positive constants $t_{\mathbf{B}}$, C and α such that for any $\varepsilon > 0$ there exists a finite dimensional subspace $\mathbf{X}_1 \subset \mathbf{X}$ satisfies

$$\{\|PQ(t)\mathbf{B}\|\}_{t \geq t_{\mathbf{B}}} \text{ is bounded,}$$

$$\|(I - P)Q(t)\mathbf{B}\| < Ce^{-\alpha t} + \varepsilon \text{ for } t \geq t_{\mathbf{B}},$$

where $P : \mathbf{X} \rightarrow \mathbf{X}_1$ is a orthogonal projector.

In the following, we denote that $\mathbf{H} = L^2(\Omega) \cap C^{2,1}(\Omega)$, $\mathbf{H}_1 = \mathbf{H}_0^1(\Omega) \cap C^{2,1}(\Omega)$, $\mathbf{H}^4 = \mathbf{H} \times \mathbf{H} \times \mathbf{H} \times \mathbf{H}$ and $\mathbf{H}_1^4 = \mathbf{H}_1 \times \mathbf{H}_1 \times \mathbf{H}_1 \times \mathbf{H}_1$. Note that \mathbf{H}^4 and \mathbf{H}_1^4 are Banach spaces equipped with norm

$$\|(S, V, I, R)^T\|_{\mathbf{H}^4} := \max \{\|S\|_{\mathbf{H}}, \|V\|_{\mathbf{H}}, \|I\|_{\mathbf{H}}, \|R\|_{\mathbf{H}}\}.$$

and

$$\|(S, V, I, R)^T\|_{\mathbf{H}_1^4} := \max \{\|S\|_{\mathbf{H}_1}, \|V\|_{\mathbf{H}_1}, \|I\|_{\mathbf{H}_1}, \|R\|_{\mathbf{H}_1}\}.$$

Definition 3.1. ([49, Definition A.2.1]) A mapping $L + G : \mathbf{H}_1 \rightarrow \mathbf{H}$ is called a gradient-type operator if there exists a C^1 functional $F : \mathbf{H}_1 \rightarrow \mathbf{R}^1$ and for some constant $C > 0$ such that $DF(u) \subset \mathbf{H}$ for every $u \in \mathbf{H}_\beta$ for some $\beta \leq 1$, and

$$\langle DF(u), Lu + G(u) \rangle_{\mathbf{H}} \leq -C_1 \|DF(u)\|_{\mathbf{H}}^2, \quad \forall u \in \mathbf{H}_1,$$

$$DF(u_0) = 0 \Leftrightarrow Lu_0 + G(u_0) = 0.$$

In this case, F is called the energy functional of $L + G$.

Theorem 3.2. System (2.1) exists a global exponential attract set \mathcal{A}^* , it exponential attracts any bounded set in \mathbf{H}^4 .

Proof. It is known by the existence of the global solution that if $\forall \varphi = u(0) = (S_0(x), V_0(x), I_0(x), R_0(x))^T \in \mathbf{H}_1^4$, then system (2.1) has a global solution $u = (S, V, I, R)^T \in \mathbf{C}^0([0, \infty), \mathbf{H}^4)$. It means that system (2.1) generates a operator semigroup $Q(t) = (Q_1(t), Q_2(t), Q_3(t), Q_4(t))^T$ and $Q(t)\varphi = u(t, \varphi)$.

We first prove that the operator semigroup $Q(t)$ has an absorbing set $\mathbf{B}_{\bar{R}} \subset \mathbf{H}^4$. It follows from the inner product of the first equation of system (2.1) in \mathbf{H} with S , we have that

$$\begin{aligned} & \left\langle d_S \Delta S + \Lambda(x) - \beta_1(x) \frac{SI}{S+I} - [\mu(x) + \alpha(x)] S, S \right\rangle_{\mathbf{H}} \\ &= d_S \int_{\Omega} S \Delta S dx + \int_{\Omega} \Lambda(x) S dx - \int_{\Omega} \beta_1(x) \frac{S^2 I}{S+I} dx - \int_{\Omega} [\mu(x) + \alpha(x)] S^2 dx \\ &\leq -d_S \|S\|_{\mathbf{H}_{\frac{1}{2}}}^2 + \int_{\Omega} \Lambda(x) S dx \\ &\leq -d_S \|S\|_{\mathbf{H}_{\frac{1}{2}}}^2 + \Lambda^* \int_{\Omega} S dx, \end{aligned}$$

where $\mathbf{H}_{\frac{1}{2}}$ is the fractional power subspace generated by the sectorial operator L . From Theorem 2.2 and Theorem 2.4, we can obtain that S is positive and bounded. Hence, there exists a $C_1 > 0$ such that

$$\Lambda^* \int_{\Omega} S dx \leq C_1.$$

Since $\mathbf{H}_{\frac{1}{2}} \hookrightarrow \mathbf{H}$, so there exists a $C > 0$, such that

$$\|S\|_{\mathbf{H}_{\frac{1}{2}}} \geq C \|S\|_{\mathbf{H}}, \quad \forall S \in \mathbf{H}_{\frac{1}{2}}.$$

As we know that

$$\frac{1}{2} \frac{d}{dt} \langle S, S \rangle_{\mathbf{H}} = \langle S_t, S \rangle_{\mathbf{H}},$$

hence,

$$\frac{1}{2} \frac{d}{dt} \|S\|_{\mathbf{H}}^2 \leq -d_S C^2 \|S\|_{\mathbf{H}}^2 + C_1,$$

By Lemma 2.3, we obtain that

$$\|S\|_{\mathbf{H}}^2 \leq e^{-\gamma_1 t} \|S_0(x)\|_{\mathbf{H}}^2 + \frac{2C_1}{\gamma_1} (1 - e^{-\gamma_1 t}),$$

where $\gamma_1 = 2d_S C^2$. It follows from the inner product of the second equation of system (2.1) in \mathbf{H} with V , we have that

$$\begin{aligned} & \left\langle d_V \Delta V + \alpha(x) S - \beta_2(x) \frac{VI}{V+I} - \mu(x) V, V \right\rangle_{\mathbf{H}} \\ &= d_V \int_{\Omega} V \Delta V dx + \int_{\Omega} \alpha(x) S V dx - \int_{\Omega} \beta_2(x) \frac{V^2 I}{V+I} dx - \int_{\Omega} \mu(x) V^2 dx \end{aligned}$$

$$\leq -d_V \|V\|_{\mathbf{H}_{\frac{1}{2}}}^2 + \alpha^* \int_{\Omega} S V dx,$$

From Theorem 2.2 and Theorem 2.4, we can obtain that S, V are positive and bounded. Hence, there exists a $C_2 > 0$ such that

$$\alpha^* \int_{\Omega} S V dx \leq C_2.$$

Since $\mathbf{H}_{\frac{1}{2}} \hookrightarrow \mathbf{H}$, so there exists a $C > 0$, such that

$$\|V\|_{\mathbf{H}_{\frac{1}{2}}} \geq C \|V\|_{\mathbf{H}}, \quad \forall V \in \mathbf{H}_{\frac{1}{2}}.$$

