Research article

Mathematical analysis on deterministic and stochastic lake ecosystem models

Zhiwei Huang and Gang Huang*

School of Mathematics and Physics, China University of Geosciences, 430074, Wuhan, P.R. China

* **Correspondence:** Email: huanggang@cug.edu.cn.

Abstract: In this paper, we propose and study the deterministic and stochastic lake ecosystem models to investigate the impact of terrestrial organic matter upon persistence of the plankton populations. By constructing Lyapunov function and using the LaSalle's Invariance Principle, we establish global properties of the deterministic model. The dynamical behavior of solutions fits well with some experimental results. It is concluded that the terrestrial organic matter plays an important role in influencing interactions between phytoplankton and zooplankton. Based on the fluctuations of lake ecosystem, we further develop a stochastically perturbed model. Theoretic analysis implies that the stochastic model exists a stationary distribution which is ergodic. The key point of our analysis is to enhance our knowledge of the factors governing the dynamics of plankton population models.

Keywords: terrestrial organic matter; Lyapunov function; Itô's formula; stationary distribution

1. Introduction

Phytoplankton-derived carbon (internal primary production) is known to be essential for the somatic growth and reproduction of zooplankton and fish. For decades, aquatic food webs were taken as systems where carbon transfer was linear from phytoplankton to zooplankton to fish [1].

With the application of isotope labeling, substantial research shows that terrestrial organic matter (TOM) plays an important role in the lake food webs. TOM can not only affect the lake ecosystem in physical and chemical way but also can be exploited by consumer as a resource [2]. And particulate organic matter (POM) of terrestrial origin can be the key factor controlling whole-lake productivity in lakes where phytoplankton productivity is low [1, 3]. Besides, in lakes, the inputs of TOM often equal to or exceed internal primary production [4].

Zooplankton composed of cladocerans, cyclopoids, and calanoids represents a critical food chain link between phytoplankton and larger fish in lakes. Lot of experimental results show that zooplankton can directly consume allochthonous POM that either entered the lake in particulate form

or was formed through flocculation of allochthonous dissolved organic matter (DOM) [5]. Nevertheless, it is mentioned that Daphnia only supported by allochthonous carbon can survive and give birth to offspring [6]. For the reason that phytoplankton and zooplankton are important components of lakes, plankton mechanism considering TOM is widely studied. Lots of experiments have been done by isotope labeling [5,7,8].

Mathematical modelling is an important tool to investigate the plankton mechanism [9–14]. Nutrient-phytoplankton-zooplankton models were studied by many researchers [15–17]. In order to enhance the knowledge of plankton mechanism, we propose phytoplankton-zooplankton models considering TOM. We focus on the ecological function of TOM in the plankton mechanism. The results of the deterministic model can fit well with some experimental results and two hypotheses supported by experimental results in [18]. The related two hypotheses are given in the following part:

(I) Catchment deposition hypothesis: Allochthony which is the portion of a zooplankter's body carbon content that is of terrestrial origin increases as more TOM is exported from the surrounding catchment.

(II) Algal subtraction hypothesis: Allochthony increases with the availability of TOM, where algal production becomes limited by shading more than it benefits from the nutrients associated with TOM.

The rest of the paper is organized as follows: in the next section, by taking TOM into account, formulation of two deterministic models are discussed. In Section 3, global dynamical properties of the three-dimensional ODE model are completely established using Lyapunov function method. In Section 4, by including stochastic perturbation of the white noise type, we develop a stochastic model and show that there is a stationary distribution in the stochastic model. In Section 5, the numerical simulation is given to verify the theoretical predictions. In the biological sense, the numerical results are analyzed in detail. In the last section, the conclusion is given.

2. Deterministic models

The research on the interaction between phytoplankton and zooplankton is important. The phytoplankton is not only the producer of the marine ecosystem but also the base of every food web. Phytoplankton is consumed by zooplankton, which is resource for consumers of higher tropical levels. By transferring organic matter and energy into higher tropical levels, zooplankton plays a critical role in lake ecosystem. Usually, the interaction between the phytoplankton and zooplankton can be described by the Prey-Predator system,

$$
\begin{aligned}
\frac{dP(t)}{dt} &= P(t)(r_1 - a_1 P(t) - bZ(t)), \\
\frac{dZ(t)}{dt} &= Z(t)(-r_2 + cP(t) - a_2 Z(t)).
\end{aligned}
\tag{2.1}
$$

Here, $P(t)$ and $Z(t)$ are the population densities of phytoplankton and zooplankton, respectively. The phytoplankton grows with rate r_1. The interaction term taken here is Holling type-I with b and c as the respective rates of grazing and biomass conversion by the zooplankton for its growth such that $b > c$. The natural mortality of zooplankton is r_2. a_1 reflects the density dependence of phytoplankton. The density dependence of predator species is wildly studied because of the environment factor [14]. In lakes where phytoplankton productivity is low, zooplankton will compete each other for the resources to survive, which means that it is more realistic to consider the density dependence of zooplankton

population in system (2.1). Furthermore, the zooplankton is resource for higher tropical levels. The mortality caused by higher trophic levels is generally modeled by a nonlinear term [9]. Hence, a_2 reflects the density dependence of zooplankton and the mortality of zooplankton caused by higher tropic. Here, it is easy to know that system (2.1) has two equilibria: $O(0,0)$, $A(\frac{r_1}{a_1},0)$. Besides, when $cr_1 - a_1r_2 > 0$, there is a positive equilibrium $E^*(P^*,Z^*)$, where $P^* = \frac{r_1a_2+r_2b}{a_1a_2+bc}$, $Z^* = \frac{cr_1-a_1r_2}{a_1a_2+bc}$, which is globally asymptotically stable. If $cr_1 - a_1r_2 \le 0$, equilibrium $A(\frac{r_1}{a_1},0)$ is globally asymptotically stable.

Model (2.1) ignores TOM in the lake ecosystem. Since TOM has critical influence in the lake food webs, it is more realistic to incorporate it into the phytoplankton-zooplankton model. Based on the important role of TOM in lake ecosystem, we propose the terrestrial organic matter-phytoplankton-zooplankton model. The meaning of the parameters and some assumptions are proposed in following part, firstly.

$C(t)$ is the concentration of TOM. $P(t)$, $Z(t)$ are the population densities of phytoplankton and zooplankton, respectively. s is the constant input rate of TOM and δ is its sinking rate. The interaction term between the TOM and zooplankton is Holling type-I. β_0 is TOM uptake rate by zooplankton. TOM is converted for the growth of zooplankton with rate β_3. The phytoplankton grows with growth rate r. K is the carrying capacity. The interaction term between the phytoplankton and zooplankton is also Holling type-I with β_1,β_2 as the respective rates of grazing and biomass conversion by the zooplankton for their growth. γ is the natural mortality of zooplankton. m measures the strength of competition among zooplankton and the mortality of zooplankton caused by higher tropic. Above statements can be seen clearly in Figure 1.
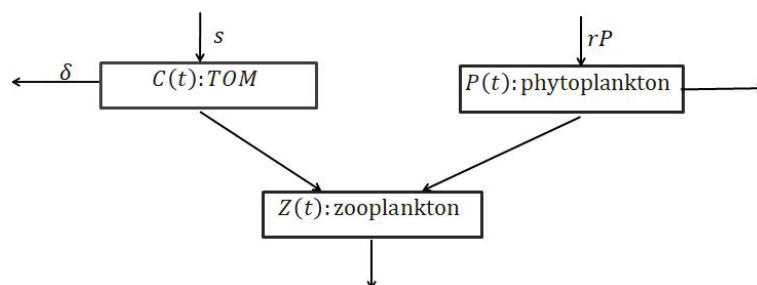

Figure 1. $C(t)$ is the concentration of TOM. $P(t)$, $Z(t)$ are the population densities of phytoplankton and zooplankton, respectively. s is the constant input rate of TOM and δ is its sinking rate. The phytoplankton grows with growth rate r. γ is the natural mortality of zooplankton. K is the carrying capacity. m measures the strength of competition among zooplankton and the mortality of zooplankton caused by higher tropic. The biomass conversion between the phytoplankton and zooplankton is $\beta_2 PZ$. The biomass conversion between TOM and zooplankton is $\beta_3 PZ$.

