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Research article

Exploring the dynamics of a tritrophic food chain model with multiple gestation periods

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Abstract: This work is mainly focused on the series of dynamical analysis of tritrophic food chain model with Sokol-Howell functional response, incorporating the multiple gestation time delays for more realistic formulation. Basic properties of the proposed model are studied with the help of boundedness, stability analysis, and Hopf-bifurcation theory. By choosing the fixed parameter set and varying the value of time delay, the stability of the model has been studied. There is a critical value for the delay parameter. Steady state is stable when the value of delay is less than the critical value and a further increase in the value of delay beyond the critical value makes the system oscillatory through Hopf-bifurcation. Whereas, another delay parameter has a stabilizing effect on the system dynamics. Chaotic dynamics has been explored in the model with the help of phase portrait and sensitivity on initial condition test. Numerical simulations are performed to validate the effectiveness of the derived theoretical results and to explore the various dynamical structures such as Hopf-bifurcation, periodic solutions, and chaotic dynamics.

Keywords: food chain; Sokol-Howell functional response; gestation delay; Hopf-bifurcation; chaos

1. Introduction

Time delays are ubiquitous in all biological situations, as species require some time in order to complete various biological activities such as digestion, gestation, maturation, incubation, etc. Also, the present growth of species may be affected by the past generations through delay mechanism [1, 2]. Introduction of time delay may change qualitatively the dynamical behaviors of a predator-prey interaction model, therefore, it is important to investigate the dynamical properties of a predator-prey model with time delays in both theoretical research and practical applications. One of the interesting observation of inclusion of time delay is the appearance of oscillatory behavior in single species models [3,4]. Time delay can produce chaotic oscillations even in simple predator-prey models [5,6]. The introduction of time delay leads to rich dynamical behaviors such as periodic orbits, stability switching, chaotic dynamics, and multiple stable coexistence through the different bifurcation routes [7]. Mathematical models incorporating the time delays are widely discussed in the books of MacDonald [8], Kuang [3] and Cushing [9].

Recently, the impact of multiple delays on the dynamics of ecological systems has caught the attention of many researchers. In particular, Gakkhar and Singh [5] have studied the predator-prey model with Holling type II response function and discrete delays. They have observed that the stable species coexistence undergoes Hopf-bifurcation for the critical value of delay and a further increase in delay beyond the Hopf-bifurcation leads the system dynamics to the chaotic state. Also in two neuron system with multiple delays, Song et al. [10, 11] have obtained stability switching, multiple stable coexistence of two resting states, two anti-symmetric periodic activity with period three, one self-symmetric periodic activity with period one, one quasiperiodic spiking and chaotic behavior. Jiang et al. [12] have considered the Phytoplankton-Zooplankton model with Holling III functional response and discrete delay. They have shown that the oscillations can be prevented by adjusting the magnitude of delay. Song et al. [13] have discussed the food chain system with multiple digestion delays and predicted that the multiple delays can generate and suppress the higher order limit cycles and chaos. Lotka-Volterra food chain system with two discrete delays has been investigated by Cui and Yan [14]. They have determined the linear stability, Hopf-bifurcation, direction and stability of bifurcating period solutions by considering the sum of two delays as a bifurcating parameter. Jiang and Wang [15] have investigated the predator-prey model with three delays and discussed the delay induced destabilization and stability analysis of periodic solutions. Further, Ghosh et al. [16] have studied the stability and bifurcation analysis of an eco-epidemiology model with multiple delays. Three species Leslie-Gower type food chain model with resource digestion delay and consumer digestion delay is analyzed by Guo et al. [17]. It has been noted that the multiple delays lead to the stability switching, generate or terminate the recurrent bloom and help control the species population to the stable coexistence.

Delay induced destabilization is a common finding. A lot of work has been done with the objective to find the critical value of the delay parameter at which system bifurcates from its stable state and starts showing the oscillatory behaviour [18–23]. However, Sen et al. [24] have provided the necessary and sufficient conditions for stability of interior equilibria in a ratio-dependent predator-prey model with Allee effect and maturation delay. Stabilizing effect of maturation delay in a ratio-dependent predator-prey model with Allee effect is investigated by Banerjee and Takeuchi [25]. Wang and Jiang [26] have demonstrated that different values of delay can induce or eradicate chaotic dynamics in the predator-prey system with dormancy in predator. Motivated by these research work, we have asked the following research questions in the current manuscript:

(i) How do multiple gestation delays affect the stable and oscillatory coexistence of species? Do several delays behave in a similar fashion?
(ii) Is it possible to obtain some parameter sets so that stable coexistence of species is not affected by the introduction of delays?
(iii) Is it possible to obtain various complex dynamical behaviors such as higher order limit cycles and chaos? If yes, what is the impact of delays on these complex dynamical structures?

For ecological forecasting, it is necessary to understand the predator-prey linkage in food chain or web systems. Different functional and numerical responses are used for modeling the trophic interactions and they are one of the most important components in the study of interacting populations. In most of the studies, population dynamics is modeled with the help of monotonic response functions (Holling type I, II, III). Observational and experimental results show that these types of response functions are not appropriate for modeling the situations with group defence and inhibitory effects [27]. More suitable in these situations is Holling type IV functional response [28, 29]. In their experiment of uptake of phenol by pure culture of *Pseudomonas putida* growing on phenol in continuous culture, Sokol and Howell [30] proposed the simplified form of Monod-Haldane functional response as $p(x) = \frac{mx}{a+x^2}$. They obtained that it better fits the experimental data and simple as involving only two parameters. Edwards [31], Boon and Laudelout [32], Xiao and Ruan [27] suggested that this type of functional response takes place at the microbial level: when the nutrient concentration attains a high value, an inhibitory effect on the specific growth rate may occur. Recently, Ali et al. [33, 34] proposed a three species food chain system with Sokol-Howell functional response. They have studied the boundedness, local and global stability of the system. Dynamical behavior is also explored by using the numerical simulations. Explosive instabilities in three species food chain model with this functional response have been investigated by Parshad et al. [35]. A three species Rosenzweig-MacArthur food chain model with this functional response has been investigated by Ali et al. [36].

In the current work, we have studied a food chain model with multiple gestation delays. As assimilation of prey into the predator biomass is a complex phenomenon and completed through various bio-physiological activities, which require time, therefore, time lags in predators gestation process have been considered. Effect of gestation delay in the system of interacting populations is studied by many researchers [18–20, 22, 23, 37–39]. Patra et al. [40] have analyzed the effect of discrete delay in a three species food chain model with ratio-dependent type functional response. Pal et al. [41] have studied the tritrophic food chain model with gestation delay, where species interacts with Holling type II response function. Here, gestation delays are incorporated in the model using the Wangersky-Cunningham delay formulation [42]. The conventional way of delay formulation has been extensively studied in literature [5, 24, 40, 43]. In recent years, Wangersky-Cunningham delay formulation is used prominently due to its clear biological explanation [19, 24, 25, 41].

The organization of paper is as follows. Formulation of the model is given in section 2. In section 3, positive invariance, boundedness, equilibria and stability analysis have been discussed. Local stability analysis and Hopf-bifurcation about the interior equilibrium point for all possible cases to incorporate gestation delays have been derived in section 4. Numerical simulation results have been presented in section 5. Finally, discussion and conclusion are given in the last section.

2. Formulation of the mathematical model

The model has been developed under the following assumptions.

(1) The behavior of whole community arises due to the coupling of three types of interacting populations: prey $X(t)$, intermediate predator $Y(t)$ and top predator $Z(t)$.
(2) Prey population grows logistically with intrinsic growth rate r and carrying capacity K. Thus, the per capita growth rate of prey in the absence of predator is given by $r\left(1 - \frac{X(t)}{K}\right)$.

(3) Intermediate and top predators consume their sole food (prey and intermediate predator respectively) according to Sokol-Howell functional response.
(4) In the absence of their only foods intermediate and top predators die out with their natural death rates.
(5) Consumption of prey by the predator is not an instantaneous process. However, predator requires some period of time to convert the prey density into itself due to gestation.

Under the above assumptions, interactions between the species are modeled by the following system of DDEs:

$$\begin{aligned}
\frac{dX}{dT} &= rX\left(1-\frac{X}{K}\right)-\frac{\omega XY}{X^2+a_1},\\
\frac{dY}{dT} &= -bY+\frac{\omega_1 X(T-T_1)Y(T-T_1)}{X^2(T-T_1)+a_1}-\frac{\omega_2 YZ}{Y^2+a_2},\\
\frac{dZ}{dT} &= -cZ+\frac{\omega_3 Y(T-T_2)Z(T-T_2)}{Y^2(T-T_2)+a_2}.
\end{aligned} \tag{2.1}$$

All the parameters $r,\ K,\ \omega,\ a_1,\ b,\ \omega_1,\ \omega_2,\ a_2,\ c,\ \omega_3,\ T_1$ and T_2 are positive and brief description about these parameters is given in Table 1.

Table 1. Parameters used in the model (2.1).

Parameters	Meaning
r	Intrinsic growth rate of prey population X
K	Carrying capacity of prey X in the absence of predator Y
$\omega,\ \omega_2$	Maximum values which per capita reduction rate of prey and intermediate predator can attain respectively
ω_1	Conversion coefficient from individual of prey to individual of intermediate predator
$b,\ c$	Death rates of intermediate predator Y and top predator Z in the absence of their sole foods X and Y respectively
ω_3	Conversion coefficient from individual of intermediate predator to the top predator
$a_1,\ a_2$	Measure the level of protection provided by environment to the prey and intermediate predator respectively
$T_1,\ T_2$	Gestation delays for intermediate and top predator respectively

Model (2.1) involves 12 parameters, which complicates the system analysis. Thus, to reduce the complexity of model (2.1), we non-dimensionalize it by using the following transformations:

$$\begin{aligned}
&\frac{X}{K}=x,\quad rT=t,\quad \frac{a_1}{K^2}=\omega_4,\quad \frac{\omega Y}{rK^2}=y,\quad \frac{b}{r}=\omega_5,\quad \frac{\omega_1}{rK}=\omega_6,\\
&\frac{\omega_2\omega^2 Z}{r^3K^4}=z,\quad \frac{a_2\omega^2}{r^2K^4}=\omega_7,\quad rT_1=\tau_1,\quad \frac{c}{r}=\omega_8,\quad \frac{\omega_3\omega}{r^2K^2}=\omega_9,\quad rT_2=\tau_2.
\end{aligned}$$

Then model (2.1) is reduced in the following dimensionless form:

$$\begin{aligned}\frac{dx}{dt} &= x(1-x) - \frac{xy}{x^2+\omega_4},\\ \frac{dy}{dt} &= -\omega_5 y + \frac{\omega_6 x(t-\tau_1)y(t-\tau_1)}{x^2(t-\tau_1)+\omega_4} - \frac{yz}{y^2+\omega_7},\\ \frac{dz}{dt} &= -\omega_8 z + \frac{\omega_9 y(t-\tau_2)z(t-\tau_2)}{y^2(t-\tau_2)+\omega_7}.\end{aligned} \tag{2.2}$$

All the variables and parameters of dimensionless system (2.2) are positive. We denote by C the Banach space of continues functions $\phi : [-\tau, 0] \to \Re^3$ with norm

$$\|\phi\| = \sup_{-\tau\leq\theta\leq 0} \{|\phi_1(\theta)|, |\phi_2(\theta)|, |\phi_3(\theta)|\}, \quad \tau = \max[\tau_1,\ \tau_2], \quad \phi = (\phi_1,\ \phi_2,\ \phi_3).$$

The initial conditions are given as

$$x(\theta) = \phi_1(\theta),\ y(\theta) = \phi_2(\theta),\ z(\theta) = \phi_3(\theta), \quad \theta \in [-\tau, 0]. \tag{2.3}$$

For biological reasons, it is assumed that

$$\phi_1(\theta) \geq 0,\ \phi_2(\theta) \geq 0,\ \phi_3(\theta) \geq 0, \quad \theta \in [-\tau, 0].$$

By the fundamental theorem of differential equations [44], there exists a unique solution $(x(t),\ y(t),\ z(t))$ of the model (2.2) with initial conditions (2.3).

3. Preliminaries

In this section, we present some basic results such as positive invariance, boundedness of the solutions, equilibria analysis and characteristic equation of model (2.2).

3.1. Positive invariance

It is important to discuss the positivity of solutions of model (2.2) as they represent the populations of prey, intermediate predator and top predator at any time. In the biological sense, positivity makes sure that the population never becomes negative and always survives in the finite time. We have established the positivity through the following theorem.

Theorem 3.1. Every solution of the system (2.2) with initial conditions (2.3) is positive.

Proof. The model (2.2) with the initial conditions (2.3) can be written in the following form:

$$\begin{aligned}\text{Consider } W =&\text{col}(x,\ y,\ z) \in \Re^3_+,\ (\phi_1(\theta),\ \phi_2(\theta),\ \phi_3(\theta)) \in C_+ = ([-\tau,\ 0],\ \Re^3_+),\\ &\phi_1(0),\ \phi_2(0),\ \phi_3(0) > 0.\end{aligned} \tag{3.1}$$

$$F(W) = \begin{pmatrix} F_1(W)\\ F_2(W)\\ F_3(W)\end{pmatrix} = \begin{pmatrix} x(1-x) - \frac{xy}{x^2+\omega_4}\\ -\omega_5 y + \frac{\omega_6 x(t-\tau_1)y(t-\tau_1)}{x^2(t-\tau_1)+\omega_4} - \frac{yz}{y^2+\omega_7}\\ -\omega_8 z + \frac{\omega_9 y(t-\tau_2)z(t-\tau_2)}{y^2(t-\tau_2)+\omega_7}\end{pmatrix}.$$

The model (2.2) becomes

$$\dot{W} = F(W), \tag{3.2}$$

with $W(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C_+$ and $\phi_1(0), \phi_2(0), \phi_3(0) > 0$. It is easy to check in system (3.2) that whenever choosing $W(\theta) \in \Re^3_+$ such that $x = y = z = 0$, then

$$F_i(W) \mid_{w_i=0,\ W\in\Re^3_+} \geq 0,$$

with $w_1(t) = x(t),\ w_2(t) = y(t),\ w_3(t) = z(t)$. Using the lemma 4 given in [45], any solution of (3.2) with $W(\theta) \in C_+$ saying $W(t) = W(t, W(\theta))$, is such that $W(t) \in \Re^3_+$ for all $t \geq 0$. Hence the solution of the system (3.2) exists in the region $\Re^3_+$ and all solutions remain nonnegative for all $t > 0$. Therefore, the positive orthant $\Re^3_+$ is an invariant region. □

3.2. Boundedness

Theorem 3.2. Let $(x(t), y(t), z(t))$ be any positive solution of the model (2.2), then there exists a time $\widetilde{T} > 0$, such that $0 \leq x(t) \leq M_1$, $0 \leq y(t) \leq M_2$ and $0 \leq z(t) \leq M_3$ for $t > \widetilde{T}$, where $M_1 = 1$, $M_2 = \frac{\omega_6}{4\omega_5}$, $M_3 = \frac{\omega_6\omega_9(\omega_5-\delta)}{4\omega_5\delta}$, δ is any positive constant satisfying $\delta < \min\{\omega_5, \omega_8\}$.