Hence,

$$\frac{1}{2} \frac{d}{dt} \|V\|_{\mathbf{H}}^2 \leq -d_V C^2 \|V\|_{\mathbf{H}}^2 + C_2,$$

By Lemma 2.3, we obtain that

$$\|V\|_{\mathbf{H}}^2 \leq e^{-\gamma_2 t} \|V_0(x)\|_{\mathbf{H}}^2 + \frac{2C_2}{\gamma_2} (1 - e^{-\gamma_2 t}),$$

where $\gamma_2 = 2d_V C^2$. It follows from the inner product of the third equation of system (2.1) in \mathbf{H} with I , we have that

$$\begin{aligned} & \left\langle d_I \Delta I + \beta_1(x) \frac{SI}{S+I} + \beta_2(x) \frac{VI}{V+I} + \rho(x)R - [\mu(x) + \eta_1(x) + \phi(x)]I, I \right\rangle_{\mathbf{H}} \\ &= d_I \int_{\Omega} I \Delta I dx + \int_{\Omega} \beta_1(x) \frac{SI^2}{S+I} dx + \int_{\Omega} \beta_2(x) \frac{VI^2}{V+I} dx + \int_{\Omega} \rho(x) IR dx \\ & \quad - \int_{\Omega} [\mu(x) + \eta_1(x) + \phi(x)] I^2 dx \\ & \leq -d_I \|I\|_{\mathbf{H}_{\frac{1}{2}}}^2 + \int_{\Omega} \beta_1(x) \frac{SI^2}{S+I} dx + \int_{\Omega} \beta_2(x) \frac{VI^2}{V+I} dx + \int_{\Omega} \rho(x) IR dx \\ & \leq -d_I \|I\|_{\mathbf{H}_{\frac{1}{2}}}^2 + (\beta_1^* + \beta_2^*) \int_{\Omega} I^2 dx + \rho^* \int_{\Omega} IR dx, \end{aligned}$$

From Theorem 2.2 and Theorem 2.4, we can obtain that I, R are positive and bounded. Hence, there exists a $C_3 > 0$ such that

$$(\beta_1^* + \beta_2^*) \int_{\Omega} I^2 dx + \rho^* \int_{\Omega} IR dx \leq C_{35}.$$

Since $\mathbf{H}_{\frac{1}{2}} \hookrightarrow \mathbf{H}$, so there exists a $C > 0$, such that

$$\|I\|_{\mathbf{H}_{\frac{1}{2}}} \geq C \|I\|_{\mathbf{H}}, \quad \forall I \in \mathbf{H}_{\frac{1}{2}}.$$

Hence,

$$\frac{1}{2} \frac{d}{dt} \|I\|_{\mathbf{H}}^2 \leq -d_I C^2 \|I\|_{\mathbf{H}}^2 + C_3,$$

By Lemma 2.3, we obtain that

$$\|I\|_{\mathbf{H}}^2 \leq e^{-\gamma_3 t} \|I_0(x)\|_{\mathbf{H}}^2 + \frac{2C_3}{\gamma_3} (1 - e^{-\gamma_3 t}),$$

where $\gamma_3 = 2d_1C^2$. It follows from the inner product of the forth equation of system (2.1) in \mathbf{H} with R , we have that

$$\begin{aligned} & \langle d_R \Delta R + \phi(x)I - [\rho(x) + \mu(x) + \eta_2(x)]R, R \rangle_{\mathbf{H}} \\ &= d_R \int_{\Omega} R \Delta R dx + \int_{\Omega} \phi(x)IR dx - \int_{\Omega} [\rho(x) + \mu(x) + \eta_2(x)]R^2 dx \\ &\leq -d_R \|R\|_{\mathbf{H}_{\frac{1}{2}}}^2 + \phi^* \int_{\Omega} IR dx. \end{aligned}$$

From Theorem 2.2 and Theorem 2.4, we can obtain that there exists a $C_4 > 0$ such that

$$\phi^* \int_{\Omega} IR dx \leq C_4.$$

Since $\mathbf{H}_{\frac{1}{2}} \hookrightarrow \mathbf{H}$, so there exists a $C > 0$, such that

$$\|R\|_{\mathbf{H}_{\frac{1}{2}}} \geq C \|R\|_{\mathbf{H}}, \quad \forall R \in \mathbf{H}_{\frac{1}{2}}.$$

Hence,

$$\frac{1}{2} \frac{d}{dt} \|R\|_{\mathbf{H}}^2 \leq -d_R C^2 \|R\|_{\mathbf{H}}^2 + C_4.$$

By Lemma 2.3, we obtain that

$$\|R\|_{\mathbf{H}}^2 \leq e^{-\gamma_4 t} \|R_0(x)\|_{\mathbf{H}}^2 + \frac{2C_4}{\gamma_4} (1 - e^{-\gamma_4 t}),$$

where $\gamma_4 = 2d_R C^2$. Therefore, it implies if $\widehat{R}^2 > \max\{\frac{2C_1}{\gamma_1}, \frac{2C_2}{\gamma_2}, \frac{2C_3}{\gamma_3}, \frac{2C_4}{\gamma_4}\}$, then there exists a $t_0 > 0$, such that for any $t \geq t_0$, there has $u(t, \varphi) \in \mathbf{B}_{\widehat{R}}$. Thence, $\mathbf{B}_{\widehat{R}} \subset \mathbf{H}^4$ is the absorbing set.

Next we verify **Condition (C*)**. Since $L_1 = d_S \Delta : \mathbf{H}_1 \rightarrow \mathbf{H}$ is a symmetrical sectorial operator, so the eigenvectors $\{e_j\}_{j \in \mathbb{N}}$ corresponding to the eigenvalues $\{\lambda_{S,j}\}_{j \in \mathbb{N}}$ are the complete orthonormal basis of \mathbf{H} , that is for any $S \in \mathbf{H}$,

$$S = \sum_{k=1}^{\infty} x_k e_k, \quad \|S\|_{\mathbf{H}}^2 = \sum_{k=1}^{\infty} x_k^2.$$

In addition, $\forall N_S > 0, \exists K_S \geq 1$ such that $-N_S \geq \lambda_{S,j}, \forall j \geq K_S + 1$.

Let

$$\mathbf{H}_1^{K_S} = \text{span}\{e_1, e_2, \dots, e_{K_S}\} \text{ and } \mathbf{H}_2^{K_S} = (\mathbf{H}_1^{K_S})^{\perp}.$$

Then, each $S \in \mathbf{H}$ can be decomposed as

$$\begin{aligned} S &= PS + (I - P)S := S_1 + S_2, \\ S_1 &= \sum_{i=1}^{K_S} x_i e_i \in \mathbf{H}_1^{K_S}, \quad S_2 = \sum_{j=K_S+1}^{\infty} x_j e_j \in \mathbf{H}_2^{K_S}, \end{aligned}$$

where $P : \mathbf{H} \rightarrow \mathbf{H}_1^{K_S}$ is the orthogonal projector. There are similar decomposition forms for each $V, I, R \in \mathbf{H}$.

