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Research article

Existence and stability of traveling wavefronts for discrete three species competitive-cooperative systems

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Abstract: The purpose of this work is to investigate the existence and stability of traveling wavefronts for competitive-cooperative systems with three species. The existence result can be derived by using the technique of monotone method with the help of a pair of explicit supersolution and subsolution. Moreover, some sufficient conditions ensure the linear determinacy for the minimal speed is given. Then, applying the weighted energy method, we prove that the traveling wavefronts are asymptotically stable in the weighted Banach spaces provided that the initial perturbations of the traveling wavefronts also belong to the same spaces.

Keywords: traveling wavefronts; monotone system; supersolution; subsolution; weighted energy estimate

1. Introduction

This paper is concerned with the existence and stability of traveling wavefronts for the following discrete three species competitive-cooperative systems of Lotka-Volterra type:

$$\begin{cases} u_t = d_1 D[u](t,x) + u(r_1 - b_{11}u - b_{12}v + b_{13}w), \\ v_t = d_2 D[v](t,x) + v(r_2 - b_{21}u - b_{22}v - b_{23}w), \\ w_t = d_3 D[w](t,x) + w(r_3 + b_{31}u - b_{32}v - b_{33}w), \end{cases} \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1.1}$$

and

$$\begin{cases} u_i'(t) = d_1 \triangle[u_i](t) + u_i(r_1 - b_{11}u_i - b_{12}v_i + b_{13}w_i), \\ v_i'(t) = d_2 \triangle[v_i](t) + v_i(r_2 - b_{21}u_i - b_{22}v_i - b_{23}w_i), \\ w_i'(t) = d_3 \triangle[w_i](t) + w_i(r_3 + b_{31}u_i - b_{32}v_i - b_{33}w_i), \end{cases} \quad \forall (t,i) \in \mathbb{R}^+ \times \mathbb{Z}. \tag{1.2}$$

Here $D[u](t,x)$ and $\triangle[u_i](t)$ mean the discrete diffusive operators given by

$$D[u](t,x) := u(t,x+1) + u(t,x-1) - 2u(t,x)$$
and
$$\Delta[u_i](t) := u_{i+1}(t) + u_{i-1}(t) - 2u_i(t).$$

Systems (1.1) and (1.2) can be considered as discrete versions of the following continuous system:

$$\begin{cases} u_t = d_1 u_{xx} + u(r_1 - b_{11}u - b_{12}v + b_{13}w), \\ v_t = d_2 v_{xx} + v(r_2 - b_{21}u - b_{22}v - b_{23}w), \qquad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \\ w_t = d_3 w_{xx} + w(r_3 + b_{31}u - b_{32}v - b_{33}w), \end{cases} \tag{1.3}$$

In system (1.3), $u(\cdot), v(\cdot)$ and $w(\cdot)$ represent the population density of the species. Each $d_i > 0 (i = 1, 2, 3)$ stands for the diffusion rate of each species, and $r_i > 0 (i = 1, 2, 3)$ is the growth rate of species. The parameter $b_{ii} > 0 (i = 1, 2, 3)$ means the intraspecific competition rates of a species, and $b_{12}, b_{21}, b_{23}, b_{32} > 0$ describe the interspecific competition rates between species. Noting that b_{13} and b_{31} maybe positive or negative constants. If b_{13} and $b_{31} < 0$ then (1.3) is a competitive system among three species and any two of the three species u, v and w are in a competitive manner. On the other hand, if b_{13} and $b_{31} > 0$, then (1.3) becomes the competitive-cooperative system of three species. That is, u and v compete and w and v also compete with each other, while u and w are in a cooperative way to help each other.

Due to different signs of the parameters, the interacting behavior between the species of (1.3) are quite complicated and different. In biology, one of the important issue is to investigate the invasion phenomenon for system (1.3). Thus it is very nature to study the propagation of traveling wave solutions. The concept of traveling wave solutions was introduced by Fisher [1] in 1937 in reaction diffusion equations, which represents a segregated spatial pattern propagating through the spatial domain at a constant speed. In addition, such solutions are natural phenomena ubiquitously for many reaction-diffusion systems, e.g., biophysics, population genetics, mathematical ecology, chemistry, chemical physics, and so on. In past years, there have many progresses on this topic in various fields. Here we only illustrate some literature for system (1.3) in the sequel.

For the competitive case, Chen et al. [2, 3] and Mimura and Tohma [4] used numerical approaches or the construction of exact traveling wave solutions to establish many kinds of pattern formulations. In addition, when the diffusion coefficients are small, Ikeda [5, 6] considered traveling wave solutions and dynamics of weakly interacting front and back waves. Other related works, we refer Kan-on and Mimura [7], Miller [8] and Mimura and Fife [9]. On the other hand, for the competitive-cooperative case, one can see that system (1.3) is a monotone system which has some ordering structures. Based on the monotone structure, Guo et al. [10] proved the existence of traveling wave solutions under the assumption $b_{13} = b_{31} = 0$. Hung [11] further considered the existence of traveling wave solutions in the case b_{13}, $b_{31} > 0$, $d_1 = d_2 = d_3$ and $r_1 = r_2 = r_3$. Recently, Chang [12] improved the results of [10, 11] to more general parameters. Motivated by the above mentioned literature, it is natural and important to study the same problems for the discrete systems (1.1) and (1.2). In this paper, we first establish the existence of traveling wavefronts for discrete systems (1.1) and (1.2). However, when these solutions are disturbed under small perturbations, only stable such solutions can be visualized in the real world. Therefore, it is quite important to study the stability problem of the traveling wavefronts. We also focus on the stability problem in this work.

Since there are many parameters appearing in the above systems, we first rescale the systems (1.1)–(1.2) into the following simpler forms:

$$\begin{cases} u_t = d_1 D[u](t,x) + u(r_1 - u - c_{12}v + c_{13}w), \\ v_t = d_2 D[v](t,x) + v(r_2 - c_{21}u - v - c_{23}w), \qquad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ w_t = d_3 D[w](t,x) + w(r_3 + c_{31}u - c_{32}v - w), \end{cases} \tag{1.4}$$

and

$$\begin{cases} u_i'(t) = d_1 \triangle[u_i](t) + u_i(r_1 - u_i - c_{12}v_i + c_{13}w_i), \\ v_i'(t) = d_2 \triangle[v_i](t) + v_i(r_2 - c_{21}u_i - v_i - c_{23}w_i), \qquad \forall (t,i) \in \mathbb{R}^+ \times \mathbb{Z}. \\ w_i'(t) = d_3 \triangle[w_i](t) + w_i(r_3 + c_{31}u_i - c_{32}v_i - w_i). \end{cases} \tag{1.5}$$

Furthermore, replacing (u, v, w) and (u_i, v_i, w_i) by $(u, r_2 - v, w)$ and $(u_i, r_2 - v_i, w_i)$ respectively, we can transform systems (1.4)–(1.5) into the following systems

$$\begin{cases} u_t = d_1 D[u](t,x) + u(r_1 - c_{12}r_2 - u + c_{12}v + c_{13}w), \\ v_t = d_2 D[v](t,x) + (v - r_2)(-c_{21}u + v - c_{23}w), \qquad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \\ w_t = d_3 D[w](t,x) + w(r_3 - c_{32}r_2 + c_{31}u + c_{32}v - w), \end{cases} \tag{1.6}$$

and

$$\begin{cases} u_i'(t) = d_1 \triangle[u_i](t) + u_i(r_1 - c_{12}r_2 - u_i + c_{12}v_i + c_{13}w_i), \\ v_i'(t) = d_2 \triangle[v_i](t) + (v_i - r_2)(-c_{21}u_i + v_i - c_{23}w_i), \qquad \forall (t,i) \in \mathbb{R}^+ \times \mathbb{Z}. \\ w_i'(t) = d_3 \triangle[w_i](t) + w_i(r_3 - c_{32}r_2 + c_{31}u_i + c_{32}v_i - w_i), \end{cases} \tag{1.7}$$

Since systems (1.6)–(1.7) are monotone systems, for simplicity, hereinafter we will consider our subject on the systems (1.6)–(1.7). By elementary computations, systems (1.6) or (1.7) have the following eight equilibria:

$$\begin{aligned} &\mathbf{E}_1 = (0,0,0),\ \mathbf{E}_2 = (u_*, r_2, w_*) = \left(\frac{r_1 + r_3 c_{13}}{1 - c_{31}c_{13}}, r_2, \frac{r_1 c_{31} + r_3}{1 - c_{31}c_{13}}\right),\ \mathbf{E}_3 = (r_1, r_2, 0), \\ &\mathbf{E}_4 = (0, r_2, 0),\ \mathbf{E}_5 = (0, r_2, r_3),\ \mathbf{E}_6 = \left(0, \frac{c_{23}(r_3 - c_{32}r_2)}{1 - c_{23}c_{32}}, \frac{r_3 - c_{32}r_2}{1 - c_{23}c_{32}}\right), \\ &\mathbf{E}_7 = \left(\frac{r_1 - c_{12}r_2}{1 - c_{12}c_{21}}, \frac{c_{21}(r_1 - c_{12}r_2)}{1 - c_{12}c_{21}}, 0\right),\ \mathbf{E}_8 = (e_1, e_2, e_3), \end{aligned}$$

where

$$\begin{aligned} e_1 &:= -\left[(r_1 - c_{12}r_2) + c_{13}(r_3 - c_{32}r_2) - c_{23}(r_1 c_{32} - r_3 c_{12})\right]/\Theta, \\ e_2 &:= -\left[(c_{21} + c_{31}c_{23})(r_1 - c_{12}r_2) + (c_{23} + c_{21}c_{13})(r_3 - c_{32}r_2)\right]/\Theta, \\ e_3 &:= -\left[(r_3 - c_{32}r_2) + c_{31}(r_1 - c_{12}r_2) + c_{21}(r_1 c_{32} - r_3 c_{12})\right]/\Theta, \\ \Theta &:= c_{13}c_{31} + c_{12}c_{23}c_{31} + c_{21}c_{13}c_{32} + c_{12}c_{21} + c_{23}c_{32} - 1. \end{aligned}$$

A traveling wave solution $(u(t,x), v(t,x), w(t,x))$ for (1.6) means that

$$(u(t,x), v(t,x), w(t,x)) = (\phi_1(x + ct), \phi_2(x + ct), \phi_3(x + ct))$$

for some smooth functions $\phi_i(\cdot)$, $i = 1, 2, 3$ with wave speed $c \in \mathbb{R}$. If $\Phi(\cdot) = (\phi_1(\cdot), \phi_2(\cdot), \phi_3(\cdot))$ is monotone, then it is called a traveling wavefront. Then, taking the moving coordinate $\xi := x + ct$, we see the profile function $(\phi_1(\xi), \phi_2(\xi), \phi_3(\xi))$ for system (1.6) satisfy the system

$$\begin{cases} c\phi_1'(\xi) = d_1\mathcal{D}[\phi_1](\xi) + \phi_1(r_1 - c_{12}r_2 - \phi_1 + c_{12}\phi_2 + c_{13}\phi_3), \\ c\phi_2'(\xi) = d_2\mathcal{D}[\phi_2](\xi) + (\phi_2 - r_2)(-c_{21}\phi_1 + \phi_2 - c_{23}\phi_3), \qquad \forall \xi \in \mathbb{R}, \\ c\phi_3'(\xi) = d_3\mathcal{D}[\phi_3](\xi) + \phi_3(r_3 - c_{32}r_2 + c_{31}\phi_1 + c_{32}\phi_2 - \phi_3), \end{cases} \tag{1.8}$$

where

$$\mathcal{D}[\phi_i](\xi) := \phi_i(\xi + 1) + \phi_i(\xi - 1) - 2\phi_i(\xi),\ i = 1, 2, 3.$$