Proof. From the positive invariance theorem, we have $x(t) \geq 0$, $y(t) \geq 0$ and $z(t) \geq 0$. Therefore, we only need to show that $x(t) \leq M_1$, $y(t) \leq M_2$ and $z(t) \leq M_3$. From the prey equation, we obtain that

$$\frac{dx}{dt} \leq x(1-x),$$

thus

$$x(t) \leq \frac{x(0)}{x(0) + (1 - x(0))e^{-t}},$$

therefore,

$$\limsup_{t\to+\infty} x(t) \leq 1 = M_1.$$

Now, we construct a new function

$$\sigma(t) = x(t-\tau_1) + \frac{y(t)}{\omega_6}.$$

By differentiating $\sigma(t)$ with respect to time t, we obtain

$$\begin{aligned}\frac{d\sigma}{dt} &= \frac{dx(t-\tau_1)}{dt} + \frac{1}{\omega_6}\frac{dy}{dt}\\ &= x(t-\tau_1)(1 - x(t-\tau_1)) - \frac{x(t-\tau_1)y(t-\tau_1)}{x^2(t-\tau_1)+\omega_4} - \frac{\omega_5 y}{\omega_6}\\ &\quad + \frac{x(t-\tau_1)y(t-\tau_1)}{x^2(t-\tau_1)+\omega_4} - \frac{y(t)z(t)}{\omega_6(y^2+\omega_7)}.\end{aligned}$$

And by using the positivity of solutions, we get

$$\frac{d\sigma}{dt} \leq x(t-\tau_1)(1 - x(t-\tau_1)) - \frac{\omega_5 y}{\omega_6}.$$

Then adding $\omega_5\sigma(t)$ on the both side of above inequality, we get

$$\frac{d\sigma}{dt} + \omega_5\sigma(t) \le x(t-\tau_1)(1-x(t-\tau_1)) + \omega_5 x(t-\tau_1).$$

Since, $\max\{x(t-\tau_1)(1-x(t-\tau_1))\} = \frac{1}{4}$, implies,

$$\begin{aligned}\frac{d\sigma}{dt} + \omega_5\sigma(t) &\le \frac{1}{4} + \omega_5 x(t-\tau_1)\\ &\le \frac{1}{4} + \omega_5.\end{aligned}$$

Therefore, by using the lemma (2) given in [46], we obtain

$$\sigma(t) \le \left(\frac{1}{4\omega_5} + 1\right) - \left(\frac{1}{4\omega_5} + 1 - \sigma(\widetilde{T_1})\right) e^{-\omega_5(t-\widetilde{T_1})}, \quad \text{for } \ t \ge \widetilde{T_1} \ge 0.$$

If $\widetilde{T_1} = 0$, then

$$\sigma(t) \le \left(\frac{1}{4\omega_5} + 1\right) - \left(\frac{1}{4\omega_5} + 1 - \sigma(0)\right) e^{-\omega_5(t-0)},$$

therefore

$$\limsup_{t\to+\infty} \sigma(t) \le \left(\frac{1}{4\omega_5} + 1\right),$$

i.e.

$$x(t-\tau_1) + \frac{y(t)}{\omega_6} \le \frac{1}{4\omega_5} + 1, \quad \text{for large } t > 0,$$

thus,

$$y(t) \le \frac{\omega_6}{4\omega_5} = M_2, \quad \text{for large } t.$$

Again for boundedness of $z(t)$, we construct another function

$$\gamma(t) = x(t-\tau_1-\tau_2) + \frac{y(t-\tau_2)}{\omega_6} + \frac{z(t)}{\omega_6\omega_9}.$$

By differentiating above equation with respect to time t, we have

$$\frac{d\gamma(t)}{dt} = \frac{dx(t-\tau_1-\tau_2)}{dt} + \frac{1}{\omega_6}\frac{dy(t-\tau_2)}{dt} + \frac{1}{\omega_6\omega_9}\frac{dz(t)}{dt}.$$

Now, by using system (2.2), we obtain

$$\frac{d\gamma(t)}{dt} = x(t-\tau_1-\tau_2)(1-x(t-\tau_1-\tau_2)) - \frac{\omega_5}{\omega_6}y(t-\tau_2) - \frac{\omega_8}{\omega_6\omega_9}z(t).$$

And by adding $\delta\gamma(t)$ on the both side of above inequality, where $\delta < \min\{\omega_5, \omega_8\}$, we obtain

$$\begin{aligned}\frac{d\gamma(t)}{dt} + \delta\gamma(t) &\le x(t-\tau_1-\tau_2)(1-x(t-\tau_1-\tau_2)) + \delta x(t-\tau_1-\tau_2)\\ &\le \frac{1}{4} + \delta x(t-\tau_1-\tau_2).\end{aligned}$$

Then using the boundedness of $x(t)$, we obtain

$$\frac{d\gamma}{dt} + \delta\gamma(t) \leq \frac{1}{4} + \delta.$$

Therefore, from lemma 2 given in [46], we have

$$\gamma(t) \leq \left(\frac{1}{4\delta} + 1\right) - \left(\frac{1}{4\delta} + 1 - \gamma(\widetilde{T_2})\right) e^{-\delta(t-\widetilde{T_2})}, \quad \text{for} \;\; t \geq \widetilde{T_2} \geq 0.$$

If $\widetilde{T_2} = 0$, then,

$$\gamma(t) \leq \left(\frac{1}{4\delta} + 1\right) - \left(\frac{1}{4\delta} + 1 - \gamma(0)\right) e^{-\delta(t-0)}.$$

Therefore

$$\limsup_{t\to+\infty} \gamma(t) \leq \left(\frac{1}{4\delta} + 1\right),$$

where

$$\gamma(t) = x(t-\tau_1-\tau_2) + \frac{y(t-\tau_2)}{\omega_6} + \frac{z(t)}{\omega_6\omega_9} \leq \left(\frac{1}{4\delta} + 1\right), \quad \text{for large } t > 0.$$

Thus

$$z(t) \leq \frac{\omega_6\omega_9}{4}\left(\frac{\omega_5 - \delta}{\omega_5\delta}\right) = M_3.$$

□

3.3. *Equilibria analysis*

Steady state solutions are obtained analytically by putting $\dot{x} = 0$, $\dot{y} = 0$ and $\dot{z} = 0$, which are independent of time delays τ_1 and τ_2. The model has four equilibrium points.

(i) Trivial equilibrium point $P_0(0,0,0)$ always exists.
(ii) Predators free axial equilibrium point $P_1(1,0,0)$ exists.
(iii) Top predator free planar equilibrium point $P_2(\bar{x},\bar{y},0)$, where $\bar{y} = (1-\bar{x})(\bar{x}^2+\omega_4)$, and $\bar{x}$ is a solution of the equation

$$f(x) = \omega_5 x^2 - \omega_6 x + \omega_4\omega_5 = 0,$$

which has positive solution $\bar{x}$ if

$$\omega_6^2 \geq 4\omega_4\omega_5^2. \tag{3.3}$$

Notice that $\bar{y} > 0$ iff $\bar{x} < 1$.
Following we discuss the existence conditions of P_2 for three cases.
(a)

$$\begin{aligned} f(1) > 0 \text{ and } \frac{\omega_6}{2\omega_5} < 1 &\Leftrightarrow \omega_5 - \omega_6 + \omega_4\omega_5 > 0,\ \omega_6 < 2\omega_5 \\ &\Leftrightarrow \omega_6 < \min\{2\omega_5,\ \omega_5(1+\omega_4)\} = \begin{cases} 2\omega_5, & \text{if } \omega_4 \geq 1, \\ \omega_5(1+\omega_4), & \text{if } \omega_4 < 1. \end{cases} \\ &\Leftrightarrow 0 < \bar{x}_{\pm} < 1,\ \bar{y}_{\pm} > 0. \end{aligned} \tag{3.4}$$

(b)

$$f(1) < 0 \Leftrightarrow \omega_6 > \omega_5(1+\omega_4)$$
$$\Leftrightarrow 0 < \bar{x}_- < 1 < \bar{x}_+$$
$$\Rightarrow 0 < \bar{x}_- < 1,\ \bar{y}_- > 0.$$

(c)

$$f(1) > 0 \text{ and } \frac{\omega_6}{2\omega_5} > 1 \Leftrightarrow \omega_5 - \omega_6 + \omega_4\omega_5 > 0,\ \omega_6 > 2\omega_5$$
$$\Leftrightarrow 2\omega_5 < \omega_6 < \omega_5(1+\omega_4) \text{ (implies } \omega_5 < \omega_4\omega_5, \text{ i.e. } \omega_4 > 1)$$
$$\Rightarrow \bar{x}_\pm > 1, \text{ so no } P_2 \text{ exists.}$$

(iv) Positive coexistence equilibrium point $P_3(x^*, y^*, z^*)$ exists provided

$$\omega_9^2 > 4\omega_7\omega_8^2 \quad \text{and} \quad \omega_5 < \frac{\omega_6 x^*}{x^{*2}+\omega_4}, \tag{3.5}$$

where $y^* = \frac{\omega_9 \pm \sqrt{\omega_9^2 - 4\omega_8^2\omega_7}}{2\omega_8}$, $z^* = (y^{*2}+\omega_7)(-\omega_5 + \frac{\omega_6 x^*}{x^{*2}+\omega_4})$, and x^* is the positive root of following equation

$$x^{*3} - x^{*2} + \omega_4 x^* + y^* - \omega_4 = 0. \tag{3.6}$$

Let $f(x^*) = x^{*3} - x^{*2} + \omega_4 x^* + y^* - \omega_4$, then $f(0) = (y^* - \omega_4)$, $f(0) < 0$ if $y^* < \omega_4$, i.e. $\omega_4\omega_9 < \omega_8(\omega_7 + \omega_4^2)$ and $f(1) = y^* > 0$. Since $f(0)f(1) < 0$, by intermediate value theorem, Eq. (3.6) has a positive root lies in (0,1) when

$$y^* < \omega_4, \quad i.e. \quad \omega_4\omega_9 < \omega_8(\omega_7 + \omega_4^2) \text{ and } \omega_9 < 2\omega_4\omega_8. \tag{3.7}$$

Positive equilibrium point $P_3(x^*, y^*, z^*)$ exists if the conditions (3.5) and (3.7) hold.

In the absence of both delays, the general variational matrix of model (2.2) at any arbitrary point (x, y, z) is given by

$$A = \begin{pmatrix} 1 - 2x - \frac{y(\omega_4 - x^2)}{(x^2+\omega_4)^2} & -\frac{x}{x^2+\omega_4} & 0 \\ \frac{\omega_6 y(\omega_4 - x^2)}{(x^2+\omega_4)^2} & -\omega_5 - \frac{z(\omega_7 - y^2)}{(y^2+\omega_7)^2} + \frac{\omega_6 x}{x^2+\omega_4} & -\frac{y}{y^2+\omega_7} \\ 0 & \frac{\omega_9 z(\omega_7 - y^2)}{(y^2+\omega_7)^2} & -\omega_8 + \frac{\omega_9 y}{y^2+\omega_7} \end{pmatrix}.$$

The behaviour of equilibrium points is summarized as follows.

(i) Eigenvalues of variational matrix at $P_0(0,0,0)$ are $1,\ -\omega_5,\ -\omega_8$. Therefore, P_0 is a saddle point having unstable manifold along x-direction and stable manifold along y and z-direction.

(ii) Eigenvalues of variational matrix at $P_1(1,0,0)$ are $-1,\ -\omega_5 + \frac{\omega_6}{1+\omega_4},\ -\omega_8$. Therefore, P_1 is locally asymptotically stable (LAS) provided $\frac{\omega_6}{1+\omega_4} < \omega_5$.

(iii) At $P_2(\bar{x}, \bar{y}, 0)$, the variational matrix becomes

$$A = \begin{pmatrix} 1 - 2\bar{x} - \frac{\bar{y}(\omega_4 - \bar{x}^2)}{(\bar{x}^2+\omega_4)^2} & -\frac{\bar{x}}{\bar{x}^2+\omega_4} & 0 \\ \frac{\omega_6 \bar{y}(\omega_4 - \bar{x}^2)}{(\bar{x}^2+\omega_4)^2} & 0 & -\frac{\bar{y}}{\bar{y}^2+\omega_7} \\ 0 & 0 & -\omega_8 + \frac{\omega_9 \bar{y}}{\bar{y}^2+\omega_7} \end{pmatrix}.$$

A is stable iff

$$1 - 2\bar{x} - \frac{\bar{y}(\omega_4 - \bar{x}^2)}{(\bar{x}^2 + \omega_4)^2} < 0, \tag{3.8}$$

$$\omega_4 - \bar{x}^2 > 0 \Leftrightarrow \bar{x} < \sqrt{\omega_4}, \tag{3.9}$$

$$-\omega_8 + \frac{\omega_9 \bar{y}}{\bar{y}^2 + \omega_7} < 0. \tag{3.10}$$

Define $f(x) = \omega_5 x^2 - \omega_6 x + \omega_4\omega_5$. Then we have $f(\bar{x}) = 0$, and

$$\bar{x} > 0 \Leftrightarrow \omega_6^2 \geq 4\omega_4\omega_5^2. \tag{3.11}$$

Since $f(\sqrt{\omega_4}) = 2\omega_4\omega_5 - \omega_6\sqrt{\omega_4} \leq 0$, we have $\bar{x}_- < \sqrt{\omega_4} < \bar{x}_+$. Hence by (3.9), $P_2^+ = (\bar{x}_+, \bar{y}_+, 0)$ is always unstable. From (3.10), we have $\omega_8\bar{y}^2 - \omega_9\bar{y} + \omega_7\omega_8 > 0$. Hence $\bar{y} > 0$ if $\omega_9^2 < 4\omega_7\omega_8^2$. That is, (3.10) is satisfied if there exists no $P_3(x^*, y^*, z^*)$.
Now let's consider the case where no P_3 exists. For this case, (3.10) is satisfied for any $\bar{y} > 0$ and (3.9) is satisfied for $\bar{x}_-$. Since $\bar{y}_- = (1 - \bar{x}_-)(\bar{x}_-^2 + \omega_4)$, we have

$$\begin{aligned} 1 - 2\bar{x}_- - \frac{\bar{y}_-(\omega_4 - \bar{x}_-^2)}{(\bar{x}_-^2 + \omega_4)^2} &= 1 - 2\bar{x}_- - \frac{(1 - \bar{x}_-)(\omega_4 - \bar{x}_-^2)}{\bar{x}_-^2 + \omega_4} < 0 \\ &\Leftrightarrow 3\bar{x}_-^2 - 2\bar{x}_- + \omega_4 > 0. \end{aligned} \tag{3.12}$$

(a) When $\omega_4 > 1/3$, (3.12) is satisfied for any $\bar{x}_- \geq 0$. Hence, P_2^- is LAS when $\omega_4 > 1/3$ and no P_3 exists.
(b) When $\omega_4 < 1/3$,

$$(3.12) \Leftrightarrow \bar{x}_- < \frac{1 - \sqrt{1 - 3\omega_4}}{3} \text{ or } \bar{x}_- > \frac{1 - \sqrt{1 + 3\omega_4}}{3}, \tag{3.13}$$

where $\bar{x}_- = \frac{\omega_6 - \sqrt{\omega_6^2 - 4\omega_4\omega_5^2}}{2\omega_5}$.
Hence P_2^- is LAS when $\omega_4 < 1/3$ and no P_3 exists and (3.13) is satisfied.