Since $Q(t)$ has a bounded absorbing set $\mathbf{B}_{\widehat{R}} \subset \mathbf{H}^4$, then for any bounded set $\mathbf{B} \subset \mathbf{H}^4$, there exists a $t_{\mathbf{B}} > 0$, without loss of generality, we assume that $t_{\mathbf{B}} > t_0$ such that $(S(t, S_0(x)), V(t, V_0(x)), I(t, I_0(x)), R(t, R_0(x))) \subset \mathbf{B}_{\widehat{R}}$, where

$$\begin{aligned} S(t, S_0(x)) &= Q_1(t) S_0(x), & V(t, V_0(x)) &= Q_2(t) V_0(x), \\ I(t, I_0(x)) &= Q_3(t) I_0(x), & R(t, R_0(x)) &= Q_4(t) R_0(x), \end{aligned}$$

$\forall (S_0(x), V_0(x), I_0(x), R_0(x)) \in \mathbf{B}, t \geq t_{\mathbf{B}}$, then

$$\begin{aligned} \|S(t, S_0(x))\|_{\mathbf{H}}^2 &= \|Q_1(t) S_0(x)\|_{\mathbf{H}}^2 \leq \widehat{R}^2, \forall t \geq t_{\mathbf{B}}, \\ \|V(t, V_0(x))\|_{\mathbf{H}}^2 &= \|Q_2(t) V_0(x)\|_{\mathbf{H}}^2 \leq \widehat{R}^2, \forall t \geq t_{\mathbf{B}}, \\ \|I(t, I_0(x))\|_{\mathbf{H}}^2 &= \|Q_3(t) I_0(x)\|_{\mathbf{H}}^2 \leq \widehat{R}^2, \forall t \geq t_{\mathbf{B}}, \\ \|R(t, R_0(x))\|_{\mathbf{H}}^2 &= \|Q_4(t) R_0(x)\|_{\mathbf{H}}^2 \leq \widehat{R}^2, \forall t \geq t_{\mathbf{B}}. \end{aligned}$$

Hence,

$$\|PQ(t)\varphi\|_{\mathbf{H}^4} \leq \widehat{R}, \forall t \geq t_{\mathbf{B}}.$$

It means that $\{\|PQ(t)\mathbf{B}\|_{\mathbf{H}^4}\}_{t \geq t_{\mathbf{B}}}$ is bounded.

It follows from the inner product of the first equation of system (2.1) in \mathbf{H} with S_2 , we have that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \langle S, S_2 \rangle_{\mathbf{H}} = \langle S_t, S_2 \rangle_{\mathbf{H}} \\ &= \left\langle d_S \Delta S + \Lambda(x) - \beta_1(x) \frac{SI}{S+I} - [\mu(x) + \alpha(x)] S, S_2 \right\rangle_{\mathbf{H}} \\ &= \langle d_S \Delta S, S_2 \rangle_{\mathbf{H}} + \left\langle \Lambda(x) - \beta_1(x) \frac{SI}{S+I} - [\mu(x) + \alpha(x)] S, S_2 \right\rangle_{\mathbf{H}} \\ &= \langle d_S \Delta S_1 + d_S \Delta S_2, S_2 \rangle_{\mathbf{H}} + \left\langle \Lambda(x) - \beta_1(x) \frac{SI}{S+I} - [\mu(x) + \alpha(x)] S, S_2 \right\rangle_{\mathbf{H}} \\ &\leq \langle d_S \Delta S_2, S_2 \rangle_{\mathbf{H}} + \left\langle \Lambda(x) - \beta_1(x) \frac{SI}{S+I} - [\mu(x) + \alpha(x)] S, S_2 \right\rangle_{\mathbf{H}}, \end{aligned}$$

where

$$\begin{aligned} \langle d_S \Delta S_2, S_2 \rangle_{\mathbf{H}} &= -d_S \|S_2\|_{\mathbf{H}^{\frac{1}{2}}}^2 \\ &= d_S \sum_{j=K_S+1}^{\infty} \lambda_{S,j} x_j^2 \\ &\leq -d_S N_S \sum_{j=K_S+1}^{\infty} x_j^2 = -d_S N_S \|S_2\|_{\mathbf{H}}^2. \end{aligned}$$

Since $t \geq t_{\mathbf{B}}$, $\|S\|_{\mathbf{H}} \leq \widehat{R}$. Then

$$\begin{aligned} &\left\langle \Lambda(x) - \beta_1(x) \frac{SI}{S+I} - [\mu(x) + \alpha(x)] S, S_2 \right\rangle_{\mathbf{H}} \\ &\leq \left\| \Lambda(x) - \beta_1(x) \frac{SI}{S+I} - [\mu(x) + \alpha(x)] S \right\|_{\mathbf{H}} \|S_2\|_{\mathbf{H}} \\ &\leq (\|\Lambda(x)\|_{\mathbf{H}} + \|\beta_1(x) S\|_{\mathbf{H}} + \|[\mu(x) + \alpha(x)] S\|_{\mathbf{H}}) \widehat{R} \end{aligned}$$

$$\leq \left[\Lambda^* + (\beta_1^* + \mu^* + \alpha^*) \widehat{R} \right] \widehat{R} := C_{S, \widehat{R}}.$$

Hence,

$$\frac{d}{dt} \|S_2\|_{\mathbf{H}}^2 \leq -2d_S N_S \|u_2\|_{\mathbf{H}}^2 + 2C_{S, \widehat{R}}.$$

By Lemma 2.3,

$$\|S_2\|_{\mathbf{H}}^2 \leq e^{-2d_S N_S (t-t_{\mathbf{B}})} \|S_2(t_{\mathbf{B}})\|_{\mathbf{H}}^2 + \frac{C_{S, \widehat{R}}}{d_S N_S} (1 - e^{-2d_S N_S (t-t_{\mathbf{B}})}), \quad \forall t \geq t_{\mathbf{B}}$$

and $N_S > 0$ is arbitrary. Similarly,

$$\|V_2\|_{\mathbf{H}}^2 \leq e^{-2d_V N_V (t-t_{\mathbf{B}})} \|V_2(t_{\mathbf{B}})\|_{\mathbf{H}}^2 + \frac{C_{V, \widehat{R}}}{d_V N_V} (1 - e^{-2d_V N_V (t-t_{\mathbf{B}})}), \quad \forall t \geq t_{\mathbf{B}},$$

$$\|I_2\|_{\mathbf{H}}^2 \leq e^{-2d_I N_I (t-t_{\mathbf{B}})} \|I_2(t_{\mathbf{B}})\|_{\mathbf{H}}^2 + \frac{C_{I, \widehat{R}}}{d_I N_I} (1 - e^{-2d_I N_I (t-t_{\mathbf{B}})}), \quad \forall t \geq t_{\mathbf{B}},$$

$$\|R_2\|_{\mathbf{H}}^2 \leq e^{-2d_R N_R (t-t_{\mathbf{B}})} \|R_2(t_{\mathbf{B}})\|_{\mathbf{H}}^2 + \frac{C_{R, \widehat{R}}}{d_R N_R} (1 - e^{-2d_R N_R (t-t_{\mathbf{B}})}), \quad \forall t \geq t_{\mathbf{B}}$$

and $N_V, N_I, N_R > 0$ are arbitrary. Therefore, we deduce that the **Condition (C*)** holds. By [53, Theorem 4.1], we obtain that system (2.1) has a global exponential attract set \mathcal{A}^* . \square

After getting the global exponential attract set, we discuss the stability and uniform persistence of the epidemic disease.

