Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/MBE

MBE, 16(5): 5551–5583.
DOI: 10.3934/mbe.2019276
Received: 18 March 2019
Accepted: 06 June 2019
Published: 17 June 2019

Research article

Analysis of a mathematical model with nonlinear susceptibles-guided interventions

Qian Li and Yanni Xiao*

Department of Applied Mathematics, Xi'an Jiaotong University, Xi'an 710049, P.R. China

* **Correspondence:** Email: yxiao@mail.xjtu.edu.cn.

Abstract: In this paper, we considered a mathematical model describing the nonlinear susceptibles-guided vaccination and isolation strategies, incorporating the continuously saturated treatment. In this strategy, we find that the disease-free periodic solution can always exist, and consequently the control reproduction number can be defined through analyzing the stability of the disease-free periodic solution. Also, we discussed the existence and stability of the positive order-1 periodic solution from two points of view. Initially, we investigated the transcritical and pitchfork bifurcation of the Poincaré map with respect to key parameters, and proved the existence of a stable or an unstable positive order-1 periodic solution near the disease-free periodic solution. For another aspect, by studying the properties of the Poincaré map, we verified the existence of the positive order-1 periodic solution in a large range of the control parameters, especially, we verified the co-existence of finite or infinite countable different positive order-1 periodic solutions. Furthermore, numerical simulations show that the unstable order-1 periodic solution can co-exist with the stable order-1, or order-2, or order-3 periodic solution. The finding implies that the nonlinear susceptibles-triggered feedback control strategy can induce much rich dynamics, which suggests us to carefully choose key parameters to ensure the stability of the disease-free periodic solution, indicating that infectious diseases die out.

Keywords: SIR model; nonlinear state-dependent feedback control; Poincaré map; disease-free periodic solution; transcritical and pitchfork bifurcation; positive order-k periodic solution

1. Introduction

In recent decades, the public health system is severely affected by the outbreak and re-occurrence of infectious diseases, which also causes social turbulence and economic retrogression. Many mathematical models are proposed and analyzed to investigate the dynamics of infectious diseases [1–9]. Comprehensive interventions, such as vaccination, treatment and isolation, are estimated to be effective for controlling the spread of infectious diseases [10–17], among which many researches studied the saturated continuous treatment related to limited medical resources [12–14]. The SIR model with continuously saturated treatment gives:

$$\begin{cases} \frac{dS(t)}{dt} &= A - \beta S I - \delta_1 S, \\ \frac{dI(t)}{dt} &= \beta S I - \delta_2 I - \gamma I - \frac{\epsilon I}{1+\omega I}, \\ \frac{dR(t)}{dt} &= \gamma I - \delta_1 R + \frac{\epsilon I}{1+\omega I}, \end{cases} \tag{1.1}$$

where S, I and R are the populations of susceptible, infected, and recovered, respectively. A represents the constant recruitment rate, β is the transmission rate, γ is the recovery rate, δ_1 denotes the natural death rate, and δ_2 denotes the death rate of class I including both the natural death rate and the disease-related death rate, hence, it is reasonable to assume $\delta_1 < \delta_2$. The term $\frac{\epsilon I}{1+\omega I}$ represents the saturated treatment. Note that the above model assumes that the recovered individuals cannot be infected again, hence the class R doesn't affect the dynamics of system (1.1). Therefore, one only needs to consider the following reduced model:

$$\begin{cases} \frac{dS(t)}{dt} &= A - \beta S I - \delta_1 S, \\ \frac{dI(t)}{dt} &= \beta S I - \delta_2 I - \gamma I - \frac{\epsilon I}{1+\omega I}. \end{cases} \tag{1.2}$$

Impulsive differential equations, including fixed-moments and state-dependent impulsive strategies, were widely used and have raised human's concern. Fixed-moments impulsive models assume that measures are carried out at fixed discrete times. Using this type of models [17–21, 33, 34, 36, 37], researchers can investigate the existence and stability of the disease-free periodic solution. However, these models described that control measures were implemented every fixed time without knowing the number of infected and susceptible individuals and the prevalence of infectious diseases, which may waste the medicine resources [17, 19, 20, 24]. Therefore, it is more reasonable to propose state-dependent impulsive models, in which the implementation of vaccination and isolation is determined by whether the size of infected or susceptible population reaches the threshold level. Traditional state-dependent impulsive mathematical models [16, 31, 35] considered the size of infected population as an index to trigger impulsive interventions, in which no disease-free periodic solution is feasible and this strategy is unable to eradicate infectious diseases. Moreover, this makes it challengeable to define the basic (or control) reproduction number for impulsive models.

Therefore, a natural consideration is whether or not the susceptibles-guided impulsive interventions can successfully control and finally eradicate infectious diseases, and how this strategy affects the dynamical behaviors. The novel idea comes from the control of measles infection, in which the number of susceptible individuals (or the level of susceptibility) is higher than or exceeds a certain level, then the vaccination will then be implemented [22, 23]. Moreover, there are some researches investigating the effectiveness of the susceptible-triggered interventions and showing that the susceptible-triggered interventions are promising and effective strategies [24–27]. Particularly, studies [24, 25] have considered the susceptibles-triggered impulsive interventions on SIR models. They assumed that the vaccination rate and isolation rate are linearly dependent on the number of susceptible and infected individuals, respectively. However, in reality, vaccination and isolation are often restricted by limited medical resources [28, 29], which can be expressed as saturation functions:

$$p_1(t) = \frac{pS(t)}{h_1 + S(t)}, \quad q_1(t) = \frac{qI(t)}{h_2 + I(t)},$$

where $p \in (0, 1)$ denotes the maximal vaccination rate of susceptible population, and $q \in (0, 1)$ is the maximal isolation rate of infected individuals. h_1 and h_2 denote the half-saturation constants of susceptible and infected individuals, respectively. Therefore, based on (1.2), we propose the following state-dependent impulsive model with susceptibles-guided comprehensive saturated interventions:

$$\begin{cases} \left.\begin{aligned} \frac{dS(t)}{dt} &= A - \beta SI - \delta_1 S, \\ \frac{dI(t)}{dt} &= \beta SI - \delta_2 I - \gamma I - \frac{\epsilon I}{1+\omega I}, \end{aligned}\right\} S(t) < S_T, \\ \left.\begin{aligned} S(t^+) &= \left(1 - \frac{pS(t)}{h_1 + S(t)}\right) S(t), \\ I(t^+) &= \left(1 - \frac{qI(t)}{h_2 + I(t)}\right) I(t), \end{aligned}\right\} S(t) = S_T, \end{cases} \tag{1.3}$$

where S_T represents the threshold level of the number of susceptible individuals determining whether to implement the impulsive control strategies or not. The main purpose of this study is to analyze the mathematical model describing the susceptibles-guided comprehensive saturated interventions (including impulsive vaccination and isolation, and continuous treatment), and further evaluate the effectiveness of this strategy for controlling the spread of infectious diseases.

The rest of this paper is organized as follows. In the next section, we give some basic definitions of the planer impulsive semi-dynamical system. In Section 3, we discuss the existence and stability of the disease-free periodic solution. Then, in the next two sections, we investigate the dynamic behaviors of our proposed model through discussing the existence and stability of the positive order-1 periodic solution. Specifically, in Section 4, we study the existence and stability of the positive order-1 periodic solutions through investigating the bifurcations near the disease-free periodic solution. In Section 5, we define the impulsive set and phase set of the Poincaré map of our proposed model and further discuss the positive order-1 periodic solutions in a large range of the control parameters by examining the properties of the Poincaré map including monotoniciity, continuity, discontinuity and convexity. In section 6, we finally give some conclusions and discussions.

2. Preliminaries and Poincaré map

We describe the generalized planer impulsive semi-dynamical system with state-dependent feedback control as:

$$\begin{cases} \frac{dx}{dt} = P(x, y), \ \frac{dy}{dt} = Q(x, y), & \text{if } \ \phi(x, y) \neq 0, \\ \Delta x = a(x, y), \ \Delta y = b(x, y), & \text{if } \ \phi(x, y) = 0. \end{cases} \tag{2.1}$$

Here $(x, y) \in R_+^2 = \{(x, y) : x \geq 0, y \geq 0\}$, $\Delta x = x^+ - x$ and $\Delta y = y^+ - y$. P, Q, a, b are continuous functions from R_+^2 to R. The impulsive function $\psi : R_+^2 \to R_+^2$ can be defined as

$$\psi(x, y) = (x^+, y^+) = (x + a(x, y), y + b(x, y)),$$

and $z^+ = (x^+, y^+)$ is called an impulsive point of $z = (x, y)$. In this study, we focus on the special state-dependent impulsive model (1.3). We start with concluding the main dynamics of the ODE subsystem.

The dynamical behaviors of subsystem (1.2) have been discussed in [14], here we just recall them briefly. Consider the region $\Omega = \{(S, I) : S + I \leq \frac{A}{\delta_1}, \ S, I \geq 0\}$ as a positively invariant set of system (1.2), and denote the basic reproduction number of system (1.2) as:

$$R_0 = \frac{A\beta}{\delta_1(\delta_2 + \gamma + \epsilon)}. \tag{2.2}$$

It is easy to see that system (1.2) always has a disease-free equilibrium $E_0 = (A/\delta_1, 0)$, which is globally stable if there is no endemic equilibrium. The existence of the endemic equilibrium depends on the solutions of the following equations:

$$\begin{cases} A - \beta SI - \delta_1 S = 0, \\ \beta SI - \delta_2 I - \gamma I - \frac{\epsilon I}{1+\omega I} = 0. \end{cases}$$

Solving above equations yields

$$I^2 + b_1 I + b_2 = 0,$$

with

$$b_1 = \tfrac{(\delta_2+\gamma)(\beta+\omega\delta_1)+\beta\epsilon-A\beta\omega}{\beta\omega(\delta_2+\gamma)}, \; b_2 = \tfrac{\delta_1(\delta_2+\gamma+\epsilon)-A\beta}{\beta\omega(\delta_2+\gamma)} = \tfrac{\delta_1(\delta_2+\gamma+\epsilon)}{\beta\omega(\delta_2+\gamma)}(1 - R_0).$$

As we can see, $b_2 \le 0$ holds true if and only if $R_0 \ge 1$.
Denote

$$\begin{aligned} I_1 &= \tfrac{-b_1+\sqrt{\Delta}}{2}, \;\; S_1 = \tfrac{A}{\beta I_1+\delta_1}, \text{ and} \\ I_2 &= \tfrac{-b_1-\sqrt{\Delta}}{2}, \;\; S_2 = \tfrac{A}{\beta I_2+\delta_1}, \text{ with } \Delta = b_1^2 - 4b_2, \end{aligned}$$

and solve $\Delta = 0$ in terms of R_0, we obtain $R_0 = \widetilde{R}_0$ with

$$\widetilde{R}_0 = \frac{A(\delta_2+\gamma)\left(\delta_2+\gamma+\left(\omega\sqrt{A}+\sqrt{\epsilon\omega}\right)^2\right)}{(\delta_2+\gamma+\epsilon)\left((\delta_2+\gamma)\left(\frac{\delta_2+\gamma}{\omega}+2\left(A+\frac{\epsilon}{\omega}\right)\right)+\omega\left(A-\frac{\epsilon}{\omega}\right)^2\right)}.$$

Therefore, we obtain the following results regarding the existence of the endemic equilibria.

Proposition 2.1. *For subsystem (1.2):*
(1) When $R_0 > 1$, there exists a unique endemic equilibrium $E_1 = (S_1, I_1)$, as shown in Figure 1;
(2) When $b_1 \ge 0$, subsystem (1.2) can undergo a forward bifurcation at $R_0 = 1$, and there exists no endemic equilibrium if $R_0 \le 1$;
(3) When $b_1 < 0$, subsystem (1.2) undergoes a backward bifurcation at $R_0 = 1$ with a saddle-node bifurcation happening at $R_0 = \widetilde{R}_0$. Specifically, there exist two endemic equilibria $E_1 = (S_1, I_1)$ and $E_2 = (S_2, I_2)$ if $\widetilde{R}_0 < R_0 < 1$ while the two equilibria coincide into one endemic equilibrium when $R_0 = \widetilde{R}_0$, and there exists no endemic equilibrium if $R_0 < \widetilde{R}_0$.

Next, we show the stability and bifurcation phenomenons of the endemic equilibria of subsystem (1.2). The characteristic equation at the endemic equilibria is shown as:

$$\lambda^2 + H(I_i)\lambda + G(I_i) = 0, \; i = 1, 2,$$

where

$$H(I_i) = \delta_1 + \beta I_i - \frac{\epsilon\omega I_i}{(1+\omega I_i)^2}, \;\; G(I_i) = \frac{A\beta^2 I_i}{\delta_1 + \beta I_i} - \frac{(\delta_1+\beta I_i)\varepsilon\omega I_i}{(1+\omega I_i)^2}.$$

Based on the main conclusions in [14], we obtain that equilibrium E_2 is always an unstable saddle point if it exists, and we conclude the results for the stability of equilibrium E_1 as follows.

Proposition 2.2. *When $R_0 > 1$ or $1 > R_0 > \widetilde{R}_0$ and $b_1 < 0$, subsystem (1.2) can undergo a Hopf bifurcation around equilibrium E_1 at the surface $H(I_1) = 0$. Corresponding to the Hopf bifurcation,* *subsystem (1.2) can either have a stable or an unstable limit cycle, as shown in Figure 1(C) and Figure 1(D). Moreover, the endemic equilibrium E_1 of subsystem (1.2) is a stable node (Figure 1(A)) or focus (Figure 1(B)) if $H(I_1) > 0$, while E_1 is an unstable node or focus if $H(I_1) < 0$, and subsystem (1.2) has at least one closed orbit in region Ω.*

Therefore, from Proposition 2.2, we obtain that when $R_0 > 1$ and $H(I_1) > 0$, then the endemic equilibrium E_1 is stable, while when $R_0 > 1$ and $H(I_1) < 0$, the endemic equilibrium E_1 is unstable and there is at least one closed orbit. Particularly, if there is a unique closed orbit, it is stable as shown in Figure 1(C). In order to address the dynamics of system (1.3), we conduct the Poincaré map. Denote the two isolines of subsystem (1.2) as follows:

$$\begin{aligned} l_1 &: \dot{S} = A - \beta SI - \delta_1 S \doteq P(S,I) = 0, \\ l_2 &: \dot{I} = \beta SI - \delta_2 I - \gamma I - \tfrac{\epsilon I}{1+\omega I} \doteq Q(S,I) = 0. \end{aligned}$$

Furthermore, we define two sections as:

$$l_3 : S_{S_T} = \{(S,I) | S = S_T, I \geq 0\},\ l_4 : S_{S_v} = \left\{(S,I) | S = \left(1 - \tfrac{pS_T}{h_1+S_T}\right) S_T \doteq S_v, I \geq 0\right\}.$$

Thus, we can define the impulsive function $\psi(S,I)$ as:

$$\psi_1(S,I) = \left(1 - \tfrac{pS(t)}{h_1+S(t)}\right) S(t),\ \ \psi_2(S,I) = \left(1 - \tfrac{qI(t)}{h_2+I(t)}\right) I(t) \doteq w_1(I).$$

In the current study, we set the section S_{S_v} as a Poincaré section. Choose an initial point $P_k^+ = (S_v, I_k^+)$ on the Poincaré section. If the orbit starting from P_k^+ reaches S_{S_T} at a finite time, we denote the intersection point as $P_{k+1} = (S_T, I_{k+1})$, then after the impulsive intervention, the trajectory will jump to $P_{k+1}^+ = (S_v, I_{k+1}^+)$ on section S_{S_v} with $I_{k+1}^+ = w_1(I_{k+1})$. Following from the existence and uniqueness of solutions, I_{k+1} is uniquely determined by I_k^+, thus we can define a function g with $g\left(I_k^+\right) = I_{k+1}$. Therefore, we can define the Poincaré map $\mathcal{P}_M$ for system (1.3) as:

$$\mathcal{P}_M :\ I_{k+1}^+ = w_1(I_{k+1}) = w_1(g(I_k^+)) \doteq \mathcal{P}_M(I_k^+).$$

It is worth noting that the domain and range of Poincaré map $\mathcal{P}_M$, which we will give detail analyses in Section 5, are strictly determined by the dynamical behaviors of ODE subsystem (1.2). From the main results in Proposition 2.1 and Proposition 2.2, we can conclude the four cases of the dynamics of subsystem (1.2) as follows:

(C_1) $R_0 < 1$ and $b_1 \geq 0$ or $R_0 < \widetilde{R}_0$ (i.e., there is no endemic equilibrium);

(C_2) $\widetilde{R}_0 < R_0 < 1$ and $b_1 < 0$ (i.e., there are two endemic equilibria);

(C_3) $R_0 > 1$ and $H(I_1) > 0$ (i.e., there is a unique endemic equilibrium, which is globally stable);

(C_4) $R_0 > 1$ and $H(I_1) < 0$ (i.e., there is a unique endemic equilibrium, which is unstable. Further, there exists at least one limit cycle).

Then, in the next section, we first investigate the dynamic behaviours of system (1.3) through discussing the existence and stability of the disease-free periodic solution.