4. Local stability analysis and Hopf-bifurcation

In this section, we discuss the effect of discrete delays on the dynamics of model (2.2).
At P_0, characteristic equation is

$$(\lambda - 1)(\lambda + \omega_5)(\lambda + \omega_8) = 0.$$

There are one positive root $\lambda_1 = 1$ and two negative roots $\lambda_2 = -\omega_5,\ \lambda_3 = -\omega_8$, which are independent of τ_1 and τ_2. Hence P_0 is unstable for any $\tau_1 \geq 0$ and $\tau_2 \geq 0$.
At P_1, characteristic equation is

$$(\lambda + \omega_5 - \frac{\omega_6}{1 + \omega_4} e^{-\lambda\tau_1})(\lambda + 1)(\lambda + \omega_8) = 0.$$

There are one root $\lambda_1 = -\omega_5 + \frac{\omega_6}{1+\omega_4} e^{-\lambda\tau_1}$ and two negative roots $\lambda_2 = -1,\ \lambda_3 = -\omega_8$.

Following we consider the equation

$$\lambda + \omega_5 - \frac{\omega_6}{1+\omega_4} e^{-\lambda\tau_1} = 0. \tag{4.1}$$

If $\tau_1 = 0$, then $\lambda_1 = -\omega_5 + \frac{\omega_6}{1+\omega_4}$. It shows that P_1 is LAS when $\frac{\omega_6}{1+\omega_4} < \omega_5$. If $\tau_1 > 0$, let us suppose that $\lambda_1 = i\omega\,(\omega > 0)$ is a pure imaginary root of equation (4.1). Separating the real and imaginary parts, we have

$$\begin{aligned} \omega_5 &= \frac{\omega_6}{1+\omega_4}\cos\omega\tau_1, \\ -\omega &= \frac{\omega_6}{1+\omega_4}\sin\omega\tau_1. \end{aligned} \tag{4.2}$$

Squaring and adding both sides of above equations lead to the following equation

$$\omega^2 = \frac{\omega_6^2 - \omega_5^2(1+\omega_4)^2}{(1+\omega_4)^2}. \tag{4.3}$$

This shows that P_1 is LAS for any $\tau_1, \tau_2 \geq 0$ if $\frac{\omega_6}{1+\omega_4} < \omega_5$.

If $\frac{\omega_6}{1+\omega_4} > \omega_5$, then equation (4.3) has one positive root $\omega_1^- = \frac{\sqrt{(\omega_6+\omega_5(1+\omega_4))(\omega_6-\omega_5(1+\omega_4))}}{1+\omega_4}$.

Solving equation (4.2) for τ_1 yields

$$\tau_1^{(j)} = \frac{1}{\omega_1^-}\left(2\pi - \arccos\frac{\omega_5(1+\omega_4)}{\omega_6} + 2j\pi\right), \quad j = 0, 1, 2, \cdots. \tag{4.4}$$

The minimum value of $\tau_1^{(j)}$ is renamed as

$$\tau_1^- = \tau_1^{(0)} = \frac{1}{\omega_1^-}\left(2\pi - \arccos\frac{\omega_5(1+\omega_4)}{\omega_6}\right). \tag{4.5}$$

Then, we have the following theorem.

Theorem 4.1. (i) P_0 is unstable for any $\tau_1 \geq 0$ and $\tau_2 \geq 0$.
(ii) P_1 is LAS for any $\tau_1, \tau_2 \geq 0$ if $\frac{\omega_6}{1+\omega_4} < \omega_5$.
(iii) P_1 is LAS for $0 \leq \tau_1 < \tau_1^-$ and any $\tau_2 \geq 0$ if $\frac{\omega_6}{1+\omega_4} > \omega_5$, where τ_1^- is given by (4.5). Furthermore, model (2.2) undergoes Hopf-bifurcation to periodic solutions at P_1 when $\tau_1 = \tau_1^-$.

For P_2, we consider two cases.
Case a: $\tau_1 > 0,\ \tau_2 = 0$.
At P_2, characteristic equation is

$$(\lambda + \omega_8 - \frac{\omega_9\bar{y}}{\bar{y}^2+\omega_7})\left(\lambda^2 - e_{11}\lambda + e_{11}d_{22} + e^{-\lambda\tau_1}(-e_{22}\lambda + e_{11}e_{22} - e_{12}e_{21})\right) = 0, \tag{4.6}$$

where

$$\begin{aligned} e_{11} &= \frac{-\bar{x}(3\bar{x}^2 - 2\bar{x} + \omega_4)}{\bar{x}^2+\omega_4},\ e_{12} = -\frac{\bar{x}}{\bar{x}^2+\omega_4} < 0,\ e_{21} = \frac{\omega_6(1-\bar{x})(\omega_4-\bar{x}^2)}{\bar{x}^2+\omega_4}, \\ e_{22} &= \frac{\omega_6\bar{x}}{\bar{x}^2+\omega_4} > 0,\ e_{33} = -\omega_8 + \frac{\omega_9\bar{y}}{\bar{y}^2+\omega_7},\ d_{22} = -\omega_5 < 0. \end{aligned}$$

One characteristic root is $\lambda_1 = -\omega_8 + \frac{\omega_9 \bar{y}}{\bar{y}^2+\omega_7}$. It is easy to show that $\lambda_1 < 0$ for any $\tau_1 \geq 0$ and $\tau_2 \geq 0$ if and only if $\frac{\omega_9 \bar{y}}{\bar{y}^2+\omega_7} < \omega_8$.
Following we consider the equation

$$\lambda^2 - e_{11}\lambda + e_{11}d_{22} + e^{-\lambda\tau_1}(-e_{22}\lambda + e_{11}e_{22} - e_{12}e_{21}) = 0. \tag{4.7}$$

Let $i\omega$ ($\omega > 0$) be a root of equation (4.7), then we have

$$-\omega^2 - e_{11}i\omega + e_{11}d_{22} + (\cos\omega\tau_1 - i\sin\omega\tau_1)(-e_{22}i\omega + e_{11}e_{22} - e_{12}e_{21}) = 0. \tag{4.8}$$

Simplifying and equating real and imaginary part of equation (4.8), we get

$$e_{22}\omega\sin\omega\tau_1 - (e_{11}e_{22} - e_{12}e_{21})\cos\omega\tau_1 = -\omega^2 + e_{11} + d_{22}, \tag{4.9}$$

$$e_{22}\omega\cos\omega\tau_1 + (e_{11}e_{22} - e_{12}e_{21})\sin\omega\tau_1 = -e_{11}\omega. \tag{4.10}$$

Squaring and adding equations (4.9) and (4.10) we get

$$\omega^4 + p_0\omega^2 + q_0 = 0, \tag{4.11}$$

where

$$p_0 = {e_{11}}^2 - {e_{22}}^2 - 2(e_{11} + d_{22}),$$
$$q_0 = (e_{11} + d_{22})^2 - (e_{11}e_{22} - e_{12}e_{21})^2.$$

We define

$$G_0(\omega) = \omega^4 + p_0\omega^2 + q_0, \tag{4.12}$$

$G_0(0) = q_0 = (e_{11} + d_{22})^2 - (e_{11}e_{22} - e_{12}e_{21})^2, \quad G_1(\infty) = \infty.$
Let

$$(e_{11} + d_{22})^2 - (e_{11}e_{22} - e_{12}e_{21})^2 < 0, \tag{4.13}$$

then $G_0(0) < 0$ and $G_0(\infty) = \infty$. Thus, equation (4.31) has at least one positive root. Without loss of generality, we assume that it has finite number of positive roots saying $\omega_1,\ \omega_2,\ \omega_3, \cdots, \omega_N$. For every fixed $\omega_k,\ k = 1,\ 2,\ 3, \cdots, N$, there exist a sequence $\{\tau_{10}^{k,j} \mid\ j = 0, 1, 2, \cdots\}$, where

$$\tau_{10}^{(k,j)} = \begin{cases} \dfrac{1}{\omega_k}\left(\arccos\dfrac{-e_{12}e_{21}\omega^2 - (e_{11} + d_{22})(e_{11}e_{22} - e_{12}e_{21})}{{e_{22}}^2\omega^2 + (e_{11}e_{22} - e_{12}e_{21})^2} + 2j\pi\right),\ j = 0, 1, 2, \cdots, \\ \text{if } -e_{22}\omega^2 + (e_{11} + d_{22})e_{22} - e_{11}(e_{11}e_{22} - e_{12}e_{21}) \geq 0, \\ \dfrac{1}{\omega_k}\left(2\pi - \arccos\dfrac{-e_{12}e_{21}\omega^2 - (e_{11} + d_{22})(e_{11}e_{22} - e_{12}e_{21})}{{e_{22}}^2\omega^2 + (e_{11}e_{22} - e_{12}e_{21})^2} + 2j\pi\right),\ j = 0, 1, 2, \cdots, \\ \text{if } -e_{22}\omega^2 + (e_{11} + d_{22})e_{22} - e_{11}(e_{11}e_{22} - e_{12}e_{21}) < 0, \end{cases} \tag{4.14}$$

Let

$$\tau_{10}^* = \tau_{10}^{(k_0,0)} = \min_{k\in\{1,\cdots,N\}}\left\{\tau_{10}^{(k,0)}\right\},\ \omega^* = \omega_{k_0}. \tag{4.15}$$

Case b: $\tau_1 = 0,\ \tau_2 > 0$.
At P_2, characteristic equation is

$$\left(\lambda + \omega_8 - \frac{\omega_9 \bar{y}}{\bar{y}^2 + \omega_7} e^{-\lambda\tau_2}\right)\left(\lambda^2 + h_1\lambda + h_2\right) = 0, \tag{4.16}$$

where $h_1 = \frac{\bar{y}(\omega_4 - \bar{x}^2)}{(\bar{x}^2+\omega_4)^2} + 2\bar{x} - 1 > 0$ if the condition (3.8) is satisfied, $h_2 = \frac{\bar{x}}{\bar{x}^2+\omega_4}\frac{\omega_6(1-\bar{x})(\omega_4-\bar{x}^2)}{\bar{x}^2+\omega_4} > 0$ if the condition (3.9) is satisfied. Hence the equation $\lambda^2 + h_1\lambda + h_2 = 0$ have two negative roots if and only if the conditions (3.8) and (3.9) are satisfied. Following we consider the equation

$$\lambda + \omega_8 - \frac{\omega_9 \bar{y}}{\bar{y}^2 + \omega_7} e^{-\lambda\tau_2} = 0. \tag{4.17}$$

Following a similar analysis of P_1, we obtain the critical value

$$\tau_2^{(j)} = \frac{1}{\omega_2^-}\left(2\pi - \arccos\frac{\omega_8(\bar{y}^2+\omega_7)}{\omega_9\bar{y}} + 2j\pi\right), \quad j = 0, 1, 2, \cdots, \tag{4.18}$$

where $\omega_2^- = \frac{\sqrt{(\omega_9\bar{y}+\omega_8(\bar{y}^2+\omega_7))(\omega_9\bar{y}-\omega_8(\bar{y}^2+\omega_7))}}{\bar{y}^2+\omega_7}$. The minimum value of $\tau_2^{(j)}$ is renamed as

$$\tau_2^- = \tau_2^{(0)} = \frac{1}{\omega_2^-}\left(2\pi - \arccos\frac{\omega_8(\bar{y}^2+\omega_7)}{\omega_9\bar{y}}\right). \tag{4.19}$$

Then, we have the following theorem.

Theorem 4.2. (i) Suppose that $\tau_1 > 0,\ \tau_2 = 0$ and $\frac{\omega_9\bar{y}}{\bar{y}^2+\omega_7} < \omega_8$ are satisfied. Then $P_2(\bar{x}, \bar{y}, 0)$ is LAS for $\tau_1 < \tau_{10}^*$ and unstable for $\tau_1 > \tau_{10}^*$. Further, the system (2.2) undergoes the Hopf-bifurcation about P_2 when $\tau_1 = \tau_{10}^*$.
(ii) Suppose that $\tau_1 = 0,\ \tau_2 > 0$, the conditions (3.8) and (3.9) are satisfied. Then $P_2(\bar{x}, \bar{y}, 0)$ is LAS for any $\tau_2 > 0$ if $\frac{\omega_9\bar{y}}{\bar{y}^2+\omega_7} < \omega_8$. And $P_2(\bar{x}, \bar{y}, 0)$ is LAS for $\tau_2 < \tau_2^-$ if $\frac{\omega_9\bar{y}}{\bar{y}^2+\omega_7} > \omega_8$, where τ_2^- is given by (4.19). Furthermore, model (2.2) undergoes Hopf-bifurcation to periodic solutions at P_2 when $\tau_2 = \tau_2^-$.

Next we obtain the characteristic equation of model (2.2) about interior equilibrium point $P_3(x^*, y^*, z^*)$ by linearizing the model (2.2). Let $\bar{x}(t) = x(t) - x^*,\ \bar{y}(t) = y(t) - y^*,\ \bar{z}(t) = z(t) - z^*$ be the perturbed variables about $P_3(x^*, y^*, z^*)$. Then the linearized form of model (2.2) is given by (bar signed are dropped for simplicity)

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = A_1\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} + A_2\begin{pmatrix} x(t-\tau_1) \\ y(t-\tau_1) \\ z(t-\tau_1) \end{pmatrix} + A_3\begin{pmatrix} x(t-\tau_2) \\ y(t-\tau_2) \\ z(t-\tau_2) \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} 1 - 2x^* - \frac{y^*(\omega_4 - x^{*2})}{\alpha^2} & -\frac{x^*}{\alpha} & 0 \\ 0 & -\omega_5 - \frac{z^*(\omega_7 - y^{*2})}{\beta^2} & -\frac{y^*}{\beta} \\ 0 & 0 & -\omega_8 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ \frac{\omega_6 y^*(\omega_4 - x^{*2})}{\alpha^2} & \frac{\omega_6 x^*}{\alpha} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{\omega_9 z^*(\omega_7 - y^{*2})}{\beta^2} & \frac{\omega_9 y^*}{\beta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{32} & c_{33} \end{pmatrix},$$

and $\alpha = x^{*2} + \omega_4,\ \beta = y^{*2} + \omega_7$.
Thus, the characteristic equation of the linearized system is given by

$$\det(A_1 + e^{-\lambda\tau_1}A_2 + e^{-\lambda\tau_2}A_3 - \lambda I_3) = 0, \tag{4.20}$$

where I_3 is an identity matrix of order 3.