Theorem 3.3. *The following statements are valid.*

(1) *If $\lambda^* < 0$, then*

$$\lim_{t \rightarrow \infty} S(x, t) = S^0(x), \quad \lim_{t \rightarrow \infty} V(x, t) = V^0(x), \quad \lim_{t \rightarrow \infty} I(x, t) = 0, \quad \lim_{t \rightarrow \infty} R(x, t) = 0$$

in \mathbf{H} , and hence, the disease-free equilibrium is globally asymptotically stable.

(2) *If $\lambda^* > 0$, then there exists a function $\gamma(x) > 0$ independent of the initial data, such that any solution (S, V, I, R) satisfies*

$$\begin{aligned} \liminf_{t \rightarrow \infty} S(x, t) &\geq \gamma(x), \quad \liminf_{t \rightarrow \infty} V(x, t) \geq \gamma(x), \\ \liminf_{t \rightarrow \infty} I(x, t) &\geq \gamma(x), \quad \liminf_{t \rightarrow \infty} R(x, t) \geq \gamma(x) \end{aligned}$$

for $x \in \overline{\Omega}$, and hence, the disease persists uniformly.

Proof. (1) Suppose $\lambda^* < 0$. Using the comparison principle, we can show that $I(x, t) \rightarrow 0, R(x, t) \rightarrow 0$ as $t \rightarrow \infty$ for every $x \in \Omega$. First, we observe from the system (2.1) that

$$\begin{cases} \frac{\partial I}{\partial t} \leq d_I \Delta I + \{\beta_1(x) + \beta_2(x) - [\mu(x) + \eta_1(x) + \phi(x)]\} I + \rho(x) R, \\ \quad x \in \Omega, t > 0, \\ \frac{\partial R}{\partial t} \leq d_R \Delta R + \phi(x) I - [\rho(x) + \mu(x) + \eta_2(x)] R, \quad x \in \Omega, t > 0. \end{cases}$$

Next, let us define $(\tilde{I}(x, t), \tilde{R}(x, t)) = (Me^{\lambda^* t} \varphi^*(x), Me^{\lambda^* t} \psi^*(x))$ where $\lambda^* < 0, \varphi^*(x) \gg 0, \psi^*(x) \gg 0$ are the eigenvalue and eigenvectors in equations (3.2) and M is chosen so large that $I(x, 0) \leq \tilde{I}(x, 0), R(x, 0) \leq \tilde{R}(x, 0)$ for every $x \in \Omega$. It can be easily shown that $(\tilde{I}(x, t), \tilde{R}(x, t))$ satisfies

$$\begin{cases} \frac{\partial \tilde{I}}{\partial t} = d_I \Delta \tilde{I} + \{\beta_1(x) + \beta_2(x) - [\mu(x) + \eta_1(x) + \phi(x)]\} \tilde{I} + \rho(x) \tilde{R}, \\ x \in \Omega, t > 0, \\ \frac{\partial \tilde{R}}{\partial t} = d_R \Delta \tilde{R} + \phi(x) \tilde{I} - [\rho(x) + \mu(x) + \eta_2(x)] \tilde{R}, x \in \Omega, t > 0, \\ \frac{\partial \tilde{I}}{\partial n} = \frac{\partial \tilde{R}}{\partial n} = 0, x \in \partial \Omega. \end{cases}$$

By the comparison principle Lemma 2.1, $I(x, t) \leq \tilde{I}(x, t), R(x, t) \leq \tilde{R}(x, t)$ for every $x \in \Omega$ and $t \geq 0$. Since $\tilde{I}(x, t) \rightarrow 0, \tilde{R}(x, t) \rightarrow 0$ as $t \rightarrow \infty$ for every $x \in \Omega$, we also have that $I(x, t) \rightarrow 0, R(x, t) \rightarrow 0$ as $t \rightarrow \infty$ for every $x \in \Omega$.

Next we claim $S(\cdot, t) \rightarrow S^0(x)$ uniformly on as $t \rightarrow \infty$. Given any small constant $\varepsilon > 0$, there exists a large time $T > 0$ such that $0 \leq I(x, t) \leq \varepsilon$ for all $x \in \bar{\Omega}, t \geq T$. From the first equation in system (2.1), it is easily observed that S is a super-solution to

$$\begin{cases} \frac{\partial w}{\partial t} - d_S \Delta w = \Lambda(x) - \beta_1^* \varepsilon - [\mu(x) + \alpha(x)] w, x \in \Omega, t \geq T, \\ \frac{\partial w}{\partial n} = 0, x \in \partial \Omega, \\ w(x, T) = S(x, T), x \in \Omega \end{cases} \quad (3.4)$$

and a sub-solution to

$$\begin{cases} \frac{\partial v}{\partial t} - d_S \Delta v = \Lambda(x) - [\mu(x) + \alpha(x)] v, x \in \Omega, t \geq T, \\ \frac{\partial v}{\partial n} = 0, x \in \partial \Omega, \\ v(x, T) = S(x, T), x \in \Omega. \end{cases} \quad (3.5)$$

Denote by w and v the solution of system (3.4) and system (3.5), respectively. The parabolic comparison principle gives

$$w(x, t) \leq S(x, t) \leq v(x, t) \text{ for all } x \in \bar{\Omega}, t \geq T.$$

For system (3.4), we can verify that

$$\begin{aligned} & \langle d_S \Delta w + \Lambda(x) - \beta_1^* \varepsilon - [\mu(x) + \alpha(x)] w, w \rangle_{\mathbf{H}} \\ &= d_S \int_{\Omega} w \Delta w dx + \int_{\Omega} \Lambda(x) w dx - \int_{\Omega} \beta_1^* \varepsilon w dx - \int_{\Omega} [\mu(x) + \alpha(x)] w^2 dx \\ &\leq -d_S \|w\|_{\mathbf{H}_{\frac{1}{2}}}^2 + \Lambda^* \int_{\Omega} w dx. \end{aligned}$$

It is similar to the proof of Theorem 3.2, we can obtain that system (3.4) exists a global exponential attract set \mathcal{A}_w . In addition, system (3.4) has a variational structure, the corresponding functional of the variational structure is

$$F(w) = \int_{\Omega} \left[\frac{d_S}{2} |\nabla w|^2 - g(x, w) \right] dx,$$

where

$$g(x, w) = \int_0^w [\Lambda(x) - \beta_1^* \varepsilon - [\mu(x) + \alpha(x)] w] dw.$$