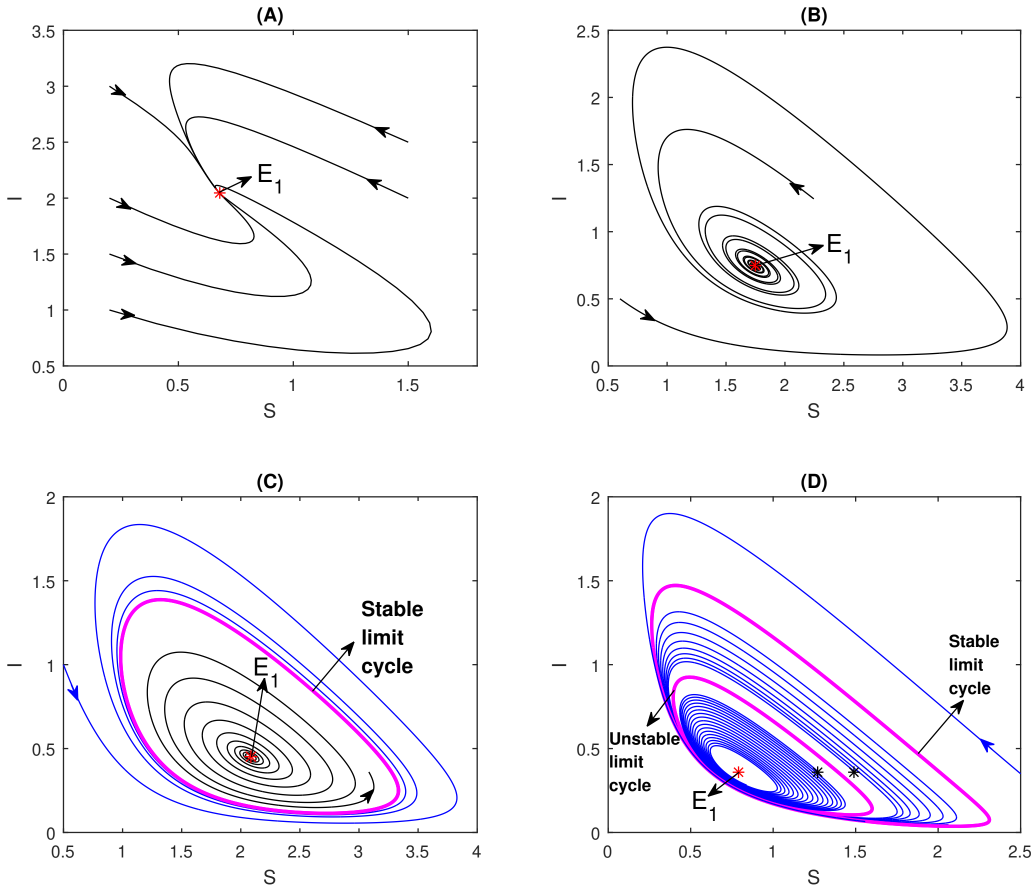

Figure 1. Dynamical behaviours of ODE subsystem (1.2) when $R_0 > 1$. (A) E_1 is a globally stable node with $A = 2.6, \beta = 1.8, \epsilon = 5, \omega = 2.9$. (B) E_1 is a globally stable focus with $A = 2.6, \beta = 1.8, \epsilon = 5, \omega = 1.2$. (C) E_1 is unstable and there is a stable limit cycle. Here, $A = 2, \beta = 1.8, \epsilon = 5, \omega = 1.2$. (D) E_1 is locally stable and there are two limit cycles, of which one is stable and the other one is unstable. Here, $A = 0.7, \beta = 2.021, \epsilon = 5.1, \omega = 10$. The other parameter values are: $\delta_1 = 0.15, \delta_2 = 0.4, \gamma = 0.1$.

3. Existence and stability of the disease-free periodic solution of system (1.3)

Letting $I(t) = 0$ for all $t \geq 0$, then we consider the following subsystem

$$\begin{cases} \frac{dS(t)}{dt} = A - \delta_1 S, & S(t) < S_T, \\ S(t^+) = \left(1 - \frac{pS(t)}{h_1 + S(t)}\right) S(t), & S(t) = S_T. \end{cases} \tag{3.1}$$

Solving Eq (3.1) with initial condition $S(0) = S_v \left(\text{i.e.,} \left(1 - \frac{pS_T}{h_1 + S_T}\right) S_T\right)$, we obtain

$$S(t) = \frac{A - (A - \delta_1 S_v)\exp(-\delta_1 t)}{\delta_1}$$

with period

$$T = \frac{1}{\delta_1} \ln \frac{A - \delta_1 S_v}{A - \delta_1 S_T}.$$

This indicates that system (1.3) has a disease-free periodic solution with period T, denoted as $(\xi(t), 0)$, with

$$\xi(t) = \frac{A - (A - \delta_1 S_v)\exp(-\delta_1 (t - (k-1)T))}{\delta_1}, \ (k-1)T < t \leq kT, \ k \in N. \tag{3.2}$$

Then we discuss the stability of the disease-free periodic solution $(\xi(t), 0)$. There are

$$\begin{aligned} &a(S, I) = -\frac{pS^2(t)}{h_1 + S(t)}, \ \ b(S, I) = -\frac{qI^2(t)}{h_2 + I(t)}, \ \ \phi(S, I) = S - S_T, \\ &(\xi(T), \eta(T)) = (S_T, 0), \ \ (\xi(T^+), \eta(T^+)) = (S_v, 0). \end{aligned}$$

Using Lemma A.1 in Appendix A, we obtain

$$\begin{aligned} \Delta_1 &= \frac{P_+\left(\frac{\partial b}{\partial I}\frac{\partial \phi}{\partial S} - \frac{\partial b}{\partial S}\frac{\partial \phi}{\partial I} + \frac{\partial \phi}{\partial S}\right) + Q_+\left(\frac{\partial a}{\partial S}\frac{\partial \phi}{\partial I} - \frac{\partial a}{\partial I}\frac{\partial \phi}{\partial S} + \frac{\partial \phi}{\partial I}\right)}{P\frac{\partial \phi}{\partial S} + Q\frac{\partial \phi}{\partial I}} = \frac{P_+\left(1 - \frac{qI(2h_2+I)}{(h_2+I)^2}\right)}{P} \\ &= \frac{P(\xi(T^+), \eta(T^+))\left(1 - \frac{qI(2h_2+I)}{(h_2+I)^2}\right)}{P(\xi(T), \eta(T))} = \left(1 - \frac{qI(2h_2+I)}{(h_2+I)^2}\right)\frac{A - \delta_1 S_v}{A - \delta_1 S_T}, \end{aligned}$$

and

$$\begin{aligned} &\exp\left(\int_0^T \left(\frac{\partial P}{\partial S}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial I}(\xi(t), \eta(t))\right) dt\right) \\ &= \exp\left(\int_0^T (-\delta_1 - \delta_2 - \gamma - \epsilon + \beta\xi(t))\, dt\right) \\ &= \exp\left(\int_0^T \left(-\delta_1 - \delta_2 - \gamma - \epsilon + \frac{\beta A}{\delta_1} - \frac{\beta(A - \delta_1 S_v)\exp(-\delta_1 t)}{\delta_1}\right) dt\right) \\ &= \exp\left(\frac{\beta A - \delta_1(\delta_1 + \delta_2 + \gamma + \epsilon)}{\delta_1^2} \ln \frac{A - \delta_1 S_v}{A - \delta_1 S_T} - \frac{\beta p S_T^2}{\delta_1(h_1 + S_T)}\right) \\ &= \left(\frac{A - \delta_1 S_v}{A - \delta_1 S_T}\right)^{\frac{\beta A - \delta_1(\delta_1 + \delta_2 + \gamma + \epsilon)}{\delta_1^2}} \exp\left(-\frac{\beta p S_T^2}{\delta_1(h_1 + S_T)}\right). \end{aligned}$$

Therefore, there is

$$\begin{aligned} \mu_2 &= \Delta_1 \exp\left(\int_0^T \left(\frac{\partial P}{\partial S}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial I}(\xi(t), \eta(t))\right) dt\right) \\ &= \left(1 - \frac{\partial b}{\partial I}|_{I=0}\right)\left(\frac{A - \delta_1 S_v}{A - \delta_1 S_T}\right)^{\frac{\beta A - \delta_1(\delta_2 + \gamma + \epsilon)}{\delta_1^2}} \exp\left(-\frac{\beta p S_T^2}{\delta_1(h_1 + S_T)}\right) \doteq R_b. \end{aligned} \tag{3.3}$$

Note that the relationship between μ_2 and 1 determines the stability of the disease-free periodic solution, thus the Floquet multiplier μ_2 can be defined as the control reproduction number of the state-dependent impulsive model (1.3), denoted by R_b, which is crucial to study the development of infectious diseases. From Eq (3.3), it is clear to see that $\frac{A-\delta_1 S_v}{A-\delta_1 S_T} > 1$. Furthermore, we can verify that if $h_2 > 0$, then $\frac{\partial b}{\partial I}|_{I=0} = 0$ with

$$R_b = \left(\frac{A-\delta_1 S_v}{A-\delta_1 S_T}\right)^{\frac{\beta A-\delta_1(\delta_2+\gamma+\epsilon)}{\delta_1^2}} \exp\left(-\frac{\beta p S_T^2}{\delta_1(h_1+S_T)}\right),$$

while if $h_2 = 0$, then $\frac{\partial b}{\partial I}|_{I=0} = -q$ with

$$R_b = (1-q)\left(\frac{A-\delta_1 S_v}{A-\delta_1 S_T}\right)^{\frac{\beta A-\delta_1(\delta_2+\gamma+\epsilon)}{\delta_1^2}} \exp\left(-\frac{\beta p S_T^2}{\delta_1(h_1+S_T)}\right).$$

For convenient, we denote

$$J \doteq \frac{\beta A-\delta_1(\delta_2+\gamma+\epsilon)}{\delta_1^2} \ln\frac{A-\delta_1 S_v}{A-\delta_1 S_T} + \frac{\beta(S_v-S_T)}{\delta_1} = \int_{S_v}^{S_T} \frac{\beta s-\delta_2-\gamma-\epsilon}{A-\delta_1 s} ds,$$

thus,

$$R_b = \begin{cases} (1-q)*\exp(J), & \text{if } h_2 = 0, \\ \exp(J), & \text{if } h_2 > 0. \end{cases} \tag{3.4}$$

Based on above discussions, we have the following conclusions.

Theorem 3.1. *If $R_b < 1$ holds true, then the disease-free periodic solution of system (1.3) is locally stable, while if $R_b > 1$, then the disease-free periodic solution of system (1.3) is unstable. Particularly, for cases (C_1) and (C_2), inequality $R_b < 1$ always holds true, further, the disease-free periodic solution is globally stable for case (C_1). For cases (C_3) and (C_4), the disease-free periodic solution is locally stable when $S_T \leq \overline{S}$. Furthermore, for case (C_3), the disease-free periodic solution is globally stable when $S_T \leq min\{\overline{S}, S_1\}$.*

Proof We have $R_0 < 1$ for cases (C_1) and (C_2), then there are $\frac{\beta A-\delta_1(\delta_2+\gamma+\epsilon)}{\delta_1^2} < 0$ and $0 < \left(\frac{A-\delta_1 S_v}{A-\delta_1 S_T}\right)^{\frac{\beta A-\delta_1(\delta_2+\gamma+\epsilon)}{\delta_1^2}} < 1$. Therefore, $R_b < 1$ holds, which indicates that the disease-free periodic solution is orbitally asymptotically stable. For the global stability, we need to prove that the disease-free periodic solution $(\xi(t), 0)$ is globally attractive. It follows from the definition of the Poincaré map and the property of subsystem (1.2) that Poincaré map $\mathcal{P}_M$ satisfies $\mathcal{P}_M(I_0) < I_0$ for $I_0 \geq 0$ for case (C_1). Therefore, the disease-free periodic solution $(\xi(t), 0)$ is globally attractive for case (C_1). For cases (C_3) and (C_4), letting

$$V(s) = \frac{\beta s-\delta_2-\gamma-\epsilon}{A-\delta_1 s},$$

we obtain

$$\frac{dV(s)}{ds} = \frac{\beta A-\delta_1(\delta_2+\gamma+\epsilon)}{(A-\delta_1 s)^2} > 0.$$

Thus, $V(s)$ is increasing for $s \in (0, \frac{A}{\delta_1})$ and $V(\overline{S}) = 0$ with $\overline{S} = \frac{\delta_2+\gamma+\epsilon}{\beta}$, which means that $V(s) < 0$ and $J < 0$ always hold for $S_T \leq \overline{S} < \frac{A}{\delta_1}$. Thus, when $S_T \leq \overline{S}$, we have $R_b < 1$, correspondingly, the disease-free periodic solution is locally stable for cases (C_3) and (C_4). In addition, when $S_T \leq \min\{\overline{S}, S_1\}$, we can similarly verify that the disease-free periodic solution $(\xi(t), 0)$ is globally attractive for case (C_3). This completes the proof.

In the next two sections, we discuss the existence and stability of the positive order-1 periodic solution from two points of view: through investigating the bifurcations near the disease-free periodic solution and examining the properties of the Poincaré map including monotonicity, continuity, discontinuity and convexity.

4. Bifurcations near the disease-free periodic solution

Based on the discussions in the last section, for case (C_3) or (C_4), the sign of J can vary when $S_T > \overline{S}$, which indicates that system (1.3) may undergo bifurcations near the disease-free periodic solution as the parameter values vary. Therefore, we can discuss the bifurcations near the disease-free periodic solution by assuming $R_0 > 1$ and $S_T > \overline{S}$. Consider subsystem (1.2) in the phase space, we define a scalar differential equation

$$\begin{cases} \frac{dI}{dS} = \frac{Q(S,I)}{P(S,I)} \doteq W(S, I), \\ I(S_v) = I_0^+. \end{cases} \tag{4.1}$$

For system (4.1), we focus on region

$$\Omega_1 = \left\{(S, I)|S > 0, I > 0, I < \frac{A - \delta_1 S}{\beta S}\right\},$$

in which function $W(S, I)$ is continuously differentiable. Given initial condition (S_0, I_0), which belongs to the phase set on the Poincaré section, one obtains

$$I(S; S_0, I_0) = I_0 + \int_{S_v}^{S} W(s, I(s; S_v, I_0))ds.$$

Then, $\mathcal{P}_M$ takes the following form:

$$\mathcal{P}_M(I_0, \alpha) = w_1(I(S_T; S_v, I_0)),$$

where α represents a bifurcation parameter. Through some straightforward calculations, we get

$$\begin{aligned} \frac{\partial I(S;S_v,I_0)}{\partial I_0} &= \exp\left(\int_{S_v}^{S} \frac{\partial W(s,I(s;S_v,I_0))}{\partial I} ds\right), \\ \frac{\partial^2 I(S;S_v,I_0)}{\partial I_0^2} &= \frac{\partial I(S;S_v,I_0)}{\partial I_0} \int_{S_v}^{S} \frac{\partial^2 W(s,I(s;S_v,I_0))}{\partial I^2} \frac{\partial I(s;S_v,I_0)}{\partial I_0} ds. \end{aligned}$$

Denoting

$$\frac{\partial I(S_T;S_v,I_0)}{\partial I_0} = \frac{\partial g(I_0;\alpha)}{\partial I_0} \doteq g'(I_0;\alpha),$$

then, we have

$$\begin{aligned} \frac{\partial \mathcal{P}_M}{\partial I_0}(0, \alpha) &= \left[\left(1 - \frac{qI(S_T;S_v,I_0)(2h_2+I(S_T;S_v,I_0))}{(h_2+I(S_T;S_v,I_0))^2}\right) g'(I_0;\alpha)\right]\Big|_{I_0=0} \\ &= w_1'(I(S_T;S_v,0))g'(0;\alpha) = R_b, \\ \frac{\partial^2 \mathcal{P}_M}{\partial I_0^2}(0, \alpha) &= \left[\left(1 - \frac{qI(S_T;S_v,I_0)(2h_2+I(S_T;S_v,I_0))}{(h_2+I(S_T;S_v,I_0))^2}\right) g''(I_0;\alpha) - \frac{2h_2^2 q}{(h_2+I(S_T;S_v,I_0))^3}(g'(I_0;\alpha))^2\right]\Big|_{I_0=0} \\ &= w_1'(I(S_T;S_v,0))g''(0;\alpha) - \frac{2h_2^2 q}{(h_2+I(S_T;S_v,0))^3}(g'(0;\alpha))^2, \\ \frac{\partial^3 \mathcal{P}_M}{\partial I_0^3}(0, \alpha) &= \left[\left(1 - \frac{qI(S_T;S_v,I_0)(2h_2+I(S_T;S_v,I_0))}{(h_2+I(S_T;S_v,I_0))^2}\right) g'''(I_0;\alpha) - \frac{6h_2^2 q g'(I_0;\alpha)g''(I_0;\alpha)}{(h_2+I(S_T;S_v,I_0))^3} + \frac{6h_2^2 q(g'(I_0;\alpha))^3}{(h_2+I(S_T;S_v,I_0))^4}\right]\Big|_{I_0=0} \\ &= w_1'(I(S_T;S_v,0))g'''(0;\alpha) - \frac{6h_2^2 q g'(0;\alpha)g''(0;\alpha)}{(h_2+I(S_T;S_v,0))^3} + \frac{6h_2^2 q(g'(0;\alpha))^3}{(h_2+I(S_T;S_v,0))^4}, \end{aligned}$$

where

$$\begin{aligned}
g'(0;\alpha) &= \exp\left(\int_{S_v}^{S_T} \frac{\partial W(s,I(s;S_v,0))}{\partial I} ds\right) = \exp\left(\int_{S_v}^{S_T} \frac{\beta s-\delta_2-\gamma-\epsilon}{A-\delta_1 s} ds\right) \\
&= \left(\frac{A-\delta_1 S_v}{A-\delta_1 S_T}\right)^{\frac{\beta A-\delta_1(\delta_2+\gamma+\epsilon)}{\delta_1^2}} \exp\left(\frac{\beta(S_v-S_T)}{\delta_1}\right), \\
g''(0;\alpha) &= g'(0;\alpha) \int_{S_v}^{S_T} \frac{\partial^2 W(s,I(s;S_v,0))}{\partial I^2} \frac{\partial I(s;S_v,0)}{\partial I_0} ds = g'(0;\alpha) \int_{S_v}^{S_T} m(s) \frac{\partial I(s;S_v,0)}{\partial I_0} ds, \\
g'''(0;\alpha) &= g''(0;\alpha) \int_{S_v}^{S_T} m(s) \frac{\partial I(s;S_v,0)}{\partial I_0} ds + g'(0;\alpha) \frac{\partial}{\partial I_0} \left(\int_{S_v}^{S_T} m(s) \frac{\partial I(s;S_v,0)}{\partial I_0} ds\right),
\end{aligned}$$

with

$$\begin{aligned}
m(s) &= \frac{\partial^2 W(s,I(s;S_v,0))}{\partial I^2} = \frac{2\omega\epsilon(A-\delta_1 s)+2\beta s(\beta s-\delta_2-\gamma-\epsilon)}{(A-\delta_1 s)^2}, \\
\frac{\partial I(s;S_v,0)}{\partial I_0} &= \left(\frac{A-\delta_1 S_v}{A-\delta_1 s}\right)^{\frac{\beta A-\delta_1(\delta_2+\gamma+\epsilon)}{\delta_1^2}} \exp\left(\frac{\beta(S_v-s)}{\delta_1}\right).
\end{aligned}$$

Based on above calculations, we mainly focus on discussing the transcritical and pitchfork bifurcations near the disease-free periodic solution with respect to the key parameters for $h_2 > 0$. Note that all of the parameters appearing in the expression of R_b can be chosen as bifurcation parameters. In what follows, we choose control parameters, such as ϵ, p, S_T and h_1 to investigate the bifurcation near the disease-free periodic solution and the bifurcation with respect to other parameters can be studied by using similar method. Furthermore, the bifurcation near the disease-free periodic solution for $h_2 = 0$ can be investigated similarly, and we study it by taking the parameter related to impulsive isolation strategy q as an example in such case.