Furthermore, equation (4.20) can be rewritten as the simplified form,

$$\lambda^3 + B_2\lambda^2 + B_1\lambda + B_0 + e^{-\lambda\tau_1}(C_2\lambda^2 + C_1\lambda + C_0) + e^{-\lambda\tau_2}(D_2\lambda^2 + D_1\lambda + D_0) + e^{-\lambda(\tau_1+\tau_2)}(E_1\lambda + E_0) = 0, \tag{4.21}$$

where

$$B_2 = -(a_{11} + a_{22} + a_{33}),\ B_1 = a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33},\ B_0 = -a_{11}a_{22}a_{33},$$
$$C_2 = -b_{22},\ C_1 = a_{11}b_{22} + b_{22}a_{33} - a_{12}b_{21},\ C_0 = a_{12}b_{21}a_{33} - a_{11}b_{22}a_{33},$$
$$D_2 = -c_{33},\ D_1 = a_{11}c_{33} + a_{22}c_{33} - a_{23}c_{32},\ D_0 = a_{11}a_{23}c_{32} - a_{11}a_{22}c_{33},$$
$$E_1 = b_{22}c_{33},\ E_0 = a_{12}b_{21}c_{33} - a_{11}b_{22}c_{33}.$$

Now we discuss the following cases.

Case I: $\tau_1 = 0 = \tau_2$.
In this case, equation (4.21) becomes

$$\lambda^3 + F_{12}\lambda^2 + F_{11}\lambda + F_{10} = 0, \tag{4.22}$$

where $F_{12} = B_2 + C_2 + D_2,\ F_{11} = B_1 + C_1 + D_1 + E_1,\ F_{10} = B_0 + C_0 + D_0 + E_0$. Therefore, by Routh-Hurwitz criterion, $P_3 = (x^*, y^*, z^*)$ is LAS in the absence of delay if

$$F_{12} > 0, \quad F_{10} > 0, \quad F_{12}F_{11} - F_{10} > 0.$$

Straight forward calculation shows that $F_{12} > 0$, if

$$\frac{2x^{*2}y^*}{\alpha^2} + \frac{2y^{*2}z^*}{\beta^2} < x^*. \tag{4.23}$$

$F_{10} > 0$, if

$$y^{*2} < \omega_7. \tag{4.24}$$

$(F_{12}F_{11} - F_{10}) > 0$, if

$$\begin{aligned} &\frac{y^*}{\alpha^5\beta^4}\left\{4y^{*3}z^{*2}\alpha^3(x^*\alpha^2 - 2x^{*2}y^*) + \beta^2\omega_6 x^*(x^*\alpha^2\beta^2 - 2x^{*2}y^*\beta^2 - 2y^{*2}z^*\alpha^2)(\omega_4 - x^{*2})\right\} \\ &> \frac{2y^{*2}z^*}{\alpha^4\beta^5}\left\{(-x^*\alpha^2 + 2x^{*2}y^*)^2\beta^3 + \alpha^4\omega_9 y^* z^*(\omega_7 - y^{*2})\right\} \end{aligned} \tag{4.25}$$

and

$$x^{*2} < \omega_4. \tag{4.26}$$

Based on the above analysis, we have constructed the following theorem for stability of model (2.2) about $E^*(x^*, y^*, z^*)$ in the absence of delay.

Theorem 4.3. Suppose that the interior equilibrium point $P_3(x^*, y^*, z^*)$ exists. Then, P_3 is LAS provided the conditions (4.23)–(4.26) hold.

Case II: $\tau_1 > 0,\ \tau_2 = 0$.
In this case, equation (4.21) becomes

$$\lambda^3 + (B_2 + D_2)\lambda^2 + (B_1 + D_1)\lambda + B_0 + D_0 + e^{-\lambda\tau_1}\left(C_2\lambda^2 + (C_1 + E_1)\lambda + C_0 + E_0\right) = 0. \tag{4.27}$$

Let $i\omega$ ($\omega > 0$) be a root of equation (4.27), then we have

$$\begin{aligned} &- i\omega^3 + (B_2 + D_2)(-\omega^2) + (B_1 + D_1)(i\omega) + (B_0 + D_0) \\ &+ (\cos\omega\tau_1 - i\sin\omega\tau_1)\left(-C_2\omega^2 + i(C_1 + E_1)\omega + (C_0 + E_0)\right) = 0. \end{aligned} \tag{4.28}$$

Simplifying and equating real and imaginary part of equation (4.28), we get

$$-\omega^3 + \omega(B_1 + D_1) = (-C_2\omega^2 + C_0 + E_0)\sin\omega\tau_1 - \omega(C_1 + E_1)\cos\omega\tau_1, \tag{4.29}$$

$$-\omega^2(B_2 + D_2) + (B_0 + D_0) = -(-C_2\omega^2 + C_0 + E_0)\cos\omega\tau_1 - \omega(C_1 + E_1))\sin\omega\tau_1. \tag{4.30}$$

Squaring and adding equations (4.29) and (4.30) we get

$$\omega^6 + p_1\omega^4 + q_1\omega^2 + r_1 = 0, \tag{4.31}$$

where

$$\begin{aligned} p_1 &= (B_2 + D_2)^2 - 2(B_1 + D_1) - C_2^2, \\ q_1 &= (B_1 + D_1)^2 - 2(B_2 + D_2)(B_0 + D_0) + 2C_2(C_0 + E_0) - (C_1 + E_1)^2, \\ r_1 &= (B_0 + D_0)^2 - (C_0 + E_0)^2. \end{aligned}$$

We define

$$G_1(\omega) = \omega^6 + p_1\omega^4 + q_1\omega^2 + r_1. \tag{4.32}$$

Then $G_1(0) = r_1 = (B_0 + D_0)^2 - (C_0 + E_0)^2, \quad G_1(\infty) = \infty$.
Let

$$(B_0 + D_0)^2 - (C_0 + E_0)^2 < 0, \tag{4.33}$$

then $G_1(0) < 0$ and $G_1(\infty) = \infty$. Thus, equation (4.31) has at least one positive root. Without loss of generality, we assume that it has finite number of positive roots saying $\omega_1,\ \omega_2,\ \omega_3, \cdots, \omega_N$. For every fixed $\omega_k,\ k = 1,\ 2,\ 3, \cdots, N$, there exist a sequence $\{\tau_1^{k,j} \mid\ j = 0, 1, 2, \cdots\}$, where

$$\tau_1^{(k,j)} = \begin{cases} \dfrac{1}{\omega_k}\left(\arccos \dfrac{\begin{array}{c}[(C_1+E_1)-C_2(B_2+D_2)]{\omega_k}^4+[(B_2+D_2)(C_0+E_0)\\ -(B_1+D_1)(C_1+E_1)]\omega_k^2+(B_0+D_0)(C_0+E_0)\end{array}}{(-C_2\omega_k^2+C_0+E_0)^2+\omega_k^2(C_1+E_1)^2}+2j\pi\right),\ j=0,1,2,\cdots, \\ \text{if } C_2{\omega_k}^4+((B_2+D_2)(C_1+E_1)-C_2(B_1+D_1)-(C_0+E_0)){\omega_k}^2 \\ \ +C_0+E_0-(B_0+D_0)(C_1+E_1)\geq 0, \\ \dfrac{1}{\omega_k}\left(2\pi-\arccos \dfrac{\begin{array}{c}[(C_1+E_1)-C_2(B_2+D_2)]{\omega_k}^4+[(B_2+D_2)(C_0+E_0)\\ -(B_1+D_1)(C_1+E_1)]\omega_k^2+(B_0+D_0)(C_0+E_0)\end{array}}{(-C_2\omega_k^2+C_0+E_0)^2+\omega_k^2(C_1+E_1)^2}+2j\pi\right),\ j=0,1,2,\cdots, \\ \text{if } C_2{\omega_k}^4+((B_2+D_2)(C_1+E_1)-C_2(B_1+D_1)-(C_0+E_0)){\omega_k}^2 \\ \ +C_0+E_0-(B_0+D_0)(C_1+E_1)< 0. \end{cases} \tag{4.34}$$

Let

$$\tau_1^* = \tau_1^{(k_0,0)} = \min_{k\in\{1,\cdots,N\}} \left\{\tau_1^{(k,0)}\right\},\ \omega^* = \omega_{k_0}. \tag{4.35}$$

By differentiating the equation (4.27) with respect to τ_1, we have the following transversality condition

$$\left\{Re\left(\frac{d\lambda}{d\tau_1}\right)^{-1}\right\}\Big|_{\tau_1=\tau_1^*} = \frac{L_1(\omega_k)S_1(\omega_k)+R_1(\omega_k)T_1(\omega_k)}{(L_1(\omega_k))^2+(R_1(\omega_k))^2} > 0,$$

provided $L_1(\omega_k)S_1(\omega_k)+R_1(\omega_k)T_1(\omega_k) > 0$, where

$$\begin{aligned}
&L_1(\omega_k) = \omega^*(C_0+E_0) - C_2\omega^{*3},\\
&R_1(\omega_k) = -\omega^{*2}(C_1+E_1),\\
&S_1(\omega_k) = (-3\omega^{*2}+B_1+D_1)\sin\omega^*\tau_1 + 2\omega^*(B_2+D_2)\cos\omega^*\tau_1 + 2\omega^* C_2,\\
&T_1(\omega_k) = (-3\omega^{*2}+B_1+D_1)\cos\omega^*\tau_1 - 2\omega^*(B_2+D_2)\sin\omega^*\tau_1 + (C_1+E_1).
\end{aligned}$$

Then, we have the following theorem.

Theorem 4.4. Suppose that $\tau_1 > 0,\ \tau_2 = 0$ and conditions (4.23)–(4.26) are satisfied. Then the interior equilibrium point $P_3(x^*, y^*, z^*)$ is LAS for $\tau_1 < \tau_1^*$ and unstable for $\tau_1 > \tau_1^*$. Further, the system (2.2) undergoes the Hopf-bifurcation about $P_3(x^*, y^*, z^*)$ when $\tau_1 = \tau_1^*$.

Case III: $\tau_1 \in (0,\ \tau_1^*),\ \ \tau_2 > 0$.
In this case, we assume that τ_1 is arbitrarily fixed within the stable interval $(0, \tau_1^*)$, while consider τ_2 as free parameter. Let $i\omega$ ($\omega > 0$) be a root of equation (4.21), then we have

$$\begin{aligned}
&-i\omega^3 - B_2\omega^2 + B_1\omega i + B_0 + e^{-i\omega\tau_1}(-C_2\omega^2 + C_1\omega i + C_0)\\
&+ e^{-i\omega\tau_2}(-D_2\omega^2 + D_1\omega i + D_0) + e^{-i\omega(\tau_1+\tau_2)}(E_1\omega i + E_0) = 0.
\end{aligned} \tag{4.36}$$

Simplifying and equating real and imaginary part, we obtain

$$\begin{aligned}&-\omega^3 + B_1\omega + C_1\omega\cos\omega\tau_1 - (-C_2\omega^2 + C_0)\sin\omega\tau_1\\&= (-D_1\omega - E_1\omega\cos\omega\tau_1 + E_0\sin\omega\tau_1)\cos\omega\tau_2 + (-D_2\omega^2 + D_0 + E_1\omega\sin\omega\tau_1 + E_0\cos\omega\tau_1)\sin\omega\tau_2,\end{aligned} \tag{4.37}$$

$$\begin{aligned}&B_2\omega^2 - B_0 - (-C_2\omega^2 + C_0)\cos\omega\tau_1 - C_1\omega\sin\omega\tau_1\\&= (-D_2\omega^2 + D_0 + E_0\cos\omega\tau_1 + E_1\omega\sin\omega\tau_1)\cos\omega\tau_2 - (-D_1\omega - E_1\omega\cos\omega\tau_1 + E_0\sin\omega\tau_1)\sin\omega\tau_2.\end{aligned} \tag{4.38}$$

Squaring and adding above equations (4.37) and (4.38), we obtain

$$\omega^6 + \widetilde{p_2}\omega^4 + \widetilde{q_2}\omega^2 + 2\widetilde{r_2}\sin\omega\tau_1 + 2\widetilde{s_2}\cos\omega\tau_1 + \widetilde{t_2} = 0, \tag{4.39}$$

where

$$\begin{aligned}\widetilde{p_2} &= B_2^2 + C_2^2 - 2B_1 - D_2^2,\\ \widetilde{q_2} &= B_1^2 - 2C_2C_0 - 2B_2B_0 + C_1^2 - D_1^2 - E_1^2 + 2D_2D_0,\\ \widetilde{r_2} &= \omega^3(-C_2\omega^2 + C_0) - B_1\omega(-C_2\omega^2 + C_0) - B_2C_1\omega^3 + C_1B_0\omega + D_1E_0\omega - E_1\omega(-D_2\omega^2 + D_0),\\ \widetilde{s_2} &= -C_1\omega^4 + B_1C_1\omega^2 - B_2\omega^2(-C_2\omega^2 + C_0) + B_0(-C_2\omega^2 + C_0) - D_1E_1\omega^2 - E_0(-D_2\omega^2 + D_0),\\ \widetilde{t_2} &= B_0^2 + C_0^2 - E_0^2 - D_0^2.\end{aligned}$$

Following the same analysis as in Case II, equation (4.39) has finite number of positive roots, saying $\omega_1, \omega_2, \omega_3, \cdots, \omega_N$, when

$$(B_0 + C_0)^2 - (D_0 + E_0)^2 < 0. \tag{4.40}$$

For every fixed $\omega_k,\ k = 1,\ 2,\ 3, \cdots, N$, there exist a sequence $\{\widetilde{\tau}_2^{k,j} \mid j = 0, 1, 2, \cdots\}$, where

$$\widetilde{\tau}_2^{k,j} = \begin{cases}\dfrac{1}{\omega_k}\left(\arccos\dfrac{M_2(\omega_k)K_2(\omega_k) + F_2(\omega_k)Q_2(\omega_k)}{(M_2(\omega_k))^2 + (F_2(\omega_k))^2} + 2j\pi\right),\ j = 0, 1, 2, \cdots,\\ \text{if } F_2(\omega_k)K_2(\omega_k) \geq M_2(\omega_k)Q_2(\omega_k),\\ \dfrac{1}{\omega_k}\left(2\pi - \arccos\dfrac{M_2(\omega_k)K_2(\omega_k) + F_2(\omega_k)Q_2(\omega_k)}{(M_2(\omega_k))^2 + (F_2(\omega_k))^2} + 2j\pi\right),\ j = 0, 1, 2, \cdots,\\ \text{if } F_2(\omega_k)K_2(\omega_k) < M_2(\omega_k)Q_2(\omega_k),\end{cases} \tag{4.41}$$

where

$$\begin{aligned}M_2(\omega_k) &= (-D_2\omega_k^2 + D_0) + E_0\cos\omega_k\tau_1 + E_1\omega_k\sin\omega_k\tau_1,\\ F_2(\omega_k) &= -D_1\omega_k - E_1\omega_p\cos\omega_k\tau_1 + E_0\sin\omega_k\tau_1,\\ K_2(\omega_k) &= B_2\omega_k^2 - B_0 - (-C_2\omega_k^2 + C_0)\cos\omega_k\tau_1 - C_1\omega_k\sin\omega_k\tau_1,\\ Q_2(\omega_k) &= -\omega_k^3 + B_1\omega_k + C_1\omega_k\cos\omega_k\tau_1 - (-C_2\omega_k^2 + C_0)\sin\omega_k\tau_1.\end{aligned}$$

Let

$$\widetilde{\tau}_2^* = \tau_2^{(k_0,0)} = \min_{k\in\{1,\cdots,N\}}\left\{\tau_2^{(k,0)}\right\},\ \widetilde{\omega}^* = \omega_{k_0}. \tag{4.42}$$

And assuming that

$$\left\{ Re\left(\frac{d\lambda(\tau_2)}{d\tau_2}\right)^{-1} \right\}\Big|_{\lambda=i\widetilde{\omega}^*} \neq 0. \tag{4.43}$$

Then, we have the following theorem.