Based on the above statements, the following ordinary differential equations can be derived,

$$
\begin{aligned}
\frac{dC(t)}{dt} &= -\beta_0 C(t)Z(t) - \delta C(t) + s, \\
\frac{dP(t)}{dt} &= rP(t)\left(1 - \frac{P(t)}{K}\right) - \beta_1 P(t)Z(t), \\
\frac{dZ(t)}{dt} &= \beta_2 P(t)Z(t) + \beta_3 C(t)Z(t) - \gamma Z(t) - mZ^2(t).
\end{aligned}
\tag{2.2}
$$

In the biological context, the population densities of phytoplankton and zooplankton are nonnegative at any time t. The concentration of TOM is also nonnegative at any time t. We get the following initial conditions,

$$C(0) = C_0 \geq 0, \ P(0) = P_0 \geq 0, \ Z(0) = Z_0 \geq 0. \tag{2.3}$$

Proposition 2.1. *The solution $(C(t), P(t), Z(t))$ of (2.2) with the initial condition (2.3) is nonnegative and bounded in R_+^3.*

Proof. From the equations of (2.2), it is easy to get that $C(t) \geq C(0)e^{-\int_0^t \beta_0 Z(s) + \delta ds} \geq 0$, $P(t) = P(0)e^{\int_0^t r\left(1 - \frac{P(s)}{K}\right) - \beta_1 Z(s) ds} \geq 0$ and $Z(t) = Z(0)e^{\int_0^t \beta_2 P(s) + \beta_3 C(s) - \gamma - mZ(s) ds} \geq 0$. Hence, the solutions are non-negative. Clearly, $\frac{dC}{dt} \leq -\delta C + s$ yielding $C(t) \leq \max\left\{P_0, \frac{s}{\delta}\right\} := C_{max}$. $\frac{dP}{dt} \leq rP(1 - \frac{P}{K})$ yielding $P(t) \leq \max\{K, P_0\} := P_{max}$. Therefore, $Z(t) \leq \beta_2 P_{max} Z + \beta_3 C_{max} Z - mZ^2$. Hence, $Z(t) \leq \max\left\{Z_0, \frac{\beta_2 P_{max} + \beta_3 C_{max}}{m}\right\}$. It implies the solutions $(C(t), P(t), Z(t))$ of (2.2) are nonnegative and bounded for all $t \geq 0$. $\qquad\square$

The nonnegativity of the solution ensures that the model has practical significance. In the following part, we can get the equilibria of (2.2). By direct calculation, it can be known that (2.2) always has two equilibria,

$$E_1 = (\frac{s}{\delta}, 0, 0), \ E_2 = (C_2, P_2, 0) = (\frac{s}{\delta}, K, 0).$$

Proposition 2.2. *If $s > \frac{\delta\gamma}{\beta_3}$, there exists unique semi-trivial equilibrium $E_3(C_3, 0, Z_3)$ of (2.2). If $K \geq \frac{\gamma}{\beta_2}$ and $s \in (0, s_1)$, or $K < \frac{\gamma}{\beta_2}$ and $s \in (s_0, s_1)$, where*

$$s_0 = \frac{\delta\gamma}{\beta_3} - \frac{\beta_2 K \delta}{\beta_3},$$
$$s_1 = \frac{\delta\gamma}{\beta_3} + \frac{r^2 m \beta_0}{\beta_3 \beta_1^2} + \frac{r(\gamma\beta_0 + \delta m)}{\beta_1 \beta_3},$$

there exists a unique positive equilibrium $E_4(C_4, P_4, Z_4)$.

Proof. The equilibria satisfy the following system of algebraic equations,

$$-\beta_0 CZ - \delta C + s = 0,$$
$$rP\left(1 - \frac{P}{K}\right) - \beta_1 PZ = 0, \tag{2.4}$$
$$\beta_2 PZ + \beta_3 CZ - \gamma Z - mZ^2 = 0.$$

If $P = 0, Z \neq 0$, the equilibrium is given by

$$C^* = \frac{\gamma + mZ^*}{\beta_3},$$
$$Z^* = \frac{-\delta C^* + s}{\beta_0 C^*}, \tag{2.5}$$

where C^* and Z^* are constants. From the second equation of (2.5), we can get

$$a_3 Z^2 + b_3 Z + c_3 = 0, \tag{2.6}$$

where $a_3 = -m\beta_0, b_3 = -(\gamma\beta_0 + \delta m), c_3 = \beta_3 s - \gamma\delta$. If $s > \frac{\delta\gamma}{\beta_3}$, then $c_3 > 0$. Thus, $\Delta = b_3^2 - 4a_3 c_3 > 0$, $Z_{31} + Z_{32} = -\frac{b_3}{a_3} < 0, Z_{31}Z_{32} = \frac{c_3}{a_3} < 0$, which means that (2.6) exists only one positive root. It is easy to get that $Z_3 = \frac{-b_3 + \sqrt{b_3^2 - 4a_3 c_3}}{2a_3}$. Then $C_3 = \frac{\gamma + mZ_3}{\beta_3}$.

If $P \neq 0, Z \neq 0$, the equilibria of (2.4) are given by

$$H(P) = a_4 P^2 + b_4 P + c_4, \tag{2.7}$$

where

$$a_4 = -\beta_0 r \left(\beta_2 + \frac{rm}{\beta_1 K} \right),$$

$$b_4 = (\beta_0 r K + \delta\beta_1 K) \left(\beta_2 + \frac{rm}{\beta_1 K} + \beta_0 r \left(\gamma + \frac{rm}{\beta_1} \right) \right),$$

$$c_4 = K(\beta_1 \beta_3 s - \left(\gamma + \frac{rm}{\beta_1} \right)(\beta_0 r + \delta\beta_1)).$$

If $s < s_1$, then $c_4 < 0$. Hence, $H(0) = c_4 < 0$. If $K \geq \frac{\gamma}{\beta_2}$, or $K < \frac{\gamma}{\beta_2}$ and $s > s_0$, $H(K) = \beta_3 s + \beta_2 K\delta - \gamma\delta > 0$. And $\lim_{P \to +\infty} H(P) < 0$. Therefore, function $H(P)$ has two positive roots, one of which is larger than K. However, $Z = \frac{r}{\beta_1}\left(1 - \frac{P}{K}\right)$. The root which is bigger than K should be discarded. Hence, there is a unique positive equilibrium. $\qquad \square$

Here, in biologic context, E_1 means that both the phytoplankton and zooplankton go extinct eventually. E_2 means that the zooplankton goes extinct, while the phytoplankton will exist at the density of K eventually. E_3 means that the phytoplankton goes extinct, while the zooplankton will exist at the density of Z_3 eventually. E_4 means that the phytoplankton and zooplankton coexist. The existence of E_3 and E_4 is mainly determined by s, the constant input rate of TOM.