Different to system (1.6), a traveling wave solution $(u_i(t), v_i(t), w_i(t))$ for (1.7) means that

$$(u_i(t), v_i(t), w_i(t)) = (\phi_1(i + ct), \phi_2(i + ct), \phi_3(i + ct))$$

for some smooth functions $\phi_i(\cdot)$, $i = 1, 2, 3$ with wave speed $c \in \mathbb{R}$. Then, taking the moving coordinate $\xi := i + ct$, we see the profile function $(\phi_1(\xi), \phi_2(\xi), \phi_3(\xi))$ for system (1.7) is the same as system (1.8). From the viewpoint of biology, we are interested in the existence and stability of solutions for system (1.8) connecting the trivial equilibria $\mathbf{E}_1$ and positive co-exist equilibrium $\mathbf{E}_2$, that is satisfy the following conditions:

$$\lim_{\xi \to -\infty} (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)) = \mathbf{E}_1 \text{ and } \lim_{\xi \to \infty} (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)) = \mathbf{E}_2. \tag{1.9}$$

It is easy to see that $\mathbf{E}_2 \gg \mathbf{0}$ when $c_{31}c_{13} < 1$. Here and in the sequel, we always use the usual notations for the standard ordering in $\mathbb{R}^3$.

In this article we first consider the existence problem of traveling wavefronts for systems (1.6) and (1.7), i.e., looking for solutions of (1.8) satisfying the condition (1.9). Since (1.8) is a monotone system, the existence problem could be reduced to find a pair of supersolution and subsolution of system (1.8). To this end, throughout this article, we assume the following assumption:

(H1) $d_1 \geq d_3 \geq d_2$, $c_{32} < 1 < c_{21} + c_{23}$ and $(c_{21} + c_{23})r_2 \leq r_3 - c_{32}r_2 \leq r_1 - c_{12}r_2$.
(H2) $(c_{21} + c_{31}c_{23})r_1 + (c_{23} + c_{21}c_{13})r_3 > r_2(1 - c_{13}c_{31}) > 0$.

Note that (H2) holds when r_2 is small enough. The assumption (H1) will be used in proving the existence of traveling wavefronts. In addition, one can verify that

$$e_2 - u_2^* = -\frac{(c_{21} + c_{31}c_{23})r_1 + (c_{23} + c_{21}c_{13})r_3 - r_2(1 - c_{13}c_{31})}{\Theta}.$$

If $\Theta > 0$ then (H2) implies that $e_2 < u_2^*$. On the other hand, if $\Theta < 0$ then (H2) implies that $e_2 > u_2^*$. Hence, under the assumptions (H1)–(H2), we know that $\mathbf{E}_8 \notin [\mathbf{0}, \mathbf{E}_2]$.

Based on the above assumptions, we can establish a pair of supersolution and subsolution of system (1.8). Then, applying the monotone iteration method, we show that (1.8) admits a strictly increasing solution satisfying (1.9) as long as the wave speed is greater than the minimum wave speed (see Theorem 3.1). That is the existence of monotonic traveling wave solutions connecting two equilibria for systems (1.1) and (1.2). In addition, we show that (H1) and (H2) are sufficient conditions which ensure the linear determinacy for the minimal speed is given, i.e., the minimal speed is determined by the linearization of the problem at some unstable equilibrium.

Next, we consider the stability of traveling wavefronts derived in Theorem 3.1. In past years, there have been extensive investigations on the stability of traveling wave solutions for reaction-diffusion systems, see e.g., [13–15], the monographs [16, 17], the survey paper [18] and the references therein. For examples, Mei et al. [14] used the weighted energy method and the Green function technique to study the global stability of monostable traveling wave solutions for nonlocal time-delayed reaction-diffusion equations. Recently, by using the monotone scheme and spectral analysis, Chang [12] considered the existence and stability of traveling wave solutions for system (1.3). More precisely, the author showed that the traveling wave solutions of (1.3) are essentially unstable in the uniform continuous function space. On the other hand, if the initial perturbations of the traveling wave solutions belong to certain exponentially weighted Banach space, then the traveling wave solutions are asymptotically stable in the weighted Banach space. However, due to the discrete diffusion operator in (1.8), the method of spectral analysis used in Chang [12] no longer works in investigating the stability problems of the discrete systems (1.6) and (1.7). Motivated by the works [14, 19], we will establish the comparison principle for systems (1.6) and (1.7). And then use the the weighted energy method (see [14, 19–22]) to show that the traveling wave solutions of (1.6) and (1.7) with large wave speed are exponentially stable when the initial perturbation around them decay exponentially as the spatial variable tending to $-\infty$ (see Theorems 4.1 and 5.1). Moreover, using different weighted functions, we improve the stability results of Theorems 4.1 and 5.1 to any wave speed greater than the minimum wave speed (see Theorems 6.1 and 6.2).

2. Construction of sub-super solutions for (1.8)

For convenience, we write $\mathbf{E}_2 = (u_1^*, u_2^*, u_3^*)$ in this section, and $F(u,v,w) := (f_1(u,v,w), f_2(u,v,w), f_3(u,v,w))$ where

$$\begin{aligned} f_1(u,v,w) &:= u(r_1 - c_{12}r_2 - u + c_{12}v + c_{13}w), \\ f_2(u,v,w) &:= (v - r_2)(-c_{21}u + v - c_{23}w), \\ f_3(u,v,w) &:= w(r_3 - c_{32}r_2 + c_{31}u + c_{32}v - w). \end{aligned}$$

Then the profile system (1.8) can be written into the form:

$$c\phi_i'(\xi) = d_i\mathcal{D}[\phi_i](\xi) + f_i(\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)), \text{ for } i = 1,2,3. \tag{2.1}$$

To establish the existence of solutions for system (2.1) by using the technique of sub-super solutions, we first give the following definition.

Definition 2.1. *A continuous function $(\phi_1(\xi), \phi_2(\xi), \phi_3(\xi))$ is called a subsolution or supersolution of* (2.1)*, if each $\phi_i(\xi)$ is continuously differentiable in $\mathbb{R}$ except at finite points and satisfies* (*resp.*)

$$c\phi_i'(\xi) \le d_i\mathcal{D}[\phi_i](\xi) + f_i(\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)), \ a.e.\ \xi \in \mathbb{R}, \tag{2.2}$$

or

$$c\phi_i'(\xi) \ge d_i\mathcal{D}[\phi_i](\xi) + f_i(\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)), \ a.e.\ \xi \in \mathbb{R}. \tag{2.3}$$

Before constructing a pair of sub-super solutions for system (1.8), we first consider the characteristic polynomials of system (1.8) at $\mathbf{E}_1$ given by

$$\det\begin{bmatrix}\Gamma_1(\mu;c) & 0 & 0\\ c_{21}r_2 & \Gamma_2(\mu;c) & c_{23}r_2\\ 0 & 0 & \Gamma_3(\mu;c)\end{bmatrix} = \Gamma_1(\mu;c)\Gamma_2(\mu;c)\Gamma_3(\mu;c),$$

where

$$\begin{aligned}\Gamma_1(\mu;c) &:= d_1(e^{\mu}+e^{-\mu}-2)-c\mu+r_1-c_{12}r_2,\\ \Gamma_2(\mu;c) &:= d_2(e^{\mu}+e^{-\mu}-2)-c\mu-r_2,\\ \Gamma_3(\mu;c) &:= d_3(e^{\mu}+e^{-\mu}-2)-c\mu+r_3-c_{32}r_2.\end{aligned}$$

It is clear that $\Gamma_2(\mu;c)=0$ have a positive root for any $c>0$. For $\Gamma_1(\mu;c)$ and $\Gamma_3(\mu;c)$, we have the following properties.

Lemma 2.1. *There exist $c_1^*\geq c_3^*>0$ such that (for $i=1,3$)*

(1) *if $c>c_i^*$, there exist $0<\mu_i^-<\mu_i^+$ such that*

$$\Gamma_i(\mu_i^{\pm};c)=0,\ \Gamma_i(\mu;c)<0,\ \forall\mu\in(\mu_i^-,\mu_i^+)\ \text{and}\ \Gamma_i(\mu;c)>0,\ \forall\mu\in[\mu_i^-,\mu_i^+]^c;$$

(2) *if $c=c_i^*$, there exists a unique $\mu_i^*\in(\mu_i^-,\mu_i^+)$ such that*

$$\Gamma_i(\mu_i^*;c_i^*)=0\ \text{and}\ \Gamma_i(\mu;c_i^*)>0,\ \forall\mu\neq\mu_i^*;$$

(3) *if $0<c<c_i^*$, then $\Gamma_i(\mu;c)>0$ for all $\mu\in\mathbb{R}$.*

In addition, we have $\mu_3^-\leq\mu_1^-<\mu_1^+\leq\mu_3^+$ when $c>c_1^*$.

By Lemma 2.1, we can construct a pair of sub-super solutions for (1.8) in the sequel.

Lemma 2.2. *Assume $c>c_1^*$. Let's set*

$$\hat{u}_1(\xi):=\begin{cases}e^{\mu_1^-\xi}+qu_1^*e^{\eta\mu_1^-\xi}, & \text{if } \xi<\xi_1,\\ u_1^*, & \text{if } \xi\geq\xi_1,\end{cases}\quad \text{and}\quad \hat{u}_i(\xi):=\begin{cases}e^{\mu_3^-\xi}+qu_i^*e^{\eta\mu_1^-\xi}, & \text{if } \xi<\xi_i,\\ u_i^*, & \text{if } \xi\geq\xi_i,\end{cases}$$

for $i=2,3$, where $\hat{u}_i(\xi_i)=u_i^*$ $(i=1,2,3)$, q and η are positive constants with

$$\mu_1^-<\eta\mu_1^-<\min\{\mu_1^+,\mu_3^+,\mu_1^-+\mu_3^-\}. \tag{2.4}$$

Then $\hat{U}(\xi)=(\hat{u}_1(\xi),\hat{u}_2(\xi),\hat{u}_3(\xi))$ is a supersolution of (1.8) when q is large enough.