Theorem 4.5. Suppose that conditions (4.23)–(4.26) are satisfied and $\tau_1 \in (0,\ \tau_1^*)$. Then the coexisting equilibrium point $P_3(x^*, y^*, z^*)$ is LAS when $\tau_2 < \widetilde{\tau}_2^*$ and it is unstable when $\tau_2 > \widetilde{\tau}_2^*$. Moreover, Hopf-bifurcation occurs when $\tau_2 = \widetilde{\tau}_2^*$.

Case IV: $\tau_1 = 0,\ \tau_2 > 0.$
From the equation (4.21), we have

$$\lambda^3 + (B_2 + C_2)\lambda^2 + (B_1 + C_1)\lambda + (B_0 + C_0) + e^{-\lambda\tau_2}(D_2\lambda^2 + (D_1 + E_1)\lambda + (D_0 + E_0)) = 0. \tag{4.44}$$

Let $i\omega$ $(\omega > 0)$ be a root of equation (4.44), then we have

$$-i\omega^3 - (B_2 + C_2)\omega^2 + (B_1 + C_1)i\omega + (B_0 + C_0) + e^{-i\omega\tau_2}(-D_2\omega^2 + (D_1 + E_1)i\omega + (D_0 + E_0)) = 0. \tag{4.45}$$

Equating real and imaginary part of equation (4.45), we obtain

$$\begin{aligned} &-\omega^3 + \omega(B_1 + C_1) + \omega(D_1 + E_1)\cos\omega\tau_2 - (-D_2\omega^2 + D_0 + E_0)\sin\omega\tau_2 = 0, \\ &-\omega^2(B_2 + C_2) + (B_0 + C_0) + (-D_2\omega^2 + D_0 + E_0)\cos\omega\tau_2 + \omega(D_1 + E_1)\sin\omega\tau_2 = 0. \end{aligned} \tag{4.46}$$

Squaring and adding both equations of system (4.46), we obtain

$$\omega^6 + p_2\omega^4 + q_2\omega^2 + r_2 = 0, \tag{4.47}$$

where

$$\begin{aligned} p_2 &= (B_2 + C_2)^2 - 2(B_1 + C_1) - {D_2}^2, \\ q_2 &= (B_1 + C_1)^2 - 2(B_2 + C_2)(B_0 + C_0) + 2D_2(D_0 + E_0) - (D_1 + E_1)^2, \\ r_2 &= (B_0 + C_0)^2 - (D_0 + E_0)^2. \end{aligned}$$

Now, similarly as in the Case II, we define

$$G_2(\omega) = \omega^6 + p_2\omega^4 + q_2\omega^2 + r_2,$$

$G_2(0) = r_2 = (B_0 + C_0)^2 - (D_0 + E_0)^2$ and $G_2(\infty) = \infty$.
Let

$$\left\{(B_0 + C_0)^2 - (D_0 + E_0)^2\right\} < 0. \tag{4.48}$$

Then $G_2(0) < 0$ and $G_2(\infty) > 0$, thus equation (4.47) has atleast one positive root. Without loss of generality, we have assume that it has finite number of positive roots say $\omega_1, \omega_2, \omega_3, \cdots, \omega_N$. For every

$\omega_k,\ k = 1, 2, 3, \cdots, N$, there exists a sequence $\{\tau_2^{k,j} \mid\ j = 0, 1, 2, \cdots\}$, where

$$\tau_2^{(k,j)} = \begin{cases} \dfrac{1}{\omega_k}\left(\arccos \dfrac{\begin{array}{c}[(D_1 + E_1) - D_2(B_2 + C_2)]{\omega_k}^4 + [(B_2 + C_2)(D_0 + E_0) \\ - (B_1 + C_1)(D_1 + E_1)]\omega_k^2 + (B_0 + C_0)(D_0 + E_0)\end{array}}{(-D_2\omega_k^2 + D_0 + E_0)^2 + \omega_k^2(D_1 + E_1)^2} + 2j\pi\right),\ j = 0, 1, 2, \cdots, \\ \text{if } D_2{\omega_k}^4 + ((B_2 + C_2)(D_1 + E_1) - D_2(B_1 + C_1) - (D_0 + E_0)){\omega_k}^2 \\ + D_0 + E_0 - (B_0 + C_0)(D_1 + E_1) \geq 0, \\ \dfrac{1}{\omega_k}\left(2\pi - \arccos \dfrac{\begin{array}{c}[(D_1 + E_1) - D_2(B_2 + C_2)]{\omega_k}^4 + [(B_2 + C_2)(D_0 + E_0) \\ - (B_1 + C_1)(D_1 + E_1)]\omega_k^2 + (B_0 + C_0)(D_0 + E_0)\end{array}}{(-D_2\omega_k^2 + D_0 + E_0)^2 + \omega_k^2(D_1 + E_1)^2} + 2j\pi\right),\ j = 0, 1, 2, \cdots, \\ \text{if } D_2{\omega_k}^4 + ((B_2 + C_2)(D_1 + E_1) - D_2(B_1 + C_1) - (D_0 + E_0)){\omega_k}^2 \\ + D_0 + E_0 - (B_0 + C_0)(D_1 + E_1) < 0. \end{cases} \tag{4.49}$$

Let

$$\tau_2^* = \tau_2^{(k_0,0)} = \min_{k\in\{1,\cdots,N\}} \left\{\tau_2^{(k,0)}\right\},\ \omega^* = \omega_{k_0}. \tag{4.50}$$

Differentiating the equation (4.44) with respect to τ_2, we have the following transversality condition

$$\left\{Re\left(\frac{d\lambda}{d\tau_2}\right)^{-1}\right\}\Big|_{\tau_2=\tau_2^*} = \frac{L_2(\omega_k)S_2(\omega_k) + R_2(\omega_k)T_2(\omega_k)}{(L_2(\omega_k))^2 + (R_2(\omega_k))^2} > 0,$$

provided $L_2(\omega_k)S_2(\omega_k) + R_2(\omega_k)T_2(\omega_k) > 0$, where

$$\begin{aligned} L_2(\omega_k) &= -D_2\omega_k^3 + (D_0 + E_0)\omega_k, \\ R_2(\omega_k) &= -(D_1 + E_1)\omega_k^2, \\ S_2(\omega_k) &= (B_1 + C_1 - 3\omega_k^2)\sin\omega_k\tau_2 + 2\omega_k(B_2 + C_2)\cos\omega_k\tau_2 + 2D_2\omega_k, \\ T_2(\omega_k) &= (B_1 + C_1 - 3\omega_k^2)\cos\omega_k\tau_2 - 2\omega_k(B_2 + C_2)\sin\omega_k\tau_2 + (D_1 + E_1). \end{aligned}$$

Then, we have the following theorem.

Theorem 4.6. Suppose that $\tau_1 = 0, \tau_2 > 0$ and conditions (4.23)–(4.26) are satisfied. Then the equilibrium point P_3 is LAS for $\tau_2 < \tau_2^*$ and unstable for $\tau_2 > \tau_2^*$. Further, the system (2.2) undergoes the Hopf-bifurcation around $P_3(x^*, y^*, z^*)$ when $\tau_2 = \tau_2^*$.

Case V: $\tau_2 \in (0,\ \tau_2^*),\ \tau_1 > 0$.
In this case, we fix τ_2 at some value from its stability range $(0,\ \tau_2^*)$ and choose τ_1 as free parameter, by the similar procedure used in Case III. Stability results are summarized in the following theorem.

Theorem 4.7. Suppose that the parameters of model (2.2) are such that conditions (4.23)-(4.26) are satisfied, $\tau_2 \in (0,\ \tau_2^*)$ and condition $(B_0+D_0)^2 < (E_0+C_0)^2$ also holds. Then the coexisting equilibrium point P_3 is LAS, when $\tau_1 \in (0,\ \widetilde{\tau}_1^*)$ and it is unstable when $\tau_1 > \widetilde{\tau}_1^*$. Moreover, Hopf-bifurcation occurs when $\tau_1 = \widetilde{\tau}_1^*$, where

$$\widetilde{\tau}_1^* = \tau_1^{(k_0,0)} = \min_{k\in\{1,\cdots,N\}} \left\{\tau_1^{(k,0)}\right\},\ \widetilde{\omega}^* = \omega_{k_0}, \tag{4.51}$$

with

$$\widetilde{\tau}_1^{k,j} = \begin{cases} \dfrac{1}{\omega_k}\left(\arccos\dfrac{M_1(\omega_k)K_1(\omega_k)+F_1(\omega_k)Q_1(\omega_k)}{(M_1(\omega_k))^2+(F_1(\omega_k))^2}+2j\pi\right),\ j=0,1,2,\cdots, \\ \text{if } F_1(\omega_k)K_1(\omega_k)\geq M_1(\omega_k)Q_1(\omega_k), \\ \dfrac{1}{\omega_k}\left(2\pi-\arccos\dfrac{M_1(\omega_k)K_1(\omega_k)+F_1(\omega_k)Q_1(\omega_k)}{(M_1(\omega_k))^2+(F_1(\omega_k))^2}+2j\pi\right),\ j=0,1,2,\cdots, \\ \text{if } F_1(\omega_k)K_1(\omega_k)<M_1(\omega_k)Q_1(\omega_k), \end{cases} \tag{4.52}$$

and where

$$\begin{aligned} M_1(\omega_k) &= (-C_2\omega_k^2+C_0)+E_1\omega_k\sin\omega_k\tau_2+E_0\cos\omega_k\tau_2, \\ F_1(\omega_k) &= -C_1\omega_k+E_0\sin\omega_k\tau_2-E_1\omega_k\cos\omega_k\tau_2, \\ K_1(\omega_k) &= B_2\omega_k^2-B_0-(-D_2\omega_k^2+D_0)\cos\omega_k\tau_2-D_1\omega_k\sin\omega_k\tau_2, \\ Q_1(\omega_k) &= -\omega_k^3+B_1\omega_k+D_1\omega_k\cos\omega_k\tau_2-(-D_2\omega_k^2+D_0)\sin\omega_k\tau_2. \end{aligned}$$

5. Numerical experiment results

In this section, analytical findings and the various dynamics of model (2.2) have been illustrated with the help of numerical examples. In the following, we present three examples corresponding to stable positive equilibrium, limit cycles and chaos of the non-delay model, and we show how time delays influence the non-delay model.

Example 1. Taking $\omega_4 = 1.4$, $\omega_5 = 0.22$, $\omega_6 = 0.8$, $\omega_7 = 2.29$, $\omega_8 = 0.09$, $\omega_9 = 0.6$ in the model (2.2), yields the following system:

$$\begin{aligned} \frac{dx}{dt} &= x(1-x)-\frac{xy}{x^2+1.4}, \\ \frac{dy}{dt} &= -0.22y+\frac{0.8x(t-\tau_1)y(t-\tau_1)}{x^2(t-\tau_1)+1.4}-\frac{yz}{y^2+2.29}, \\ \frac{dz}{dt} &= -0.09z+\frac{0.6y(t-\tau_2)z(t-\tau_2)}{y^2(t-\tau_2)+2.29}. \end{aligned} \tag{5.1}$$

Initial densities of species are taken as $(x_0, y_0, z_0) = (0.3, 0.3, 0.3)$. Simulations are carried out in the non-delay system by Matlab. Unique positive interior equilibrium point is obtained as $P_3 = (0.825453, 0.363298, 0.235593)$. Here we have taken all numerical numbers with 6 digits after decimal to unify the results obtained from Mathematica.

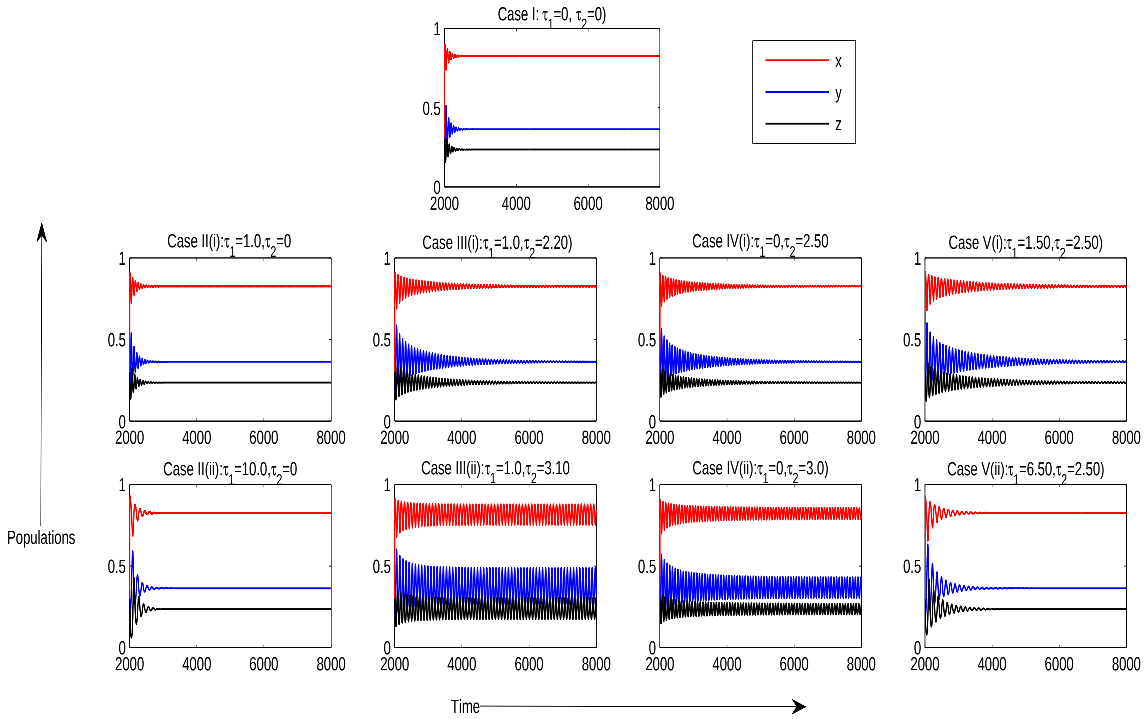

Figure 1. Time evolution of species x, y, z for system (5.1). Case I, when $\tau_1 = \tau_2 = 0$, positive interior equilibrium P_3 $(0.825453, 0.363298, 0.235593)$ is LAS. In Case II, system remains stable for all $\tau_1 \geq 0, \tau_2 = 0$. In Case III, system is LAS for $\tau_2 = 2.20 < \widetilde{\tau}_2^* = 3.0405$ and unstable for $\tau_2 = 3.10 > \widetilde{\tau}_2^* = 3.0405$ by choosing $\tau_1 = 1.0$. In Case IV ($\tau_1 = 0$), system remains LAS for $\tau_2 = 2.50 < \tau_2^* = 2.88823$ and shows oscillatory behaviour for $\tau_2 = 3.0 > \tau_2^* = 2.88823$. In Case V, system is LAS for all $\tau_1 \geq 0$ at $\tau_2 = 2.50$.