In the next section, we will analyse the stability of those equilibria. It should be pointed out that (2.2) is totally different from the nutrient-phytoplankton-zooplankton model, for the reason that s stands for TOM, which is ingested by zooplankton.

3. Dynamical properties of (2.2)

In this section, global dynamical properties of (2.2) are established.

Firstly, at the trivial equilibrium point $E_1(\frac{s}{\delta}, 0, 0)$ in (2.2), the corresponding characteristic equation is

$$(\lambda + \delta)(\lambda - r)\left(\lambda + \gamma - \frac{\beta_3 s}{\delta}\right) = 0. \tag{3.1}$$

It is apparent that equation (3.1) has one positive eigenvalue, which implies that $E_1(\frac{s}{\delta}, 0, 0)$ is always unstable.

Theorem 3.1. *If $s \geq s_1$, then $E_3(C_3, 0, Z_3)$ is globally asymptotically stable.*

Proof. Consider the following Lyapunov function

$$V_1(t) = \frac{\beta_3}{2\beta_0 C_3}(C - C_3)^2 + Z - Z_3 - Z_3 \ln \frac{Z}{Z_3} + \frac{\beta_2}{\beta_1} P. \tag{3.2}$$

Since the function $h(z) = z - 1 - \ln z$ is always nonnegative, it is easy to know that $V_1(t)$ is also nonnegative at any time t. Then the derivative of $V_1(t)$ along the solution of (2.2) is given by

$$\begin{aligned}
\frac{dV_1}{dt} &= \frac{\beta_3}{\beta_0 C_3}(C - C_3)(-\beta_0 CZ - \delta C + s) + \frac{\beta_2 r}{\beta_1} P - \beta_2 PZ - \frac{\beta_2 r}{\beta_1 K} P^2 \\
&\quad + (Z - Z_3)(\beta_2 P + \beta_3 C - mZ - \gamma) \\
&= \frac{\beta_3}{\beta_0 C_3}(C - C_3)(-\beta_0 CZ - \delta C + \beta_0 C_3 Z_3 + \delta C_3) - m(Z_3 - Z)^2 \\
&\quad + \beta_3(C - C_3)(Z - Z_3) + \beta_2\left(\frac{r}{\beta_1} - Z_3\right) P - \frac{\beta_2 r}{\beta_1 K} P^2 \\
&= -\frac{\beta_3 \delta}{\beta_0 C_3}(C - C_3)^2 - \frac{\beta_3 Z}{C_3}(C - C_3)^2 - m(Z - Z_3)^2 - \frac{\beta_2 r}{\beta_1 K} P^2 + \beta_2\left(\frac{r}{\beta_1} - Z_3\right) P.
\end{aligned}$$

If $s \geq s_1$, then $\beta_2\left(\frac{r}{\beta_1} - Z_3\right) \leq 0$. It follows that $\frac{dV_1}{dt} \leq 0$. Furthermore, $\frac{dV_1}{dt} = 0$ if and only if $C(t) = C_3$, $P(t) = 0$ and $Z(t) = Z_3$. Thus, the largest invariant set on which $\frac{dV_1}{dt}$ is zero consists of just equilibrium E_3. Therefore, by LaSalle's Invariance Principle, equilibrium E_3 is globally asymptotically stable. $\quad\square$

Theorem 3.2. *If $K \geq \frac{\gamma}{\beta_2}$ and $s \in (0, s_1)$, or $K < \frac{\gamma}{\beta_2}$ and $s \in (s_0, s_1)$, then $E_4(C_4, P_4, Z_4)$ is globally asymptotically stable.*

Proof. Consider the following Lyapunov function

$$V_2(t) = \frac{\beta_3}{2\beta_0 C_4}(C - C_4)^2 + Z - Z_4 - Z_4 \ln \frac{Z}{Z_4} + \frac{\beta_2}{\beta_1}\left(P - P_4 - P_4 \ln \frac{P}{P_4}\right). \tag{3.3}$$

Similarly, the derivative of $V_2(t)$ along the solution of (2.2) is given by

$$\begin{aligned}
\frac{dV_2}{dt} &= \frac{\beta_3}{\beta_0 C_4}(C - C_4)(-\beta_0 CZ - \delta C + s) + \frac{\beta_2}{\beta_1}(P - P_4)\left(r - \frac{rP}{K} - \beta_1 Z\right) \\
&\quad + (Z - Z_4)(\beta_2 P + \beta_3 C - mZ - \gamma) \\
&= \frac{\beta_3}{\beta_0 C_4}(C - C_4)(-\beta_0 CZ - \delta C + \beta_0 C_4 Z_4 + \delta C_4) - m(Z_4 - Z)^2 \\
&\quad + \beta_3(C - C_4)(Z - Z_4) - \frac{r\beta_2}{K\beta_1}(P - P_4)^2 \\
&= -\frac{\beta_3 \delta}{\beta_0 C_4}(C - C_4)^2 - \frac{\beta_3 Z}{C_4}(C - C_4)^2 - m(Z - Z_4)^2 - \frac{r\beta_2}{K\beta_1}(P - P_4)^2.
\end{aligned}$$

It is easy to know $\frac{dV_2}{dt} \leq 0$. $\frac{dV_2}{dt} = 0$ if and only if $C(t) = C_4$, $P(t) = P_4$ and $Z(t) = Z_4$, and hence the largest invariant set of (2.2) in the set $\left\{(C(t), P(t), Z(t))|\frac{dV_2}{dt} = 0\right\}$ is the singleton $\{E_4\}$. Therefore, by the LaSalle's Invariance Principle, equilibrium E_4 is globally asymptotically stable. $\quad\square$

Theorem 3.3. *If $K < \frac{\gamma}{\beta_2}$ and $s \in (0, s_0]$, then $E_2(\frac{s}{\delta}, K, 0)$ is globally asymptotically stable.*

Proof. Consider the following Lyapunov function

$$V_3(t) = \frac{\beta_3}{2\beta_0 C_2}(C - C_2)^2 + Z + \frac{\beta_2}{\beta_1}\left(P - P_2 - P_2 \ln \frac{P}{P_2}\right). \tag{3.4}$$