Proof. Let us write $\xi_i=\xi_i(q)$ $(i=1,2,3)$ as a function of q. Since $\hat{u}_i(\xi_i)=u_i^*$, one can easily verify that

$$\lim_{q\to\infty}\xi_i(q)=-\infty,\ \text{for } i=1,2,3. \tag{2.5}$$

Then, for $\xi\geq\xi_1(q)$, it is clear that

$$d_1\mathcal{D}[\hat{u}_1](\xi)-c\hat{u}_1'(\xi)+f_1(\hat{U}(\xi))\leq f_1(\mathbf{E}_2)=0. \tag{2.6}$$

If $\xi<\xi_1(q)$, by (2.4) and (2.5) and elementary computations, we have

$$d_1\mathcal{D}[\hat{u}_1](\xi)-c\hat{u}_1'(\xi)+f_1(\hat{U}(\xi))$$

$$
\begin{aligned}
&= q u_1^* e^{\eta\mu_1^-\xi}\Gamma_1(\eta\mu_1^-;c) + \hat{u}_1(\xi)\big(-\hat{u}_1(\xi) + c_{12}\hat{u}_2(\xi) + c_{13}\hat{u}_3(\xi)\big)\\
&\le q u_1^* e^{\eta\mu_1^-\xi}\Gamma_1(\eta\mu_1^-;c) + \hat{u}_1(\xi)\big(-e^{\mu_1^-\xi} + (c_{12}+c_{13})e^{\mu_3^-\xi} + q(-u_1^* + c_{12}u_2^* + c_{13}u_3^*)e^{\eta\mu_1^-\xi}\big)\\
&\le q u_1^* e^{\eta\mu_1^-\xi}\Gamma_1(\eta\mu_1^-;c) + (e^{\mu_1^-\xi} + q u_1^* e^{\eta\mu_1^-\xi})(c_{12}+c_{13})e^{\mu_3^-\xi} \le 0,
\end{aligned}
\tag{2.7}
$$

provided that q is large enough.

Next, we set

$$
u^* := \max\{u_1^*, u_2^*, u_3^*\} \text{ and } \hat{u}(\xi) := e^{\mu_3^-\xi} + q u^* e^{\eta\mu_1^-\xi}.
$$

Then, for all $\xi \in \mathbb{R}^-$, it is clear that $\max\{\hat{u}_1(\xi), \hat{u}_2(\xi), \hat{u}_3(\xi)\} \le \hat{u}(\xi)$ and

$$
\begin{aligned}
f_2(\hat{U}(\xi)) &= (\hat{u}_2(\xi) - r_2)\big(-c_{21}\hat{u}_1(\xi) + \hat{u}_2(\xi) - c_{23}\hat{u}_3(\xi)\big)\\
&\le (\hat{u}_2(\xi) - r_2)\big(-c_{21}\hat{u}(\xi) + \hat{u}_2(\xi) - c_{23}\hat{u}(\xi)\big)\\
&\le -r_2(-c_{21} - c_{23})\hat{u}(\xi).
\end{aligned}
\tag{2.8}
$$

Assuming that q is large enough, we have $\xi_2(q) < 0$. For $\xi > \xi_2(q)$, it is clear that

$$
d_2\mathcal{D}[\hat{u}_2](\xi) - c\hat{u}_2'(\xi) + f_2(\hat{U}(\xi)) \le f_2(\mathbf{E}_2) = 0. \tag{2.9}
$$

If $\xi < \xi_2(q)$, by the fact $\hat{u}_2(\xi) \le r_2$, (2.4), (2.8) and (H1), we can obtain

$$
\begin{aligned}
&d_2\mathcal{D}[\hat{u}_2](\xi) - c\hat{u}_2'(\xi) + f_2(\hat{U}(\xi))\\
\le& d_2\mathcal{D}[\hat{u}_2](\xi) - c\hat{u}_2'(\xi) - r_2(-c_{21} - c_{23})\hat{u}(\xi)\\
=& e^{\mu_1^-\xi}[d_2(e^{\mu_1^-} + e^{-\mu_1^-} - 2) - c\mu_1^- + r_2(c_{21}+c_{23})]\\
&+ q u^* e^{\eta\mu_1^-\xi}[d_2(e^{\eta\mu_1^-} + e^{-\eta\mu_1^-} - 2) - c\eta\mu_1] + q u^* r_2(c_{21}+c_{23})e^{\eta\mu_1^-\xi}\\
\le& e^{\mu_3^-\xi}\Gamma_3(\mu_3^-;c) + q u^* e^{\eta\mu_1^-\xi}\big(d_3(e^{\eta\mu_1^-} + e^{-\eta\mu_1^-} - 2) - c\eta\mu_1 + r_2(c_{21}+c_{23})\big)\\
\le& q u^* e^{\eta\mu_1^-\xi}\Gamma_3(\eta\mu_1^-;c) \le 0.
\end{aligned}
\tag{2.10}
$$

Finally, for $\xi > \xi_3(q)$, it is clear that

$$
d_3\mathcal{D}[\hat{u}_3](\xi) - c\hat{u}_3'(\xi) + f_3(\hat{U}(\xi)) \le f_3(\mathbf{E}_2) = 0. \tag{2.11}
$$

If $\xi < \xi_3(q)$, then (2.4) implies that

$$
\begin{aligned}
&d_3\mathcal{D}[\hat{u}_3](\xi) - c\hat{u}_3'(\xi) + f_3(\hat{U}(\xi))\\
=& e^{\mu_3^-\xi}\Gamma_3(\mu_3^-;c) + q u_3^* e^{\eta\mu_1^-\xi}\Gamma_3(\eta\mu_1^-;c) + \hat{u}_3(\xi)\big(c_{31}\hat{u}_1(\xi) + c_{32}\hat{u}_2(\xi) - \hat{u}_3(\xi)\big)\\
\le& q u_3^* e^{\eta\mu_1^-\xi}\Gamma_1(\eta\mu_1;c) + \hat{u}_3(\xi)\big(c_{31}e^{\mu_1^-\xi} + c_{32}e^{\mu_3^-\xi} - e^{\mu_3^-\xi} + q(c_{31}u_1^* + c_{21}u_2^* - u_1^*)e^{\eta\mu_1^-\xi}\big)\\
\le& q u_3^* e^{\eta\mu_1^-\xi}\Gamma_1(\eta\mu_1;c) + c_{31}(e^{\mu_3^-\xi} + q u_3^* e^{\eta\mu_1^-\xi})e^{\mu_1^-\xi} \le 0,
\end{aligned}
\tag{2.12}
$$

provided that q is large enough. Hence, it follows from (2.6)–(2.12) that $\hat{U}(\xi)$ is a supersolution of system (1.8) when q is large enough. The proof is complete. □

Lemma 2.3. *Assume $c > c_1^*$. Let's set $\bar{u}_2(\xi) :\equiv 0$,*

$$\bar{u}_1(\xi) := \begin{cases} e^{\mu_1^-\xi} - qu_1^* e^{\eta\mu_1^-\xi}, & \text{if } \xi < \bar{\xi}_1, \\ 0, & \text{if } \xi \geq \bar{\xi}_1, \end{cases} \quad \text{and } \bar{u}_3(\xi) := \begin{cases} e^{\mu_3^-\xi} - qu_3^* e^{\eta\mu_3^-\xi}, & \text{if } \xi < \bar{\xi}_3, \\ 0, & \text{if } \xi \geq \bar{\xi}_3, \end{cases}$$

where $\bar{u}_i(\bar{\xi}_i) = 0$ for $i = 1, 3$; q and η are positive constants with

$$1 < \eta < \min\{\mu_3^+/\mu_3^-, \mu_1^+/\mu_1^-, 2\}. \tag{2.13}$$

Then $\bar{U}(\xi) = (\bar{u}_1(\xi), \bar{u}_2(\xi), \bar{u}_3(\xi))$ is a subsolution of (1.8) *when q is large enough.*

Proof. Let us also write $\bar{\xi}_i = \bar{\xi}_i(q)$ as a function of q. Similarly, $\bar{\xi}_i(\infty) = -\infty$, for $i = 1, 3$. According to the definition of $\bar{u}_i(\xi)$, we only need to consider the cases $\xi < \bar{\xi}_1(q)$ and $\xi < \bar{\xi}_3(q)$ for $\bar{u}_1(\xi)$ and $\bar{u}_3(\xi)$, respectively.

If $\xi < \bar{\xi}_1(q)$, by (2.13), we have

$$\begin{aligned}
& d_1\mathcal{D}[\bar{u}_1](\xi) - c\bar{u}_1'(\xi) + f_1(\bar{U}(\xi)) \\
=& e^{\mu_1^-\xi}\Gamma_1(\mu_1^-; c) - qu_1^* e^{\eta\mu_1^-\xi}\Gamma_1(\eta\mu_1^-; c) + \bar{u}_1(\xi)\big(- \bar{u}_1(\xi) + c_{13}\bar{u}_3(\xi)\big) \\
\geq& e^{\mu_1^-\xi}\Gamma_1(\mu_1; c) - qu_1^* e^{\eta\mu_1^-\xi}\Gamma_1(\eta\mu_1^-; c) - \bar{u}_1(\xi)\bar{u}_1(\xi) \\
=& - qu_1^* e^{\eta\mu_1^-\xi}\Gamma_1(\eta\mu_1^-; c) - (e^{\mu_1^-\xi} - qu_1^* e^{\eta\mu_1^-\xi})(e^{\mu_1^-\xi} - qu_1^* e^{\eta\mu_1^-\xi}) \\
\geq& - qu_1^* e^{\eta\mu_1^-\xi}\Gamma_1(\eta\mu_1^-; c) - (e^{\mu_1^-\xi} - qu_1^* e^{\eta\mu_1^-\xi})e^{\mu_1^-\xi} \geq 0,
\end{aligned} \tag{2.14}$$

provided that q is large enough.

For $\xi < \xi_3(q)$, by (2.13) again, one can see that

$$\begin{aligned}
& d_3\mathcal{D}[\bar{u}_3](\xi) - c\bar{u}_3'(\xi) + f_3(\bar{U}(\xi)) \\
=& e^{\mu_3^-\xi}\Gamma_3(\mu_3^-; c) - qu_3^* e^{\eta\mu_3^-\xi}\Gamma_3(\eta\mu_3^-; c) + \bar{u}_3(\xi)\big(c_{31}\bar{u}_1(\xi) - \bar{u}_3(\xi)\big) \\
\geq& e^{\mu_3^-\xi}\Gamma_3(\mu_3^-; c) - qu_3^* e^{\eta\mu_3^-\xi}\Gamma_3(\eta\mu_3^-; c) - \bar{u}_3(\xi)\bar{u}_3(\xi) \\
=& - qu_3^* e^{\eta\mu_3^-\xi}\Gamma_3(\eta\mu_3^-; c) - (e^{\mu_3^-\xi} - qu_3^* e^{\eta\mu_3^-\xi})(e^{\mu_3^-\xi} - qu_3^* e^{\eta\mu_3^-\xi}) \\
\geq& - qu_3^* e^{\eta\mu_3^-\xi}\Gamma_3(\eta\mu_3^-; c) - (e^{\mu_3^-\xi} - qu_3^* e^{\eta\mu_3^-\xi})e^{\mu_3^-\xi} \geq 0,
\end{aligned} \tag{2.15}$$

if q is large enough. Hence, it follows from (2.14) and (2.15) that $\bar{U}(\xi)$ is a subsolution of (1.8) when q is large enough. The proof is complete. □

3. Existence of traveling wavefronts for (1.6) and (1.7)

Based on the supersolution and subsolution derived in previous section, we can apply the the monotone iteration method to obtain the following existence result.