Now, for system (5.1) we have verified all five cases with the help of numerical simulations.

(i) Case I ($\tau_1 = \tau_2 = 0$): $F_{12} = 0.700569 > 0$, $F_{10} = 0.005547 > 0$ and $F_{12}F_{11} - F_{10} = 0.008031 > 0$. Thus, all the conditions of Case I are satisfied. Numerical simulation results show that the interior equilibrium point, $P_3 = (0.825453, 0.363298, 0.235593)$ is LAS (see Figure 1 (Case I: $\tau_1 = \tau_2 = 0$)).

(ii) Case II ($\tau_1 > 0, \tau_2 = 0$): Conditional stability condition (4.33) is not satisfied, as $(B_0 + D_0)^2 - (C_0 + E_0)^2 = 0.000031 > 0$. We have not obtained any positive root of equation (4.31). Thus, we are not able to find any value of τ_1, where system experiences Hopf-bifurcation. System is LAS for all $\tau_1 > 0$ (see Figure 1 (Case II(i): $\tau_1 = 1.0$, $\tau_2 = 0$ and Case II(ii): $\tau_1 = 10.0$, $\tau_2 = 0$)).

(iii) Case III ($\tau_1 \in (0, \tau_1^*), \tau_2 > 0$): As in Case II, system not bifurcates for any value of τ_1, thus all values of τ_1 comes in its stability range. $(B_0 + C_0)^2 - (E_0 + D_0)^2 = -0.000019 < 0$, so conditional stability condition (4.40) is satisfied. In particular, we have taken $\tau_1 = 1.0$, for this value of τ_1, $\omega = 0.069338$ and critical value of τ_2 is obtained as $\widetilde{\tau_2}^* = 3.0405$. System is LAS for $\tau_2 = 2.2 < \widetilde{\tau_2} = 3.0405$ and unstable for $\tau_2 = 3.1 > \widetilde{\tau_2}^* = 3.0404$. Hopf-bifurcation occurs at $\widetilde{\tau_2}^* = 3.0404$ (see Figure 1 (Case III(i): $\tau_1 = 1.0$, $\tau_2 = 2.20$ and Case III(ii):$\tau_1 = 1.0$, $\tau_2^* = 3.10$)).

(iv) Case IV ($\tau_1 = 0, \tau_2 > 0$): $(B_0 + C_0)^2 - (D_0 + E_0)^2 = -0.000019 < 0$, thus conditional stability condition (4.48) is satisfied. Critical value of τ_2 for $\omega = 0.0794052$ is obtained as $\tau_2^* = 2.88823$.

$$\left\{Re\left(\frac{d\lambda(\tau_2)}{d\tau_2}\right)^{-1}\right\}\Big|_{\tau_2=\tau_2^*=2.88823} = 135.057 > 0, \tag{5.2}$$

thus transversality condition is also satisfied at $\tau_2^* = 2.88823$. System is LAS for $\tau_2 = 2.5 < \tau_2^* = 2.88823$ and shows oscillations for $\tau_2 = 3.0 > \tau_2^* = 2.88823$ (see Figure 1 (Case IV(i): $\tau_1 = 0, \tau_2 = 2.50$ and Case IV(ii): $\tau_1 = 0, \tau_2 = 3.0$)). Hopf-bifurcation occurs at critical value of $\tau_2 = \tau_2^* = 2.88823$.

(v) Case V ($\tau_1 > 0, \tau_2 \in (0, \tau_2^*)$): In this case, for $\tau_2 = 2.5$ from its stability range (0, 2.88823), conditional stability condition is not satisfied as $(B_0 + D_0)^2 - (E_0 + C_0)^2 = 0.000031 > 0$. Also we have not get any critical value of τ_1, in the stability range of τ_2. Thus, system not bifurcates for any value of τ_1 in the stability range of τ_2. System is LAS for all $\tau_1 \geq 0$, $\tau_2 \in (0, 2.88823)$ (see Figure 1 Case V(i): $\tau_1 = 1.50, \tau_2 = 2.50$, V(ii) $\tau_1 = 6.50, \tau_2 = 2.50$).

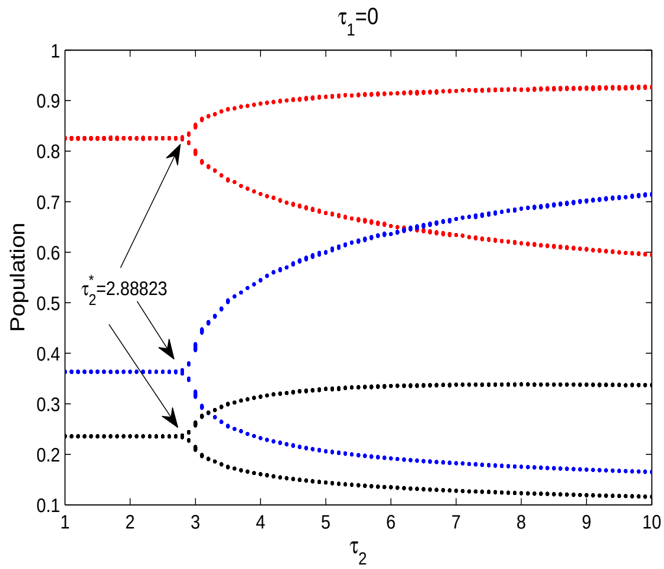

Figure 2. Bifurcation diagram for system (5.1) showing the effect of gestation delay τ_2 (for top predator z) at $\tau_1 = 0$. Figure shows that the system is stable for $\tau_2 < \tau_2^* = 2.88823$ and unstable for $\tau_2 > \tau_2^* = 2.88823$. Hopf-bifurcation occurs at $\tau_2^* = 2.88823$.

Bifurcation diagram with respect to τ_2 keeping $\tau_1 = 0$ is shown in Figure 2. This illustrates that the bifurcation occurs at critical value of $\tau_2^* = 2.88823$, and below this value system (5.1) is stable and above this value system (5.1) shows oscillatory behaviour.

The two dimensional bifurcation diagram for the system (5.1) in $\tau_1 - \tau_2$ plane has been presented in Figure 3. In this figure, the blue line denotes Hopf-bifurcation line i.e., at any point (τ_1, τ_2) on this blue line, system experiences Hopf-bifurcation. The regions which lie below and above this line are stability and instability regions, respectively. For $\tau_2 < \tau_2^* = 2.88823$, system (5.1) remains stable for all $\tau_1 \geq 0$ and for $\tau_2 > \tau_2^* = 2.88823$, system (5.1) becomes unstable for all values of $\tau_1 \geq 0$. Here, we have only one critical value τ_2^*, below which the system is stable and above which system becomes unstable and it remains unstable, thus the system does not exhibit stability switching [47] with further increase in values of delay parameters.

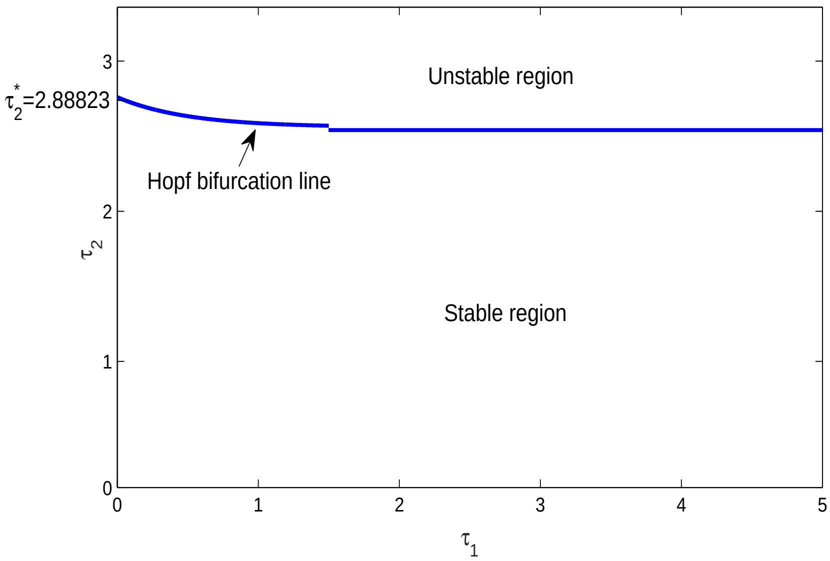

Figure 3. Hopf-bifurcation diagram for the system (5.1) in $\tau_1 - \tau_2$ plane. The blue line is the Hopf-bifurcation line on which system switches its stability via Hopf-bifurcation.

Table 2. Stability results of system (5.1) with $\omega_7 = 3.29$.

Case	Conditions	Critical value of delay	Delay value	Status	Figure (3)
I	$F_{12} = 0.556105$ $F_{10} = 0.003450$ $F_{12}F_{11} - F_{10} = 0.017281$	NA	$\tau_1 = 0, \tau_2 = 0$	Stable	Case I
II	$(B_0 + D_0)^2 - (C_0 + E_0)^2$ $= 0.000012 > 0$ with conditions of Case I	NA	$\tau_1 = 0.50, \tau_2 = 0$ $\tau_1 = 5.0, \tau_2 = 0$	Stable	Case II(i) Case II(ii)
III	$(B_0 + C_0)^2 - (E_0 + D_0)^2$ $= 7.48222 \times 10^{-6} > 0$ with the conditions of Case I	NA	$\tau_1 = 0.50, \tau_2 = 0.50$ $\tau_1 = 0.50, \tau_2 = 5.0$	Stable	Case III(i) Case III(ii)
IV	$(B_0 + C_0)^2 - (D_0 + E_0)^2$ $= 7.48222 \times 10^{-6} > 0$ with the conditions of Case I	NA	$\tau_1 = 0, \tau_2 = 0.5$ $\tau_1 = 0, \tau_2 = 5.0$	Stable	Case IV(i) Case IV(ii)
V	$(B_0 + D_0)^2 - (C_0 + E_0)^2$ $= 0.000011 > 0$ with the conditions of Case I	NA	$\tau_1 = 0.50, \tau_2 = 1.0$ $\tau_1 = 0.50, \tau_2 = 5.0$	Stable	Case V(i) Case V(ii)

NA stands for not applicable.

Further, in the numerical Example 1, we have taken $\omega_7 = 3.29$, i.e. slightly increase the value of ω_7 (protection provided by environment to the middle predator). Coexistence equilibrium point is obtained as $P_3 = (0.720293, 0.536708, 0.287340)$. Then, in the absence of delay, system in Example 1 is LAS (see Figure 4 (Case I)). We have numerically discussed all the Cases (II-V) for different values of τ_1, τ_2 and observed that system remains stable (see Figure 4 (Case II-V)). Thus, for sufficiently high value of environmental protection to the intermediate predator, system remains stable around coexistence equilibrium point and not bifurcates for any value of τ_1 and τ_2 (gestation delays for middle and top predators respectively). Detail description of all the Cases (I-V) is given in Table 2.

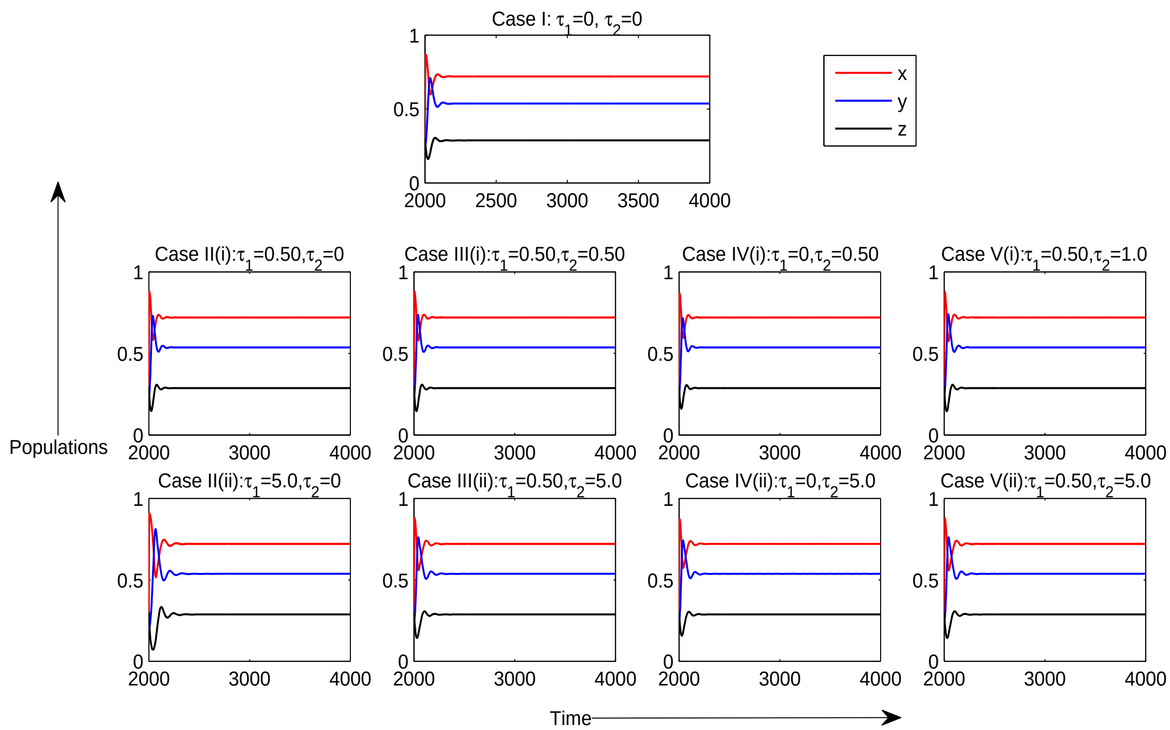

Figure 4. Time evolution of species x, y, z for model (5.1) with $\omega_7 = 3.29$, in Case I ($\tau_1 = \tau_2 = 0$), system is LAS around coexistence equilibrium point $P_3 = (0.720293, 0.536708, 0.287340)$ and remains stable for all possible values of τ_1 and τ_2, Case II-V.