Similarly, the derivative of $V_3(t)$ along the solution of (2.2) is given by

$$\begin{aligned}
\frac{dV_3}{dt} ={}& \frac{\beta_3}{\beta_0 C_2}(C - C_2)(-\beta_0 CZ - \delta C + s) + \frac{\beta_2}{\beta_1}(P - P_2)\left(r - \frac{rP}{K} - \beta_1 Z\right) \\
& + \beta_2 PZ + \beta_3 CZ - \gamma Z - mZ^2 \\
={}& \frac{\beta_3}{\beta_0 C_2}(C - C_2)(-\beta_0 CZ - \delta C + \delta C_2) + \frac{\beta_2}{\beta_1}(P - P_2)\left(\frac{rP_2}{K} - \frac{rP}{K} - \beta_1 Z\right) \\
& + \beta_2 PZ + \beta_3 CZ - \gamma Z - mZ^2 \\
={}& -\frac{\beta_3 \delta}{\beta_0 C_2}(C - C_2)^2 + \frac{\beta_3}{\beta_0 C_2}(C - C_2)(-\beta_0 CZ + \beta_0 C_2 Z - \beta_0 C_2 Z) \\
& - \frac{r\beta_2}{K\beta_1}(P - P_2)^2 + \beta_2 P_2 Z + \beta_3 CZ - \gamma Z - mZ^2 \\
={}& -\frac{\beta_3 \delta}{\beta_0 C_2}(C - C_2)^2 - \frac{\beta_3 Z}{C_2}(C - C_2)^2 - \frac{r\beta_2}{K\beta_1}(P - P_4)^2 - mZ^2 \\
& + (\beta_3 C_2 + \beta_2 P_2 - \gamma)Z.
\end{aligned}$$

If $s \le \frac{\delta\gamma}{\beta_3} - \frac{\beta_2 K\delta}{\beta_3}$, then $\beta_3 C_2 + \beta_2 P_2 - \gamma = \frac{\beta_3 s}{\delta} + \beta_2 K - \gamma \le 0$. It is easy to know $\frac{dV_3}{dt} \le 0$. $\frac{dV_3}{dt} = 0$ if and only if $C(t) = \frac{s}{\delta}$, $P(t) = K$ and $Z(t) = 0$. Similarly, by the LaSalle's Invariance Principle, every solution of (2.2) tends to M, where $M = \left\{(C(t), P(t), Z(t))|\frac{dV_3}{dt} = 0\right\} = \{E_2\}$ is the largest invariant set of (2.2). It implies that E_2 is globally asymptotically stable. □

Global dynamical properties of (2.2) are shown clearly in Table 1. From biological viewpoint, Theorem 3.1 implies that if the constant input of TOM is larger than s_1, it will result in the extinction of phytoplankton eventually which is impossible in the system (2.1). It also means that the increasing input of the TOM will inhibit the growth of phytoplankton, which fits well with the hypotheses(II). In addition, in the extinct process of the phytoplankton, in order to survive, the zooplankton will uptake more TOM which is easy to get. That is the Allochthony of zooplankton will increase, which fits well with the hypotheses(I). The coexistence of TOM and zooplankton is mentioned in [6]. It also means that the producers of catchment can support both the landscape and lake ecosystem, as told in [2]. Furthermore, from Theorem 3.2, it can be known that different from the system (2.1), with the enough constant input of the TOM($s > s_0$), zooplankton in the deterministic setting can produce and give offsprings at small carrying capacity(K). The result fits well with the experimental result in [19]. Besides, the threshold value s_1 is positively related to the birth rate of phytoplankton, which means that phytoplankton with high birth rate is less possible to become extinct. The result of Theorem 3.3 implies that zooplankton goes extinct if both the carrying capacity and constant input rate of TOM are small.

Table 1. Stability of equilibria in (2.2).

$K \geq \frac{d}{\beta_2}$	$s \in [s_1, +\infty)$	E_3 is G.A.S.
	$s \in (0, s_1)$	E_4 is G.A.S.
$K < \frac{d}{\beta_2}$	$s \in [s_1, +\infty)$	E_3 is G.A.S.
	$s \in (s_0, s_1)$	E_4 is G.A.S.
	$s \in (0, s_0]$	E_2 is G.A.S.

4. Mathematical model with stochastic perturbation

In lake ecosystem, the weather, temperature and other physical factors are hard to predict. TOM, phytoplankton and zooplankton are easily affected by these factors. The deterministic model studied in the previous section does not take environmental randomness into consideration. Taking the randomness into account is more realistic. Recently, many researchers studied the stochastic biological models. Different types of stochastic perturbations are introduced into the models. Prey-predator models with fluctuations around the positive equilibrium were studied [20, 21]. The stochastic perturbations which are proportional to the variables were discussed [22, 23]. Fluctuations manifesting in the transmission coefficient rate were discussed [24, 25].

In this section, by considering role of unpredictable environmental factors in plankton dynamics, we get following stochastic equations,

$$\begin{cases} dC(t) = (-\beta_0 C(t)Z(t) - \delta C(t) + s)dt + \sigma_1 C(t)dB_1(t), \\ dP(t) = \left(rP(t)\left(1 - \frac{P(t)}{K}\right) - \beta_1 P(t)Z(t)\right)dt + \sigma_2 P(t)dB_2(t), \\ dZ(t) = (\beta_2 P(t)Z(t) - \gamma Z(t) + \beta_3 C(t)Z(t) - mZ^2(t))dt + \sigma_3 Z(t)dB_3(t), \end{cases} \tag{4.1}$$

where $B_i(t), i = 1, 2, 3$, are independent Brownian motions and σ_i, $i = 1, 2, 3$ are the corresponding intensities of stochastic perturbations. Obviously, all the equilibria of (2.2) are no longer the equilibria of the stochastic model (4.1).

In this paper, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions(i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B_i(t)$, $i = 1, 2, 3$, be a standard one-dimensional Brownian motion defined on this complete probability space. Let $R_+^3 = \{x \in R^3 : x_i > 0, 1 \leq i \leq 3\}$.

Definition 4.1 ([26])**.** A non-negative random variable $\tau(\omega)$, which is allowed to the value ∞, is called stopping time(with respect to filtration $\mathcal{F}_t$), if for each t, the event $\{\omega : \tau \leq t\} \in \mathcal{F}_t$.

Lemma 4.2 ([26])**.** *Let $\{X_t\}_t$ be right-continuous on R^n and adapted to $\mathcal{F}_t$. Define $\inf \emptyset = \infty$. If D is an open interval on R^n, $\tau_D = \inf\{t \geq 0 : X_t \notin D\}$ is a $\mathcal{F}_t$ stopping time.*

Lemma 4.3 ([26])**.** *Let $x(t) = (x_1(t), ..., x_n(t))$ be a regular adapted process n-dimensional vector process. We can have any number n of process driven by a d-dimensional Brownian motion*

$$dx(t) = b(t)dt + \sigma(t)dB_t,$$

where $\sigma(t)$ is $n \times d$ matrix valued function, B_t is d-dimensional Brownian motion, $x(t)$, $b(t)$ are n-dim vector-valued functions, the integrals with respect to Brownian motion are Itô integrals. Then $x(t)$ is

called an Itô process. The only restriction is that this dependence results in:

(i) for any $i = 1, 2...n$, $b_i(t)$ is adapted and $\int_0^T |b_i(t)| < \infty$ a.s..

(ii) for any $i = 1, 2, ...n$, $\sigma_{ij}(t)$ is adapted and $\int_0^T \sigma_{ij}^2(t) < \infty$ a.s.