Then

$$\begin{aligned} & \langle DF(w), Lw + G(w) \rangle_{\mathbf{H}} \\ &= \langle DF(w), d_S \Delta w + \Lambda(x) - \beta_1^* \varepsilon - [\mu(x) + \alpha(x)] w \rangle_{\mathbf{H}} \\ &\leq -C \|DF(w)\|_{\mathbf{H}}^2, \end{aligned}$$

where $C > 0$, so $L + G$ is a gradient type operator. From [49, Theorem A.2.2], we can prove that

$$\lim_{t \rightarrow \infty} w(x, t) = S_-^0(\varepsilon, x) \text{ in } \mathbf{H},$$

where $S_-^0(\varepsilon, x)$ is the unique positive steady state of problems (3.4). Similarly, for system (3.5), we can obtain

$$\lim_{t \rightarrow \infty} v(x, t) = S^0(x) \text{ in } \mathbf{H},$$

where $S^0(x)$ is the unique positive steady state of problems (3.5). Furthermore, due to the arbitrariness of ε , it is checked that

$$S_-^0(\varepsilon, x) \rightarrow S^0(x) \text{ in } \mathbf{H}, \text{ as } \varepsilon \rightarrow 0.$$

Thus, our analysis implies that $S(\cdot, t) \rightarrow S^0(x)$ uniformly on as $t \rightarrow \infty$. Similarly, we can also prove that $V(\cdot, t) \rightarrow V^0(x)$ uniformly on as $t \rightarrow \infty$.

(2) Since $\lambda^* > 0$. It is observed that the solution of

$$\begin{cases} \frac{\partial S_-}{\partial t} - d_S \Delta S_- = \Lambda(x) - [\beta_1(x) + \mu(x) + \alpha(x)] S_-, & x \in \Omega, t \geq T, \\ \frac{\partial S_-}{\partial n} = 0, & x \in \partial\Omega, \\ S_-(x, T) = S(x, T), & x \in \Omega \end{cases} \quad (3.6)$$

is a sub-solution of the first equation in (2.1). Similar to the proof of conclusion (1), system (3.6) is also a gradient type equation. From [49, Theorem A.2.2], we can prove that

$$\lim_{t \rightarrow \infty} S_-(x, t) = S_-^*(x) \text{ in } \mathbf{H},$$

By weak maximum principle, we know that $S_-^*(x) > 0$ for all $x \in \Omega$. Similarly, we can also prove that $V_-^*(x) > 0$ for all $x \in \Omega$. Next, let us define $(L_-(x, t), R_-(x, t)) = (\varepsilon\varphi^*(x), \varepsilon\psi^*(x))$ where $\varphi^*(x) \gg 0, \psi^*(x) \gg 0$ are the eigenvalue and eigenvectors in equations (3.2) and $\varepsilon > 0$ is a sufficiently small constant. Substituting $\varepsilon\varphi^*, \varepsilon\psi^*$ into the third and the fourth equations of system (2.1), we know

$$\begin{aligned} & \varepsilon d_I \Delta \varphi^* + \beta_1(x) \frac{S \varepsilon \varphi^*}{S + \varepsilon \varphi^*} + \beta_2(x) \frac{V^* \varepsilon \varphi^*}{V^* + \varepsilon \varphi^*} + \varepsilon \rho(x) \psi^* \\ & - \varepsilon [\mu(x) + \eta_1(x) + \phi(x)] \varphi^* - \frac{\partial(\varepsilon \varphi^*)}{\partial t} \\ & = \varepsilon d_I \Delta \varphi^* + \beta_1(x) \frac{S \varepsilon \varphi^*}{S + \varepsilon \varphi^*} + \beta_2(x) \frac{V^* \varepsilon \varphi^*}{V^* + \varepsilon \varphi^*} + \varepsilon \rho(x) \psi^* \end{aligned}$$

$$\begin{aligned}
& -\varepsilon [\mu(x) + \eta_1(x) + \phi(x)] \varphi^* + \varepsilon \beta_1(x) \varphi^* - \varepsilon \beta_1(x) \varphi^* + \varepsilon \beta_2(x) \varphi^* - \varepsilon \beta_2(x) \varphi^* \\
& = \varepsilon \{d_I \Delta \varphi^* + [\beta_1(x) + \beta_2(x)] \varphi^* - [\mu(x) + \eta_1(x) + \phi(x)] \varphi^* + \rho(x) \psi^*\} \\
& \quad + \varepsilon \beta_1(x) \varphi^* \left[\frac{S^*}{S^* + \varepsilon \varphi^*} - 1 \right] + \varepsilon \beta_2(x) \varphi^* \left[\frac{V^*}{V^* + \varepsilon \varphi^*} - 1 \right] \\
& = \varepsilon \varphi^* \left[\lambda^* + \beta_1(x) \left(\frac{S^*}{S^* + \varepsilon \varphi^*} - 1 \right) + \beta_2(x) \left(\frac{V^*}{V^* + \varepsilon \varphi^*} - 1 \right) \right] > 0
\end{aligned}$$

and

$$\begin{aligned}
& \varepsilon d_R \Delta \psi^* + \varepsilon \phi(x) \varphi^* - \varepsilon [\rho(x) + \mu(x) + \eta_2(x)] \psi^* - \frac{\partial(\varepsilon \psi^*)}{\partial t} \\
& = \varepsilon [d_R \Delta \psi^* + \phi(x) \varphi^* - [\rho(x) + \mu(x) + \eta_2(x)] \psi^*] \\
& = \varepsilon \lambda^* \psi^* > 0.
\end{aligned}$$

Therefore, $(\varepsilon \varphi^*, \varepsilon \psi^*)$ is the sub-solution of the third and the fourth equations of system (2.1). We choose $0 < \gamma(x) < \min \{S_-(x), V_-(x), \varepsilon \varphi^*(x), \varepsilon \psi^*(x)\}$, we can obtain that

$$\begin{aligned}
\liminf_{t \rightarrow \infty} S(x, t) & \geq \gamma(x), \quad \liminf_{t \rightarrow \infty} V(x, t) \geq \gamma(x), \\
\liminf_{t \rightarrow \infty} I(x, t) & \geq \gamma(x), \quad \liminf_{t \rightarrow \infty} R(x, t) \geq \gamma(x)
\end{aligned}$$

for $x \in \bar{\Omega}$, then it shows that the disease persists. □

The synthesis of Theorem 3.2 and Theorem 3.3 can give the following results: if $\lambda^* > 0$, then the positive solution of system (2.1) is globally exponentially attractive and the attraction domain is \mathcal{A}^* . From an epidemiological point of view, the disease persists in this situation. The epidemic is globally asymptotically stable or attracted by the global exponential attract set.