Example 2. Consider the following model with a new parameter set

$$\begin{aligned} \frac{dx}{dt} &= x(1-x) - \frac{xy}{x^2+0.5}, \\ \frac{dy}{dt} &= -0.22y + \frac{0.8x(t-\tau_1)y(t-\tau_1)}{x(t-\tau_1)^2+0.5} - \frac{yz}{y^2+1.76}, \\ \frac{dz}{dt} &= -0.09z + \frac{0.6y(t-\tau_2)z(t-\tau_2)}{y(t-\tau_2)^2+1.76}. \end{aligned} \tag{5.3}$$

Note that parameter values of system (5.3) are the same as those of system (5.1), except for ω_4 and ω_7. That is, the protection provided by the environment to the prey and intermediate predator is decreased in Example 2. Dynamics of the original system (2.2) have been explored with the help of Example 2. It is observed that system (5.3) shows the limit cycle behaviour in the absence of delay (see Figure 5(a)). Now, we have investigated the effect of both delays τ_1 and τ_2 individually on the dynamics of system (5.3). It is clear from the Figure 5(b), when τ_1 crosses the value $\tau_1^* = 4.85$ for $\tau_2 = 0$, system (5.3) becomes stable and remains stable for $\tau_1 > \tau_1^* = 4.85$, $\tau_2 = 0$. Thus, oscillatory behaviour of coexisting equilibrium point is settled down to the stable dynamics. Again, effect of delay τ_2 for $\tau_1 = 0$, is determined by the Figure 5(c), which shows that increasing value of τ_2 makes the system dynamics chaotic through the period doubling sequences. The combined effect of both the gestation delay τ_1 and τ_2 on the system dynamics is given in the Figure 5(d). Figure 5(d) is plotted at $\tau_1 = 7.1$ (value of τ_1 at which system (5.3) is stable in the absence of τ_2) while taking τ_2 as bifurcating parameter. It is observed that the stable coexistence of species is lost by increasing the value of τ_2 and oscillatory coexistence is obtained for higher value of τ_2.

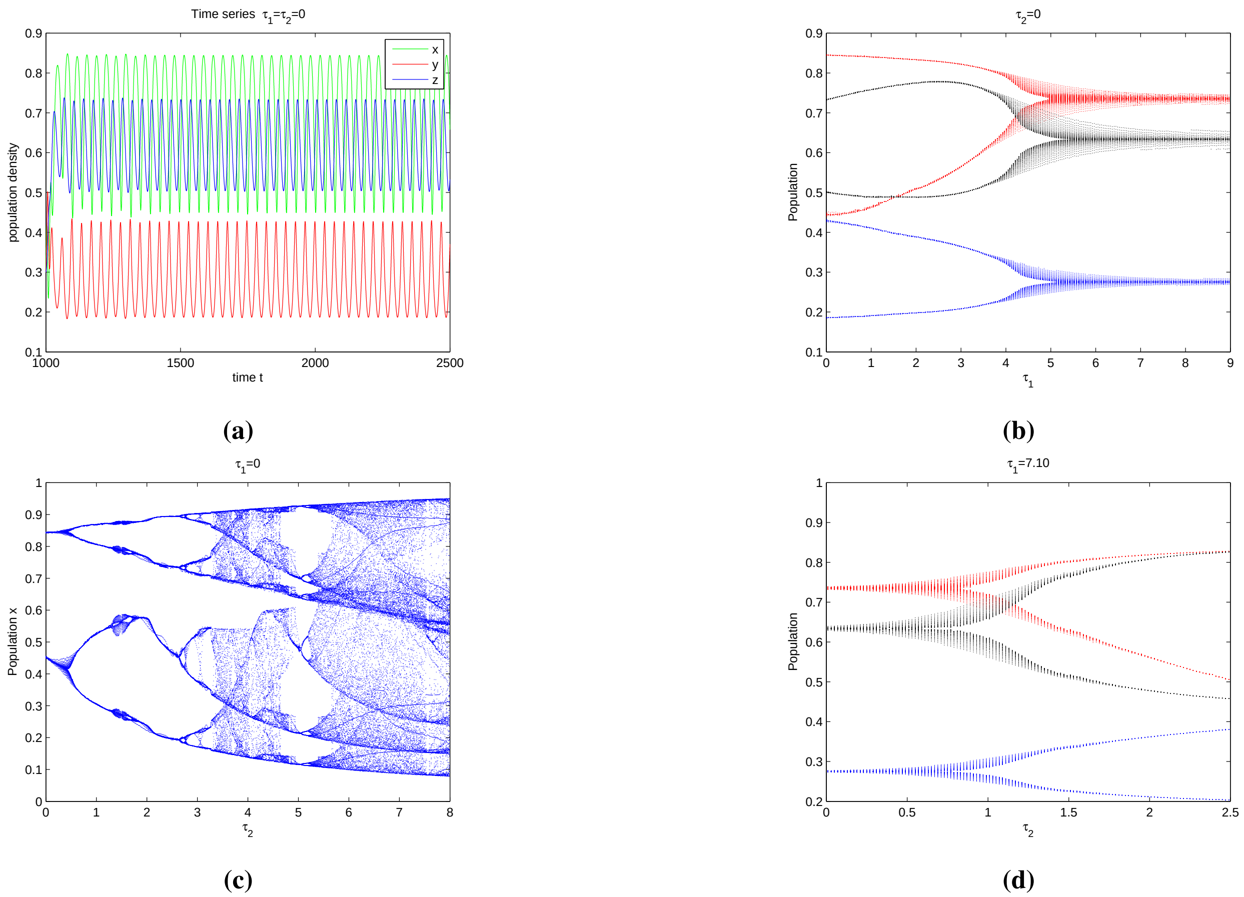

Figure 5. (a) Time evolution of model (5.3) in the absence of delay. (b) Bifurcation diagram as a function of τ_1 keeping $\tau_2 = 0$. (c) Bifurcation diagram as a function of τ_2 keeping $\tau_1 = 0$. (d) Bifurcation diagram as a function of τ_2 for $\tau_1 = 7.10$.

Example 3. Again, consider the following example with new set of parameter values

$$\begin{aligned}\frac{dx}{dt} &= x(1-x) - \frac{xy}{x^2+0.3},\\ \frac{dy}{dt} &= -0.348y + \frac{0.59x(t-\tau_1)y(t-\tau_1)}{x^2(t-\tau_1)+0.3} - \frac{yz}{y^2+0.74},\\ \frac{dz}{dt} &= -0.126z + \frac{0.573y(t-\tau_2)z(t-\tau_2)}{y^2(t-\tau_2)+0.74}.\end{aligned} \tag{5.4}$$

Chaotic behaviour of model (2.2) is illustrated with the help of Example 3. To determine the chaotic behaviour of system, we have plotted the 3D phase portrait of the system. A chaotic attractor is obtained around which system tends to evolve for wide variety of initial conditions and for given sufficient time (see Figure 6(a)). Sensitivity on initial condition (SIC) test is one of the most intuitive tool to check the chaotic behaviour. SIC test tells that if the trajectories owning the slightly different initial conditions grow until their differences become as large as the signal then this ensures the existence of chaotic dynamics in the system. SIC test is given by Figure 6(b).

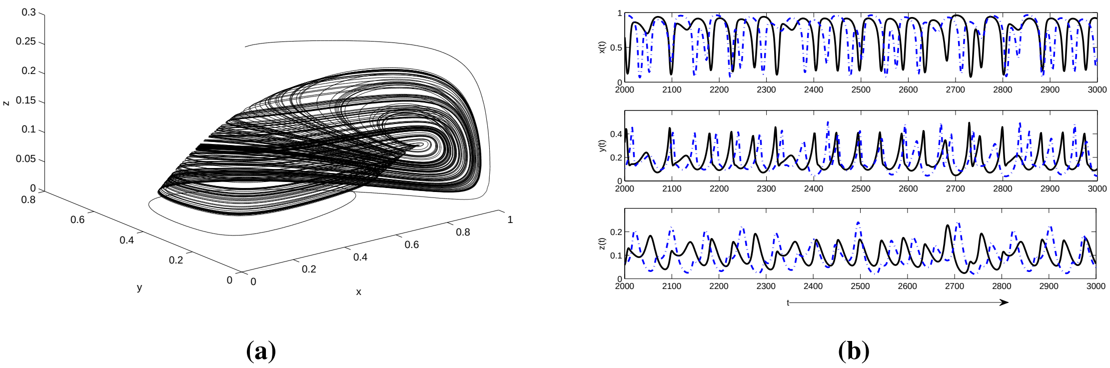

Figure 6. (a) 3D Phase portrait in xyz space showing the chaotic behaviour of system. (b) SIC test: dot-dashed lines for initial density of x, increased by 0.005 keeping y, z fixed, similarly for y and z plot.

To describe the effect of gestation delay τ_1 and τ_2 on the chaotic dynamics of system, bifurcation diagram for x as a function of τ_1 keeping $\tau_2 = 0.8$ is given by Figure 7(d), which shows that the period halving Hopf-bifurcation phenomenon. Therefore, increase in the value of τ_1 leads to the stable limit cycle dynamics through sequence of chaotic dynamics and different order limit cycles (see Figure 7(a)–(c)).

Bifurcation diagrams of the species x, y, z are also presented taking ω_7 as bifurcating parameter for system (5.4) in the absence of delay. Period halving Hopf-bifurcation phenomenon is observed. It is clear from the bifurcation figure that, for higher value of ω_7, i.e., protection provided by environment to the intermediate predator, species x, y will survive and remain stable, moreover species z will extinct (see Figure 8).

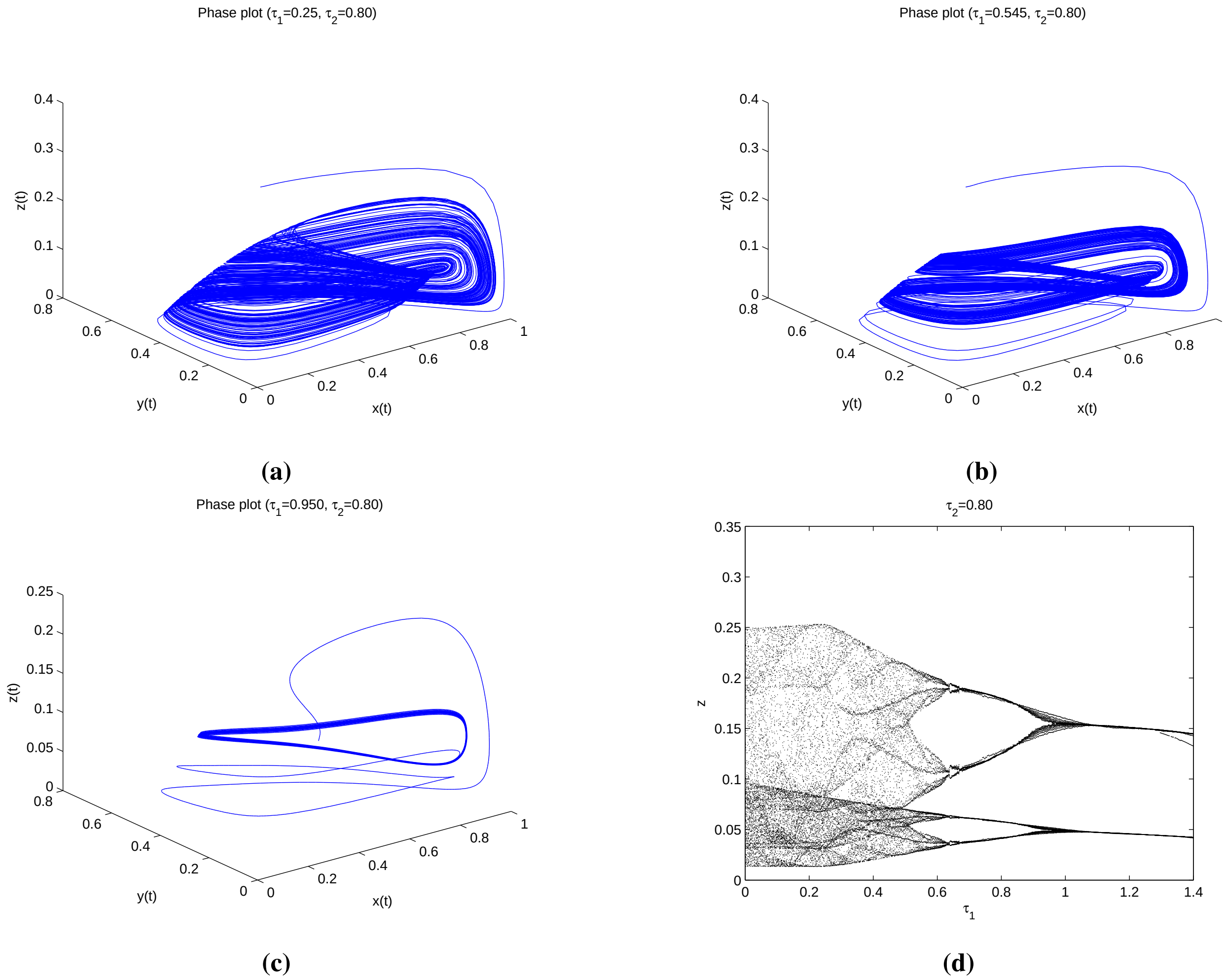

Figure 7. Phase portrait in *xyz* space and bifurcation diagram for the system (5.4) showing the influence of τ_1 on the system dynamics for fixed value of $\tau_2 = 0.80$. (a) $\tau_1 = 0.25$ (chaotic dynamics). (b) $\tau_1 = 0.545$ (higher order limit cycle). (c)$\tau_1 = 0.95$ (1-period limit cycle). (d) Bifurcation diagram as a function of τ_1, varied in the range $[0.001, 1.4]$.

6. Discussion and Conclusion

Empirical results supporting the existence of chaos in real ecological systems are very rare. McCann et al. [48] suggested that it might be due to weak links of species, which may provide stability to these systems. In addition, it is difficult to obtain accurate data on the intrinsic role of species interactions due to measurement error, weather fluctuation, and seasonal disturbances. Experimental demonstrations of chaos in a three species food chain system of ciliate *Tetrahymena pyriformis*, rod-shaped *Pedobacter* and coccus *Brevundimonas* were given by Becks et al. [49] and Becks and Arndt [50]. In contrast to experimental evidences, chaotic behavior can be observed in interacting population model of species predation, competition, etc. [46, 51–53].