4.1. Bifurcation with respect to ϵ

In this subsection, ϵ is chosen as a bifurcation parameter. For $h_2 > 0$, taking the derivative of $R_b(\epsilon)$ with respect to ϵ yields

$$\frac{\partial R_b(\epsilon)}{\partial \epsilon} = -\frac{R_b(\epsilon)}{\delta_1} * \ln\left(\frac{A-\delta_1 S_v}{A-\delta_1 S_T}\right) < 0,$$

which means that $R_b(\epsilon)$ is decreasing for $\epsilon \in (0, +\infty)$. It is easy to verify that

$$\lim_{\epsilon\to+\infty} R_b(\epsilon) = 0.$$

Furthermore, if

$$R_b(0) = \left(\frac{A-\delta_1 S_v}{A-\delta_1 S_T}\right)^{\frac{\beta A-\delta_1(\delta_2+\gamma)}{\delta_1^2}} \exp\left(-\frac{\beta p S_T^2}{\delta_1 (h_1 + S_T)}\right) > 1,$$

then we have that there is a unique $\epsilon^* \in (0, +\infty)$ such that $R_b(\epsilon^*) = 1$ and $\frac{\partial R_b(\epsilon^*)}{\partial \epsilon} < 0$ with ϵ^* satisfying

$$\left(\frac{A-\delta_1 S_v}{A-\delta_1 S_T}\right)^{\frac{\beta A-\delta_1(\delta_2+\gamma+\epsilon^*)}{\delta_1^2}} \exp\left(-\frac{\beta p S_T^2}{\delta_1(h_1+S_T)}\right) = 1.$$

Therefore, we have the main results as follows.

Proposition 4.1. *Suppose $h_2 > 0$, $R_0 > 1$ and $S_T > \overline{S}$. If $R_b(0) > 1$ holds true, then there exists a unique $\epsilon^* \in (0, +\infty)$ such that $R_b(\epsilon^*) = 1$ with $\frac{\partial R_b(\epsilon^*)}{\partial \epsilon} < 0$. And the disease-free periodic solution $(\xi(t), 0)$ of system (1.3) is orbitally asymptotically stable for $\epsilon \in (\epsilon^*, +\infty)$ and unstable for $\epsilon \in (0, \epsilon^*)$.*

Next, we consider the bifurcation near the disease-free periodic solution at $\epsilon = \epsilon^*$. We have that

$\mathcal{P}_M(0,\epsilon)=0$ always holds, further,

$$\frac{\partial \mathcal{P}_M}{\partial I_0}(0,\epsilon^*)=1,\ \ \frac{\partial^2 \mathcal{P}_M}{\partial I_0 \partial \epsilon}(0,\epsilon^*)<0,\ \ \frac{\partial^2 \mathcal{P}_M}{\partial I_0^2}(0,\epsilon^*)=g''(0;\epsilon^*)-\frac{2q}{h_2}.$$

Note that if $g''(0;\epsilon^*)\neq \frac{2q}{h_2}$, then $\frac{\partial^2 \mathcal{P}_M}{\partial I_0^2}(0,\epsilon^*)\neq 0$. Furthermore, $g''(0;\epsilon^*)>\frac{2q}{h_2}$ indicates $\frac{\partial^2 \mathcal{P}_M}{\partial I_0^2}(0,\epsilon^*)>0$, while $g''(0;\epsilon^*)<\frac{2q}{h_2}$ means $\frac{\partial^2 \mathcal{P}_M}{\partial I_0^2}(0,\epsilon^*)<0$. As for the special condition $\frac{\partial^2 \mathcal{P}_M}{\partial I_0^2}(0,\epsilon^*)=0$ $\left(\text{i.e., } g''(0;\epsilon^*)=\frac{2q}{h_2}\right)$, we further consider the sign of $\frac{\partial^3 \mathcal{P}_M}{\partial I_0^3}(0,\epsilon^*)$. Note that

$$\frac{\partial^3 \mathcal{P}_M}{\partial I_0^3}(0,\epsilon^*)=g'''(0;\epsilon^*)-\frac{6q(2q-1)}{h_2^2},$$

thus, $\frac{\partial^3 \mathcal{P}_M}{\partial I_0^3}(0,\epsilon^*)\neq 0$ when $g'''(0;\epsilon^*)\neq \frac{6q(2q-1)}{h_2^2}$. Based on above discussions and Lemma $A.2$ and Lemma $A.3$ presented in Appendix A, we have the following conclusions.

Theorem 4.1. *Suppose $h_2>0$, $R_0>1$, $S_T>\overline{S}$ and $R_b(0)>1$. We have:*

(a) If $g''(0;\epsilon^*)>\frac{2q}{h_2}$ holds true, then the Poinceré map $\mathcal{P}_M(I_0,\epsilon)$ undergoes a transcritical bifurcation at $\epsilon=\epsilon^*$. Further, an unstable positive fixed point appears when ϵ passes through $\epsilon=\epsilon^*$ from left to right. Correspondingly, system (1.3) has an unstable positive periodic solution for $\epsilon\in(\epsilon^*,\epsilon^*+\varepsilon)$ with $\varepsilon>0$ small enough;

(b) If $g''(0;\epsilon^*)<\frac{2q}{h_2}$ holds true, then a stable positive fixed point appears when ϵ passes through $\epsilon=\epsilon^*$ from right to left. Correspondingly, system (1.3) has a stable positive periodic solution for $\epsilon\in(\epsilon^*-\varepsilon,\epsilon^*)$ with $\varepsilon>0$ small enough;

(c) If $g''(0;\epsilon^*)=\frac{2q}{h_2}$ and $g'''(0;\epsilon^*)>\frac{6q(2q-1)}{h_2^2}$, then the Poincaré map $\mathcal{P}_M(I_0,\epsilon)$ undergoes a pitchfork bifurcation at $\epsilon=\epsilon*$. Accordingly, system (1.3) has an unstable positive periodic solution for $\epsilon\in(\epsilon^*,\epsilon^*+\varepsilon)$ with $\varepsilon>0$ small enough;

(d) If $g''(0;\epsilon^*)=\frac{2q}{h_2}$ and $g'''(0;\epsilon^*)<\frac{6q(2q-1)}{h_2^2}$, then $\mathcal{P}_M(I_0,\epsilon)$ undergoes a pitchfork bifurcation at $\epsilon=\epsilon^*$. Accordingly, system (1.3) has a stable positive periodic solution for $\epsilon\in(\epsilon^*-\varepsilon,\epsilon^*)$ with $\varepsilon>0$ small enough.

4.2. Bifurcation with respect to p

When $h_2>0$, R_b can be written as a function with respect to parameter p, given as:

$$R_b(p)=\left(\frac{A-\delta_1 S_v}{A-\delta_1 S_T}\right)^{\frac{\beta A-\delta_1(\delta_2+\gamma+\epsilon)}{\delta_1^2}}\exp\left(-\frac{\beta p S_T^2}{\delta_1(h_1+S_T)}\right).$$

Taking the derivative of $R_b(p)$ with respect to p, we obtain

$$\frac{\partial R_b(p)}{\partial p}=\frac{R_b(p)S_T^2}{(A-\delta_1 S_v)(h_1+S_T)}\left[\beta S_v-(\delta_2+\gamma+\epsilon)\right].$$

It is clear that $\frac{R_b(p)S_T^2}{(A-\delta_1 S_v)(h_1+S_T)}>0$, thus the sign of $\frac{\partial R_b(p)}{\partial p}$ is determined by $\beta S_v-\delta_2-\gamma-\epsilon$. Solving $\frac{\partial R_b(p)}{\partial p}=0$, we obtain a unique root, denoted by $\overline{p}$, with

$$\overline{p}=\left(1+\frac{h_1}{S_T}\right)\left(1-\frac{\overline{S}}{S_T}\right).$$

We further assume $\frac{h_1 S_T}{h_1+S_T} \leq \overline{S}$ to ensure that $\overline{p} \in (0,1)$. As a result, there is a unique $\overline{p}$ such that $S_v < \overline{S}$ and $\frac{\partial R_b(p)}{\partial p} < 0$ for $p > \overline{p}$, while $S_v > \overline{S}$ and $\frac{\partial R_b(p)}{\partial p} > 0$ for $p < \overline{p}$, which means that $R_b(p)$ is increasing on the interval $(0, \overline{p}]$ and decreasing on the interval $[\overline{p}, 1)$. Furthermore,

$$R_b(0) = 1, \quad R_b(\overline{p}) = \exp\left(\int_{\overline{S}}^{S_T} \frac{\beta s - \delta_2 - \gamma - \epsilon}{A - \delta_1 s} ds\right) > 1.$$

Therefore, considering the monotonicity of $R_b(p)$, we have
(1) If $p \in (0, \overline{p})$, then $R_b(p) > 1$ always holds, which indicates that the disease-free periodic solution $(\xi(t), 0)$ is unstable.
(2) If $p \in (\overline{p}, 1)$ and $R_b(1) > 1$, then $R_b(p) > 1$ for $p \in (0,1)$, indicating that $(\xi(t), 0)$ is always unstable.
(3) If $p \in (\overline{p}, 1)$ and $R_b(1) < 1$, then there is a unique p^* satisfying $R_b(p^*) = 1$. This means that $(\xi(t), 0)$ is unstable for $p \in (\overline{p}, p^*)$, while $(\xi(t), 0)$ is stable for $p \in (p^*, 1)$, indicating that the bifurcations could occur at $p = p^*$.

Proposition 4.2. *Suppose $h_2 > 0$, $R_0 > 1$ and $S_T > \overline{S}$. If $R_b(1) > 1$ holds true, then the disease-free periodic solution $(\xi(t), 0)$ is always unstable for $p \in (0,1)$; If $R_b(1) < 1$ holds, then the disease-free periodic solution $(\xi(t), 0)$ is unstable for $p \in (0, p^*]$ and orbitally asymptotically stable for $p \in [p^*, 1)$.*

Based on above discussions, we next consider the bifurcations with respect to p. We have $\mathcal{P}_M(0, p) = 0$ for all $p \in (0,1)$, and it is easy to see that

$$\frac{\partial \mathcal{P}_M}{\partial I_0}(0, p^*) = R_b(p^*) = 1, \quad \frac{\partial^2 \mathcal{P}_M}{\partial I_0 \partial p}(0, p^*) = \frac{\partial R_b(p^*)}{\partial p} < 0.$$

Moreover, there are

$$\begin{aligned} g''(0; p^*) &= g'(0; p^*) \int_{S_{vp^*}}^{S_T} m(s) \frac{\partial I(s; S_v, 0)}{\partial I_0} ds = \int_{S_{vp^*}}^{S_T} m(s) \frac{\partial I(s; S_v, 0)}{\partial I_0} ds, \\ g'''(0; p^*) &= \frac{4q^2}{h_2^2} + \frac{\partial}{\partial I_0}\left(\int_{S_{vp^*}}^{S_T} m(s) \frac{\partial I(s; S_v, 0)}{\partial I_0} ds\right), \end{aligned} \tag{4.2}$$

with $S_{vp^*} = \left(1 - \frac{p^* S_T}{h_1 + S_T}\right) S_T$. Thus,

$$\begin{aligned} \frac{\partial^2 \mathcal{P}_M}{\partial I_0^2}(0, p^*) &= g''(0; p^*) - \frac{2q}{h_2}. \\ \frac{\partial^3 \mathcal{P}_M}{\partial I_0^3}(0, p^*) &= g'''(0; p^*) - \frac{6q(2q-1)}{h_2^2}. \end{aligned} \tag{4.3}$$

Based on above discussions and Lemma $A.2$ and Lemma $A.3$ presented in Appendix A, we conclude as follows.

Theorem 4.2. *Suppose $h_2 > 0$, $R_0 > 1$, $S_T > \overline{S}$ and $R_b(1) < 1$. We have:*

(a) If $g''(0; p^*) > \frac{2q}{h_2}$ holds true, then the Poincaré map $\mathcal{P}_M(I_0, p)$ undergoes a transcritical bifurcation at p^*. Moreover, an unstable positive fixed point appears when p changes through $p = p^*$ from left to right. Then system (1.3) accordingly has an unstable positive periodic solution if $p \in (p^*, p^* + \varepsilon)$ with $\varepsilon > 0$ small enough;

(b) If $g''(0; p^*) < \frac{2q}{h_2}$ holds true, then a stable positive fixed point of map $\mathcal{P}_M(I_0, p)$ appears when p changes through $p = p^*$ from right to left. System (1.3) accordingly has a stable positive periodic solution if $p \in (p^* - \varepsilon, p^*)$ with $\varepsilon > 0$ small enough;

(c) If $g''(0;p^*) = \frac{2q}{h_2}$ and $g'''(0;p^*) > \frac{6q(2q-1)}{h_2^2}$ hold, then the Poincaré map $\mathcal{P}_M(I_0, p)$ undergoes a pitchfork bifurcation at $p = p^*$. Correspondingly, system (1.3) has an unstable positive periodic solution if $p \in (p^*, p^* + \varepsilon)$ with $\varepsilon > 0$ small enough;

(d) If $g''(0;p^*) = \frac{2q}{h_2}$ and $g'''(0;p^*) < \frac{6q(2q-1)}{h_2^2}$ hold, then $\mathcal{P}_M(I_0, p)$ undergoes a pitchfork bifurcation at $p = p^*$. Correspondingly, system (1.3) has a stable positive periodic solution if $p \in (p^* - \varepsilon, p^*)$ with $\varepsilon > 0$ small enough.

4.3. Bifurcation with respect to S_T

In this subsection, we choose S_T as a bifurcation parameter. When $h_2 > 0$, we take the derivative of $R_b(S_T)$ with respect to S_T and obtain

$$\frac{\partial R_b(S_T)}{\partial S_T} = \exp(J(S_T))\frac{\partial J(S_T)}{\partial S_T},$$

with $\frac{\partial J(S_T)}{\partial S_T} = \frac{\beta S_T-(\delta_2+\gamma+\epsilon)}{A-\delta_1 S_T} - \left(1 - \frac{pS_T(2h_2+S_T)}{(h_2+S_T)^2}\right)\frac{\beta S_v-(\delta_2+\gamma+\epsilon)}{A-\delta_1 S_v}$. Denote $f(x) = \frac{\beta s-(\delta_2+\gamma+\epsilon)}{A-\delta_1 s}$, we have

$$\frac{\partial J(S_T)}{\partial S_T} = f(S_T) - \left(1 - \frac{pS_T(2h_2 + S_T)}{(h_2 + S_T)^2}\right)f(S_v).$$

Furthermore, there is

$$f'(x) = \frac{\beta A-\delta_1(\delta_2+\gamma+\epsilon)}{(A-\delta_1 x)^2} > 0.$$

Thus, $f(x)$ is monotonically increasing with respect to x. In what follows, we discuss the sign of $\frac{\partial J(S_T)}{\partial S_T}$:
(1) If $S_v \le \overline{S}$, then $f(S_v) \le 0$, which indicates that $\frac{\partial J(S_T)}{\partial S_T} > 0$ always holds;
(2) If $S_v > \overline{S}$, then $f(S_v) > 0$, and one has

$$\frac{\partial J(S_T)}{\partial S_T} > f(S_v) - \left(1 - \frac{pS_T(2h_2 + S_T)}{(h_2 + S_T)^2}\right)f(S_v) = \frac{pS_T(2h_2 + S_T)}{(h_2 + S_T)^2}f(S_v) > 0.$$

This means that $\frac{\partial J(S_T)}{\partial S_T} > 0$ holds under both conditions. Hence, $\frac{\partial R_b(S_T)}{\partial S_T} > 0$ holds, i.e., $R_b(S_T)$ is monotonically increasing with respect to S_T. Denoting $K \doteq \frac{A}{\delta_1}$ for convenience, then we have

$$R_b(\overline{S}) < 1, \quad \lim_{S_T \to K^-} R_b(S_T) = +\infty.$$

Thus, there is a unique $S_T^* \in (\overline{S}, K)$ such that $R_b(S_T^*) = 1$. Based on above discussions, we conclude the following main results.

Proposition 4.3. *Suppose $h_2 > 0$ and $R_0 > 1$. There is a unique $S_T^* \in (\overline{S}, K)$ satisfying $R_b(S_T^*) = 1$. The disease-free periodic solution $(\xi(t), 0)$ of system (1.3) is orbitally asymptotically stable for $S_T \in (\overline{S}, S_T^*)$ and unstable for $S_T \in (S_T^*, K)$.*

In what follows, we discuss the bifurcation near the disease-free periodic solution at $S_T = S_T^*$. Similarly, $\mathcal{P}_M(0, S_T) = 0$ holds for all $S_T \in (\overline{S}, K)$, and

$$\frac{\partial \mathcal{P}_M}{\partial I_0}(0, S_T^*) = 1, \quad \frac{\partial^2 \mathcal{P}_M}{\partial I_0 \partial S_T}(0, S_T^*) > 0,$$
$$\frac{\partial^2 \mathcal{P}_M}{\partial I_0^2}(0, S_T^*) = g''(0; S_T^*) - \frac{2q}{h_2}, \quad \frac{\partial^3 \mathcal{P}_M}{\partial I_0^3}(0, S_T^*) = g'''(0; S_T^*) - \frac{6q(2q-1)}{h_2^2}.$$

Therefore, we obtain the following results.