Lemma 4.4 ([27])**.** *Let $x(t)$ be a l-dimensional Itô process on $t \geq 0$ with stochastic differential*

$$dx(t) = f(t)dt + g(t)dB_t,$$

where $f \in \mathcal{L}(R_+; R^l)$ and $g \in \mathcal{L}^2(R_+; R^{l \times m})$. Let $V \in C^{2,1}(R^l \times R_+; R)$. Then $V(x(t), t)$ is till an Itô process with the stochastic differential given by

$$dV(x, t) = \left[V_t + V_x(t, x)f(t) + \frac{1}{2}Tr[g^T(t)V_{xx}g(t)] \right] dt + V_x(x, t)g(t)dB(t) \ a.s.,$$

where

$$V_t = \frac{\partial V}{\partial t}, \ V_x = \left(\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_l} \right), \ V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{l \times l}.$$

Suppose $V \in C^{2,1}(R^l \times R_+; R)$, define a operator LV from $R^l \times R_+$ to R by

$$LV(t, x) = V_t + V_x(t, x)f(t) + \frac{1}{2}Tr[g^T(t)V_{xx}g(t)].$$

Theorem 4.5. *For any given initial value $(C(0), P(0), Z(0)) \in R_+^3$, (4.1) has almost surely (a.s.) a unique positive solution $(C(t), P(t), Z(t))$ for $t \geqslant 0$, and the solution remains in R_+^3 with probability 1.*

Proof. Since the coefficients of the equation are locally Lipschitz continuous, for any given initial value $(C(0), P(0), Z(0)) \in R_+^3$, it can be obtained that there is a unique local solution $(C(t), P(t), Z(t))$ on $t \in [0, \tau_e)$ where τ_e is the explosion time. Explosion refers to the situation when the process reaches infinite values in finite time. Solution can be considered until the time of explosion [26]. If we can check that $\tau_e = \infty$ a.s., then the solution is global. Let h_0 be sufficient large, such that every coordinate of (C(0), P(0), Z(0)) lies within the interval $\left[\frac{1}{h_0}, h_0 \right]$. For each integer $h \geq h_0$, stopping time is defined by

$$\tau_h = \inf \left\{ t \in [0, \tau_e) : P(t) \notin \left(\frac{1}{h}, h \right) \text{ or } C(t) \notin \left(\frac{1}{h}, h \right) \text{ or } Z(t) \notin \left(\frac{1}{h}, h \right) \right\},$$

where we set $\inf \emptyset = \infty$. Clearly, τ_h is increasing as $h \to \infty$. Let $\tau_\infty = \lim_{h \to \infty} \tau_h$, where $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and $(C(t), P(t), Z(t)) \in R_+^3$ a.s. for all $t \geq 0$. If not, then it exists constants $T > 0$ and $\epsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \epsilon$. Hence, there is integer $h_1 \geq h_0$ such that

$$P\{\tau_h \leq T\} > \epsilon, \ h \geq h_1.$$

Define a function $V_4: R_+^3 \to R_+$ as

$$V_4(C, P, Z) = \frac{\beta_3}{\beta_0}(C + 1 - \ln C) + \frac{\beta_2}{\beta_1}(P + 1 - \ln P) + Z + 1 - \ln Z. \tag{4.2}$$

We can get the non-negativity of this function from $f(u) = u + 1 - \ln u \geq 0, \forall u > 0$. Applying Itô formula, it follows that

$$
\begin{aligned}
dV_4(C, P, Z) =& \left(\frac{\beta_3}{\beta_0}\left(1 - \frac{1}{C}\right)(-\beta_0 CZ - \delta C + s) + \frac{\beta_2}{\beta_1}\left(1 - \frac{1}{P}\right)\left(rP\left(1 - \frac{P}{K}\right) - \beta_1 PZ\right)\right. \\
&+ \left(1 - \frac{1}{Z}\right)(\beta_2 PZ + \beta_3 CZ - \gamma Z - mZ^2) + \frac{\beta_3 \sigma_1^2}{2\beta_0} + \frac{\beta_2 \sigma_2^2}{2\beta_1} + \left.\frac{\sigma_3^2}{2}\right)dt \\
&+ \frac{\beta_3 \sigma_1}{\beta_0}(C - 1)dB_1(t) + \frac{\beta_2 \sigma_2}{\beta_1}(P - 1)dB_2(t) + \sigma_3(Z - 1)dB_3(t) \\
=& \left(\frac{\beta_3}{\beta_0}\left(-\delta C + s + \beta_0 Z + \delta - \frac{s}{C}\right) + \frac{\beta_2}{\beta_1}\left(rP\left(1 - \frac{P}{K}\right) - r + \frac{rP}{K}\right) + \gamma\right. \\
&+ Z(\beta_2 - \gamma - mZ) - \beta_2 P - \beta_3 C + mZ + \frac{\beta_3 \sigma_1^2}{2\beta_0} + \frac{\beta_2 \sigma_2^2}{2\beta_1} + \left.\frac{\sigma_3^2}{2}\right)dt \\
&+ \frac{\beta_3 \sigma_1}{\beta_0}(C - 1)dB_1(t) + \frac{\beta_2 \sigma_2}{\beta_1}(P - 1)dB_2(t) + \sigma_3(Z - 1)dB_3(t).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
dV_4(C, P, Z) \leq& \left(\frac{\beta_3}{\beta_0}(s + \delta) + \gamma + \beta_3 C + \frac{\beta_2}{\beta_1}\left(r + \frac{r}{K} + \beta_2\right)P + (\beta_2 + \beta_3 + m)Z\right. \\
&+ \left.\frac{\beta_3 \sigma_1^2}{2\beta_0} + \frac{\beta_2 \sigma_2^2}{2\beta_1} + \frac{\sigma_3^2}{2}\right)dt + \frac{\beta_3 \sigma_1}{\beta_0}(C - 1)dB_1(t) \\
&+ \frac{\beta_2 \sigma_2}{\beta_1}(P - 1)dB_2(t) + \sigma_3(Z - 1)dB_3(t).
\end{aligned}
$$

Let

$$
\begin{aligned}
q_1 &= \frac{\beta_3}{\beta_0}(s + \delta) + \gamma + \frac{\beta_3 \sigma_1^2}{2\beta_0} + \frac{\beta_2 \sigma_2^2}{2\beta_1} + \frac{\sigma_3^2}{2}, \\
q_2 &= \max\left\{\beta_0, r + \frac{r}{K} + \beta_2, \beta_2 + \beta_3 + m\right\}, \\
q_3 &= \beta_2 + \beta_3 + m.
\end{aligned}
$$