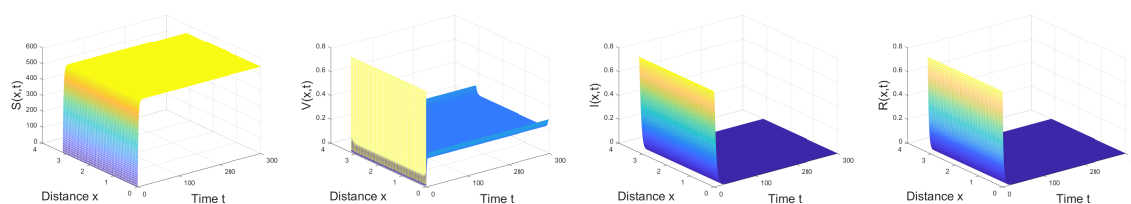
4. Numerical simulation

In this section, we hope to select a specific epidemic disease with some more practical and more definitive data to simulate the stability of our *SVIR* model with relapse. As we know, HBV epidemic model is a kind of infectious disease model with relapse and vaccination is the most effective way to prevent hepatitis. WHO also releases global hepatitis report every year. From ‘‘Global Hepatitis Report, 2017’’ [63], we know that worldwide, approximately 240 million people have chronic hepatitis B virus infection. Without an expanded and accelerated response, the number of people living with hepatitis B virus is projected to remain at the current, high levels for the next 40-50 years, with a cumulative 20 million deaths occurring between 2015 and 2030. A stepped-up global response can no longer be delayed. From World Health Statistics [64], we can find that the data of global natural mortality, the male natural mortality rate is 19% and female natural mortality rate is 12.9% respectively. After a simple calculation we can get the natural mortality rate of the global is 15.95%. In conclusion, we can get the data in Table 2.

According to the data in Table 2, we will first simulate the stability of the constant coefficient model about system (2.1) when $\Lambda, \beta_1, \beta_2, \alpha, \rho, \phi, \mu, \eta_1$ and η_2 are assumed to be positive constants. Then we can obtain the stability of disease-free equilibrium of constant coefficient model (Figure 2).

Table 2. The parameters description of HBV model.

Parameter	Description	Data estimated	Data sources
Λ	Recruitment scale of the population.	300year^{-1}	[63]
β_1	Estimated HBV prevalence (Transmission rate) without vaccination.	0.062year^{-1}	[63]
β_2	Estimated HBV prevalence (Transmission rate) with vaccination.	0.035year^{-1}	[63]
α	vaccinate rate.	0.39year^{-1}	[63]
ρ	Relapse rate.	0.3year^{-1}	Estimate
ϕ	The per-capita recovery (treatment) rate.	0.08year^{-1}	Estimate
μ	Natural mortality rate.	0.1595year^{-1}	[64]
η_1	The disease-related death rate of the HBV.	0.2year^{-1}	[63]
η_2	The disease-related death rate of being treated.	0.1year^{-1}	Estimate
d_S	The diffusion coefficient of susceptible individuals.	2year^{-1}	Estimate
d_V	The diffusion coefficient of vaccinated individuals.	2.5year^{-1}	Estimate
d_I	The diffusion coefficient of infected individuals.	1.5year^{-1}	Estimate
d_R	The diffusion coefficient of restored individuals.	5year^{-1}	Estimate

**Figure 2.** The global stability of disease-free equilibrium of constant coefficient model.

From the numerical simulation, it can be seen that although hepatitis B is still the primary disease endangering human health, however, the global hepatitis B disease has shown a trend of extinction.

In this article, we discuss the dynamics of the epidemic model with a spatial heterogeneous environment, so we then simulate the stability and persistence of the spatial system (2.1). From the system (2.1) we can see that all the parameters are x -related functions, so we choose different functions will directly lead to different stability results. We choose $\beta_1(x) = \frac{1}{1+x}$, $\mu(x) = e^{-2x}$, $\phi(x) = e^{-x}$, $\rho(x) = 0.4 \sin x$, $\eta_1(x) = 0.2$, $\eta_2(x) = 0.1$ and select other parameters from Table 2. We can show the global stability of non-constant disease-free equilibrium $E^0(x)$ of system (2.1) (Figure 3).

We choose $\mu(x) = e^{-x}$, $\phi(x) = \frac{1}{1+x}$, $\beta_1(x) = 0.62$, $\beta_2(x) = 0.33e^{-x}$ and select other parameters from Table 2. We can obtain that the spatial heterogeneity epidemic is global stability (Figure 4).

We choose $\mu(x) = e^{-x}$, $\phi(x) = \frac{1}{1+x}$, $\rho(x) = 0.4 \sin x$, $\beta_1(x) = 0.62$, $\beta_2(x) = 0.33e^{-x}$ and select other parameters from Table 2. We can clearly see that the spatial heterogeneity epidemic is persists uniformly (Figure 5).

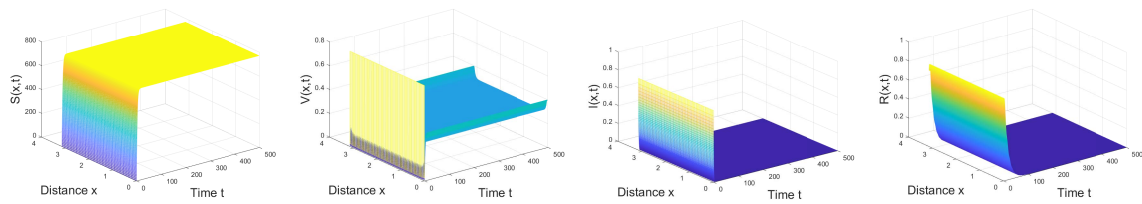


Figure 3. The global stability of non-constant disease-free equilibrium $E^0(x)$ of system (2.1) when $\beta_1(x) = \frac{1}{1+x}$, $\mu(x) = e^{-2x}$, $\phi(x) = e^{-x}$, $\rho(x) = 0.4 \sin x$, $\eta_1(x) = 0.2$, $\eta_2(x) = 0.1$.

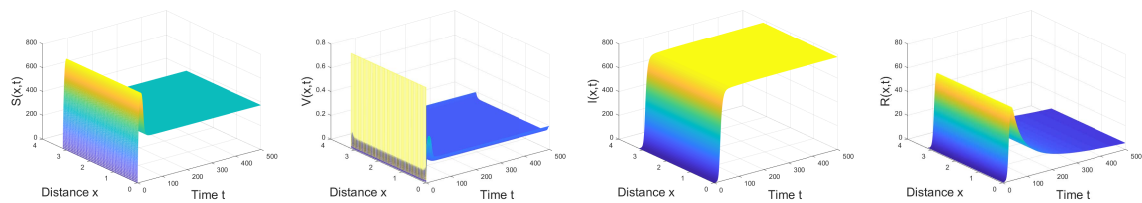


Figure 4. The global stability of endemic of spatial heterogeneity system (2.1) when $\mu(x) = e^{-x}$, $\phi(x) = \frac{1}{1+x}$, $\beta_1(x) = 0.62$, $\beta_2(x) = 0.33e^{-x}$.

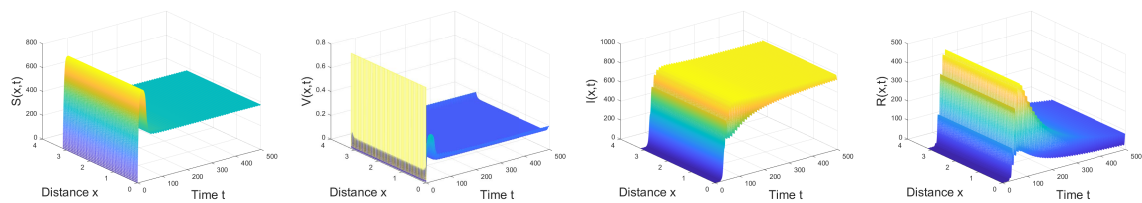


Figure 5. The spatial heterogeneity epidemic is persists uniformly when $\mu(x) = e^{-x}$, $\phi(x) = \frac{1}{1+x}$, $\rho(x) = 0.4 \sin x$, $\beta_1(x) = 0.62$, $\beta_2(x) = 0.33e^{-x}$.