Theorem 4.3. *Suppose $h_2 > 0$ and $R_0 > 1$. We have:*

(a) If $g''(0; S_T^*) > \frac{2q}{h_2}$ holds true, then an unstable positive fixed point appears when S_T goes through $S_T = S_T^*$ from right to left. Correspondingly, system (1.3) has an unstable positive periodic solution if $S_T \in (S_T^* - \varepsilon, S_T^*)$ with $\varepsilon > 0$ small enough;

(b) If $g''(0; S_T^*) < \frac{2q}{h_2}$ holds true, then a stable positive fixed point appears when S_T goes through $S_T = S_T^*$ from left to right. Correspondingly, system (1.3) has a stable positive periodic solution if $S_T \in (S_T^*, S_T^* + \varepsilon)$ with $\varepsilon > 0$ small enough.

(c) If $g''(0; S_T^*) = \frac{2q}{h_2}$ and $g'''(0; S_T^*) > \frac{6q(2q-1)}{h_2^2}$, then system (1.3) has an unstable positive periodic solution if $S_T \in (S_T^* - \varepsilon, S_T^*)$ with $\varepsilon > 0$ small enough;

(d) If $g''(0; S_T^*) = \frac{2q}{h_2}$ and $g'''(0; S_T^*) < \frac{6q(2q-1)}{h_2^2}$ hold true, then system (1.3) has a stable positive periodic solution if $S_T \in (S_T^*, S_T^* + \varepsilon)$ with $\varepsilon > 0$ small enough.

4.3.1. Bifurcation with respect to h_1

In this subsection, we choose h_1 as a bifurcation parameter and consider R_b as a function of h_1, which can help us to reveal the impact of the saturation phenomenon of state-dependent feedback control on infectious diseases. When $h_2 > 0$, we have $R_b(h_1) = \exp(J(h_1))$. By simple calculations we have

$$R_b(0) = \left(\frac{A - \delta_1(1-p)S_T}{A - \delta_1 S_T}\right)^{\frac{\beta A - \delta_1(\delta_2+\gamma+\epsilon)}{\delta_1^2}} \exp\left(-\frac{\beta p S_T}{\delta_1}\right), \quad \lim_{h_1 \to +\infty} R_b(h_1) = 1.$$

Moreover, taking the derivative of $R_b(h_1)$ with respect to h_1 yields

$$\frac{\partial R_b(h_1)}{\partial h_1} = \frac{pS_T^2 R_b(h_1)}{(A - \delta_1 S_v)(h_1 + S_T)^2)} * (\delta_2 + \gamma + \epsilon - \beta S_v).$$

Solving $\frac{\partial R_b(h_1)}{\partial h_1} = 0$, we obtain a unique root $\overline{h}_1$ with

$$\overline{h}_1 = \frac{S_T(\overline{S} - (1-p)S_T)}{S_T - \overline{S}}.$$

If $h_1 < \overline{h}_1$, then $\frac{\partial R_b(h_1)}{\partial h_1} > 0$ holds, while if $h_1 > \overline{h}_1$ holds, then $\frac{\partial R_b(h_1)}{\partial h_1} < 0$, indicating that $R_b(h_1)$ is increasing for $h_1 < \overline{h}_1$ and decreasing for $h_1 > \overline{h}_1$. If $\overline{S} < (1-p)S_T$, then we have $\overline{h}_1 < 0$ and correspondingly, $R_b(h_1)$ is decreasing on the interval $(0, +\infty)$. Thus, $R_b(h_1) > 1$ always holds and the disease-free periodic solution $(\xi(t), 0)$ is unstable and there is no bifurcation near $(\xi(t), 0)$. If $\overline{S} > (1-p)S_T$, then we have $\overline{h}_1 > 0$. Therefore, $R_b(h_1)$ is increasing on the interval $(0, \overline{h}_1]$ and decreasing on the interval $[\overline{h}_1, +\infty)$. Under this case, when $R_b(0) > 1$, then $R_b(h_1) > 1$ always holds for $h_1 \in (0, +\infty)$, which means that the disease-free periodic solution $(\xi(t), 0)$ is unstable and there is no bifurcation near $(\xi(t), 0)$. On the other hand, when $R_b(0) < 1$, there is a unique $h_1^* \in (0, \overline{h}_1)$ such that $R_b(h_1^*) = 1$ with $\frac{\partial R_b(h_1^*)}{\partial h_1} > 0$.
Therefore, we have the main conclusions as follows.

Proposition 4.4. *Suppose $h_2 > 0$, $R_0 > 1$ and $S_T > \overline{S} > (1-p)S_T$. If $R_b(0) < 1$ holds, then there exists a unique $h_1^* \in (0, \overline{h}_1)$ satisfying $R_b(h_1^*) = 1$ with $\frac{\partial R_b(h_1^*)}{\partial h_1} > 0$. Accordingly, the disease-free periodic* *solution $(\xi(t), 0)$ of system (1.3) is orbitally asymptotically stable for $h_1 \in (0, h_1^*)$ and unstable for $h_1 \in (h_1^*, +\infty)$.*

As for the bifurcation of the disease-free periodic solution $(\xi(t), 0)$ at h_1^*, we have $\mathcal{P}_M(0, h_1) = 0$ for all $h_1 \in (0, +\infty)$, and

$$\frac{\partial \mathcal{P}_M}{\partial I_0}(0, h_1^*) = 1, \quad \frac{\partial^2 \mathcal{P}_M}{\partial I_0 \partial h_1}(0, h_1^*) > 0,$$
$$\frac{\partial^2 \mathcal{P}_M}{\partial I_0^2}(0, h_1^*) = g''(0; h_1^*) - \frac{2q}{h_2}, \quad \frac{\partial^3 \mathcal{P}_M}{\partial I_0^3}(0, h_1^*) = g'''(0; h_1^*) - \frac{6q(2q-1)}{h_2^2}.$$

Therefore, we have the following conclusions.

Theorem 4.4. *Suppose $h_2 > 0$, $R_0 > 1$, $S_T > \overline{S}$ and $R_b(0) < 1$. We obtain:*

(a) If $g''(0; h_1^*) > \frac{2q}{h_2}$ holds, then the Poinceré map $\mathcal{P}_M(I_0, h_1)$ undergoes a transcritical bifurcation at $h_1 = h_1^*$. Further, an unstable positive fixed point appears when h_1 passes through $h_1 = h_1^*$ from right to left. Accordingly, system (1.3) has an unstable positive periodic solution for $h_1 \in (h_1^* - \varepsilon, h_1^*)$ with $\varepsilon > 0$ small enough;

(b) If $g''(0; h_1^*) < \frac{2q}{h_2}$ holds, then a stable positive fixed point appears when h_1 passes through $h_1 = h_1^*$ from left to right. Then, system (1.3) has a stable positive periodic solution for $h_1 \in (h_1^*, h_1^* + \varepsilon)$ with $\varepsilon > 0$ small enough;

(c) If $g''(0; h_1^*) = \frac{2q}{h_2}$ and $g'''(0; h_1^*) \neq \frac{6q(2q-1)}{h_2^2}$ holds, then the Poinceré map $\mathcal{P}_M(I_0, h_1)$ undergoes a pitchfork bifurcation at $h_1 = h_1^*$. Accordingly, system (1.3) has a positive periodic solution.

Note that the bifurcation with respect to the demographic parameters, such as the recruitment rate A, can also be studied. Here we only give the main conclusions for the bifurcation with respect to A, and the detailed analyses are given in Appendix B.

Theorem 4.5. *Suppose $h_2 > 0$, $R_0 > 1$, and $S_T > \overline{S} > \frac{S_v + S_T}{2}$. If $g''(0; A^*) \neq \frac{2q}{h_2}$ holds true, then the Poincaré map $\mathcal{P}_M(I_0, A)$ occurs with a transcritical bifurcation at $A = A^*$. Thus, a positive fixed point appears when A goes through $A = A^*$, and correspondingly, system (1.3) has a positive periodic solution. However, if $g''(0; A^*) = \frac{2q}{h_2}$ and $g'''(0; A^*) \neq \frac{6q(2q-1)}{h_2^2}$ hold, then the Poincaré map $\mathcal{P}_M(I_0, A)$ undergoes a pitchfork bifurcation at $A = A^*$. Thus, a positive fixed point appears when A passes through $A = A^*$, and accordingly, system (1.3) has a positive periodic solution.*

So far we have discussed the bifurcation with respect to key parameters including ϵ, p, S_T, h_1 and A for $h_2 > 0$. Similarly, we can also investigate the bifurcation with these parameters for $h_2 = 0$, and list the main results with respect to parameter q in the following and find details in Appendix B.

Theorem 4.6. *Suppose $h_2 = 0$, $R_0 > 1$, $S_T > \overline{S}$ and $J > 0$. If $g''(0; q^*) \neq 0$ holds true, then the Poincaré map $\mathcal{P}_M(I_0, q)$ undergoes a transcritical bifurcation at $q = q^*$. In fact, if $g''(0; q^*) > 0$ holds true, then an unstable positive fixed point appears when q goes through $q = q^*$ from left to right. Correspondingly, system (1.3) has an unstable positive periodic solution if $q \in (q^*, q^* + \varepsilon)$ with $\varepsilon > 0$ small enough. However, if $g''(0; q^*) < 0$, then the Poincaré map $\mathcal{P}_M(I_0, q)$ has a stable positive fixed point when p passes through $q = q^*$ from right to left. Correspondingly, system (1.3) has a stable positive periodic solution if $q \in (q^* - \varepsilon, q^*)$ with $\varepsilon > 0$ small enough.*

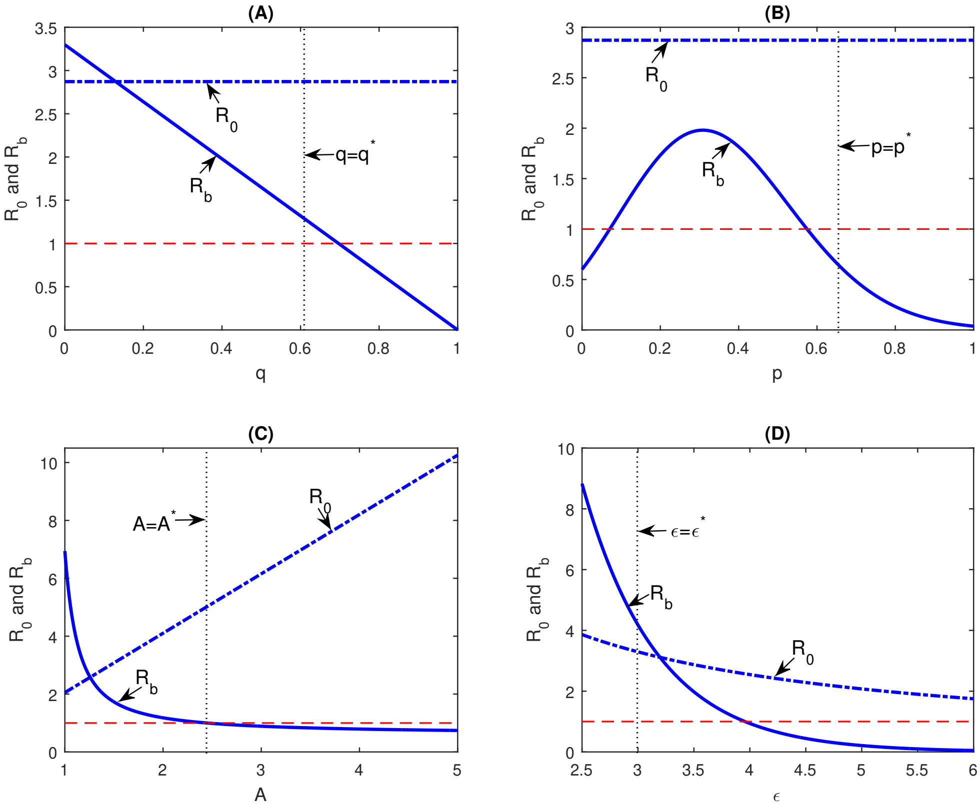

Figure 2. The one parameter bifurcation diagram of R_0 and R_b with respect to parameters q, p, A, ϵ. The baseline parameter values are $A = 1.4, \beta = 1.2, \delta_1 = 0.15, \delta_2 = 0.3, \epsilon = 3.5, \omega = 1.2, \gamma = 0.1, S_T = 4.5, p = 0.3, q = 0.4, h_1 = 0.5, h_2 = 0$.

Through numerical simulation, we verify the existence of the transcritical bifurcation with respect to some key parameters. We illustrated the relationships between R_0 and R_b with respect to parameters p, ϵ, A, q (shown in Figure 2) and parameter S_T (shown in Figure 3(A)). We find that for all these parameters, there exists a threshold value such that $R_b = 1$. This confirms the existence of the transcritical bifurcation by choosing these parameters as bifurcation parameters. As shown in Figure 3, the disease-free periodic solution is locally stable for $S_T < S_T^*$ and unstable for $S_T > S_T^*$. Correspondingly, in Figure 3(D), we choose $S_T = 3.6$ such that $S_T > S_T^*$, and show that the disease-free periodic solution is unstable and all the orbits finally tend to the positive equilibrium E_1. In Figure 3(C), as we decrease the parameter value of S_T to 2.8 such that $S_T < S_T^*$, the disease-free periodic solution becomes stable, which is bistable with the positive equilibrium E_1. It follows from Figure 3(C) that an unstable positive order-1 periodic solution appears as well via the transcritical bifurcation. Furthermore, in Figure 3(B), by choosing $S_T = 1.7$ such that $S_T < S_1$, the disease-free periodic solution becomes globally stable.

Similarly, we verified the main theoretical results and showed in Figure 4 that when the ODE subsystem has limit cycles, system (1.3) can also undergo the transcritical bifurcation with an unstable positive order-1 periodic solution appearing. Specifically, when there exists a unique stable limit cycle of subsystem (1.2) (Figure 4(A) and (B)), if we decrease the threshold value S_T from $S_T = 3.6$ (Figure 4(B)) to $S_T = 3.4$ (Figure 4(A)), then an unstable positive order-1 periodic solution appears and the limit cycle of subsystem (1.2) is bistable with the disease-free periodic solution, shown in Figure 4(A). Similar phenomenons are illustrated in Figure 4(C) and (D) when there are two limit cycles of subsystem (1.2). Note that in Figure 4, we have chosen the threshold level relatively large such that the impulsive line $S = S_T$ did not intersect with limit cycles. If the impulsive line intersects with the limit cycle, the Poincaré map of the system becomes very complex [30] while the dynamical behaviours are very rich and complicated. In Figure 5, we showed that by changing the parameter value of p, the unstable positive order-1 periodic solution (bifurcated from the disease-free periodic solution) can co-exist with a stable positive order-1 periodic solution (Figure 5(A)), or a stable positive order-2 periodic solution (Figure 5(B)), or a stable positive order-3 periodic solution (Figure 5(C)). For another aspect, the existence of order-3 periodic solution implies the existence of the phenomenon of chaos, which is illustrated in Figure 5(D).

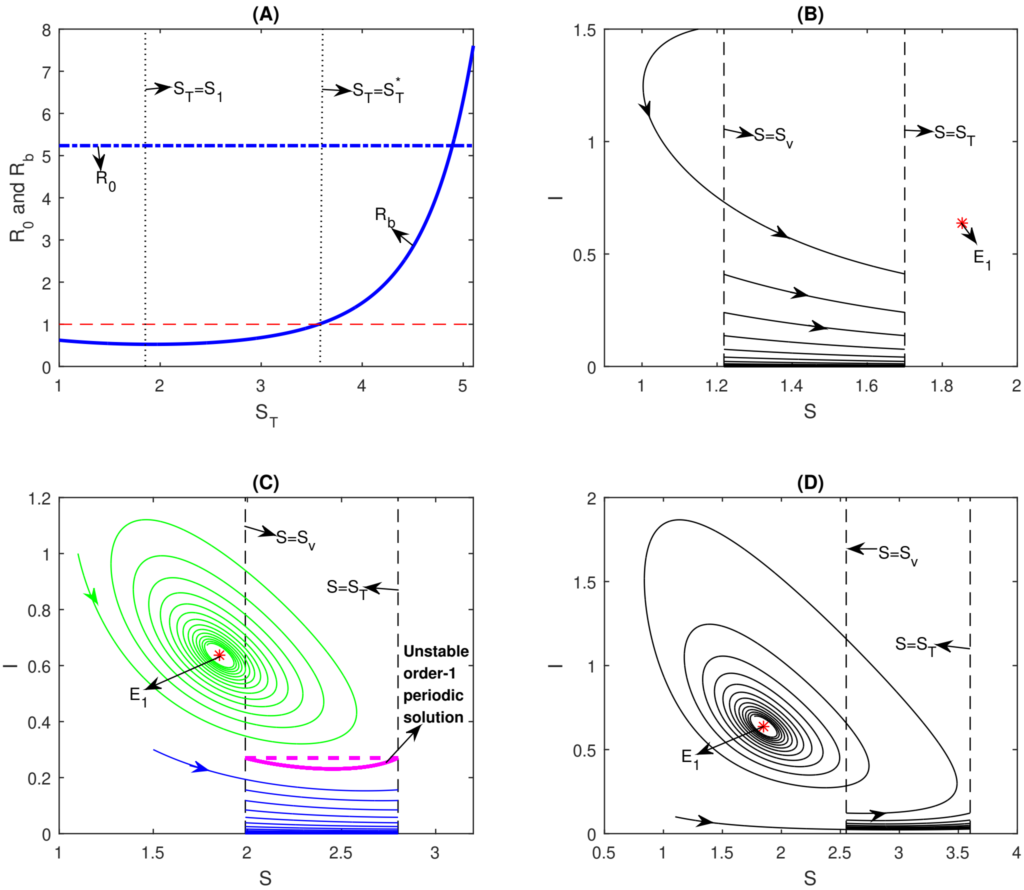

Figure 3. (A) Bifurcation diagram of R_b with respect to S_T. (B) Phase portraits of system (1.3) when $S_T < S_1$ with $S_T = 1.7$. (C) Phase portraits of system (1.3) when $S_1 < S_T < S_T^*$ with $S_T = 2.8$. (D) Phase portraits of system (1.3) when $S_T > S_T^*$ with $S_T = 3.6$. The other parameter values are: $A = 2.4, \beta = 1.8, \delta_1 = 0.15, \delta_2 = 0.4, \epsilon = 5, \omega = 1.2, \gamma = 0.1, p = 0.3, q = 0.015, h_1 = 0.1, h_2 = 1$.