Since the function f is positive, it yields $u_i \leq 2(u_i + 1 - \ln u_i)$ [20]. Therefore, $\beta_3 C + \frac{\beta_2}{\beta_1}\left(r + \frac{r}{K} + \beta_2\right)P + q_3 Z \leq 2q_2 V_4(C, P, Z)$, which implies

$$
\begin{aligned}
dV_4(C, P, Z) \leq& (q_1 + 2q_2 V_4(C, P, Z))dt + \frac{\beta_3 \sigma_1}{\beta_0}(C - 1)dB_1(t) \\
&+ \frac{\beta_2 \sigma_2}{\beta_1}(P - 1)dB_2(t) + \sigma_3(Z - 1)dB_3(t),
\end{aligned}
$$

and hence,

$$
\begin{aligned}
dV_4(C, P, Z) \leq& q_4(1 + V_4(C, P, Z))dt + \frac{\beta_3 \sigma_1}{\beta_0}(C - 1)dB_1(t) \\
&+ \frac{\beta_2 \sigma_2}{\beta_1}(P - 1)dB_2(t) + \sigma_3(Z - 1)dB_3(t),
\end{aligned}
$$

where $q_4 = \max\{q_1, 2q_2\}$. Let us define $a \wedge b = \min\{a, b\}$. If $h_1 \leq T$, then

$$\int_0^{\tau_h \wedge h_1} dV_4(C, P, Z) \leq \int_0^{\tau_h \wedge h_1} q_4(1 + V_4(C, P, Z))dt + \int_0^{\tau_h \wedge h_1} \frac{\beta_3 \sigma_1}{\beta_0}(C - 1)dB_1(t)$$
$$+ \int_0^{\tau_h \wedge h_1} \frac{\beta_2 \sigma_2}{\beta_1}(P - 1)dB_2(t) + \int_0^{\tau_h \wedge h_1} \frac{\beta_3 \sigma_3}{\beta_0}(Z - 1)dB_3(t).$$

Taking expectation of both sides, we can get that

$$EV_4(C(\tau_h \wedge h_1), P(\tau_h \wedge h_1), Z(\tau_h \wedge h_1))$$
$$\leq V_4(C(0), P(0), Z(0)) + E \int_0^{\tau_h \wedge h_1} q_4(1 + V_4(C, P, Z))dt$$
$$\leq V_4(C(0), P(0), Z(0)) + q_4 h_1 + E \int_0^{\tau_h \wedge h_1} V_4(C, P, Z)dt$$
$$\leq V_4(C(0), P(0), Z(0)) + q_4 T + \int_0^{h_1} EV_4(C(\tau_h \wedge h_1), P(\tau_h \wedge h_1), Z(\tau_h \wedge h_1))dt.$$

By the Gronwall inequality,

$$EV_4(C(\tau_h \wedge T), P(\tau_h \wedge T), Z(\tau_h \wedge T)) \leq (V_4(C(0), P(0), Z(0)) + q_4 T)e^{q_4 T}.$$

We set $\Omega_h = \{\tau_h \leq T\}$ for $h \geq h_1$, which implies $P(\Omega_h) \geq \epsilon$. Note that for every $\omega \in \Omega_h$, we can have that $C(\tau_h, \omega)$ equals either h or $\frac{1}{h}$ or $P(\tau_h, \omega)$ equals either h or $\frac{1}{h}$ or $Z(\tau_h, \omega)$ equals either h or $\frac{1}{h}$, and hence,

$$V_4(C(\tau_h \wedge T, \omega), P(\tau_h \wedge T, \omega), Z(\tau_h \wedge T, \omega)) \geq \min\left\{\frac{\beta_3}{\beta_0}(h + 1 - \ln h),\right.$$
$$\left. \frac{\beta_2}{\beta_1}(h + 1 - \ln h), \ h + 1 - \ln h, \ \frac{\beta_3}{\beta_0}\left(\frac{1}{h} + 1 + \ln h\right), \frac{\beta_2}{\beta_1}\left(\frac{1}{h} + 1 + \ln h\right), \frac{1}{h} + 1 + \ln h\right\}.$$

Then it implies that

$$(V_4(C(0), P(0), Z(0)) + q_4 T)e^{q_4 T}$$
$$\geq E[1_{\Omega_h(\omega)}V_4(C(\tau_h \wedge T, \omega), P(\tau_h \wedge T, \omega)Z(\tau_h \wedge T, \omega))]$$
$$\geq \epsilon \min\left\{\frac{\beta_3}{\beta_0}(h + 1 - \ln h), \frac{\beta_2}{\beta_1}(h + 1 - \ln h), h + 1 - \ln h,\right.$$
$$\left. \frac{\beta_3}{\beta_0}\left(\frac{1}{h} + 1 + \ln h\right), \frac{\beta_2}{\beta_1}\left(\frac{1}{h} + 1 + \ln h\right), \frac{1}{h} + 1 + \ln h\right\},$$

where 1_{Ω_h} is the indicator function of Ω_h. Then, we let $h \to \infty$, which leads to the contradiction $\infty > (V_4(C(0), P(0), Z(0)) + q_4 T)e^{q_4 T} = \infty$. Therefore, $\tau_h = \infty$ a.s. □

Now, we give the main result of this section in the following part.

Theorem 4.6. *Assume $\sigma_1^2 < \delta$, σ_2, $\sigma_3 > 0$ such that*

$$\eta < \min\left\{\frac{\beta_3}{\beta_0 C_4}(\delta - \sigma_1^2)C_4^2, \ mZ_4^2, \ \frac{r\beta_2}{K\beta_1}P_4^2\right\}.$$

Then there is a stationary distribution $\mu(\cdot)$ for (4.1) with initial value $(P_0, C_0, Z_0) \in R_+^3$, which has ergodic property.

Proof. Define a Lyapunov functional

$$V_5(C, P, Z) = \frac{\beta_3}{2\beta_0 C_4}(C - C_4)^2 + Z - Z_4 - Z_4 \ln\frac{Z}{Z_4} + \frac{\beta_2}{\beta_1}\left(P - P_4 - P_4 \ln\frac{P}{P_4}\right). \tag{4.3}$$

Then applying Itô's formula to (4.1), we obtain

$$LV_5 = -\frac{\beta_3 \delta}{\beta_0 C_4}(C - C_4)^2 - \frac{\beta_3 Z}{C_4}(C - C_4)^2 - m(Z - Z_4)^2 - \frac{r\beta_2}{K\beta_1}(P - P_4)^2$$

$$+ \frac{\beta_3 \sigma_1^2}{2\beta_0 C_4}C^2 + \frac{\beta_2 \sigma_2^2}{2\beta_1} + \frac{\sigma_3^2}{2}. \tag{4.4}$$

Since $(a + b)^2 \leq 2(a^2 + b^2)$, we have that

$$LV_5 \leq -\frac{\beta_3}{\beta_0 C_4}(\delta - \sigma_1^2)(C - C_4)^2 - \frac{\beta_3 Z}{C_4}(C - C_4)^2 - m(Z - Z_4)^2$$

$$- \frac{r\beta_2}{K\beta_1}(P - P_4)^2 + \frac{\beta_3 \sigma_1^2}{\beta_0}C_4 + \frac{\beta_2 \sigma_2^2}{2\beta_1} + \frac{\sigma_3^2}{2}$$

$$\leq -\frac{\beta_3}{\beta_0 C_4}(\delta - \sigma_1^2)(C - C_4)^2 - m(Z - Z_4)^2 - \frac{r\beta_2}{K\beta_1}(P - P_4)^2 + \eta,$$

where $\eta = \frac{\beta_3 \sigma_1^2}{\beta_0}C_4 + \frac{\beta_2 \sigma_2^2}{2\beta_1} + \frac{\sigma_3^2}{2}$. When $\eta < \min\left\{\frac{\beta_3}{\beta_0}(\delta - \sigma_1^2)C_4, \ mZ_4^2, \ \frac{r\beta_2}{K\beta_1}P_4^2\right\}$, we can get that the ellipsoid

$$-\frac{\beta_3}{\beta_0 C_4}(\delta - \sigma_1^2)(C - C_4)^2 - m(Z - Z_4)^2 - \frac{r\beta_2}{K\beta_1}(P - P_4)^2 + \eta = 0,$$

lies entirely in R_+^3. Let U be a neighborhood of the ellipsoid with $\bar{U} \subset E_l = R_+^3$, so for $x \in U \backslash E_l$, $LV_5 < -\zeta$ (ζ is a positive constant). Therefore, we have that condition (B.2) in Lemma 2.1 of [23] is satisfied(E_l denotes euclidean l-space). Besides, there is $M > 0$ such that

$$\sigma_1^2 C^2 \xi_1^2 + \sigma_2^2 P^2 \xi_2^2 + \sigma_3^2 Z^2 \xi_3^2 \geq M|\xi^2| \quad all \ (C, P, Z) \in \bar{U}, \ \xi \in R_+^3.$$