Theorem 3.1. *Given any $c \geq c_1^*$, system* (1.8) *admits a strictly increasing traveling wave solution $\Phi(\xi) = (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi))$ satisfying* (1.9) *and with wave speed c.*

Proof. Let $c > c_1^*$ and $\hat{U}(\xi)$ and $\bar{U}(\xi)$ be the supersolution and subsolution constructed in Lemmas 2.2 and 2.3 respectively. Since (1.8) is a monotone system on $[\mathbf{E}_1, \mathbf{E}_2]$, by the monotone iteration method, system (1.8) admits a non-decreasing solution $\Phi(\xi) = (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi))$ satisfying

$$\bar{U}(\xi) \leq \Phi(\xi) = (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)) \leq \hat{U}(\xi), \text{ for all } \xi \in \mathbb{R}.$$

Since $\bar{U}(-\infty) = \hat{U}(-\infty) = \mathbf{E}_1$, it follows that $\Phi(-\infty) = \mathbf{E}_1$. Moreover, we have $\Phi(\infty) = \mathbf{E}_* = (\mathbf{E}_*^1, \mathbf{E}_*^2, \mathbf{E}_*^3)$ for some equilibrium $\mathbf{E}_* \leq \mathbf{E}_2$. By the non-decreasing property of $\Phi(\xi)$ and the fact $\bar{u}_1 \not\equiv 0$ and $\bar{u}_3 \not\equiv 0$, we see that $\mathbf{E}_*^1 > 0$ and $\mathbf{E}_*^2 > 0$, and hence $\mathbf{E}_* \in \{\mathbf{E}_2, \mathbf{E}_8\}$. Since $\mathbf{E}_8 \notin [\mathbf{0}, \mathbf{E}_2]$, we conclude that $\mathbf{E}_* = \mathbf{E}_2$. Hence $\Phi(\xi)$ satisfies the condition (1.9).

Next, we consider the case $c = c_1^*$. Let $\{\ell_n\}$ be a sequence with $\ell_n > c_1^*$ for all $n \in \mathbb{N}$, which converges to c_1^*. Denoting $\Phi_n(\xi)$ by the non-decreasing solution of (1.8) satisfying (1.9) with $c = \ell_n$. Then, by the limiting arguments (cf. [23]), $\{\Phi_n(\xi)\}$ has a convergent subsequence which converges to a function $\Phi_*(\xi)$ which satisfies (1.8) and (1.9) with $c = c_1^*$.

Finally, we show that $\Phi'(\xi) \gg \mathbf{0}$ for all $\xi \in \mathbb{R}$. We first claim that $\Phi(\xi) \gg \mathbf{0}$ for all $\xi \in \mathbb{R}$. Note that $\phi_1(+\infty) = u_*$. If there exists $\xi_1 \in \mathbb{R}$ such that $\phi_1(\xi_1) = 0$, we may assume that $\phi_1(\xi) > 0$ for all $\xi > \xi_1$. Since $\phi_1(\cdot) \geq 0$, we have $\phi_1'(\xi_1) = 0$ and hence it follows the first equation of (1.8) that $\phi_1(\xi_1 + 1) = 0$, which contradicts to the definition of ξ_1. Thus, $\phi_1(\xi) > 0$ for all $\xi \in \mathbb{R}$. Similarly, we can show that $\phi_3(\xi) > 0$ for all $\xi \in \mathbb{R}$. Suppose that there exists $\xi_2 \in \mathbb{R}$ such that $\phi_2(\xi_2) = 0$ and $\phi_2(\xi) > 0$ for all $\xi > \xi_2$. By the second equation of (1.8), we have

$$0 = \phi_2'(\xi_2) = d_2[\phi_2(\xi_2 + 1) + \phi_2(\xi_2 - 1) + r_2[c_{21}\phi_1(\xi_2) + c_{23}\phi_3(\xi_2)] \geq 0,$$

which implies that $\phi_2(\xi_2 + 1) = 0$. This contradiction shows that $\phi_2(\xi) > 0$ for all $\xi \in \mathbb{R}$. Hence the claim holds.

According to (1.8), we know that

$$\Phi(\xi) = e^{-\ell\xi} \int_{-\infty}^{\xi} e^{\ell s} H(\Phi(s)) ds, \tag{3.1}$$

where ℓ is a positive constant and

$$H(\Phi(\xi)) = (H_1(\Phi(\xi)), H_2(\Phi(\xi)), H_3(\Phi(\xi))) := \ell\Phi(\xi) + F(\Phi(\xi)).$$

Choosing ℓ large enough, we know that $H(\Psi)$ is monotone increasing for any $\Psi \in [\mathbf{E}_1, \mathbf{E}_2]$. Since $\Phi(\xi)$ is non-decreasing in ξ, by differentiating (3.1) with respect to ξ, we have

$$\Phi'(\xi) = -\ell e^{-\ell\xi} \int_{-\infty}^{\xi} e^{\ell s} [H(\Phi(s)) - H(\Phi(\xi))] ds \geq \mathbf{0}. \tag{3.2}$$

Suppose that $\phi_i'(\xi_i) = 0$ for some $\xi_i \in \mathbb{R}$ ($i = 1, 2$, or 3), then (3.2) implies that $H_i(\Phi(s)) = H_i(\Phi(\xi_i))$ for all $s \leq \xi_i$. Taking $s \to -\infty$, it follows that

$$\ell\phi_i(\xi_i) + \phi_i'(\xi_i) = H_i(\Phi(\xi_i)) = H_i(\Phi(-\infty)) = 0.$$

which implies that $\phi_i(\xi_i) = 0$. This contradiction implies that $\Phi'(\xi) > \mathbf{0}$, $\forall \xi \in \mathbb{R}$. The proof is complete. □

Next, we investigate the linear determinacy for the problem (1.8). The definition of linear determinacy was first introduced in [24], which means that the minimal speed is determined by the linearization of the problem at some unstable equilibrium. In the following theorem, we show that c_1^* is the minimal speed of system (1.8).

Theorem 3.2. *Assume* $c < c_1^*$. *System* (1.8) *has no strictly increasing solution* $\Phi(\xi) = (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)) \in [\mathbf{E}_1, \mathbf{E}_2]$ *satisfying the condition* (1.9).

Proof. Suppose that (1.8) admits a strictly increasing solution $\Phi(\xi) = (\phi_1(\xi), \phi_2(\xi), \phi_3(\xi)) \in [\mathbf{E}_1, \mathbf{E}_2]$ satisfying (1.9) with $c < c_1^*$. Then we define $\psi(\xi) := \phi_1'(\xi)/\phi_1(\xi)$. From (1.8), one can verify that $\psi(\xi)$ satisfies the equation

$$c\psi(\xi) = d_1[e^{\int_\xi^{\xi-1}\psi(s)ds} + e^{\int_\xi^{\xi+1}\psi(s)ds}] - 2d_1 + f_1(\Phi(\xi))/\phi_1(\xi). \tag{3.3}$$

Since $\Phi(-\infty) = \mathbf{0}$, we have

$$\lim_{\xi\to-\infty}[-2d_1 + f_1(\Phi(\xi))/\phi_1(\xi)] = -2d_1 + r_1 - c_{12}r_2.$$

According to (3.3) and [10, Proposition 3], the limit $\psi(-\infty)$ exists and satisfies

$$\Gamma_1(\psi(-\infty); c) = 0. \tag{3.4}$$

Then it follows from the proof of Lemma 2.1 that $c \geq c_1^*$, which gives a contradiction. This completes the proof. □

4. Stability of traveling wavefronts for (1.6) with large c

In this section, we will apply the weighted energy method to study the stability of traveling wavefronts for (1.6). Inspired by [14, 19], we first introduce the following definition.

Definition 4.1. *Let I be an interval and $\omega(x) : \mathbb{R} \to \mathbb{R}$ be a differentiable function.*

(1) *Let $L^2(I)$ be the space of square integrable functions defined on I. We denote $L^2_\omega(I)$ by the weighted L^2-space with the weight function $\omega(x)$, which endows with the norm*

$$\|f(x)\|_{L^2_\omega(I)} = (\int_I \omega(x) f^2(x)dx)^{\frac{1}{2}}.$$

(2) *Let $H^k(I)$ ($k \geq 0$) be the Sobolev space of the L^2-functions $f(x)$ defined on I whose ith-derivative also belong to $L^2(I)$ for $i = 1, \cdots, k$. We denote $H^k_\omega(I)$ by the weighted Sobolev space with the weight function $\omega(x)$, which endows with the norm*

$$\|f(x)\|_{H^k_\omega(I)} = (\sum_{i=0}^{k}\int_I \omega(x)\left|\frac{d^i f(x)}{dx^i}\right|^2 dx)^{\frac{1}{2}}.$$

(3) *Let $T > 0$ and $\mathbb{B}$ be a Banach space. We denote $C([0,T];\mathbb{B})$ by the space of the $\mathbb{B}$-valued continuous functions defined on $[0,T]$, and $L^2([0,T];\mathbb{B})$ is regarded as the space of $\mathbb{B}$-valued L^2-function on $[0,T]$. The corresponding spaces of the $\mathbb{B}$-valued function on $[0,\infty)$ can be defined similarly.*

Note that we always assume (H1) and (H2) throughout this article. Moreover, we assume the parameters satisfying the following assumption:

$$(\text{S1}) \quad \begin{aligned} &\ell_1 := -2r_1 + (4 - c_{12} - c_{13})u_* - (2c_{13} + c_{31})w_* > 0,\\ &\ell_2 := -4r_2 + (2c_{21} - c_{12})u_* + (2c_{23} - c_{32})w_* > 0,\\ &\ell_3 := -2r_3 + (4 - c_{31} - c_{32})w_* - (2c_{31} + c_{13})u_* > 0. \end{aligned}$$

First, we establish the following global existence and uniqueness of solutions, and the comparison theorem for system (1.6) with initial data

$$U_0(x) = (u(0,x), v(0,x), w(0,x)) := (u_0(x), v_0(x), w_0(x))$$

satisfying the following condition:

(S2) $(u_0(x), v_0(x), w_0(x)) \in [\mathbf{E}_1, \mathbf{E}_2]$, $\forall x \in \mathbb{R}$ and $U_0(x) - \Phi(x) \in H^1_\omega(\mathbb{R})$.

Here we assume that the weight function $\omega(\xi)$ in (S2) is given by

$$\omega(\xi) := \begin{cases} e^{-\sigma(\xi-\xi_0)}, & \xi \le \xi_0, \\ 1, & \xi > \xi_0, \end{cases} \tag{4.1}$$

for some positive constants σ and ξ_0 which will be determined later.

Lemma 4.1. (See also [19].) *Assume* (S1)–(S2). *Then the following statements are valid.*

(1) *There exists a unique solution* $U(t,x) = (u(t,x), v(t,x), w(t,x))$ *of* (1.6) *with initial data* $U_0(x)$ *such that* $\mathbf{E}_1 \le U(t,x) \le \mathbf{E}_2$, $\forall t > 0$, $x \in \mathbb{R}$. *In addition,*

$$U(t,x) - \Phi(x+ct) \in C([0,+\infty); H^1_\omega(\mathbb{R})) \cap L^2([0,+\infty); H^1_\omega(\mathbb{R})). \tag{4.2}$$

(2) *Let* $U^\pm(t,x)$ *be solutions of* (1.6) *with* $U^\pm(0,x) = (u^\pm(x), v^\pm(x), w^\pm(x))$*, respectively. If* $\mathbf{E}_1 \le U^-(0,x) \le U^+(0,x) \le \mathbf{E}_2$, $\forall x \in \mathbb{R}$, *then*

$$\mathbf{E}_1 \le U^-(t,x) \le U^+(t,x) \le \mathbf{E}_2, \ \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{4.3}$$

Proof. (1) The assertion can be derived by the theory of abstract functional differential equation, see [25]. Also the standard energy method and continuity extension method, see [26]. Here we skip the details.