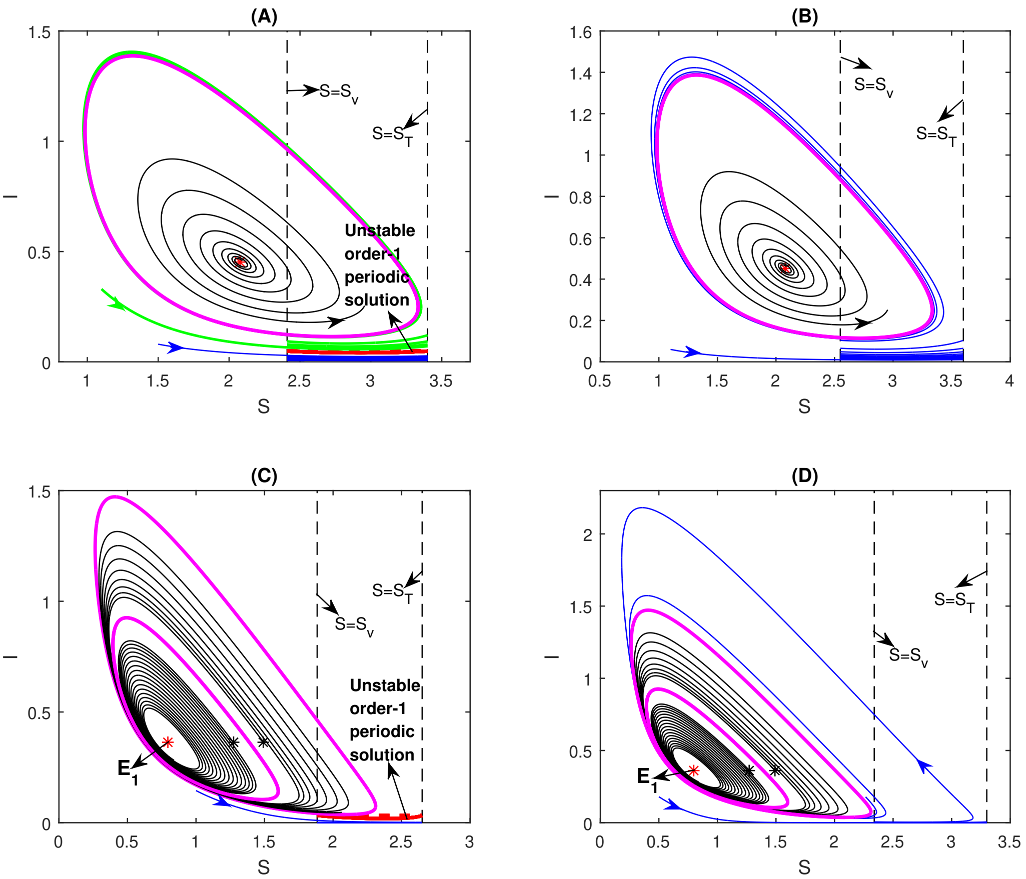

Figure 4. Phase portraits of system (1.3) when ODE subsystem (1.2) has limit cycle. The control parameter values are: $p = 0.3, q = 0.015, h_1 = 0.1, h_2 = 1$ with $S_T = 3.4$ in (A), $S_T = 3.6$ in (B), $S_T = 2.65$ in (C) and $S_T = 3.3$ in (D). The other parameter values of (A-B) and (C-D) are fixed as the same as those in Figure 1(C) and Figure 1(D), respectively.

5. Properties of the Poincaré map $\mathcal{P}_M$

5.1. Impulsive set and phase set of the Poincaré map

In order to further discuss the existence and stability of the positive order-1 periodic solution of system (1.3), we initially define the impulsive set and phase set of the Poincaré map for various cases. For case (C_1), the disease-free equilibrium $E_0\left(\frac{A}{\delta_1}, 0\right)$ is globally asymptotically stable. As shown in Figure 6(A), depending on the properties of the vector fields of subsystem (1.2), it is easily verified that there is an orbit Γ_1 tangent to S_{S_v} at point $Q_{S_v} = (S_v, I_{S_v})$ with $I_{S_v} = \frac{A-\delta_1 S_v}{\beta S_v}$. The intersection point of Γ_1 to S_{S_T} can be denoted as

$$Q^* = (S_T, I^*) = (S_T, I(S_T; S_v, I_{S_v}))\,.$$

Then the impulsive set is

$$M_1 = \{(S, I)|S = S_T, I \in [0, I^*]\},$$

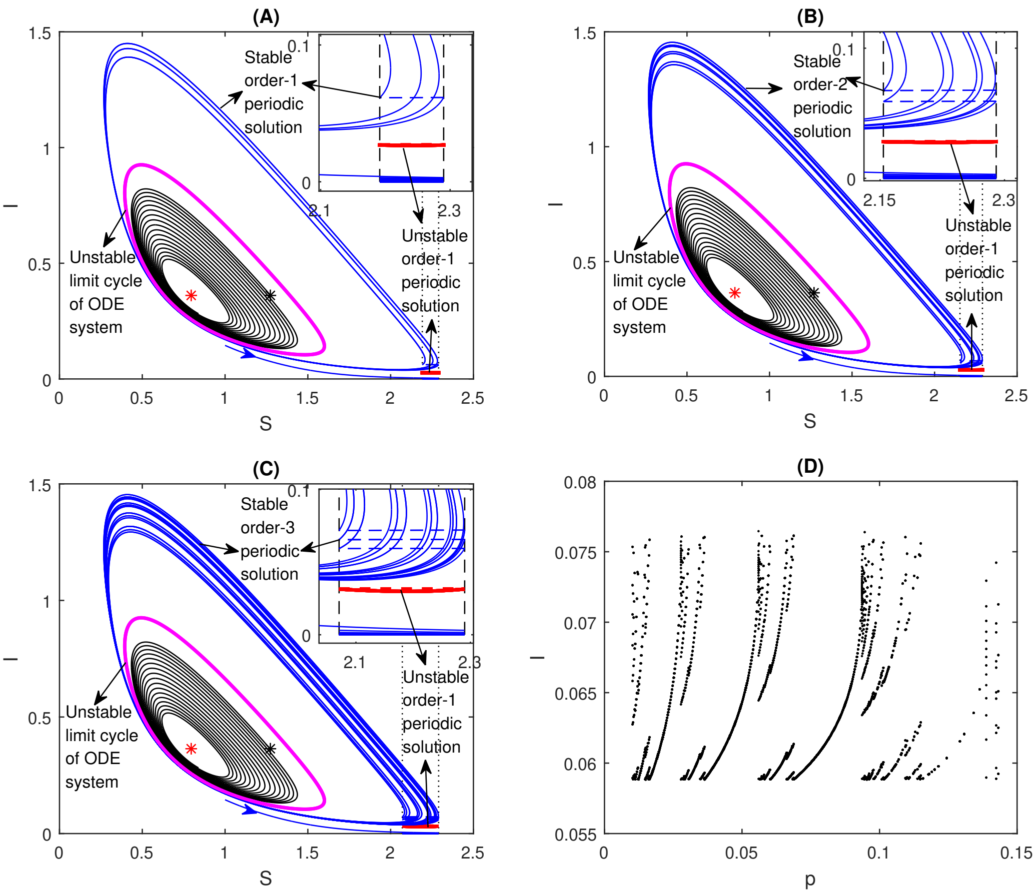

Figure 5. (A-C) Phase portraits of system (1.3) with $p = 0.045$ in (A), $p = 0.062$ in (B) and $p = 0.1$ in (C). (D) Bifurcation diagram of the positive order-k periodic solution with respect to p. The other control parameter values are $S_T = 2.29, q = 0.05, h_1 = 0.1, h_2 = 1$, and the rest parameter value are the same as those in Figure 1(D).

and the phase set can be defined as:

$$N_1 = \{(S^+, I^+)|S^+ = S_v, I^+ \in [0, w_1(I^*)]\}.$$

For case (C_2), due to the complex trajectories of subsystem (1.2), we cannot determine the exact domains of the impulsive set and phase set. Under scenario (C_3), there exists a unique endemic equilibrium $E_1(S_1, I_1)$ which is globally stable. In what follows, we consider $\Delta < 0$ implying E_1 is a focus. If $S_T < S_1$ holds, denoted as case (C_{31}), we can define the definitions of impulsive set and phase set of system (1.3) as M_1 and N_1, respectively, which is similar to case (C_1).

When $S_T > S_1$, there is an orbit Γ_2 tangent to section S_{S_T} at point $Q_{S_T} = (S_T, I_{S_T})$ with $I_{S_T} = \frac{A-\delta_1 S_T}{\beta S_T}$ and Γ_2 intersects with line l_1 at point $L(S_l, I_l)$, as shown in Figure 6(B-C). Then we consider the following two subcases:

$$(C_{32}) \quad S_v < S_l \quad \text{and} \quad (C_{33}) \quad S_v \geq S_l.$$

For subcase (C_{32}), the impulsive set and phase set are M_1 and N_1, respectively, through similar methods used for case (C_1). Note that for (C_{33}), the orbit Γ_2 intersects with line l_4 at two points $B_1(S_v, I_{b_1})$ and

$B_2(S_v, I_{b_2})$ with $I_{b_1} < I_{b_2}$, shown in Figure 6(C). Moreover, the orbit Γ_2 will reach line l_4 at $Q_{S_v} = (S_v, I_{S_v})$ with $I(S_T; S_v, I_{S_v}) = I_{S_T}$. This indicates that any solution of system (1.3) with initial value (S_v, I_0^+), where $I_0^+ \in (0, I_{S_v})$, will reach l_4 in a finite time. Thus, we can define the impulsive set and the phase set of system (1.3) as:

$$M_2 = \{(S, I)|S = S_T, I \in [0, I_{S_T}]\},$$

and

$$N_2 = \{(S^+, I^+)|S = S_v, I^+ \in [0, w_1(I_{S_T})] \cap [0, I_{b_1}]\}.$$

For case (C_4), there exists at least one limit cycle. Assuming that E_1 is an unstable focus and there is a unique stable limit cycle of subsystem (1.2), shown in Figure 1(C), then we discuss the impulsive set and the phase set for the Poincaré map $\mathcal{P}_M$ of system (1.3). In this circumstance, the limit cycle intersects with line l_1 at two points $T_1(S_{t_1}, I_{t_1})$ and $T_2(S_{t_2}, I_{t_2})$ with $S_{t_1} < S_{t_2}$. Depending on the positions between S_T, S_1 and S_{t_2}, we consider three subcases as follows:

$$(C_{41}) \quad S_T \leq S_1, \quad (C_{42}) \quad S_1 < S_T < S_{t_2}, \quad \text{and} \quad (C_{43}) \quad S_T \geq S_{t_2}.$$

When (C_{41}) holds true, by using similar methods for case (C_1), it is clear that the impulsive set and the phase set are M_1 and N_1, respectively. When $S_1 < S_T < S_{t_2}$ (i.e., subcase (C_{42})), we consider:

$$(C_{42}^a) \quad S_v \leq S_{t_1}, \quad (C_{42}^b) \quad S_{t_1} < S_v < S_1, \quad \text{and} \quad (C_{42}^c) \quad S_1 \leq S_v < S_T.$$

If $S_v \leq S_{t_1}$ (i.e., (C_{42}^a)) holds, the impulsive set and the phase set can also be defined as M_1 and N_1, respectively. For subcase (C_{42}^b), there are two possible cases depending on whether orbit Γ_2 crosses line l_4 before it is tangents to line l_3 at point Q_{S_T}. If Γ_2 crosses line l_4 before it is tangents to line l_3 and Γ_2 intersects with line l_4 at two points $\gamma_1(S_{\gamma_1}, I_{\gamma_1})$ and $\gamma_2(S_{\gamma_2}, I_{\gamma_2})$ with $I_{\gamma_1} < I_{\gamma_2}$, denoted as case $C_{42}^{b_1}$, the impulsive set is defined as M_2 and the phase set is

$$N_3 = \{(S^+, I^+)|S = S_v, I^+ \in [0, w_1(I_{S_T})]\}.$$

However, if Γ_2 crosses line l_4 after it is tangents to line l_3, denoted as case $C_{42}^{b_2}$, then the impulsive set and the phase set are M_1 and N_1, respectively.

For subcase (C_{42}^c), the impulsive set and the phase set can be similarly defined as those for subcase (C_{42}^b) with M_2 and N_3, respectively.

When $S_T \geq S_{t_2}$ (i.e., (C_{43})), depending on the position between S_v and S_{t_1}, we consider the following two subcases:

$$(C_{43}^a) \quad S_v < S_{t_1} \quad \text{and} \quad (C_{43}^b) \quad S_v \geq S_{t_1}.$$

Under scenario (C_{43}^a), the impulsive set and the phase set are defined as M_1 and N_1, respectively. However, for subcase (C_{43}^b), the limit cycle intersects with line l_4 at two points $C_1(S_{c_1}, I_{c_1})$ and $C_2(S_{c_2}, I_{c_2})$ with $I_{c_1} < I_{c_2}$ and it is clear that the impulsive set and the phase set are M_2 and N_3, respectively.

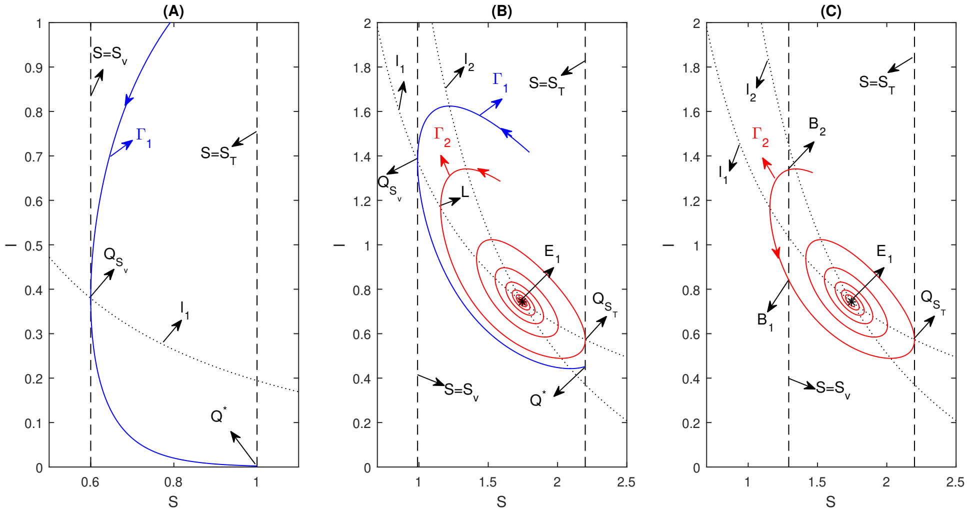

Figure 6. The illustration of the domain of the Poincaré map for the case (C_1), (C_{32}) and (C_{33}), respectively. (A) $A = 0.5, p = 0.8, S_T = 1$; (B) $A = 2.6, p = 0.8, S_T = 2.2$;(C) $A = 2.6, p = 0.6, S_T = 2.2$. The other parameter values are $\beta = 1.8, \delta_1 = 0.15, \delta_2 = 0.4, \epsilon = 5, \omega = 1.2, \gamma = 0.1, q = 0.01, h_1 = 1, h_2 = 0.5$.

5.2. *Properties of the Poincaré map $\mathcal{P}_M$*

In this subsection, based on above discussions of the impulsive set and the phase set of the Poincaré map, we further discuss the existence and stability of the positive order-1 periodic solution of system (1.3) through analyzing the properties of the Poincaré map. As we mentioned above, based on various ODE dynamical behaviors, the definition of $\mathcal{P}_M$, especially for the domain and the range of it, could be various. Thus, we also consider the properties of the Poincaré map in different cases of the dynamics of the ODE subsystem. Due to the complex trajectories of subsystem (1.2) for case (C_2), we cannot determine the exact domains of the impulsive set and the phase set, indicating that it is difficult to study the properties of the Poincaré map for case (C_2). Therefore, we focus on investigating the properties of the Poincaré map for cases (C_1), (C_3) and (C_4). For case (C_1), we have the following results.

Theorem 5.1. *For case (C_1), the Poincaré map $\mathcal{P}_M$ of system (1.3) satisfies the following properties.*
(1) The domain and range of $\mathcal{P}_M$ are $[0, +\infty)$ and $[0, w_1(I^*)]$, respectively. $\mathcal{P}_M$ is increasing on $[0, I_{S_v}]$ and decreasing on $[I_{S_v}, +\infty)$;
(2) $\mathcal{P}_M$ is continuously differentiable on its domain and convex on $[0, I_{S_v}]$ provided that $\frac{\partial^2 \mathcal{P}_M(I_0)}{\partial I_0^2} > 0$ for all $I_0 \in [0, I_{S_v}]$;
(3) There exists no positive fixed point for $\mathcal{P}_M$.