Thus, we have that the condition (B.1) of [23] is also satisfied. Therefore, there exists a stable stationary distribution $\mu(\cdot)$ which is ergodic in (4.1). □

5. Numerical simulations

In this section, the numerical simulations of (2.2) and (4.1) are given to study the proposed models in details. Different simulated results can fit with different hypotheses mentioned in the first section.

Table 2. Parameter definitions and the values used for model (2.2) and model (4.1).

Parameters	Definition	Values	Reference
β_0	Zooplankton grazing rate on C	0.15	-
δ	Sinking rate of C	0.2	-
r	Intrinsic growth rate of P	0.7 d^{-1}	[9]
β_1	Zooplankton grazing rate on P	$0.4 \, \mu\text{gcl}^{-1}\text{d}^{-1}$	-
β_2	Zooplankton grazing efficiency on P	0.25	-
β_{max}	Maximum grazing rate on P	$6 \, \mu\text{gcl}^{-1}\text{d}^{-1}$	[9]
β_3	Zooplankton grazing efficiency on C	0.05	-
γ	Zooplankton mortality	0.3 d^{-1}	[28]
s	Constant input rate of C	1	-
m	Zooplankton competitive mortality	0.1	-
K	Carrying capacity	$100 \, \mu\text{gcl}^{-1}$	[28]

Unless otherwise stated, parameter values given in Table 2 are used for the simulations. In order to get biologically plausible results, many values are taken from parameter ranges found in the literature. And the numerical simulations of (2.2) are prepared by tool kit ode45 of Matlab.

It is easy to get $s_1 = 4.4$, which is the threshold value of model (2.2). It should be noticed that since it is impossible that the absorbed energy is fully used for reproduction, the values of our parameters should satisfy $\beta_0 > \beta_3$ and $\beta_1 > \beta_2$. In the experimental results of [6], it points out that for zooplankton, the growth efficiency of assimilating TOM is lower than that of ingesting phytoplankton. In following analysis, hence, the values of our parameters should satisfy $\beta_1 > \beta_0$.

In the case where $s = 1 < s_1$, Figure 2(a) shows that solutions dampen and tend to a stable steady state. The carrying capacity varies with the geographical location of the lake. Especially, the carrying capacity of poor nutrient lake is very small. If we decrease K to 1, from Theorem 3.1, we can know dynamical behavior of (2.2) is similar. However, the solution of (2.1) with the same parameter values will tend to $A(\frac{r_1}{a_1}, 0)$, which means the extinction of zooplankton. Above statements can fit well with the experimental results of [19] that Daphnia magna(zooplankton) can use terrestrial-derived dissolved organic matter (t-DOM) to support growth and reproduction when alternative food sources are limiting.

The catchment areas around different lakes are different. The constant input of TOM is mainly determined by the catchment areas. It is reasonable to consider the variation of parameter value s. The values of other parameters are kept invariant.

In the case where $s = 5 > s_1$, we see from Figure 3(a) that the phytoplankton goes extinct eventually. In particular, because of the extinction of phytoplankton, the growth of zooplankton is determined by TOM. That is, the zooplankton shows a preference for TOM, which fits well with the hypothesis (I).

In addition, it also means the increasing input rate of TOM will hinder the growth of phytoplankton, which fits well with the hypothesis (II).

In this part, we decrease $K = 100$ to 1 and $s = 1$ to 0.1. The other parameter values are the same as before. The result of numerical simulation is shown in Figure 4(a). We can notice that the density of zooplankton decreases to zero, which means if both environmental capacity and constant input of TOM are small, the zooplankton goes extinct in the end.
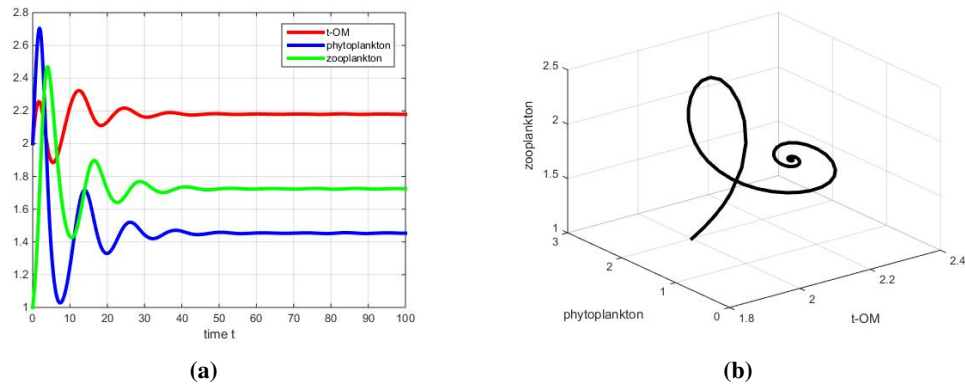
Figure 2. In the case where $s = 1 < s_1$, the solutions of (2.2) that initially oscillate will dampen and tend to a stable steady state, which means the phytoplankton and zooplankton coexist in the end.
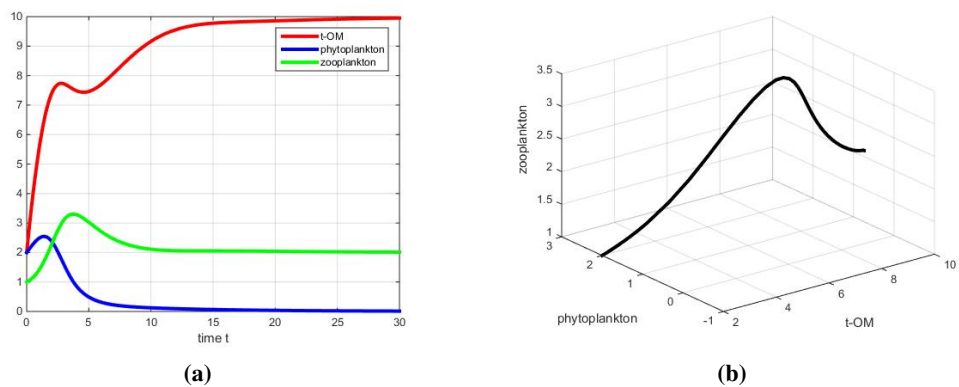