(2) The proof of this part is the same as that of [27, Lemma 3.2] and omitted. □

Then, applying the technique of weighted energy estimate, we have the following stability result.

Theorem 4.1. *Assume that* (S1)–(S2) *hold. Let* $\Phi(x+ct)$ *be a traveling wave front of* (1.6) *satisfying* (1.9) *and with speed* $c > \max\{c_1^*, c_1, c_2, c_3\}$ (*Note that* c_i, $i = 1,2,3$ *are defined in* (4.23)–(4.25)). *Let* $U(t,x) = (u(t,x), v(t,x), w(t,x))$ *be the unique solution of the initial value problem* (1.6). *In addition, there exist small* $\sigma = \sigma_0 > 0$ *and large* $\xi_0 > 0$ *such that*

$$U(t,x) - \Phi(x+ct) \in C([0,+\infty); H^1_\omega(\mathbb{R})) \cap L^2([0,+\infty); H^1_\omega(\mathbb{R})) \tag{4.4}$$

and

$$\sup_{x\in\mathbb{R}} \|U(t,x) - \Phi(x+ct)\| \le Ce^{-\mu t}, \ \forall t > 0, \tag{4.5}$$

for some positive constants C *and* μ.

To prove the result of Theorem 4.1 by using the weighted energy method, we need to establish a priori estimate for the difference of solutions of systems (1.6) and (1.8). For convenience, we denote $U(t,x) = (u(t,x), v(t,x), w(t,x))$ by the solution of system (1.6) with initial data $U_0(x) = (u_0(x), v_0(x), w_0(x))$ satisfying (S2). Then, $\forall x \in \mathbb{R}$, we set

$$
\begin{aligned}
U_0^-(x) :=& (\min\{u_0(x), \phi_1(x)\}, \min\{v_0(x), \phi_2(x)\}, \min\{w_0(x), \phi_3(x)\}),\\
U_0^+(x) :=& (\max\{u_0(x), \phi_1(x)\}, \max\{v_0(x), \phi_2(x)\}, \max\{w_0(x), \phi_3(x)\}).
\end{aligned}
$$

It is clear that $U_0^\pm(x)$ satisfy

$$\mathbf{E}_1 \le U_0^-(x) \le U_0(x), \Phi(x) \le U_0^+(x) \le \mathbf{E}_2,\ \forall x \in \mathbb{R}. \tag{4.6}$$

Let $U^\pm(t,x)$ be solutions of (1.6) with initial data $U_0^\pm(x)$, by Lemma 4.1, we have

$$\mathbf{E}_1 \le U^-(t,x) \le U(t,x), \Phi(x+ct) \le U^+(t,x) \le \mathbf{E}_2,\ \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}. \tag{4.7}$$

Then it follows from (4.7) that

$$\|U(t,x) - \Phi(x+ct)\| \le \max\{\|U^+(t,x) - \Phi(x+ct)\|, \|U^-(t,x) - \Phi(x+ct)\|\},$$

for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$. Therefore, to derive a priori estimate of $U(t,x) - \Phi(x+ct)$, it suffices to estimate the functions $U^\pm(t,x) - \Phi(x+ct)$.

4.1. Weighted energy estimate

For convenience, let's denote

$$V^\pm(t,x) := U^\pm(t,x) - \Phi(x+ct) \quad \text{and} \quad V_0^\pm(x) := U^\pm(0,x) - \Phi(x),\ \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

Then it follows from (4.6) and (4.7) that

$$\mathbf{E}_1 \le V_0^\pm(x) \le \mathbf{E}_2 \quad \text{and} \quad \mathbf{E}_1 \le V^\pm(t,x) \le \mathbf{E}_2,\ \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}.$$

In the sequel, we only estimate $V^+(t,x)$, since $V^-(t,x)$ can also be discussed in the same way. For convenience, we drop the sign " + " for $V^+(t,x)$, $U^+(t,x)$ and set

$$
\begin{aligned}
V(t,\xi) &= (V_1(t,\xi), V_2(t,\xi), V_3(t,\xi)) = V^+(t,x) := U^+(t,x) - \Phi(\xi),\\
V_0(\xi) &= (V_1^0(\xi), V_2^0(\xi), V_3^0(\xi)) := V(0,\xi) = V_0^+(x),\ \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}.
\end{aligned}
$$

By systems (1.6) and (1.8), we can obtain

$$
\begin{aligned}
V_{1t} + cV_{1\xi} =& d_1 D[V_1] + [r_1 - c_{12} r_2 - 2\phi_1 + c_{12}(V_2 + \phi_2) + c_{13}(V_3 + \phi_3)]V_1 +\\
& c_{12}\phi_1 V_2 + c_{13}\phi_1 V_3 - V_1^2, \qquad (4.8)\\
V_{2t} + cV_{2\xi} =& d_2 D[V_2] + [-r_2 + 2\phi_2 - c_{21}(V_1 + \phi_1) - c_{23}(V_3 + \phi_3)]V_2 +\\
& c_{21}(r_2 - \phi_2)V_1 + c_{23}(r_2 - \phi_2)V_3 + V_2^2, \qquad (4.9)\\
V_{3t} + cV_{3\xi} =& d_3 D[V_3] + [r_3 - c_{32} r_2 - 2\phi_3 + c_{31}(V_1 + \phi_1) + c_{32}(V_2 + \phi_2)]V_3 +
\end{aligned}
$$

$$c_{31}\phi_3 V_1 + c_{32}\phi_3 V_2 - V_3^2. \tag{4.10}$$

It is easy to see that $V_i^0(\xi) \in H_\omega^1(\mathbb{R})$, then we have $V_i(t,\xi) \in C([0,+\infty), H_\omega^1(\mathbb{R}))$, for $i = 1,2,3$. To employing the technique of energy estimate to the equations (4.8), (4.9) and (4.10), it is necessary to assure that the solutions $V_i(t,\xi)$ have sufficient regularity. To this end, we mollify the initial condition setting

$$V_i^{0\varepsilon}(\xi) := (J_\varepsilon * V_i^0)(\xi) = \int_{\mathbb{R}} J_\varepsilon(\xi - s)V_i^0(s)ds \in H_\omega^2(\mathbb{R}),\ i = 1,2,3,$$

where $J_\varepsilon(\xi)$ is the usual mollifier. Let $V^\varepsilon(t,\xi)$ be the solutions of (4.8), (4.9) and (4.10) with this mollified initial condition $V^{0\varepsilon}(\xi) = (V_1^{0\varepsilon}(\xi), V_2^{0\varepsilon}(\xi), V_3^{0\varepsilon}(\xi))$. Then, we have

$$V_i^\varepsilon(t,\xi) \in C([0,+\infty), H_\omega^2(\mathbb{R})),\ i = 1,2,3.$$

Letting $\varepsilon \to 0$, it follows that $V^\varepsilon(t,\xi) \to V(t,\xi)$ uniformly for all $(t,\xi) \in \mathbb{R}^+ \times \mathbb{R}$. Therefore, without loss of generality, we may assume $V_i(t,\xi) \in C([0,+\infty), H_\omega^2(\mathbb{R}))$, for $i = 1,2,3$ in establishing the following energy estimates (cf. [14]).

First, let's multiply both sides of (4.8), (4.9) and (4.10) by $e^{2\mu t}\omega(\xi)V_i(\xi,t)$ with $i = 1,2,3$, respectively, where $\mu > 0$ will be determined later. Direct computations give

$$\begin{aligned}
&(\frac{1}{2}e^{2\mu t}\omega V_1^2)_t + (\frac{c}{2}e^{2\mu t}\omega V_1^2)_\xi - d_1 e^{2\mu t}\omega V_1[V_1(t,\xi+1) + V_1(t,\xi-1)] \\
=&e^{2\mu t}\omega V_1^2 Q_1(t,\xi) + e^{2\mu t}\omega V_1[c_{12}\phi_1 V_2 + c_{13}\phi_1 V_3 - V_1^2], \qquad (4.11)\\
&(\frac{1}{2}e^{2\mu t}\omega V_2^2)_t + (\frac{c}{2}e^{2\mu t}\omega V_2^2)_\xi - d_2 e^{2\mu t}\omega V_2[V_2(t,\xi+1) + V_2(t,\xi-1)] \\
=&e^{2\mu t}\omega V_2^2 Q_2(t,\xi) + e^{2\mu t}\omega V_2[c_{21}(r_2 - \phi_2)V_1 + c_{23}(r_2 - \phi_2)V_3 + V_2^2], \qquad (4.12)\\
&(\frac{1}{2}e^{2\mu t}\omega V_3^2)_t + (\frac{c}{2}e^{2\mu t}\omega V_3^2)_\xi - d_3 e^{2\mu t}\omega V_3[V_3(t,\xi+1) + V_3(t,\xi-1)] \\
=&e^{2\mu t}\omega V_3^2 Q_3(t,\xi) + e^{2\mu t}\omega V_3[c_{31}\phi_3 V_1 + c_{32}\phi_3 V_2 - V_3^2], \qquad (4.13)
\end{aligned}$$

where

$$\begin{aligned}
Q_1(t,\xi) :=&\mu - 2d_1 + \frac{c}{2}\frac{\omega_\xi}{\omega} + [r_1 - 2\phi_1 + c_{12}(V_2 + \phi_2 - r_2) + c_{13}(V_3 + \phi_3)],\\
Q_2(t,\xi) :=&\mu - 2d_2 + \frac{c}{2}\frac{\omega_\xi}{\omega} + [-r_2 + 2\phi_2 - c_{21}(V_1 + \phi_1) - c_{23}(V_3 + \phi_3)],\\
Q_3(t,\xi) :=&\mu - 2d_3 + \frac{c}{2}\frac{\omega_\xi}{\omega} + [r_3 - 2\phi_3 + c_{31}(V_1 + \phi_1) + c_{32}(V_2 + \phi_2 - r_2)].
\end{aligned}$$

Applying the Cauchy–Schwarz inequality $2xy \le x^2 + y^2$, we can obtain

$$\begin{aligned}
2\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega V_i V_i(\xi \pm 1, s)d\xi ds \le& \int_0^t e^{2\mu s}\int_{\mathbb{R}} \omega(V_i^2 + V_i^2(\xi \pm 1, s))d\xi ds\\
=& \int_0^t e^{2\mu s}[\int_{\mathbb{R}} \omega V_i^2 d\xi + \int_{\mathbb{R}} \frac{\omega(\xi \mp 1)}{\omega}\omega V_i^2 d\xi]ds, \qquad (4.14)
\end{aligned}$$

$$2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega V_iV_j d\xi ds \le \int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(V_i^2+V_j^2)d\xi ds,\ i,j=1,2,3. \tag{4.15}$$