Proof (1) The vector field of system (1.3) without impulsive strategies implies that the domain of $\mathcal{P}_M$ is $[0, +\infty)$. For any $I_{k1}^+, I_{k2}^+ \in [0, I_{S_v}]$ with $I_{k1}^+ < I_{k2}^+$, it is clear that $g(I_{k1}^+) < g(I_{k2}^+)$, and consequently, $\mathcal{P}_M(I_{k1}^+) < \mathcal{P}_M(I_{k2}^+)$. For any $I_{k1}^+, I_{k2}^+ \in [I_{S_v}, +\infty)$ with $I_{k1}^+ < I_{k2}^+$, the orbits initiating from (S_v, I_{k1}^+) and (S_v, I_{k2}^+) will cross line l_4 before they hit line l_3. Denoting the vertical coordinates of the two orbits intersecting with line l_4 as I_{q1}^+ and I_{q2}^+, we note that $I_{q1}^+ > I_{q2}^+$. Similarly, we have $g(I_{q1}^+) > g(I_{q2}^+)$ and $\mathcal{P}_M(I_{k1}^+) = \mathcal{P}_M(I_{q1}^+) > \mathcal{P}_M(I_{q2}^+) = \mathcal{P}_M(I_{k2}^+)$. Therefore, $\mathcal{P}_M$ is increasing on the interval $[0, I_{S_v}]$ and decreasing on the interval $[I_{S_v}, +\infty)$. Meanwhile, The range of $\mathcal{P}_M$ is $[0, \mathcal{P}_M(I_{S_v})]$ (i.e., $[0, w_1(I^*)]$).

(2) It follows from (4.1) that

$$\begin{aligned}\frac{\partial W(S,I)}{\partial I} &= \frac{(A-\delta_1 S)\left(\beta S-\delta_2-\gamma-\frac{\epsilon}{(1+\omega I)^2}\right)}{(A-\beta SI-\delta_1 S)^2},\\ \frac{\partial^2 W(S,I)}{\partial I^2} &= \frac{(A-\delta_1 S)\left(\frac{\epsilon\omega(A-\beta SI-\delta_2 S)}{(1+\omega I)^3}+2\beta S\left(\beta S-\delta_2-\gamma-\frac{\epsilon}{(1+\omega I)^2}\right)\right)}{(A-\beta SI-\delta_1 S)^3}.\end{aligned}$$

According to the theorem of Cauchy and Lipschitz with parameters on the scalar differential equation, we obtain

$$\frac{\partial I(s,I_0)}{\partial I_0} = \exp\left(\int_{S_v}^{s}\frac{\partial}{\partial I}W(z,I(z,I_0))dz\right) > 0,$$

and

$$\frac{\partial^2 I(s,I_0)}{\partial I_0^2} = \frac{\partial I(s,I_0)}{\partial I_0}\exp\int_{S_v}^{s}\frac{\partial^2}{\partial I^2}W(z,I(z,I_0))\frac{\partial I(z,I_0)}{\partial I_0}dz.$$

Following from the definition of function $\mathcal{P}_M(I_0) = I(S_T, I_0)\left(1 - \frac{qI(S_T,I_0)}{h_2+I(S_T,I_0)}\right)$, we have

$$\frac{\partial\mathcal{P}_M(I_0)}{\partial I_0} = \frac{\partial I(S_T,I_0)}{\partial I_0}\left(1 - \frac{qI(S_T,I_0)(2h_2+I(S_T,I_0))}{(h_2+I(S_T,I_0))^2}\right),$$

and

$$\frac{\partial^2\mathcal{P}_M(I_0)}{\partial I_0^2} = \frac{\partial^2 I(S_T,I_0)}{\partial I_0^2}\left(1 - \frac{qI(S_T,I_0)(2h_2+I(S_T,I_0))}{(h_2+I(S_T,I_0))^2}\right) + \left(\frac{\partial I(S_T,I_0)}{\partial I_0}\right)^2\frac{2qh_2^2}{(h_2+I(S_T,I_0))^3}.$$

Based on above discussions, we conclude that $\frac{\partial\mathcal{P}_M(I_0)}{\partial I_0} > 0$ while the sign of $\frac{\partial^2\mathcal{P}_M(I_0)}{\partial I_0^2}$ is not determined. Therefore, if $\frac{\partial^2\mathcal{P}_M(I_0)}{\partial I_0^2} > 0$ holds true on the interval $[0, I_{S_v}]$, $\mathcal{P}_M$ is convex on the interval $[0, I_{S_v}]$.

(3) Note that $\frac{dI}{dt} < 0$ always holds due to the assumption that $S_T < \frac{A}{\delta_1}$. Therefore, for any initial point (S_v, I_0) on line l_4, there is $g(I_0) < I_0$. Furthermore, there is $\mathcal{P}_M(I_0) = w_1(g(I_0))$. Thus, we have $\mathcal{P}_M(I_0) < I_0$ for $I_0 \in [0, +\infty)$. This means that there is no positive fixed point for the Poincaré map $\mathcal{P}_M$. This completes the proof.

According to the third property in Theorem 5.1, we obtain that there is no positive order-1 periodic solution of system (1.3) for case (C_1). Furthermore, it is clear that for case (C_{31}), the properties are the same as those shown in Theorem 5.1. Correspondingly, there exists no positive order-1 periodic solution of system (1.3) for case (C_{31}). In what follows, we initially investigate the existence and stability of the positive order-1 periodic solutions under case (C_{32}). Similar to the properties proposed in Theorem 5.1, we can conclude that the domain and range of $\mathcal{P}_M$ are $[0, +\infty]$ and $[0, w_1(I^*)]$, respectively, and $\mathcal{P}_M$ is increasing on the interval $[0, I_{S_v}]$ and decreasing on the interval $[I_{S_v}, +\infty)$. Furthermore, $\mathcal{P}_M$ is convex on $[0, I_{S_v}]$ provided that $\frac{\partial^2\mathcal{P}_M(I_0)}{\partial I_0^2} > 0$ for all $I_0 \in [0, I_{S_v}]$. It is easy to see that $I^* < I_{S_T}$ and $I_{S_T} < I_{S_v}$. Thus, the relationship between I^* and I_{S_v} is $\mathcal{P}_M(I_{S_v}) = w_1(I^*) < I^* < I_{S_v}$. Combining with $\frac{\partial^2\mathcal{P}_M(I_0)}{\partial I_0^2} > 0$ for all $I_0 \in [0, I_{S_v}]$, we have that $\mathcal{P}_M(I_0) < I_0$ holds for all $I_0 \in [0, I_{S_v}]$ and there is no positive fixed point of $\mathcal{P}_M$. Accordingly, there is no positive order-1 periodic solution of system (1.3). Therefore, we have the following conclusion:

Theorem 5.2. *For case (C_{31}), there is no fixed point of the Poincaré map, hence no positive order-1 periodic solution is feasible for system (1.3). For case (C_{32}), if $\frac{\partial^2\mathcal{P}_M(I_0)}{\partial I_0^2} > 0$ holds true for all $I_0 \in [0, I_{S_v}]$, there exists no positive periodic solution of system (1.3), shown in Figure 7(A).*

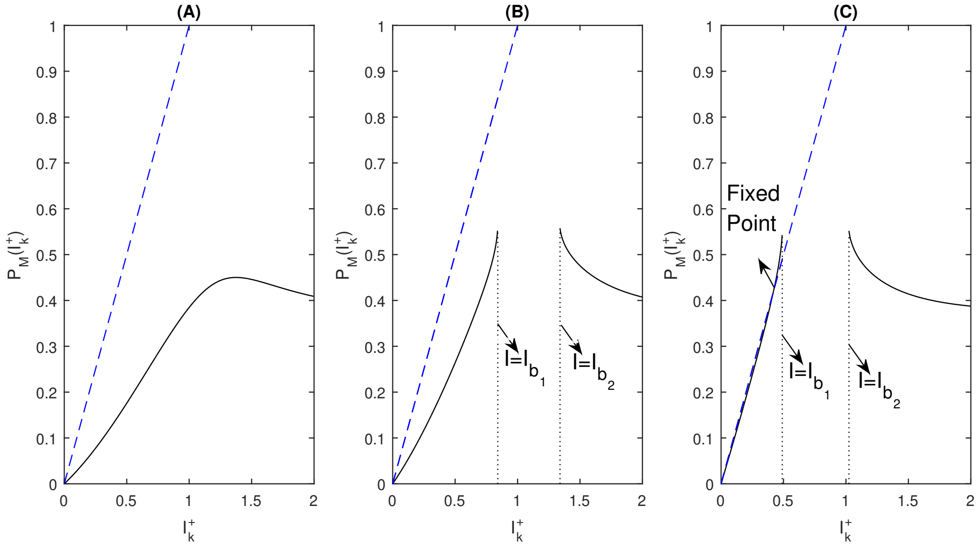

Figure 7. (A) Poincaré map of system (1.3) for the case (C_{32}) with $p = 0.8$. The dashed line represents $\mathcal{P}_M(I_k^+) = I_k^+$ and the black curve denotes $\mathcal{P}_M(I_k^+)$. (B-C) Poincaré map of system (1.3) for the case (C_{33}) with $p = 0.6$ in (B) and $p = 0.08$ in (C). The other parameter values are $A = 2.6, \beta = 1.8, \delta_1 = 0.15, \delta_2 = 0.4, \epsilon = 5, \omega = 1.2, \gamma = 0.1, S_T = 2.2, q = 0.01, h_1 = 1$, $h_2 = 0.5$.

As for case (C_{33}), we get the main properties of the Poinceré map $\mathcal{P}_M$ as follows.

Theorem 5.3. *For case* (C_{33})*, we obtain the following results of the Poinceré map* $\mathcal{P}_M$*:*
(1) The domain and range of the Poinceré map $\mathcal{P}_M$ *are* $[0, I_{b_1}] \cup [I_{b_2,+\infty})$ *and* $[0, \omega_1(I_{S_T})]$*, respectively;*
(2) $\mathcal{P}_M$ *is continuous on the two intervals* $[0, I_{b_1}]$ *and* $[I_{b_2}, +\infty)$*. Moreover, it is increasing on the interval* $[0, I_{b_1}]$ *and decreasing on the interval* $[I_{b_2}, +\infty)$*;*
(3) Suppose $\frac{\partial^2 \mathcal{P}_M(I_0)}{\partial I_0^2} > 0$ *holds true for all* $I_0 \in [0, I_{b_1}]$*. If* $\mathcal{P}_M(I_{b_1}) < I_{b_1}$*, then there is no positive fixed point of* $\mathcal{P}_M$*, shown in Figure 7 (B). Accordingly, there is no positive order-1 periodic solution of system (1.3). If* $\mathcal{P}_M(I_{b_1}) > I_{b_1}$ *holds, there exists a unique fixed points belonging to* $[0, I_{b_1}]$*, shown in Figure 7 (C). Then, system (1.3) has a unique positive order-1 periodic solution.*

Proof The methods of the proof of properties (1) and (2) are similar to the proof of Theorem 5.1. Thus, in the following we focus on proving the existence of the positive order-1 periodic solution in property (3). We know that if $\frac{\partial^2 \mathcal{P}_M(I_0)}{\partial I_0^2} > 0$ for all $I_0 \in [0, I_{b_1}]$, then $\mathcal{P}_M$ is convex on the interval $[0, I_{b_1}]$. Then there must be an interval $(0, \delta] \in [0, I_{b_1}]$ such that $\mathcal{P}_M(I_0) < I_0$ for all $I_0 \in (0, \delta]$. When $\mathcal{P}_M(I_{b_1}) < I_{b_1}$ holds, it is clear that $\mathcal{P}_M(I_0) < I_0$ for $I_0 \in [0, I_{b_1}]$. Therefore, there is no fixed point belonging to $[0, I_{b_1}]$. Moreover, as a result of $I_{b_2} > I_l > I_{S_T}$, we have $\mathcal{P}_M(I_{b_2}) < \mathcal{P}_M(I_{b_1}) < I_{S_T} < I_{b_2}$. Then $\mathcal{P}_M(I_0) < \mathcal{P}_M(I_{b_2}) < I_{b_2} < I_0$ for all $I_0 \in (I_{b_2}, +\infty)$. Thus, there exists no fixed point belonging to $[I_{b_2}, +\infty]$. Then we conclude that there exists no positive fixed point of $\mathcal{P}_M$ and there is no positive order-1 periodic solution of system (1.3). However, if $\mathcal{P}_M(I_{b_1}) > I_{b_1}$ holds, there is a unique fixed point $\bar{I} \in (\delta, I_{b_1})$ satisfying $\mathcal{P}_M(\bar{I}) = \bar{I}$ due to the continuity and convexity of $\mathcal{P}_M$. As mentioned above, there is no fixed point on the interval $[I_{b_2}, +\infty]$. Therefore, there exists a unique fixed point $\bar{I} \in (\delta, I_{b_1})$ of $\mathcal{P}_M$. Correspondingly, system (1.3) has a unique positive order-1 periodic solution. The proof is completed.

Remark 1. *Note that if* $\frac{\partial^2 \mathcal{P}_M(I_0)}{\partial I_0^2} > 0$ *for all* $I_0 \in [0, I_{b_1}]$ *and* $\mathcal{P}_M(I_{b_1}) > I_{b_1}$ *hold, then* $0 < \frac{\partial \mathcal{P}_M(I_0)}{\partial I_0} < 1$ *holds true on the interval* $[0, \bar{I}]$*. Therefore, we obtain* $|\mu_2| < 1$*. According to the properties of the Poinceré map* $\mathcal{P}_M$*, the unique positive order-1 periodic solution of system (1.3) is unstable, which matches the conclusions shown in the study of the bifurcations near the disease-free periodic solution of system (1.3).*

When there is a unique stable limit cycle of subsystem (1.2), we mainly consider the most complicated subcase, i.e., case (C_{42}^c). Although the domain of the Poinceré map $\mathcal{P}_M$ is $[0, +\infty)$ for case (C_{42}^c), the continuity and monotonicity of $\mathcal{P}_M$ can be much more complex. Therefore, we further discuss the properties of $\mathcal{P}_M$ for case (C_{42}^c) in more details. When orbit Γ_2 intersects with line l_4 (i.e., the line $S = S_v$) at a unique point $P(S_v, I_p)$ before it is tangents to line l_3 (i.e., the line $S = S_T$), shown in Figure 8 (A), we have the following conculsions.

Theorem 5.4. *For case* (C_{42}^c)*, if there exists a unique discontinuous point P, then the Poinceré map* $\mathcal{P}_M$ *satisfies the following properties:*
(1) The domain and range of the Poinceré map $\mathcal{P}_M$ *are* $[0, \infty)$ *and* $[0, w_1(I_{S_T})]$*, respectively;*
(2) $\mathcal{P}_M$ *is continuous on the intervals* $[0, I_p]$*,* $(I_p, I_{S_v}]$ *and* $[I_{S_v}, +\infty)$*. Moreover, it is increasing on the intervals* $[0, I_p]$ *and* $(I_p, I_{S_v}]$ *and decreasing on the interval* $[I_{S_v}, +\infty)$*;*
(3) Suppose $\frac{\partial^2 \mathcal{P}_M(I_0)}{\partial I_0^2} > 0$ *holds true for all* $I_0 \in [0, I_p]$*. If* $\mathcal{P}_M(I_p) < I_p$*, then there is no positive fixed point of* $\mathcal{P}_M$ *and no positive periodic solution of system (1.3). If* $\mathcal{P}_M(I_p) > I_p$ *holds, then there may exist one or two positive fixed points, shown in Figure 8 (B-C). Accordingly, system (1.3) has one or two positive order-1 periodic solutions.*

Proof The first two results can be similarly proved as before. As for the existence of the positive periodic solution of system (1.3), we give the proof as follows. When $\frac{\partial^2 \mathcal{P}_M(I_0)}{\partial I_0^2} > 0$ for all $I_0 \in [0, I_p]$ and $\mathcal{P}_M(I_p) < I_p$, we have $\mathcal{P}_M(I_0) < I_0$ for $I_0 \in [0, I_p]$. In addition, it is clear that $\mathcal{P}_M(I_0) < I_0$ for $I_0 \in (I_p, +\infty)$. Therefore, there is no positive fixed point of $\mathcal{P}_M$. However, if $\mathcal{P}_M(I_p) > I_p$, there is a unique positive fixed point $\bar{I}_1 \in (0, I_p)$. Moreover, if there exists $\bar{\delta} > 0$ small enough such that $\mathcal{P}_M(I_p + \bar{\delta}) > I_p + \bar{\delta}$, combining with $\mathcal{P}_M(I_{S_v}) < I_{S_T} < I_{S_v}$ and the monotonicity of $\mathcal{P}_M$, we obtain that there is another fixed point $\bar{I}_2 \in (I_p, I_{S_v})$. Due to the monotonically decrease of $\mathcal{P}_M$ on the interval $[I_{S_v}, +\infty)$, we have that $\mathcal{P}_M(I_0) < I_0$ for all $I_0 \in [I_{S_v}, +\infty)$. Thus, there are two positive fixed points of $\mathcal{P}_M$ and two positive periodic solutions of system (1.3). On the contrary, when there exists no $\bar{\delta}$ satisfying $\mathcal{P}_M(I_p + \bar{\delta}) > I_p + \bar{\delta}$, if $\frac{\partial^2 \mathcal{P}_M(I_0)}{\partial I_0^2} > 0$ holds for all $I_0 \in (I_p, I_{S_v}]$, then $\mathcal{P}_M(I_0) < I_0$ for $I_0 \in (I_{S_v}, +\infty)$. Then there is only one positive fixed point of $\mathcal{P}_M$ and a unique positive periodic solution of system (1.3). This completes the proof.

Next, we consider the case that orbit Γ_2 intersects with line l_4 at three points $P_1(S_v, I_{p_1})$, $P_2(S_v, I_{p_2})$ and $P_3(S_v, I_{p_3})$ before it is tangents to line l_3 with $I_{p_1} < I_{p_3} < I_{p_2}$, shown in Figure 8 (A). Therefore, the domain of $\mathcal{P}_M$ can be divided into:

$$[0, I_{p_1}], \quad (I_{p_1}, I_{p_3}], \quad (I_{p_3}, I_{S_v}], \quad [I_{S_V}, I_{p_2}), \quad [I_{p_2}, +\infty).$$

Based on above discussions, we have the main conclusions as follows.