In this work, we have examined a three species food chain system with nonmonotonic functional response. The system is highly nonlinear and applicable for modeling the large variety of natural

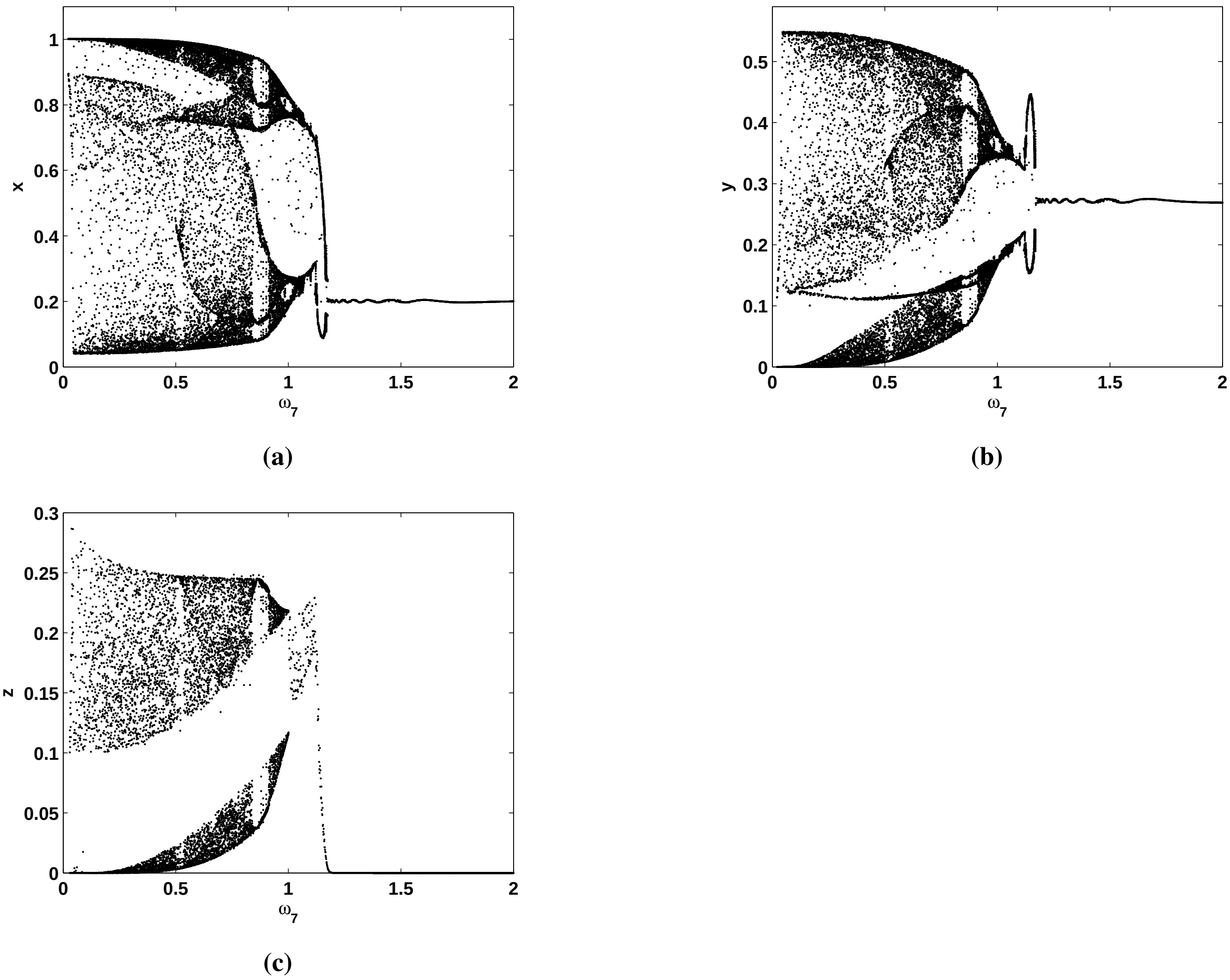

Figure 8. Bifurcation diagrams of species x, y, z for system (5.4) (taking $\tau_1 = \tau_2 = 0$) with ω_7 as bifurcating parameter.

systems. Gestation delays are incorporated in the system for more realistic consideration. Various interesting dynamical conclusions have been drawn. Stability properties of the system about the equilibrium points are discussed for both delayed and non-delayed systems. Boundedness, positive invariance and conditions for stability of the system are derived. Hopf-bifurcation analysis is discussed for all possible combinations of time delays τ_1 and τ_2. Extensive numerical simulations are performed to validate the analytical findings and to explore the various complex dynamical structures.

From numerical simulation results, we have observed that gestation delay τ_1 has stabilizing effect on the model dynamics (see Figure 5(b)), which is rare as signature feature of time delay is destabilization [18, 22, 41]. Recently, stabilizing effect of maturation time delay has been discussed by Banerjee and Takeuchi [25]. However, increasing the value of time delay τ_2 makes the model dynamics chaotic (see Figure 5(c)). Thus, gestation delay for the top predator, τ_2 has a destabilizing effect on the model dynamics. We have obtained the critical value of $\tau_2^* = 2.88823$, below which system shows stable dynamics and above this value system starts showing oscillations through Hopf-bifurcation. Hopf-bifurcation diagram for species x, y and z, taking ω_7 as bifurcating parameter are also plotted for the non-delayed system. Main findings of our work can be summarized as follows:

(i) New periodic activities are induced in the stable dynamics of the system due to the incorporation of GDTP, τ_2 and periodic activities are suppressed due to the introduction of GDMP, τ_1.
(ii) Numerically, it has been explained that for the sufficiently high value of ω_7 (environmental protection to intermediate predator), if the system is stable about coexisting equilibrium point in the absence of delay then it will remain stable for all possible combination of τ_1 and τ_2.
(iii) Existence of chaotic dynamics in the system has been observed which is confirmed by the SIC test. Effect of gestation delays τ_1 and τ_2 on the chaotic dynamics is studied with the help of bifurcation diagram and phase portrait.

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Conflict of interest

The author declares no conflicts of interest in this paper.

References

1. M. Pedraza-Garcia and L. A. Cubillos, Population dynamics of two small pelagic fish in the central-south area off Chile: delayed density-dependence and biological interaction, *Env. Biol. Fish*, **82** (2008), 111–122.
2. R. K. Walsh, C. Bradley, C. S. Apperson, et al., An experimental field study of delayed density dependence in natural populations of Aedes albopictus, *PLoS One*, **7** (2012), e35959.
3. Y. Kuang, *Delay differential equations: with applications in population dynamics*, Academic Press, 1993.
4. J. D. Murray, *Mathematical Biology. II Spatial Models and Biomedical Applications*, 3nd edition, Springer-Verlag, New York, 2003.
5. S. Gakkhar and A. Singh, Complex dynamics in a prey predator system with multiple delays, *Comm. Nonlinear Sci. Numer. Simulat.*, **7** (2012), 914–929.
6. S. Nakaoka, Y. Saito and Y. Takeuchi, Stability, delay, and chaotic behavior in a Lotka-Volterra predator-prey system, *Math. Biosci. Eng.* , **3** (2006), 173–187.
7. S. Yao, L. Ding, Z. Song, et al., Two bifurcation routes to multiple chaotic coexistence in an inertial two-neural system with time delay, *Nonlinear Dyn.*, **95** (2019), 1549–1563.
8. N. MacDonald, *Biological delay systems: linear stability theory*, Cambridge University Press, Uk, 1989.

9. J. M. Cushing, *Integro-differential equations and delay models in population dynamics*, Springer Science & Business Media, 2013.
10. Z. Song, J. Xu and B. Zhen Multitype activity coexistence in an inertial two-neuron system with multiple delays, *Int. J. Bifurc. Chaos*, **25** (2015), 1530040.
11. Z. Song, C. Wang and B. Zhen, Codimension-two bifurcation and multistability coexistence in an inertial two-neuron system with multiple delays, *Nonlinear Dyn.*, **85** (2016), 2099–2113.
12. Z. Jiang, W. Zhang, J. Zhang, et al., Dynamical analysis of a phytoplankton-zooplankton system with harvesting term and Holling III functional response, *Int. J. Bifur. Chaos*, **28** (2018), 1850162.
13. Z. G. Song, B. Zhen and J. Xu, Species coexistence and chaotic behavior induced by multiple delays in a food chain system, *Ecol. Complex*, **19** (2014), 9–17.
14. G. H. Cui and X. P. Yan, Stability and bifurcation analysis on a three-species food chain system with two delays, *Commun. Nonlinear Sci. Numer. Simulat.*, **16** (2011), 3704–3720.
15. Z. Jiang and L. Wang, Global Hopf bifurcation for a predator-prey system with three delays, *Int. J. Bifur. Chaos*, **27** (2017), 1750108.
16. K. Ghosh, S. Samanta, S. Biswas, et al., Stability and bifurcation analysis of an eco-epidemiological model with multiple delays, *Nonlinear Stud.*, **23** (2016), 167–208.
17. L. Guo, Z. G. Song and J. Xu, Complex dynamics in the Leslie–Gower type of the food chain system with multiple delays, *Comm. Nonlinear Sci. Numer.*, **19** (2014), 2850–2865.
18. R. Agrawal, D. Jana, R. K. Upadhyay, et al., Complex dynamics of sexually reproductive generalist predator and gestation delay in a food chain model: double Hopf-bifurcation to chaos, *J. Appl. Math. Comput.*, **55** (2017), 513–547.
19. N. Bairagi and D. Jana, Dynamics and responses of a predator-prey system with competitive interference and time delay, *Appl. Math. Model.*, **35** (2011), 3255–3267.
20. Y. Chen, J. Yu and C. Sun, Stability and Hopf bifurcation analysis in a three-level food chain system with delay, *Chaos Soliton Fract.*, **31** (2007), 683–694.
21. Y. Dong, Y. Takeuchi and S. Nakaoka, A mathematical model of multiple delayed feedback control system of the gut microbiota-Antibiotics injection controlled by measured metagenomic data, *Nonlinear Anal. Real World Appl.*, **43** (2018), 1–17.
22. R. K. Upadhyay and R. Agrawal, On the stability and Hopf bifurcation of a delay-induced predator-prey system with habitat complexity, *Nonlinear Dyn.*, **83** (2017), 821–837.
23. Z. Zhang, H. Yang and H. Liu, Bifurcation analysis for a delayed food chain system with two functional responses, *Electron. J. Qual. Theory Differ. Equ.*, **53** (2013), 1–13.
24. M. Sen M, M. Banerjee and A. Morozov, Stage-structured ratio-dependent predator-prey models revisited: When should the maturation lag result in systems' destabilization?, *Ecol. Complex*, **19** (2014), 23–34.
25. M. Banerjee and Y. Takeuchi, Maturation delay for the predators can enhance stable coexistence for a class of prey-predator models, *J. Theor. Biol.*, **412** (2017), 154–171.
26. J. Wang and W. Jiang, Bifurcation and chaos of a delayed predator-prey model with dormancy of predators, *Nonlinear Dyn.*, **69** (2012), 1541–1558.

27. D. Xiao and S. Ruan, Global analysis in a predator-prey system with nonmonotonic functional response, *SIAM J. Appl. Math.*, **61** (2001), 1445–1472.

28. J. F. Andrews, A mathematical model for the continuous culture of microorganisms utilizing inhibitory substrates, *Biotechnol. Bioeng.*, **10** (1968), 707–723.

29. R. Pal, D. Basu, S. Biswas, et al., Modelling of phytoplankton allelopathy with Monod–Haldane-type functional response-A mathematical study, *BioSystems*, **95** (2009), 243–253.

30. W. Sokol and J. A. Howell, Kinetics of phenol oxidation by washed cells, *Biotechnol. Bioeng.*, **23** (1981), 2039–2049.

31. V. H. Edwards, Influence of high substrate concentrations on microbial kinetics, *Biotechnol. Bioeng.*, **23** (1970), 679–712.

32. B. Boon and H. Landelout, Kinetics of nitrite oxidation by nitrobacter winogradski, *Biochem J.*, **23** (1962), 440–447.

33. S. J. Ali, N. M. Arifin, R. K. Naji, et al., Boundedness and stability of Leslie-Gower model with Sokol-Howell functional response, *Recent Advances in Math. Sci.*, **61** (2016), 13–26.

34. S. J. Ali, N. M. Arifin, R. K. Naji, et al., Dynamics of Leslie-Gower model with simplified Holling type IV functional response, *J. Nonlinear Syst. Appl.*, **5** (2016), 25–33.

35. R. D. Parshad, R. K. Upadhyay, S. Mishra, et al., On the explosive instability in a three-species food chain model with modified Holling type IV functional response, *Math. Methods Appl. Sci.*, **40** (2017), 5707–5726.

36. S. J. Ali, N. M. Arifin, R. K. Naji, et al., Analysis of ecological model with Holling type IV functional response, *Int. J. Pure Appl. Math.*, **106** (2016), 317–331.

37. G. Huang and Y. Dong, A note on global properties for a stage structured predator-prey model with mutual interference, *Adv. Differ. Equ.*, **2018** (2018), 308.

38. R. K. Upadhyay and R. Agrawal, Modeling the effect of mutual interference in a delay-induced predator-prey system, *J. Appl. Math. Comput.*, **49** (2015), 13–39.

39. T. Zhang, X. Meng, Y. Song, et al., A stage-structured predator-prey SI model with disease in the prey and impulsive effects, *Math. Model. Anal.*, **18** (2013), 505–528.

40. B. Patra, A. Maiti and G. P. Samanta, Effect of time delay on a ratio dependent food chain model, *Nonlinear Anal. Model. Control*, **14** (2009), 199–216.

41. N. Pal, S. Samanta, S. Biswas, et al., Stability and bifurcation analysis of a three-species food chain model with delay, *Int. J. Bifur. Chaos*, **25** (2015), 1550123.

42. P. J. Wangersky and W. J. Cunningham, Time lag in prey-predator population models, *Ecology*, **106** (1957), 136–139.

43. D. Jana, R. Agrawal and R. K. Upadhyay, Top-predator interference and gestation delay as determinants of the dynamics of a realistic model food chain, *Chaos Soliton Fract.*, **106** (2014), 50–63.

44. J. K. Hale and S. M. Lunel, *Introduction to functional differential equations*, Springer Science & Business Media, 2013.

45. X. Yang, L. Chen and J. Chen, Permanence and positive periodic solution for the single-species non autonomous delay diffusive models, *Comput. Math. Appl.*, **32** (1996), 109–116.

46. M. A. Aziz-Alaoui, Study of a Leslie-Gower-type tri-trophic population model, *Chaos Soliton Fract.*, **14** (2002), 1275–1293.

47. Z. Song and J. Xu, Stability switches and multistability coexistence in a delay-coupled neural oscillators system, *J. Theor. Biol.*, **313** (2012), 98–114.

48. K. McCann, A. Hastings and G. R. Huxel, Weak trophic interactions and the balance of nature, *Nature*, **395** (1998), 794.

49. L. Becks, F. M. Hilker, H. Malchow, et al., Experimental demonstration of chaos in a microbial food web, *Nature*, **435** (2005), 1226.

50. L. Becks and H. Arndt, Transitions from stable equilibria to chaos, and back, in an experimental food web, *Ecology*, **11** (2008), 3222–3226.

51. Y. Dong, M. Sen, M. Banerjee, et al., Delayed feedback induced complex dynamics in an Escherichia coli and Tetrahymena system, *Nonlinear Dyn.*, **94** (2018), 1447–1466.

52. R. K. Upadhyay and R. K. Naji, Dynamics of a three species food chain model with Crowley-Martin type functional response, *Chaos Soliton Fract.*, **42** (2009), 1337–1346.

53. R. K. Upadhyay and S. N. Raw, Complex dynamics of a three species food-chain model with Holling type IV functional response, *Nonlinear Anal. Model Control*, **16** (2011), 353–374.