Since $V_i(t,\xi)\in H^1_\omega$, we have $\left\{e^{2\mu t}\omega V_i^2\right\}\Big|_{\xi=-\infty}^{\xi=\infty}=0$, for $i=1,2,3$. Therefore, integrating both sides of (4.11), (4.12) and (4.13) over $\mathbb{R}\times[0,t]$ with respect to ξ and t and using (4.14), we can obtain

$$\begin{aligned} e^{2\mu t}\|V_1(t,\xi)\|^2_{L^2_\omega} \le & \|V_1(0,\xi)\|^2_{L^2_\omega} + d_1\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega[2+\frac{\omega(\xi+1)}{\omega}+\frac{\omega(\xi-1)}{\omega}]V_1^2 d\xi ds+ \\ & 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega Q_1(s,\xi)V_1^2 d\xi ds+\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega c_{12}\phi_1(V_1^2+V_2^2)d\xi ds+ \\ & \int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega c_{13}\phi_1(V_1^2+V_3^2)d\xi ds, \end{aligned} \tag{4.16}$$

$$\begin{aligned} e^{2\mu t}\|V_2(t,\xi)\|^2_{L^2_\omega} \le & \|V_2(0,\xi)\|^2_{L^2_\omega} + d_2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega[2+\frac{\omega(\xi+1)}{\omega}+\frac{\omega(\xi-1)}{\omega}]V_2^2 d\xi ds+ \\ & 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega Q_2(s,\xi)V_2^2 d\xi ds+\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega c_{21}(r_2-\phi_2)(V_1^2+V_2^2)d\xi ds+ \\ & \int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega c_{23}(r_2-\phi_2)(V_2^2+V_3^2)d\xi ds+2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega V_2^3 d\xi ds, \end{aligned} \tag{4.17}$$

$$\begin{aligned} e^{2\mu t}\|V_3(t,\xi)\|^2_{L^2_\omega} \le & \|V_3(0,\xi)\|^2_{L^2_\omega} + d_3\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega[2+\frac{\omega(\xi+1)}{\omega}+\frac{\omega(\xi-1)}{\omega}]V_3^2 d\xi ds+ \\ & 2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega Q_3(s,\xi)V_3^2 d\xi ds+\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega c_{31}\phi_3(V_1^2+V_3^2)d\xi ds+ \\ & \int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega c_{32}\phi_3(V_2^2+V_3^2)d\xi ds. \end{aligned} \tag{4.18}$$

Noting that $r_2-\phi_2>0$. Summing up the inequalities (4.16)–(4.18), we can derive

$$\sum_{i=1}^3 e^{2\mu t}\|V_i(t,\xi)\|^2_{L^2_\omega}+\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega\sum_{i=1}^3 R_i^\mu(s,\xi)V_i^2 d\xi ds \le \sum_{i=1}^3\|V_i(0,\xi)\|^2_{L^2_\omega}, \tag{4.19}$$

where

$$\begin{aligned} R_1^\mu(t,\xi) &:= -d_1[2+\frac{\omega(\xi+1)}{\omega(\xi)}+\frac{\omega(\xi-1)}{\omega(\xi)}]-2Q_1-(c_{12}+c_{13})\phi_1-c_{21}(r_2-\phi_2)-c_{31}\phi_3, \\ R_2^\mu(t,\xi) &:= -d_2[2+\frac{\omega(\xi+1)}{\omega(\xi)}+\frac{\omega(\xi-1)}{\omega(\xi)}]-2Q_2-c_{12}\phi_1-(c_{21}+c_{23})(r_2-\phi_2)-c_{32}\phi_3-2V_2, \\ R_3^\mu(t,\xi) &:= -d_3[2+\frac{\omega(\xi+1)}{\omega(\xi)}+\frac{\omega(\xi-1)}{\omega(\xi)}]-2Q_3-(c_{31}+c_{32})\phi_3-c_{23}(r_2-\phi_2)-c_{13}\phi_1. \end{aligned}$$

For convenience to estimate $R_1^\mu(t,\xi)$, we further set

$$\Lambda_1(\xi) := -2r_1+4\phi_1(\xi)-2c_{13}w_*-(c_{12}+c_{13})u_*-c_{21}(r_2-\phi_2(\xi))-c_{31}w_*,$$

$$\Lambda_2(\xi) := -4r_2 + (2c_{21} - c_{12})u_* + (2c_{23} - c_{32})w_* - (c_{21} + c_{23})(r_2 - \phi_2(\xi)),$$
$$\Lambda_3(\xi) := -2r_3 + 4\phi_3(\xi) - 2c_{31}u_* - (c_{31} + c_{32})w_* - c_{23}(r_2 - \phi_2(\xi)) - c_{13}u_*,$$
$$D_i(\sigma) := d_i[-2 + e^{\sigma} + e^{-\sigma}], \text{ for } i = 1, 2, 3.$$

Then we have the following properties.

Lemma 4.2. *Assume that* (S1) *holds. There exist small* $\sigma_0 > 0$ *and large* $\xi_0 > 0$ *such that, for* $i = 1, 2, 3$*,*

$$\Lambda_i(\xi) > 0 \textit{ and } d_i[e^{-\sigma_0} - 1] - D_i(\sigma_0) + \Lambda_i(\xi) > 0, \textit{ for all } \xi \geq \xi_0.$$

Proof. By (S1) and the fact

$$\lim_{\sigma\to 0}[D_i(\sigma) - d_i(e^{-\sigma} - 1)] = 0, \text{ for } i = 1, 2, 3,$$

there exists a small $\sigma_0 > 0$ such that

$$\ell_i > D_i(\sigma_0) - d_i(e^{-\sigma_0} - 1), \text{ for } i = 1, 2, 3. \tag{4.20}$$

Fixing this σ_0, then it follows from (1.9) and (4.20) that $\lim_{\xi\to\infty} \Lambda_i(\xi) = \ell_i > 0$ and

$$\lim_{\xi\to\infty}(d_i[e^{-\sigma_0} - 1] - D_i(\sigma_0) + \Lambda_i(\xi)) = d_i[e^{-\sigma_0} - 1] - D_i(\sigma_0) + \ell_i > 0, \text{ for } i = 1, 2, 3.$$

Hence, this assertion holds by the continuity argument. □

Let's choose $\omega(\xi)$ as the form (4.1), where $\sigma = \sigma_0$ and ξ_0 are the positive constants derived in Lemma 4.2. It's easy to see that

$$\frac{\omega'(\xi)}{\omega(\xi)} = \begin{cases} -\sigma_0, & \text{if } \xi < \xi_0, \\ 0, & \text{if } \xi > \xi_0, \end{cases} \qquad \frac{\omega(\xi+1)}{\omega(\xi)} = \begin{cases} e^{-\sigma_0}, & \text{if } \xi < \xi_0 - 1, \\ e^{\sigma_0(\xi-\xi_0)}, & \text{if } \xi_0 - 1 < \xi \leq \xi_0, \\ 1, & \text{if } \xi_0 < \xi, \end{cases} \tag{4.21}$$

$$\frac{\omega(\xi-1)}{\omega(\xi)} = \begin{cases} e^{\sigma_0}, & \text{if } \xi \leq \xi_0, \\ e^{-\sigma_0(\xi-1-\xi_0)}, & \text{if } \xi_0 \leq \xi < \xi_0 + 1, \\ 1, & \text{if } \xi_0 + 1 \leq \xi. \end{cases} \tag{4.22}$$

Furthermore, let's fix three wave speeds $c_i > 0$ such that

$$c_1\sigma_0 := D_1(\sigma_0) + d_1 + 2r_1 + 2c_{13}w_* + (c_{12} + c_{13})u_* + r_2c_{21} + c_{31}w_*, \tag{4.23}$$
$$c_2\sigma_0 := D_2(\sigma_0) + d_2 + 4r_2 + c_{12}u_* + r_2(c_{21} + c_{23}) + c_{32}w_*, \tag{4.24}$$
$$c_3\sigma_0 := D_3(\sigma_0) + d_3 + 2r_3 + 2c_{31}u_* + (c_{31} + c_{32})w_* + r_2c_{23} + c_{13}u_*. \tag{4.25}$$

Then we estimate $R_i^{\mu}(t, \xi)$, $i = 1, 2, 3$ in the following lemma.

Lemma 4.3. *Assume that* (S1)–(S2) *hold and* $c > \max\{c_1^*, c_1, c_2, c_3\}$. *Then there exists a small* $\mu > 0$ *such that the following statements hold:*

(1) *There exists a positive constant* C_0 *such that*

$$R_i^{\mu}(t, \xi) \geq C_0, \ \forall (t, \xi) \in \mathbb{R}^+ \times \mathbb{R}, i = 1, 2, 3. \tag{4.26}$$

(2) *There exists a positive constant C_1 such that*

$$\sum_{i=1}^{3}\|V_i(\cdot,t)\|^2_{L^2_\omega}+\int_0^t e^{-2\mu(t-s)}\sum_{i=1}^{3}\|V_i(\cdot,s)\|^2_{L^2_\omega}ds\le C_1e^{-2\mu t}\sum_{i=1}^{3}\|V_i(\cdot,0)\|^2_{L^2_\omega}. \tag{4.27}$$

Proof. (1) Noting that $(0,0,0)<(V_1+\phi_1,V_2+\phi_2,V_3+\phi_3)<(u_*,r_2,w_*)$. Let's prove the assertion by considering the following four cases.

Case 1: $\xi<\xi_0-1$. By Lemma 4.2 and (4.21)–(4.25), we have

$$\begin{aligned}
R_1^0(t,\xi)=&-D_1(\sigma_0)+c\sigma_0-2[r_1-2\phi_1+c_{12}(V_2+\phi_2-r_2)+c_{13}(V_3+\phi_3)]\\
&-(c_{12}+c_{13})\phi_1-c_{21}(r_2-\phi_2)-c_{31}\phi_3\\
>&c_1\sigma_0-D_1(\sigma_0)-d_1-2r_1-2c_{13}w_*-(c_{12}+c_{13})u_*-r_2c_{21}-c_{31}w_*=0,\\
R_2^0(t,\xi)=&-D_2(\sigma_0)+c\sigma_0-2[-r_2+2\phi_2-c_{21}(V_1+\phi_1)-c_{23}(V_3+\phi_3)]\\
&-c_{12}\phi_1-(c_{21}+c_{23})(r_2-\phi_2)-c_{32}\phi_3-2V_2,\\
>&c_2\sigma_0-D_2(\sigma_0)-d_2-4r_2-c_{12}u_*-r_2(c_{21}+c_{23})-c_{32}w_*=0,\\
R_3^0(t,\xi)>&c_3\sigma_0-D_3(\sigma_0)-d_3-2r_3-2c_{31}u_*-(c_{31}+c_{32})w_*-r_2c_{23}-c_{13}u_*=0.
\end{aligned}$$