Theorem 5.5. *For case (C_{42}^c), if there are three discontinuous points P_1, P_2 and P_3, then the Poinceré map $\mathcal{P}_M$ satisfies the following properties:*
(1) The domain and range of the Poinceré map $\mathcal{P}_M$ are $[0, \infty)$ and $[0, w_1(I_{S_T})]$, respectively;
(2) $\mathcal{P}_M$ is continuous on the five intervals $[0, I_{p_1}]$, $(I_{p_1}, I_{p_3}]$, $(I_{p_3}, I_{S_v}]$, $[I_{S_V}, I_{p_2})$ and $[I_{p_2}, +\infty)$. Moreover, it is increasing on the intervals $[0, I_{p_1}]$, $(I_{p_1}, I_{p_3}]$ and $(I_{p_3}, I_{S_v}]$ and decreasing on the intervals $[I_{S_V}, I_{p_2})$ and $[I_{p_2}, +\infty)$;
(3) Suppose $\frac{\partial^2 \mathcal{P}_M(I_0)}{\partial I_0^2} > 0$ holds true for all $I_0 \in [0, I_{p_1}]$. If $\mathcal{P}_M(I_{p_1}) < I_{p_1}$, then there exists no positive fixed point of $\mathcal{P}_M$ and no positive periodic solution of system (1.3). If $\mathcal{P}_M(I_{p_1}) > I_{p_1}$ holds, there may exist one, two or three positive fixed points of $\mathcal{P}_M$, shown in Figure 8 (D-F). Correspondingly, there may be one, two or three positive order-1 periodic solutions of system (1.3).

The properties given by Theorem 5.5 can be similarly proved by using the methods in Theorem 5.4, and we omit the details. For convenience, we just considered two conditions for (C_{42}^c) (i.e., there is one discontinuous point P or three discontinuous points P_1, P_2 and P_3) to discuss the existence of the positive periodic solution of system (1.3). It is worth noting that for case (C_{42}^c), before orbit Γ_2 reaches line $S = S_T$, it may intersect with line $S = S_v$ $2n + 1$ times, and n is increasing as S_v tend to the equilibrium E_1. Thus, the number of discontinuous points could be infinitely countable, which indicates that system (1.3) may exist an infinite number of positive order-1 periodic solutions.

Note that the properties of the Poincaré map for other subcases of case (C_4) can be discussed similarly. Specifically, we can obtain the increasing and decreasing intervals through using the same methods mentioned in above theorems. Moreover, as for the existence of the positive order-1 periodic solution, it can be verified that there may be no positive order-1 periodic solution, which is similar to the results shown in Theorem 5.1 and there may be a finite number of the positive order-1 periodic solutions which is similar to the results shown in Theorem 5.4 and Theorem 5.5, and we give the main properties of the Poincaré map for other subcases of case (C_4) in Table 1.

Table 1. Properties of the Poinceré map $\mathcal{P}_M$ for the subcases of case (C_4).

Cases	Domain and range of $\mathcal{P}_M$	Monotonicity of $\mathcal{P}_M$	The number of PPS of system (1.3)
C_{41}	$[0, +\infty)$ and $[0, w_1(I^*)]$	$\mathcal{P}_M$ increases on $[0, I_{S_v}]$ and decreases on $[I_{S_v}, +\infty)$	No PPS
C_{42}^a	$[0, +\infty)$ and $[0, w_1(I^*)]$	$\mathcal{P}_M$ increases on $[0, I_{S_v}]$ and decreases on $[I_{S_v}, +\infty)$	No PPS
$C_{42}^{b_1}$	$[0, +\infty)$ and $[0, w_1(I_{S_T})]$	$\mathcal{P}_M$ increases on $[0, I_{\gamma_1}]$ and $(I_{\gamma_1}, I_{S_v}]$ and decreases on $[I_{S_v}, I_{\gamma_2})$ and $[I_{\gamma_2}, +\infty)$	At most four PPSs
$C_{42}^{b_2}$	$[0, +\infty)$ and $[0, w_1(I_{S_T})]$	$\mathcal{P}_M$ increases on $[0, I_{S_v}]$ and decreases on $[I_{S_v}, +\infty)$	Zero or two PPSs
C_{43}^a	$[0, +\infty)$ and $[0, w_1(I^*)]$	$\mathcal{P}_M$ increases on $[0, I_{S_v}]$ and decreases on $[I_{S_v}, +\infty)$	Zero or two PPSs
C_{43}^b	$[0, I_{c_1}] \cup [I_{c_2, +\infty})$ and $[0, w_1(I_{S_T})]$	$\mathcal{P}_M$ increases on $[0, I_{c_1}]$ and decreases on $[I_{c_2}, +\infty)$	At most one PPS

Note: 'PPS' represents ' The positive order-1 periodic solution'.

6. Conclusions and discussions

Many mathematical models have assumed that there is a threshold level of the infected population determining the implementation of control methods. Unfortunately, under this assumption, no disease-free periodic solution is feasible or the control reproduction number of the state-dependent impulsive model cannot be defined. Thus, recent studies [24, 25] proposed mathematical models with susceptibles-guided linear impulsive control. In the current study, considering the limitation of

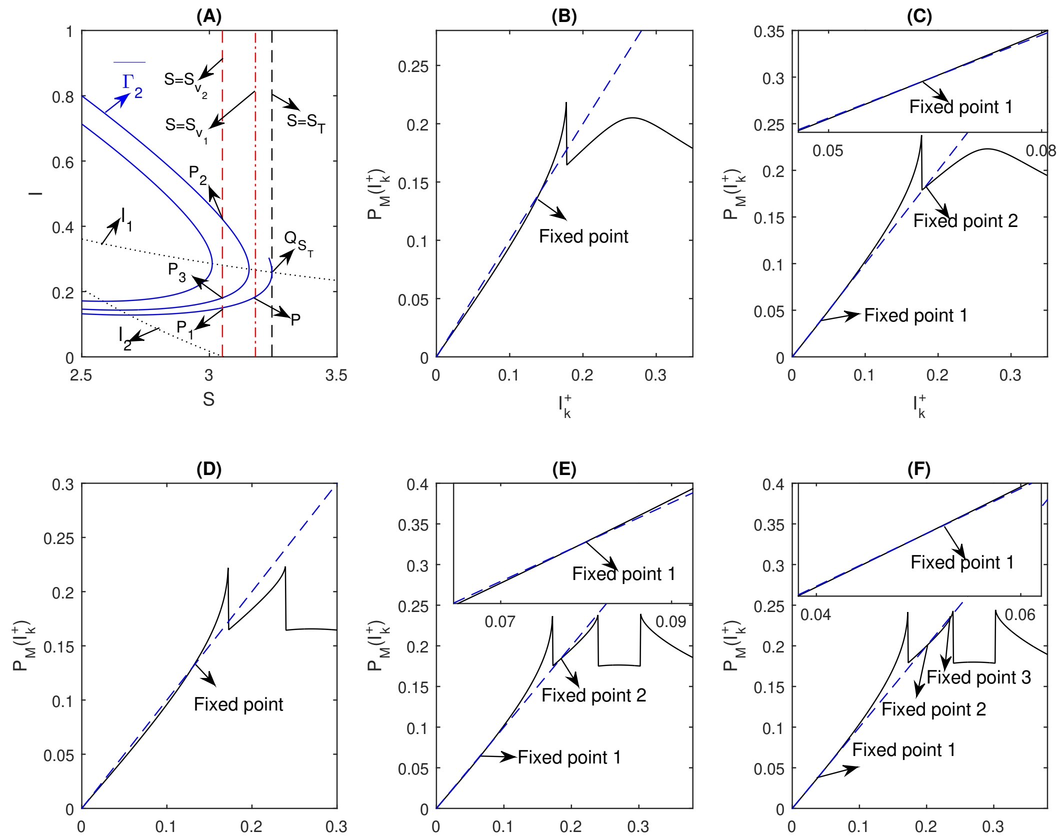

Figure 8. (A) The illustration of the domain of the Poincaré map for the case (C^c_{42}) with one discontinuous point P on the line $S = S_{v_1}$ or three discontinuous points P_1, P_2 and P_3 on the line $S = S_{v_2}$. (B-C) Poincaré map of system (1.3) with one discontinuous point for the case (C^c_{42}). (D-F) Poincaré map of system (1.3) with three discontinuous points for the case (C^c_{42}). (A) $q = 0.05$, $p = 0.02$ for the line $S = S_{v_1}$ and $p = 0.06$ for the line $S = S_{v_2}$; (B) $p = 0.025, q = 0.13$; (C) $p = 0.025, q = 0.05$; (D) $p = 0.03, q = 0.13$; (E) $p = 0.03, q = 0.07$; (F) $p = 0.03, q = 0.05$. The other parameter values are $A = 2, \beta = 1.8, \delta_1 = 0.15, \delta_2 = 0.4$, $\epsilon = 5, \omega = 1.2, \gamma = 0.1, S_T = 3.245, h_1 = 0.01, h_2 = 0.01$.

resources, we introduced the comprehensive saturated control strategies (including saturated impulsive vaccination and isolation, and saturated continuous treatment), and proposed a state-dependent impulsive model with comprehensive saturation interventions.

We first briefly concluded the main dynamics of the ODE subsystem. Based on the dynamics of the ODE subsystem, we investigated the dynamical behaviours of system (1.3). We find that under the susceptibles-guided impulsive control strategy, there always exists the disease-free periodic solution. Further, by discussing the stability of the disease-free periodic solution, we defined the control reproduction number R_b of the state-dependent feedback control system, that is, the disease-free periodic solution is locally stable when R_b is less than 1 and unstable otherwise.

Furthermore, we studied the existence and stability of the positive order-1 periodic solution through analyzing the bifurcation phenomenon near the disease-free periodic solution and discussing the properties of the Poincaré map. We proved that the system can undergo the transcritical bifurcation and the pitchfork bifurcation with respect to the key parameters, including the control parameters such as the maximal vaccination rate p, the threshold level S_T and the parameter ϵ related to saturated continuous treatment. Accordingly, it can be shown that by changing key parameter values, a stable or an unstable positive order-1 periodic solution can bifurcate from the disease-free periodic solution. On the other hand, based on the complexity of the definitions of the domain of the Poincaré map for different cases, there will be a finite number of discontinuous points or an infinitely countable number of discontinuous points for the Poincaré map. Consequently, there may exist multiple positive order-1 periodic solution of system (1.3). Comparing with the analysis of the linear susceptibles-guided impulsive control strategy in [25], our current model considered both continuous saturated treatment and nonlinear impulsive interventions, and we investigated the existence of finite or infinite countable positive order-1 periodic solutions through studying the properties of the Poincaré map. Moreover, through discussing the bifurcations near the disease-free periodic solution with respect to the half-saturation constant of susceptible individuals h_1, we concluded that the disease-free periodic solution is stable when $h_1 < h_1^*$. This implies that the saturation phenomenon of the impulsive control strategy greatly influences the spread of infectious diseases, and large half-saturation constant of susceptibles induces diseases eradication less likely.

Comparing with the model with continuous treatment (i.e., the ODE subsystem (1.2)), we proved that the disease-free periodic solution is stable provided that $S_T \leq \bar{S}$ even if $R_0 > 1$ for subsystem (1.2), implying that the susceptibles-guided impulsive strategy can eradicate infectious diseases successfully with choosing proper threshold level of susceptible population even if $R_0 > 1$ for subsystem (1.2). Moreover, comparing with the modeling approaches of the infected individuals-triggered impulsive control, there always exists the disease-free periodic solution, especially, we can also define the control reproduction number for our state-dependent impulsive model. Therefore, for our proposed model, it is essential to emphasize that the susceptibles-triggered impulsive intervention strategy leads to interesting biological implications, which is helpful to design an optimal treatment strategy. It follows from Figures 2 and 3(A) that selecting proper parameter values plays a crucial effect on controlling infectious diseases. As shown in Figure 2(A), (B) and (D), R_b decreases with respect to q, A and ϵ, which means that enhancing the maximal isolation rate or the continuous treatment is always beneficial to the control of infectious diseases. In addition, large recruitment rate is also helpful to eradicate infectious diseases. As for another key parameter p, we find that when the chosen value of p is large enough, increasing p results in the decrease of R_b, however, for a quite low level of p, R_b increases with respect to p, shown in Figure 2(B), which means that enhancing maximal vaccination rate may be a disadvantage of controlling infectious disease. These results indicate that it is important to choose proper maximal vaccination rate and we should choose relatively large vaccination rate in order to avoid this kind of paradoxical effects. Meanwhile, it is revealed that relatively large threshold level S_T is not beneficial to eradicate infectious diseases, shown in Figure 3(A). Another interesting result shown in Figure 3 reveals that if we choose a properly small threshold value S_T, infectious diseases can be eventually eradicated, which plays a significant role in mitigating the spread of infectious diseases. Therefore, we should take account of these key parameters in order to develop effective and optimal susceptibles-triggered impulsive control strategies.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (NSFCs 11631012, 11571273).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. R. M. Anderson and R. M. May, *Infectious Diseases of Humans, Dynamics and Control*, Oxford University, Oxford,1991.
2. V. Capasso, *Mathematical Structures of Epidemic Systems, Lecture Notes in Biomathematics*, vol. 97, Springer-Verlag, Berlin, 1993.
3. H. W. Hethcote, The mathematics of infectious diseases, *SIAM Rev.*, **42** (2000).
4. O. Diekmann and J. A. P. Heesterbeek, *Mathematical Epidemiology of Infectious Diseases. Model Building, Analysis and Interpretation*, Wiley Ser. Math. Comput. Biol., John Wiley and Sons, Chichester, England, 2000.
5. F. Brauer and P. van den Driessche, Models for the transmission of disease with immigration of infectives, *Math. Biosci.*, **171** (2001), 143–154.
6. M. A. Nowak and A. R. McLean, A mathematical model of vaccination against HIV to prevent the development of AIDS, *Proc. Biol. Sci.*, **246** (1991), 141–146.
7. N. M. Ferguson, D. A. T. Cummings, S. Cauchemez, et al., Strategies for containing an emerging in fuenza pandemic in Southeast Asia, *Nature*, **437** (2005), 209–214.
8. Y. N. Xiao, T. T. Zhao and S. Y. Tang, Dynamics of an infectious disease with media/ psychology induced non-sooth incidence, *Math. Biosci. Eng.*, **10** (2013), 445–461.
9. A. L. Wang and Y. N. Xiao, A Filippov system describing media effects on the spread of infectious diseases, *Nonlinear Anal. Hybri.*, **11** (2014), 84–97.
10. J. A. Cui, X. X. Mu and H. Wan, Saturation recovery leads to multiple endemic equilibria and backward bifurcation, *J. Theor. Biol.*, **254** (2008), 275–283.

11. J. L. Wang, S. Q. Liu, B. W. Zheng, et al., Qualitative and bifurcation analysis using an SIR model with a saturated treatment function, *Math. Comput. Model.*, **55** (2012), 710–722.
12. X. Zhang and X. N. Liu, Backward bifurcation of an epidemic model with saturated treatment function, *J. Math. Anal. Appl.*, **348** (2008), 433–443.
13. L. H. Zhou and M. Fan, Dynamics of an SIR epidemic model with limited medical resources revisited, *Nonlinear Anal. Real.*, **13** (2012), 312–324.
14. H. Wan and J. A. Cui, Rich dynamics of an epidemic model with saturation recovery, *J. Appl. Math.*, 314958 (2013).
15. Z. H. Zhang and Y. H. Suo, Qualitative analysis of a SIR epidemic model with saturated treatment rate, *J. Appl. Math. Comput.*, **34** (2010), 177–194.
16. S. Y. Tang and R. A. Cheke, Stage-dependent impulsive models of integrated pest management (IPM) strategies and their dynamic consequences, *J. Math. Biol.*, **50** (2005), 257–292.
17. S. Y. Tang, Y. N. Xiao and R. A. Cheke, Dynamical analysis of plant disease models with cultural control strategies and economic thresholds, *Math. Comput. Simul.*, **80** (2010), 894–921.
18. A. B. Sabin, Measles, killer of millions in developing countries: strategies of elimination and continuing control, *Eur. J. Epidemiol.*, **7** (1991), 1–22.
19. B. Shulgin, L. Stone and Z. Agur, Pulse vaccination strategy in the SIR epidemic model, *Bull. Math. Biol.*, **60** (1998), 1123–1148.
20. L. Stone, B. Shulgin and Z. Agur, Theoretical examination of the pulse vaccination policy in the SIR epidemic model, *Math. Comput. Model.*, **31** (2000), 207–215.
21. A. D'Onofrio, On pulse vaccination strategy in the SIR epidemic model with vertical transmission, *Appl. Math. Lett.*, **18** (2005), 729–732.
22. F. L. Black, Measles endemicity in insular populations: Critical community size and its evolutionary implication, *J. Theor. Biol.*, **11** (1966), 207–211.
23. W. J. Moss and P. Strebel, Biological feasibility of measles eradication, *J. Infect. Dis.*, **204** (2011), 47–53.
24. Q. Q. Zhang, B. Tang and S. Y. Tang, Vaccination threshold size and backward bifurcation of SIR model with state-dependent pulse control, *J. Theor. Biol.*, **455** (2018), 75–85.
25. Q. Li and Y. N. Xiao, Dynamical behaviour and bifurcation analysis of the SIR model with continuous treatment and state-dependent impulsive control, *Int. J. Bifurcat. Chaos*, (2019), Accepted.
26. A. L. Wang, Y. N. Xiao and R. Smith?, Using non-smooth models to determine thresholds for microbial pest management, *J. Math. Biol.*, (2019).
27. A. L. Wang, Y. N. Xiao and R. Smith?, Multiple equilibria in a non-smooth epidemic model with medical-resource constraints, *Bull. Math. Biol.*, **81** (2019), 963–994.
28. W. J. Qin, S. Y. Tang and R. A. Cheke, Nonlinear pulse vaccination in an SIR epidemic model with resource limitation, *Abstr. Appl. Anal.*, 670263 (2013).
29. J. Yang and S. Y. Tang, Holling type II predator–prey model with nonlinear pulse as state-dependent feedback control, *J. Comput. Appl. Math.*, **291** (2016), 225–241.