Figure 3. In the case where $s = 5 > s_1$, the high input rate of TOM can result in the extinction of the phytoplankton.
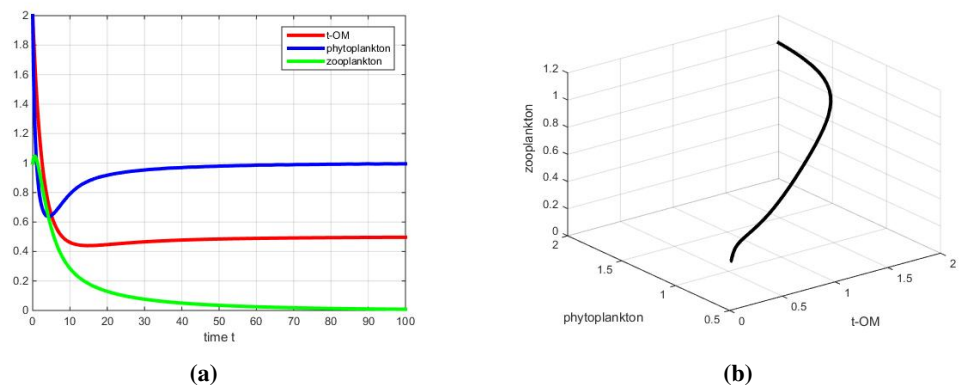

Figure 4. Here, $K = 1$, $s = 0.1$, it can be seen that the zooplankton goes extinct in the end.

Finally, we consider the stochastic model. the values of the parameters are same as those for Figure 2. And $\sigma_1 = \sigma_2 = \sigma_3 = 0.1$. The following discretization equations in Milsteins type [29] are used to iteratively calculate the approximate solutions of stochastic system (4.1) in Matlab programs:

$$C_{i+1} = C_i + (-\delta C_i - \beta_0 C_i Z_i + s)\,\Delta t + \sigma_1 C_i \xi_{1,i}\,\sqrt{\Delta t} + \frac{\sigma_1^2}{2} C_i (\xi_{1,i}^2 \Delta t - \Delta t),$$

$$P_{i+1} = P_i + \left(r\left(P_i - \frac{P_i}{K} \right) - \beta_1 P_i Z_i \right)\Delta t + \sigma_2 P_i \xi_{2,i}\,\sqrt{\Delta t} + \frac{\sigma_2^2}{2} P_i (\xi_{2,i}^2 \Delta t - \Delta t), \qquad (5.1)$$

$$Z_{i+1} = Z_i + (\beta_2 P_i Z_i + \beta_3 C_i Z_i - \gamma Z_i - m Z_i^2)\Delta t + \sigma_3 Z_i \xi_{3,i}\,\sqrt{\Delta t} + \frac{\sigma_3^2}{2} Z_i (\xi_{3,i}^2 \Delta t - \Delta t),$$

where $\xi_{1,i}, \xi_{2,i}$, and $\xi_{3,i}$ are $N(0, 1)$-distributed independent Gaussian random variable, σ_1, σ_2 and σ_3 are intensities of white noise and time increment $\Delta t > 0$. The simulation results are shown in Figure 5.

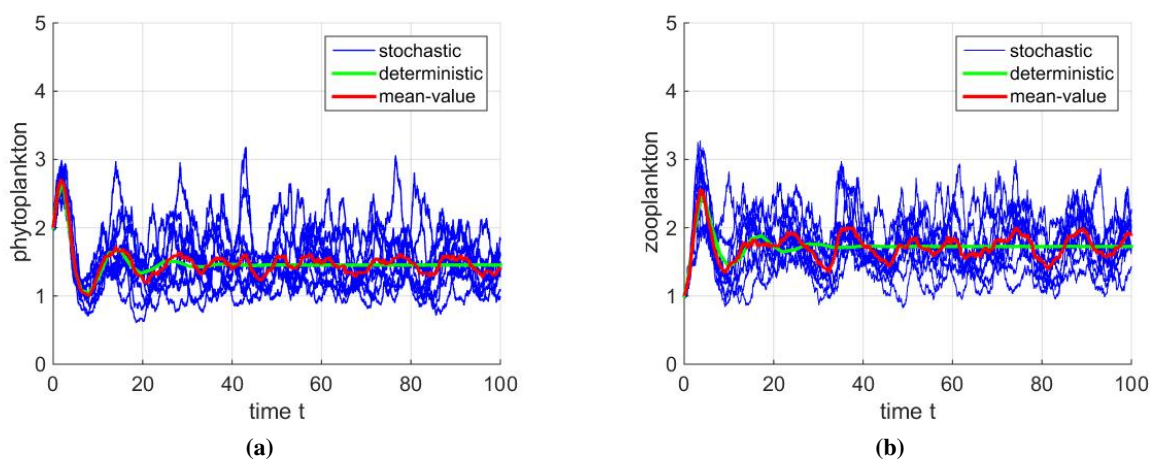(a)   (b)

Figure 5. Here, $\sigma_1 = \sigma_2 = \sigma_3 = 0.1$. The other parameter values are the same as those for Figure 2. The Figure 5*a* and Figure 5*b* are the trajectories of population densities of phytoplankton and zooplankton respectively. When the perturbation is small, the numerical simulations illustrate that mean value trajectory(the red trajectory) of (4.1) which is the average of ten trajectories(the blue trajectories) of (4.1) oscillates randomly surrounding the solution of (2.2) (the green trajectory). And the numerical simulations also show that the trajectory of (4.1) is totally different from that of (2.2).

6. Conclusion

In this paper, firstly, phytoplankton-zooplankton model including TOM is proposed to investigate effects of TOM upon planktonic dynamics. By constructing Lyapunov functions and using LaSalle's Invariance Principle, global stability of the equilibria which is determined by threshold values s_0 and s_1 is established. The analytical results of (2.2) indicate that TOM has significant effects on the lake ecosystem stability. Different constant input rates of TOM in the model can result in different results, which represents different biological meanings.

Plankton populations which reside in lake ecosystem are persistently influenced by the environmental fluctuations. In order to make more precise biological findings, we develop a stochastic model. We get that when the perturbation is small, the plankton populations reach a stationary distribution, which is ergodic. Then we simulate model (2.2) with many parameter values taken from the literature by varying the constant input rate of TOM along with the carrying capacity value. The simulations indicate that the magnitude of constant input rate of TOM and carrying capacity affect the persistence of phytoplankton populations greatly. We also simulate the stochastic model with the same parameter values. We can notice that mean value trajectory of (4.1) which is the average of ten trajectories oscillates randomly surrounding the solution of system (2.2), rather than reach the stable steady state (Figure 5). And the simulations also show that the trajectory of (4.1) is totally different from that of (2.2).

In summary, we set up exploratory models to consider the potential role of TOM that mediates interactions between trophic levels in a simple plankton food-chain. We have shown that, in principle, TOM plays an important role in influencing interactions between phytoplankton and zooplankton. The theoretical results of our deterministic model fit well with some experimental results. Further, when the perturbation is small, the stochastic model exists a stationary distribution which is ergodic. We emphasize that the qualitative behaviour of our models that we are interested in and this study are only meant as an initial exploratory attempt to consider the influence of TOM on the lake ecosystem.

Acknowledgments

Conflict of interest

The authors declare there is no conflict of interest.

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