Case 2: $\xi_0-1<\xi\le\xi_0$. In this case, $d_ie^{-\sigma_0}+d_i(1-e^{\sigma_0(\xi-\xi_0)})>0$, for $i=1,2,3$. By Lemma 4.2 and (4.21)–(4.25), we have

$$\begin{aligned}
R_1^0(t,\xi)=&-d_1[-2+e^{\sigma_0(\xi-\xi_0)}+e^{\sigma_0}]+c\sigma_0-2[r_1-2\phi_1+c_{12}(V_2+\phi_2-r_2)+c_{13}(V_3+\phi_3)]\\
&-(c_{12}+c_{13})\phi_1-c_{21}(r_2-\phi_2)-c_{31}\phi_3\\
>&d_1e^{-\sigma_0}+d_1(1-e^{\sigma_0(\xi-\xi_0)})+\\
&c_1\sigma_0-D_1(\sigma_0)-d_1-2r_1-2c_{13}w_*-(c_{12}+c_{13})u_*-r_2c_{21}-c_{31}w_*>0,\\
R_2^0(t,\xi)=&-d_2[-2+e^{\sigma_0(\xi-\xi_0)}+e^{\sigma_0}]+c\sigma_0-2[-r_2+2\phi_2-c_{21}(V_1+\phi_1)-c_{23}(V_3+\phi_3)]\\
&-c_{12}\phi_1-(c_{21}+c_{23})(r_2-\phi_2)-c_{32}\phi_3-2V_2,\\
>&d_2e^{-\sigma_0}+d_2(1-e^{\sigma_0(\xi-\xi_0)})+\\
&c_2\sigma_0-D_2(\sigma_0)-d_2-4r_2-c_{12}u_*-r_2(c_{21}+c_{23})-c_{32}w_*>0,\\
R_3^0(t,\xi)=&-d_3[-2+e^{\sigma_0(\xi-\xi_0)}+e^{\sigma_0}]+c\sigma_0-2[r_3-2\phi_3+c_{31}(V_1+\phi_1)+c_{32}(V_2+\phi_2-r_2)]\\
&-(c_{31}+c_{32})\phi_3-c_{23}(r_2-\phi_2)-c_{13}\phi_1\\
>&d_3e^{-\sigma_0}+d_3(1-e^{\sigma_0(\xi-\xi_0)})+\\
&c_3\sigma_0-D_3(\sigma_0)-d_3-2r_3-2c_{31}u_*-(c_{31}+c_{32})w_*-r_2c_{23}-c_{13}u_*>0.
\end{aligned}$$

Case 3: $\xi_0<\xi\le\xi_0+1$. In this case, one can see that $d_1[e^{\sigma_0}-e^{-\sigma_0(\xi-\xi_0-1)}]\le 0$. By Lemma 4.2, (4.21) and (4.22), we have

$$\begin{aligned}
R_1^0(t,\xi)=&-d_1[-1+e^{-\sigma_0(\xi-\xi_0-1)}]-2[r_1-2\phi_1+c_{12}(V_2+\phi_2-r_2)+c_{13}(V_3+\phi_3)]\\
&-(c_{12}+c_{13})\phi_1-c_{21}(r_2-\phi_2)-c_{31}\phi_3\\
\ge&d_1[e^{-\sigma_0}-1]-D_1(\sigma_0)+\Lambda_1(\xi)>0,\\
R_2^0(t,\xi)=&-d_2[-1+e^{-\sigma_0(\xi-\xi_0-1)}]-2[-r_2+2\phi_2-c_{21}(V_1+\phi_1)-c_{23}(V_3+\phi_3)]
\end{aligned}$$

$$
\begin{aligned}
&- c_{12}\phi_1 - (c_{21} + c_{23})(r_2 - \phi_2) - c_{32}\phi_3 - 2V_2, \\
&\geq d_2[e^{-\sigma_0} - 1] - D_2(\sigma_0) + \Lambda_2(\xi) > 0, \\
R_3^0(t,\xi) =& - d_3[-1 + e^{-\sigma_0(\xi-\xi_0-1)}] - 2[r_3 - 2\phi_3 + c_{31}(V_1 + \phi_1) + c_{32}(V_2 + \phi_2 - r_2)] \\
&- (c_{31} + c_{32})\phi_3 - c_{23}(r_2 - \phi_2) - c_{13}\phi_1 \\
&\geq d_3[e^{-\sigma_0} - 1] - D_3(\sigma_0) + \Lambda_3(\xi) > 0.
\end{aligned}
$$

Case 4: $\xi > \xi_0 + 1$. In this case, by Lemma 4.2, (4.21) and (4.22), we have

$$
\begin{aligned}
R_1^0(t,\xi) =& - 2[r_1 - 2\phi_1 + c_{12}(V_2 + \phi_2 - r_2) + c_{13}(V_3 + \phi_3)] \\
&- (c_{12} + c_{13})\phi_1 - c_{21}(r_2 - \phi_2) - c_{31}\phi_3 \\
&\geq \Lambda_1(\xi) > 0, \\
R_2^0(t,\xi) =& - 2[-r_2 + 2\phi_2 - c_{21}(V_1 + \phi_1) - c_{23}(V_3 + \phi_3)] \\
&- c_{12}\phi_1 - (c_{21} + c_{23})(r_2 - \phi_2) - c_{32}\phi_3 - 2V_2, \\
&\geq \Lambda_2(\xi) > 0, \\
R_3^0(t,\xi) =& - 2[r_3 - 2\phi_3 + c_{31}(V_1 + \phi_1) + c_{32}(V_2 + \phi_2 - r_2)] \\
&- (c_{31} + c_{32})\phi_3 - c_{23}(r_2 - \phi_2) - c_{13}\phi_1 \\
&\geq \Lambda_3(\xi) > 0.
\end{aligned}
$$

According to the above four cases, we may choose a small $\mu > 0$ such that (4.26) holds for some positive constant C_0.

(2) The inequality (4.27) is a direct consequence of (4.19) and (4.26). □

4.2. Derivative Estimates

Now we consider the derivative estimates of system (4.8). By differentiating (4.8), (4.9) and (4.10) with respect to ξ, it follows that

$$
\begin{aligned}
V_{1t\xi} + cV_{1\xi\xi} =& d_1 D[V_{1\xi}] + [r_1 - c_{12}r_2 - 2\phi_1 + c_{12}(V_2 + \phi_2) + c_{13}(V_3 + \phi_3)]V_{1\xi} + \\
&[-2\phi_{1\xi} + c_{12}(V_{2\xi} + \phi_{2\xi}) + c_{13}(V_{3\xi} + \phi_{3\xi})]V_1 + \\
&c_{12}[\phi_{1\xi}V_2 + \phi_1 V_{2\xi}] + c_{13}[\phi_{1\xi}V_3 + \phi_1 V_{3\xi}] - 2V_1 V_{1\xi}, \quad (4.28) \\
V_{2t\xi} + cV_{2\xi\xi} =& d_2 D[V_{2\xi}] + [-r_2 + 2\phi_2 - c_{21}(V_1 + \phi_1) - c_{23}(V_3 + \phi_3)]V_{2\xi} + \\
&[2\phi_{2\xi} - c_{21}(V_{1\xi} + \phi_{1\xi}) - c_{23}(V_{3\xi} + \phi_{3\xi})]V_2 + \\
&c_{21}(r_2 V_{1\xi} - \phi_{2\xi}V_1 - \phi_2 V_{1\xi}) + c_{23}(r_2 V_{3\xi} - \phi_{2\xi}V_3 - \phi_2 V_{3\xi}) + 2V_2 V_{2\xi}, \quad (4.29) \\
V_{3t\xi} + cV_{3\xi\xi} =& d_3 D[V_{3\xi}] + [r_3 - c_{32}r_2 - 2\phi_3 + c_{31}(V_1 + \phi_1) + c_{32}(V_2 + \phi_2)]V_{3\xi} + \\
&[-2\phi_{3\xi} + c_{31}(V_{1\xi} + \phi_{1\xi}) + c_{32}(V_{2\xi} + \phi_{2\xi})]V_3 + \\
&c_{31}(\phi_{3\xi}V_1 + \phi_3 V_{1\xi}) + c_{32}(\phi_{3\xi}V_2 + \phi_3 V_{2\xi}) - 2V_3 V_{3\xi}. \quad (4.30)
\end{aligned}
$$

Multiplying (4.28)–(4.30) by $e^{2\mu t}\omega(\xi)V_{i\xi}(t,\xi)$ with $i = 1, 2, 3$, respectively, we can obtain

$$
(\frac{1}{2}e^{2\mu t}\omega V_{1\xi}^2)_t + (\frac{c}{2}e^{2\mu t}\omega V_{1\xi}^2)_\xi - d_1 e^{2\mu t}\omega V_{1\xi}[V_{1\xi}(t,\xi+1) + V_{1\xi}(t,\xi-1)]
$$

$$
\begin{aligned}
=&e^{2\mu t}\omega Q_1(t,\xi)V_{1\xi}^2 + e^{2\mu t}\omega[-2\phi_{1\xi} + c_{12}(V_{2\xi} + \phi_{2\xi}) + c_{13}(V_{3\xi} + \phi_{3\xi})]V_1 V_{1\xi} + \\
&e^{2\mu t}\omega[-2V_1 V_{1\xi} + c_{12}(\phi_{1\xi}V_2 + \phi_1 V_{2\xi}) + c_{13}(\phi_{1\xi}V_3 + \phi_1 V_{3\xi})]V_{1\xi}, \qquad (4.31)\\
&(\frac{1}{2}e^{2\mu t}\omega V_{2\xi}^2)_t + (\frac{c}{2}e^{2\mu t}\omega V_{2\xi}^2)_\xi - d_2 e^{2\mu t}\omega V_2[V_{2\xi}(t,\xi+1) + V_{2\xi}(t,\xi-1)]\\
=&e^{2\mu t}\omega Q_2(t,\xi)V_{2\xi}^2 + e^{2\mu t}\omega[2\phi_{2\xi} - c_{21}(V_{1\xi} + \phi_{1\xi}) - c_{23}(V_{3\xi} + \phi_{3\xi})]V_2 V_{2\xi} + \\
&e^{2\mu t}\omega[2V_2 V_{2\xi} + c_{21}(r_2 V_{1\xi} - \phi_{2\xi}V_1 - \phi_2 V_{1\xi}) + c_{23}(r_2 V_{3\xi} - \phi_{2\xi}V_3 - \phi_2 V_{3\xi})]V_{2\xi}, \qquad (4.32)
\end{aligned}
$$

$$
\begin{aligned}
&(\frac{1}{2}e^{2\mu t}\omega V_{3\xi}^2)_t + (\frac{c}{2}e^{2\mu t}\omega V_{3\xi}^2)_\xi - d_2 e^{2\mu t}\omega V_3[V_{3\xi}(t,\xi+1) + V_{3\xi}(t,\xi-1)]\\
=&e^{2\mu t}\omega Q_3(t,\xi)V_{3\xi}^2 + e^{2\mu t}\omega[-2\phi_{3\xi} + c_{31}(V_{1\xi} + \phi_{1\xi}) + c_{32}(V_{2\xi} + \phi_{2\xi})]V_3 V_{3\xi} + \\
&e^{2\mu t}\omega[-2V_3 V_{3\xi} + c_{31}(\phi_{3\xi}V_1 + \phi_3 V_{1\xi}) + c_{32}(\phi_{3\xi}V_2 + \phi_3 V_{2\xi})]V_{3\xi}. \qquad (4.33)
\end{aligned}
$$

Then, applying the Cauchy–Schwarz inequality, it follows that

$$
\begin{aligned}
2\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega V_{i\xi}V_{i\xi}(s,\xi\pm 1)d\xi ds \le& \int_0^t e^{2\mu s}\int_{\mathbb{R}} \omega(V_{i\xi}^2 + V_{i\xi}^2(s,\xi\pm 1))d\xi ds\\
=& \int_0^t e^{2\mu s}[\int_{\mathbb{R}} \omega V_{i\xi}^2 d\xi + \int_{\mathbb{R}} \frac{\omega(\xi\mp 1)}{\omega}\omega V_{i\xi}^2 d\xi]ds, \qquad (4.34)\\
2\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega V_{i\xi}V_{j\xi} d\xi ds \le& \int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega(V_{i\xi}^2 + V_{j\xi}^2)d\xi ds,\ i,j = 1,2,3. \qquad (4.35)
\end{aligned}
$$