30. S. Y. Tang, B. Tang, A. L. Wang, et al., Holling II predator-prey impulsive semi-dynamic model with complex Poincaré map, *Nonlinear Dyn.*, **81** (2015), 1575–1596.
31. S. Y. Tang, W. H. Pang, R. A. Cheke, et al., Global dynamics of a state-dependent feedback control system, *Adv. Diff. Equ.*, **322** (2015), DOI: 10.1186/s13662-015-0661-x.
32. J. M. Grandmont, Nonlinear difference equations, bifurcations and chaos: an introduction, *Research in Economics*, **62** (2008), 122–177.
33. Y. N. Xiao, X. X. Xu and S. Y. Tang, Sliding mode control of outbreaks of emerging infectious diseases, *Bull. Math. Biol.*, **74** (2012), 2403–2422.
34. Y. P. Yang, Y. N. Xiao and J. H. Wu, Pulse HIV vaccination: feasibility for virus eradication and optimal vaccination schedule, *Bull. Math. Biol.*, **75** (2013), 725–751.
35. Y. Tian, K. B. Sun, A. Kasperski, et al., Nonlinear modelling and qualitative analysis of a real chemostat with pulse feeding, *Discrete Dyn. Nat. Soc.*, 640594 (2011), 1–18.
36. S. Y. Tang and W. H. Pang, On the continuity of the function describing the times of meeting impulsive set and its application, *Math. Biosci. Eng.*, **14** (2017), 1399–1406.
37. X. N. Liu, Y. Takeuchi and S. Iwami, SVIR epidemic model with vaccination strategy, *J. Theor. Biol.*, **253** (2008), 1–11.

Appendix A

The following lemma shows the local stability of an order-*k* periodic solution.

Lemma A.1 *The order-k periodic solution $(x, y) = (\xi(t), \eta(t))$ with period T of (1.3) is orbitally asymptotically stable if the Floquet multiplier μ_2 satisfies $|\mu_2| < 1$, where*

$$\mu_2 = \prod_{k=1}^{q} \Delta_k \exp\left[\int_0^T \left(\frac{\partial P}{\partial x}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t))\right) dt\right],$$

with

$$\Delta_k = \frac{P_+\left(\frac{\partial b}{\partial y}\frac{\partial \phi}{\partial x} - \frac{\partial b}{\partial x}\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x}\right) + Q_+\left(\frac{\partial a}{\partial y}\frac{\partial \phi}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\right)}{P\frac{\partial \phi}{\partial x} + Q\frac{\partial \phi}{\partial y}},$$

and $P, Q, \frac{\partial a}{\partial x}, \frac{\partial a}{\partial y}, \frac{\partial b}{\partial x}, \frac{\partial b}{\partial y}, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}$ are calculated at the point $(\xi(\tau_k), \eta(\tau_k))$, and $P_+ = P(\xi(\tau_k^+), \eta(\tau_k^+))$, $Q_+ = Q(\xi(\tau_k^+), \eta(\tau_k^+))$ with $\tau_k (k \in N)$ denoting the time of the k-th jump. Here, $\phi(x, y)$ is a sufficiently smooth function such that $grad\phi(x, y) \neq 0$.

Then, we give two lemmas of the transcritical bifurcation and the pitchfork bifurcation of the discrete one-parameter family of maps [32].

Lemma A.2 *(Transcritical bifurcation). Let $G : U \times I \to R$ define a one-parameter family of maps, where G is C^r with $r \geq 2$, and U, I are open intervals of the real line containing 0. Assume*

$$(1)\ G(0, \alpha) = 0\ for\ all\ \alpha; \quad (2)\ \frac{\partial G}{\partial x}(0, 0) = 1;$$
$$(3)\ \frac{\partial^2 G}{\partial x \partial \alpha}(0, 0) > 0; \quad (4)\ \frac{\partial^2 G}{\partial x^2}(0, 0) > 0.$$

Then there are $\alpha_1 < 0 < \alpha_2$ *and* $\zeta > 0$ *such that*
(1) If $\alpha_1 < \alpha < 0$*, then* G_α *has two fixed points, 0 and* $x_{1\alpha} > 0$ *in* $(-\zeta, \zeta)$*. The origin is asymptotically stable, while the other fixed point is unstable.*
(2) If $0 < \alpha < \alpha_2$*, then* G_α *has two fixed points, 0 and* $x_{1\alpha} < 0$ *in* $(-\zeta, \zeta)$*. The origin is unstable, while the other fixed point is asymptotically stable.*
Similarly, note that making the change of parameter $\alpha \to -\alpha$*, we can handle* $\frac{\partial^2 G}{\partial x^2}(0,0) < 0$*.*

Lemma A.3 *(Supercritical pitchfork bifurcation). Let* $G : U \times I \to R$ *define a one-parameter family of maps as in Lemma A.2, except that G is* C^r *with* $r \geq 3$*,* $\frac{\partial^2 G}{\partial x^2}(0,0) = 0$ *and* $\frac{\partial^3 G}{\partial x^3}(0,0) < 0$*. Then there are* $\alpha_1 < 0 < \alpha_2$ *and* $\zeta > 0$ *such that*
(1) If $\alpha_1 < \alpha \leq 0$*, then* G_α *has a unique fixed point,* $x = 0$*. And it is asymptotically stable.*
(2) If $0 < \alpha < \alpha_2$*, then* G_α *has three fixed points, 0 and* $x_{1\alpha} < 0 < x_{2\alpha}$ *in* $(-\zeta, \zeta)$*. The origin is unstable, while the other two fixed points are asymptotically stable.*
Note that the for the case $\frac{\partial^3 G}{\partial x^3}(0,0) > 0$*, we can make the change of parameter* $\alpha \to -\alpha$*, which is called the subcritical pitchfork bifurcation.*

Appendix B

(A) The bifurcation near the disease-free periodic solution with respect to A for $h_2 > 0$.
Firstly, we investigate the existence of $A^* \in (\delta_1 S_T, +\infty)$ such that $R_b(A^*) = 1$. There are

$$\lim_{A \to \delta_1 S_T^+} R_b(A) = +\infty, \quad \lim_{A \to +\infty} R_b(A) = \lim_{A \to +\infty} \exp(J(A)) = 1. \tag{6.1}$$

Taking the derivative of $R_b(A)$ with respect to A, one obtains

$$\frac{\partial R_b(A)}{\partial A} = R_b(A) \frac{\partial J(A)}{\partial A},$$

with

$$\frac{\partial J(A)}{\partial A} = \frac{\beta}{\delta_1^2} \left(\ln \frac{A - \delta_1 S_v}{A - \delta_1 S_T} + \frac{\delta_1 (A - \delta_1 \overline{S})(S_v - S_T)}{(A - \delta_1 S_v)(A - \delta_1 S_T)} \right).$$

Denoting $W_1(A) = \ln \frac{A - \delta_1 S_v}{A - \delta_1 S_T} + \frac{\delta_1 (A - \delta_1 \overline{S})(S_v - S_T)}{(A - \delta_1 S_v)(A - \delta_1 S_T)}$ and taking the derivative of $W_1(A)$ with respect to A, we get

$$\frac{\partial W_1(A)}{\partial A} = \frac{\delta_1^2 (S_v - S_T)}{(A - \delta_1 S_v)^2 (A - \delta_1 S_T)^2} \left((2\overline{S} - S_v - S_T)A + \delta_1 (2 S_v S_T - \overline{S}(S_v + S_T)) \right).$$

Note that if $2\overline{S} = S_v + S_T$, we have

$$\frac{\partial W_1(A)}{\partial A} = \frac{\delta_1^3 (S_v - S_T)}{(A - \delta_1 S_v)^2 (A - \delta_1 S_T)^2} \left(2 S_v S_T - \frac{(S_v + S_T)^2)}{2} \right).$$

As a result of $4 S_v S_T < (S_v + S_T)^2$, then $\frac{\partial W_1(A)}{\partial A} > 0$. This indicates that $W_1(A)$ is monotonically increasing for $A \in (\delta_1 S_T, +\infty)$. Combining with $\lim_{A \to +\infty} W_1(A) = 0$, we yield $\frac{\partial J(A)}{\partial A} < 0$ holds for all $A \in (\delta_1 S_T, +\infty)$, i.e., $\frac{\partial R_b(A)}{\partial A} < 0$, which means that $R_b(A)$ is monotonically decreasing. Therefore, $R_b(A) > 1$ for all $A \in (\delta_1 S_T, +\infty)$. Under this situation, the disease-free periodic solution is unstable and no bifurcation occurs with respect to parameter A.
However, if $2\overline{S} \neq S_v + S_T$, we denote

$$W_2(A) = (2\overline{S} - S_v - S_T)A + \delta_1 (2 S_v S_T - \overline{S}(S_v + S_T)) \doteq a_1 A + a_2.$$

Then, we have

$$a_1 > 0 \Leftrightarrow \overline{S} > \frac{S_v + S_T}{2},\ \ a_2 > 0 \Leftrightarrow \overline{S} < \frac{2S_vS_T}{S_v + S_T}.$$

Moreover, there is a unique $\overline{A} = -\frac{a_2}{a_1}$ such that $W_2(\overline{A}) = 0$. In what follows, we focus on discussing the bifurcation related to parameter A by considering the following cases:

(1) If $a_1 > 0$, it is clear that $a_2 < 0$ holds, thus, $\overline{A} > 0$. Then we consider two subcases as follows:

(a) If $\overline{A} \leq \delta_1 S_T$, we obtain $W_2(A) > 0$, i.e., $\frac{\partial W_1(A)}{\partial A} < 0$ holds for all $A \in (\delta_1 S_T, +\infty)$. Thus, $W_1(A)$ is monotonically decreasing on the interval $(\delta_1 S_T, +\infty)$ and $\lim_{A\to+\infty} W_1(A) = 0$, which indicates that $W_1(A) > 0$ for all $A \in (\delta_1 S_T, +\infty)$. Therefore, $R_b(A)$ is monotonically increasing on the interval $(\delta_1 S_T, +\infty)$. However, this result contradicts equations (6.1), indicating that $\overline{A} > \delta_1 S_T$ always holds.

(b) In the following, we consider the condition $\overline{A} > \delta_1 S_T$. Under this scenario, we have $W_2(A) < 0$ for $A \in (\delta_1 S_T, \overline{A})$ and $W_2(A) > 0$ for $A \in (\overline{A}, +\infty)$. Therefore, $\frac{\partial W_1(A)}{\partial A} > 0$ for $A \in (\delta_1 S_T, \overline{A})$ and $\frac{\partial W_1(A)}{\partial A} < 0$ for $A \in (\overline{A}, +\infty)$, which means that $W_1(A)$ is monotonically increasing on the interval $(\delta_1 S_T, \overline{A})$ and monotonically decreasing on the interval $(\overline{A}, +\infty)$. According to $\lim_{A\to+\infty} W_1(A) = 0$, we have $W_1(A) > 0$ for all $A \in (\overline{A}, +\infty)$, and consequently, $R_b(A)$ is monotonically increasing on the interval $(\overline{A}, +\infty)$. It is easy to verify that there is a unique $A' \in (\delta_1 S_T, \overline{A})$ satisfying $W_1(A') = 0$. In fact, if $W_1(A) > 0$ always holds for $A \in (\delta_1 S_T, \overline{A})$, then $R_b(A)$ is monotonically increasing on the interval $(\delta_1 S_T, +\infty)$, which contradicts equations (6.1). Thus, $W_1(A) < 0$ for $A \in (\delta_1 S_T, A')$ and $W_1(A) > 0$ for $A \in (A', +\infty)$. Correspondingly, $R_b(A)$ is monotonically decreasing on the interval $(\delta_1 S_T, A')$ and increasing on the interval $(A', +\infty)$. According to equations (6.1), there must be a unique $A^* \in (\delta_1 S_T, A')$ such that $R_b(A^*) = 1$ with $\frac{\partial R_b(A^*)}{\partial A} < 0$.

(2) If $a_1 < 0$ and $a_2 > 0$, we have $\overline{A} > 0$. Then we consider the following subcases:

(a) If $\overline{A} \leq \delta_1 S_T$, then we have $W_2(A) < 0$, i.e., $\frac{\partial W_1(A)}{\partial A} > 0$ holds for all $A \in (\delta_1 S_T, +\infty)$. Therefore, $W_1(A)$ is monotonically increasing on the interval $(\delta_1 S_T, +\infty)$ with $\lim_{A\to+\infty} W_1(A) = 0$, which indicates that $W_1(A) < 0$ holds true for all $A \in (\delta_1 S_T, +\infty)$. Correspondingly, $R_b(A)$ is monotonically decreasing on the interval $(\delta_1 S_T, +\infty)$. According to $\lim_{A\to+\infty} R_b(A) = 1$, we have $R_b(A) > 1$ is true for $A \in (\delta_1 S_T, +\infty)$. These results show that the disease-free periodic solution is unstable and there is no bifurcation near the disease-free periodic solution.

(b) If $\overline{A} > \delta_1 S_T$, we have $W_2(A) > 0$ for $A \in (\delta_1 S_T, \overline{A})$ and $W_2(A) < 0$ for $A \in (\overline{A}, +\infty)$. Consequently, $W_1(A)$ is monotonically decreasing on the interval $(\delta_1 S_T, \overline{A})$ and monotonically increasing on the interval $(\overline{A}, +\infty)$. According to $\lim_{A\to\infty} W_1(A) = 0$, we have that $W_1(A) < 0$ for all $A \in (\overline{A}, +\infty)$ and $R_b(A)$ is monotonically decreasing on the interval $(\overline{A}, +\infty)$. As for $A \in (\delta_1 S_T, \overline{A})$, if there exists a A'' such that $W_1(A'') = 0$, then $W_1(A) > 0$ for $A \in (\delta_1 S_T, A'')$ and $W_1(A) < 0$ for $A \in (A'', +\infty)$, which contradicts Eq (6.1). Therefore, $W_1(A) < 0$ holds for $A \in (\delta_1 S_T, +\infty)$ and $R_b(A)$ is monotonically decreasing on the interval $(\delta_1 S_T, +\infty)$. Similar to above discussions for subcase (a), we know that $R_b(A) > 1$ always holds true. Therefore, the disease-free periodic solution is unstable and there is no bifurcation near the disease-free periodic solution.

(3) If $a_1 < 0$ and $a_2 < 0$, then we have $\overline{A} < 0$. Under this scenario, $W_2(A) < 0$, i.e., $\frac{\partial W_1(A)}{\partial A} > 0$ holds for all $A \in (\delta_1 S_T, +\infty)$. Therefore, $W_1(A)$ is monotonically increasing on the interval $(\delta_1 S_T, +\infty)$ with $\lim_{A\to+\infty} W_1(A) = 0$. Therefore, $W_1(A) < 0$ always holds. Accordingly, $R_b(A)$ is monotonically decreasing on the interval $(\delta_1 S_T, +\infty)$. Combining with equations (6.1), we have that $R_b(A) > 1$ holds true for $A \in (\delta_1 S_T, +\infty)$, meaning that the disease-free periodic solution is unstable and there is no bifurcation near the disease-free periodic solution. Based on above discussions, we have conclusions

as follows.

Proposition B.1 *Assume $R_0 > 1$. If $S_T > \overline{S} > \frac{S_v+S_T}{2}$ holds, then there exists a unique $A^* \in (\delta_1 S_T, A')$ satisfying $R_b(A^*) = 1$ with $\frac{\partial R_b(A^*)}{\partial A} < 0$. And the disease-free periodic solution $(\xi(t), 0)$ of system (1.3) is orbitally asymptotically stable when $A \in (A^*, +\infty)$ and unstable when $A \in (\delta_1 S_T, A^*)$.*

As for the bifurcation of the disease-free periodic solution at A^*, we have that $\mathcal{P}_M(0, A) = 0$ always holds for $A \in (\delta_1 S_T, +\infty)$, and

$$\frac{\partial \mathcal{P}_M}{\partial I_0}(0, A^*) = 1, \quad \frac{\partial^2 \mathcal{P}_M}{\partial I_0 \partial A}(0, A^*) < 0,$$
$$\frac{\partial^2 \mathcal{P}_M}{\partial I_0^2}(0, A^*) = g''(0; A^*) - \frac{2q}{h_2}, \quad \frac{\partial^3 \mathcal{P}_M}{\partial I_0^3}(0, A^*) = g'''(0; A^*) - \frac{6q(2q-1)}{h_2^2}.$$

Therefore, we can conclude the main results for the bifurcation near the disease-free periodic solution with respect to A in Theorem 4.5.

(B) The bifurcation near the disease-free periodic solution with respect to q for $h_2 = 0$.

When $h_2 = 0$, the bifurcation near the disease-free periodic solution can be similarly studied. It is clear that $R_b(q) = (1 - q)\exp(J)$ when $h_2 = 0$. Thus, q can be chosen as a bifurcation parameter. It is easily obtained that $R_b(1) = 0$. When $J > 0$ holds, then there is a unique $q^* \in (0, 1)$ such that $R_b(q^*) = 1$ with $q^* = 1 - \exp(-J)$, which is equal to $\frac{\partial \mathcal{P}_M}{\partial I_0}(0, q^*) = 1$. Note that $\mathcal{P}_M(0, q) = 0$ always holds, and $\frac{\partial^2 \mathcal{P}_M}{\partial I_0 \partial q}(0, q^*) = -\exp(J) < 0$. Moreover, there is

$$\frac{\partial^2 \mathcal{P}_M}{\partial I_0^2}(0, q^*) = (1 - q^*)g''(0; q^*).$$

Therefore, we can obtain the conclusions given in Theorem 4.6.

AIMS Press