In Figure 5, we can find that the image of the solution is turbulent, and the amplitude decreases with time. Finally, the image of the turmoil is controlled in range. And this range is the scope of the global exponential attract set. From the comparison of Figure 4 and Figure 5, we can find that spatial heterogeneity has a great impact on the spread of disease, and different parameters lead to changes in long-term dynamic behavior.

From all above simulation we can clearly see that the disease-free equilibrium of the constant-coefficient model is global asymptotic stability. For spatial heterogeneous models, the stability of the model is dependent on the spatial parameters and diffusion coefficients. In addition, we can also see that the image whether rising or falling is very fast in the initial stage, which also confirms that the solutions of system (2.1) are global exponential attract.

5. Conclusions and discussion

We have formulated a reaction-diffusion $SVIR$ model with relapse in the spatially heterogeneous environment. We first prove the positivity of solution by using the comparison principle of monotone nonlinear reaction-diffusion dynamical system and comparison principle of parabolic equation. Most

of the models in the previous work are non-coupled models or systems that contain only one diffusion equation. The usual way to study these two problems is to study one of the equations in the model first (such as eigenvalue problems or the existence of solutions), and then to get the properties of several other equations. However, the above methods do not applicable to the coupling system of the four diffusion equations we have established in this paper. In addition, we prove that the disease-free equilibrium is globally asymptotically stable and the endemic is persisting uniformly by the global exponential attract set theory. From our proof, we can find that λ^* has threshold characteristics. From the numerical simulation we can find the effect of spatial heterogeneity on the diffusion of disease is very large. Simulation results show that the best way to eliminate the disease is to control the contact rate. Reduce the relapse rate or increase the vaccination rate can make the disease tends to a stable state. Choosing different spatial coefficients can produce a fundamental change of the global dynamics of the system.

Acknowledgements

The authors would like to thank the anonymous referees for their useful and valuable suggestions. The first author was supported by the Fundamental Research Funds for the Central Universities (JUSRP11949), NSF of China (11671180,11671181) and the third author was supported by the science and technology foundation of Suqian(Z201444), China.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. F. Brauer and C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology* (Second Edition), Springer, 2012.
2. K. Dietz, The first epidemic model: A historical note on P.D. En'ko, *Australian J. Stat.*, **30** (1988), 56–65.
3. H. F. Huo and X. M. Zhang, Complex dynamics in an alcoholism model with the impact of Twitter, *Math. Biosci.*, **281** (2016), 24–35.
4. H. F. Huo and C. C. Zhu, Influence of relapse in a giving up smoking model, *Abstr. Appl. Anal.*, (2013), 525461.
5. H. F. Huo and X. M. Zhang, Modelling effects of treatment at home on tuberculosis transmission dynamics, *Appl. Math. Model.*, **40** (2016), 9474–9484.
6. C. Vargas-De-León, On the global stability of SIS, SIR and SIRS epidemic models with standard incidence, *Chaos Solitons Fractals*, **44** (2011), 1106–1110.
7. C. C. Zhu and J. Zhu, Stability of a reaction-diffusion alcohol model with the impact of tax policy, *Comput. Math. Appl.*, **74** (2017), 613–633.
8. Z. Hu, L. Chang, Z. Teng, et al., Bifurcation analysis of a discrete SIRS epidemic model with standard incidence rate, *Adv. Differ. Equ.*, (2016) 2016, 155.

9. Z. Jiang, W. Ma and J. Wei, Global hopf bifurcation and permanence of a delayed SEIRS epidemic model, *Math. Comput. Simul.*, **122** (2016), 35–54.
10. A. M. Yousef and S. M. Salman, Backward bifurcation in a fractional-order SIRS epidemic model with a nonlinear incidence rate, *Int. J. Nonlinear Sci. Numer. Simul.*, **17** (2016), 401–412.
11. H. Zhao and M. Zhao, Global hopf bifurcation analysis of an susceptible-infective-removed epidemic model incorporating media coverage with time delay, *J. Biol. Dyn.*, **11** (2016), 8–24.
12. S. Whang, S. Choi and E. Jung, A dynamic model for tuberculosis transmission and optimal treatment strategies in South Korea, *J. Theor. Biol.*, **279** (2011), 120–131.
13. S. Choi and E. Jung, Optimal Tuberculosis Prevention and Control Strategy from a Mathematical Model Based on Real Data, *Bull. Math. Biol.*, **76** (2014), 1566–1589.
14. Y. Hosono and B. Ilyas, Travelling waves for a simple diffusive epidemic model, *Math. Model Meth. Appl. Sci.*, **5** (1995), 935–966.
15. W. T. Li, G. Lin, C. Ma, et al., Traveling wave solutions of a nonlocal delayed SIR model without outbreak threshold, *Discrete Contin. Dyn. Syst. Ser. B*, **19** (2014), 467–484.
16. R. Peng and X. Q. Zhao, A reaction–diffusion SIS epidemic model in a time-periodic environment, *Nonlinearity*, **25** (2012), 1451–1471.
17. X. Q. Zhao, Global dynamics of a reaction and diffusion model for Lyme disease, *J. Math. Biol.*, **65** (2012), 787–808.
18. Y. X. Zhang and X. Q. Zhao, A reaction-diffusion Lyme disease model with seasonality, *SIAM J. Appl. Math.*, **73** (2013), 2077–2099.
19. Y. J. Lou and X. Q. Zhao, A reaction–diffusion malaria model with incubation period in the vector population, *J. Math. Biol.*, **62** (2011), 543–568.
20. G. Webb, A reaction–diffusion model for a deterministic diffusive models, *J. Math. Anal. Appl.*, **84** (1981), 150–161.
21. W. E. Fitzgibbon, M. E. Parrott and G. F. Webb, Diffusion epidemic models with incubation and crisscross dynamics, *Math. Biosci.*, **128** (1985), 131–155.
22. H. R. Thieme, Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity, *SIAM J. Appl. Math.*, **70** (2009), 188–211.
23. W. Wang and X. Q. Zhao, A nonlocal and time-delayed reaction-diffusion model of dengue transmission, *SIAM J. Appl. Math.*, **71** (2011), 147–168.
24. W. Wang and X. Q. Zhao, Basic reproduction numbers for reaction-diffusion epidemic models, *SIAM J. Appl. Dyn. Syst.*, **11** (2012), 1652–1673.
25. Z. F. Xie, Cross-diffusion induced Turing instability for a three species food chain model, *J. Math. Anal. Appl.*, **388** (2012), 539–547.
26. K. Hattaf and N. Yousfi, Global stability for reaction–diffusion equations in biology, *Comput. Math. Appl.*, **67** (2014), 1439–1449.
27. E. Latosa and T. Suzuki, Global dynamics of a reaction–diffusion system with mass conservation, *J. Math. Anal. Appl.*, **411** (2014), 107–118.

28. J. B. Wang, W. T. Li and F. Y. Yang, Traveling waves in a nonlocal dispersal SIR model with nonlocal delayed transmission, *Commun. Nonlinear Sci. Numer. Simul.*, **27** (2015), 136–152.
29. Z. Xu, Traveling waves for a diffusive SEIR epidemic model, *Comm. Pure Appl. Anal.*, **15**(2016), 871–892.
30. Z. Xu and C. Ai, Traveling waves in a diffusive influenza epidemic model with vaccination, *Appl. Math. Model.*, **40** (2016), 7265–7280.
31. C. C. Zhu, W. T. Li and F. Y. Yang, Traveling waves in a nonlocal dispersal SIRH model with relapse, *Comput. Math. Appl.*, **73** (2017), 1707–1723.
32. C. C. Zhu, W. T. Li and F. Y. Yang, Traveling waves of a reaction-diffusion SIRQ epidemic model with relapse, *J. Appl. Anal. Comput.*, **7** (2017), 147–171.
33. X. Bao, W. Shen and Z. Shen, Spreading speeds and traveling waves for space-time periodic nonlocal dispersal cooperative systems, *Commun. Pure Appl. Anal.*, **18** (2019) 361–396.
34. K. Yamazaki and X. Wang, Global well-posedness and asymptotic behavior of solutions to a reaction-convection-diffusion cholera epidemic model, *Discrete Contin. Dyn. Syst. Ser. B*, **21** (2016), 1297–1316.
35. L. J. S. Allen, B. M. Bolker, Y. Lou, et al., Asymptotic profiles of the steady states for an SIS epidemic reaction-diffusion model, *Discrete Contin. Dyn. Syst.*, **21** (2008), 1–20.
36. R. Peng, Asymptotic profiles of the positive steady state for an SIS epidemic reaction-diffusion model. Part I, *J. Differ. Eq.*, **247** (2009), 1096–1119.
37. R. Peng and S. Liu, Global stability of the steady states of an SIS epidemic reaction-diffusion model, *Nonlinear Anal.*, **71** (2009), 239–247.
38. R. Peng and F. Yi, Asymptotic profile of the positive steady state for an SIS epidemic reaction-diffusion model: effects of epidemic risk and population movement, *Phys. D*, **259** (2013), 8–25.
39. T. Kuniya and J. Wang, Lyapunov functions and global stability for a spatially diffusive SIR epidemic model, *Appl. Anal.*, (2016) <http://dx.doi.org/10.1080/00036811.2016.1199796>.
40. K. Deng and Y. Wu, Dynamics of a susceptible–infected–susceptible epidemic reaction-diffusion model, *Proc. Roy. Soc. Edinburgh Sect. A*, **146** (2016), 929–946.
41. Y. Wu and X. Zou, Asymptotic profiles of steady states for a diffusive SIS epidemic model with mass action infection mechanism, *J. Differential Equations*, **261** (2016), 4424–4447.
42. H. Li, R. Peng and F. B. Wang, Varying total population enhances disease persistence: Qualitative analysis on a diffusive SIS epidemic model, *J. Differ. Equ.*, **262** (2017), 885–913.
43. Y. Cai, Y. Kang, M. Banerjeed, et al., Complex Dynamics of a host–parasite model with both horizontal and vertical transmissions in a spatial heterogeneous environment, *Nonlinear Anal. Real World Appl.*, **40** (2018), 444–465.
44. Y. Tong and C. Lei, An SIS epidemic reaction-diffusion model with spontaneous infection in a spatially heterogeneous environment, *Nonlinear Anal. Real World Appl.*, **41** (2018), 443–460.
45. R. Temam, *Infinite-Dimensional Systems in Mechanics and Physics*, Springer-Verlag, New York, 1997.

46. J. C. Robinson, *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge University Press, 2001.
47. M. Yang and C. Y. Sun, Exponential attractors for the strongly damped wave equations, *Nonlinear Anal. Real World Appl.*, **11** (2010), 913–919.
48. T. Ma, *The theory and method of partial differential equation*, Science Press, Beijing, 2011.
49. T. Ma and S. Wang, *Phase Transition Dynamics*, Springer Science+Business Media, LLC 2014.
50. Y. Li, H. Wu and T. Zhao, Necessary and sufficient conditions for the existence of exponential attractors for semigroups, and applications, *Nonlinear Anal.*, **75** (2012), 6297–6305.
51. Y. Zhong and C. K. Zhong, Exponential attractors for reaction-diffusion equations with arbitrary polynomial growth, *Nonlinear Anal.*, **71**(2009), 751–765.
52. J. Zhang and C. K. Zhong, The existence of global attractors for a class of reaction–diffusion equations with distribution derivatives terms in \mathbb{R}^n , *J. Math. Anal. Appl.*, **427** (2015), 365–376.
53. J. Zhang, P. E. Kloeden, M. Yang, et al., Global exponential κ -dissipative semigroups and exponential attraction, *Discrete Contin. Dyn. Syst. Ser.*, **37**(2017), 3487–3502.
54. J. Zhang, Y. Wang and C. K. Zhong, Robustness of exponentially κ -dissipative dynamical systems with perturbation, *Discrete Contin. Dyn. Syst. Ser. B*, **22** (2017), 3875–3890.
55. X. Wang and J. Wang, Analysis of cholera epidemics with bacterial growth and spatial movement, *J. Biol. Dyn.*, **9** (2015), 233–261.
56. N. K. Vaidya, F. B. Wang and X. Zou, Avian influenza dynamics in wild birds with bird mobility and spatial heterogeneous environment, *Discrete Contin. Dyn. Syst. Ser. B*, **17** (2012), 2829–2848.
57. K. Yamazaki, Global well-posedness of infectious disease models without life-time immunity: the cases of cholera and avian influenza, *Math. Med. Biol.*, **35** (2018), 428–445.
58. Q. X. Ye, Z. Y. Li, M. X. Wang, et al., *Introduction to reaction–diffusion equations*, second edition, Science Press, Beijing, 2011 (in Chinese).
59. H. L. Smith, *Monotone dynamical systems*, *Mathematical Surveys and Monographs*, vol.41, American Mathematical Society, Providence, RI, 1995.
60. I. I. Vrabie, *C_0 semigroups and application*, Elsevier Science B.V., New York, 2003.
61. C. V. Pao, *Nonlinear Parabolic and Elliptic Equations*, New York, London: Plenum, 1992.
62. Z. Du and R. Peng, A priori L^∞ estimates for solutions of a class of reaction-diffusion systems, *J. Math. Biol.*, **72** (2016), 1429–1439.
63. Global hepatitis report, 2017. Available from: <http://www.who.int/hepatitis/publications/global-hepatitis-report2017/en/>.
64. World Health Statistics, 2013. Available from: <http://www.who.int/>.



AIMS Press

©2019 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (<http://creativecommons.org/licenses/by/4.0>)