Since $V_i \in H_\omega^2$, we know that $\{e^{2\mu t}\omega V_{i\xi}^2\}\big|_{\xi=-\infty}^{\xi=\infty} = 0$, for $i = 1,2,3$. Therefore, by (4.34), (4.35) and integrating both sides of (4.31)–(4.33) over $[0,t]\times\mathbb{R}$ with respect to t and ξ, we have

$$
\begin{aligned}
e^{2\mu t}\|V_{1\xi}(t,\xi)\|_{L_\omega^2}^2 \le& \|V_{1\xi}(0,\xi)\|_{L_\omega^2}^2 + d_1\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega[2 + \frac{\omega(\xi+1)}{\omega} + \frac{\omega(\xi-1)}{\omega}]V_{1\xi}^2 d\xi ds + \\
&2\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega Q_1(s,\xi)V_{1\xi}^2 d\xi ds + \int_0^t \int_{\mathbb{R}} e^{2\mu t}\omega(c_{12}+c_{13})(V_1+\phi_1)V_{1\xi}^2 d\xi ds + \\
&\int_0^t \int_{\mathbb{R}} e^{2\mu t}\omega c_{12}(V_1+\phi_1)V_{2\xi}^2 d\xi ds + \int_0^t \int_{\mathbb{R}} e^{2\mu t}\omega c_{13}(V_1+\phi_1)V_{3\xi}^2 d\xi ds + \\
&2\int_0^t \int_{\mathbb{R}} e^{2\mu t}\omega[-2\phi_{1\xi} + c_{12}\phi_{2\xi} + c_{13}\phi_{3\xi}]V_1 V_{1\xi} ds + \\
&2\int_0^t \int_{\mathbb{R}} e^{2\mu t}\omega[c_{12}\phi_{1\xi}V_2 + c_{13}\phi_{1\xi}V_3]V_{1\xi} ds, \qquad (4.36)\\
e^{2\mu t}\|V_{2\xi}(t,\xi)\|_{L_\omega^2}^2 \le& \|V_{2\xi}(0,\xi)\|_{L_\omega^2}^2 + d_2\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega[2 + \frac{\omega(\xi+1)}{\omega} + \frac{\omega(\xi-1)}{\omega}]V_{2\xi}^2 d\xi ds + \\
&2\int_0^t \int_{\mathbb{R}} e^{2\mu s}\omega Q_2(s,\xi)V_{2\xi}^2 d\xi ds + \int_0^t \int_{\mathbb{R}} e^{2\mu t}\omega c_{21}(r_2 - \phi_2 - V_2)V_{1\xi}^2 ds + \\
&\int_0^t \int_{\mathbb{R}} e^{2\mu t}\omega(4V_2 + c_{21}(r_2 - \phi_2 - V_2) + c_{23}(r_2 - \phi_2 - V_2))V_{2\xi}^2 ds + \qquad (4.37)\\
&\int_0^t \int_{\mathbb{R}} e^{2\mu t}\omega c_{23}(r_2 - \phi_2 - V_2)V_{3\xi}^2 ds +
\end{aligned}
$$

$$2\int_0^t\int_{\mathbb{R}} e^{2\mu t}\omega[2\phi_{2\xi}-c_{21}\phi_{1\xi}-c_{23}\phi_{3\xi}]V_2V_{2\xi}ds+$$
$$2\int_0^t\int_{\mathbb{R}} e^{2\mu t}\omega[-c_{21}\phi_{2\xi}V_1-c_{23}\phi_{2\xi}V_3]V_{2\xi}ds, \tag{4.38}$$

$$\begin{aligned}e^{2\mu t}\|V_{3\xi}(t,\xi)\|^2_{L^2_\omega}\leq&\|V_{3\xi}(0,\xi)\|^2_{L^2_\omega}+d_3\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega[2+\frac{\omega(\xi+1)}{\omega}+\frac{\omega(\xi-1)}{\omega}]V^2_{3\xi}d\xi ds+\\&2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega Q_3(s,\xi)V^2_{3\xi}d\xi ds+\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega c_{31}(V_3+\phi_3)V^2_{1\xi}ds+\\&\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega c_{32}(V_3+\phi_3)V^2_{2\xi}ds+\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(c_{31}+c_{32})(V_3+\phi_3)V^2_{3\xi}ds+\\&2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega[-2\phi_{3\xi}+c_{31}\phi_{1\xi}+c_{32}\phi_{2\xi})]V_3V_{3\xi}ds+\\&2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega[c_{31}\phi_{3\xi}V_1+c_{32}\phi_{3\xi}V_2]V_{3\xi}ds.\end{aligned} \tag{4.39}$$

Summing up the inequalities (4.36)–(4.39), we can derive

$$\begin{aligned}&\sum_{i=1}^3 e^{2\mu t}\|V_{i\xi}(t,\xi)\|^2_{L^2_\omega}+\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)\sum_{i=1}^3\widehat{R}^\mu_i(s,\xi)V^2_{i\xi}d\xi ds\\\leq&\sum_{i=1}^3\|V_{i\xi}(0,\xi)\|^2_{L^2_\omega}+2\int_0^t\int_{\mathbb{R}} e^{2\mu s}\omega(\xi)H(s,\xi)d\xi ds,\end{aligned} \tag{4.40}$$

where

$$\begin{aligned}\widehat{R}^\mu_1(t,\xi):=&-d_1[2+\frac{\omega(\xi+1)}{\omega(\xi)}+\frac{\omega(\xi-1)}{\omega(\xi)}]-2Q_1\\&-(c_{12}+c_{13})(V_1+\phi_1)-c_{21}(r_2-\phi_2-V_2)-c_{31}(V_3+\phi_3),\\\widehat{R}^\mu_2(t,\xi):=&-d_2[2+\frac{\omega(\xi+1)}{\omega(\xi)}+\frac{\omega(\xi-1)}{\omega(\xi)}]-2Q_2\\&-c_{12}(V_1+\phi_1)-4V_2-(c_{21}+c_{23})(r_2-\phi_2-V_2)-c_{32}(V_3+\phi_3),\\\widehat{R}^\mu_3(t,\xi):=&-d_3[2+\frac{\omega(\xi+1)}{\omega(\xi)}+\frac{\omega(\xi-1)}{\omega(\xi)}]-2Q_3\\&-c_{13}(V_1+\phi_1)-c_{23}(r_2-\phi_2-V_2)-(c_{31}+c_{32})(V_3+\phi_3),\\H(t,\xi):=&[c_{12}\phi_{1\xi}V_2+c_{13}\phi_{1\xi}V_3]V_{1\xi}-[c_{21}\phi_{2\xi}V_1+c_{23}\phi_{2\xi}V_3]V_{2\xi}+[c_{31}\phi_{3\xi}V_1+c_{32}\phi_{3\xi}V_2]V_{3\xi}+\\&[-2\phi_{1\xi}+c_{12}\phi_{2\xi}+c_{13}\phi_{3\xi}]V_1V_{1\xi}+[2\phi_{2\xi}-c_{21}\phi_{1\xi}-c_{23}\phi_{3\xi}]V_2V_{2\xi}+\\&[-2\phi_{3\xi}+c_{31}\phi_{1\xi}+c_{32}\phi_{2\xi})]V_3V_{3\xi}.\end{aligned}$$

Similar to the discussion of Lemma 4.3, we have the following lemma.

Lemma 4.4. *Assume* (S1)–(S2) *and* $c>\max\{c_1^*,c_1,c_2,c_3\}$. *There exists a small* $\mu>0$ *such that the following statements hold:*

(1) *There exists a positive constant $\widehat{C}_0$ such that*

$$\widehat{R}_i^{\mu}(t,\xi) \geq \widehat{C}_0,\ \forall (t,\xi) \in \mathbb{R}^+ \times \mathbb{R},\ i = 1,2,3. \tag{4.41}$$

(2) *There exists a positive constant $\widehat{C}_1$ such that*

$$\sum_{i=1}^{3} \|V_{i\xi}(t,\cdot)\|^2_{L^2_\omega} + \int_0^t e^{-2\mu(t-s)} \sum_{i=1}^{3} \|V_{i\xi}(s,\cdot)\|^2_{L^2_\omega} ds \leq \widehat{C}_1 e^{-2\mu t} \sum_{i=1}^{3} \|V_{i\xi}(0,\cdot)\|^2_{L^2_\omega}. \tag{4.42}$$

Proof. (1) Using the same definitions of $\Lambda_i(\xi)$ and c_j ($i = 1,\cdots,6,\ j = 1,2,3$), the proof of this assertion is similar to that of part (1) in Lemma 4.3 and omitted.

(2) According to (4.40), we first consider the following integral:

$$2\int_0^t \int_{\mathbb{R}} e^{2\mu s} \omega H(s,\xi) d\xi ds. \tag{4.43}$$

Based on the properties of the traveling wavefront $(\phi_1(\xi), \phi_2(\xi), \phi_3(\xi))$, we can know that $(\phi_1'(\xi), \phi_2'(\xi), \phi_3'(\xi))$ is bounded for all $\xi \in \mathbb{R}$. Thus, by the Young-inequality $2xy \leq \varepsilon^{-1}x^2 + \varepsilon y^2$ with $\varepsilon > 0$, we have

$$\begin{aligned} |H(s,\xi)| \leq& C_2(V_1 + V_2 + V_3)(|V_{1\xi}| + |V_{2\xi}| + |V_{3\xi}|) \\ \leq& \bar{C}_2[\varepsilon^{-1} \sum_{i=1}^{3} V_i^2(s,\xi) + \varepsilon \sum_{i=1}^{3} V_{i\xi}^2(s,\xi)],\ \forall (s,\xi) \in (0,\infty) \times \mathbb{R}, \end{aligned}$$

for some constant $\bar{C}_2 > 0$. Then, by (4.27), one has

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} e^{2\mu s} \omega H(s,\xi) d\xi ds \leq& \bar{C}_2 \varepsilon^{-1} \int_0^t e^{2\mu s} \sum_{i=1}^{3} \|V_i(s,\cdot)\|^2_{L^2_\omega} ds + \bar{C}_2 \varepsilon \int_0^t e^{2\mu s} \sum_{i=1}^{3} \|V_{i\xi}(s,\cdot)\|^2_{L^2_\omega} ds \\ \leq& \bar{C}_2 \varepsilon^{-1} C_1 \sum_{i=1}^{3} \|V_i(0,\cdot)\|^2_{L^2_\omega} + \bar{C}_2 \varepsilon \int_0^t e^{2\mu s} \sum_{i=1}^{3} \|V_{i\xi}(s,\cdot)\|^2_{L^2_\omega} ds. \end{aligned}$$

Choosing ε small enough, it follows from (4.40) and (4.41) that the inequality (4.42) holds. The proof is complete. □

4.3. *Proof of Theorem 4.1*

Based on Lemmas 4.3 and 4.4, we know that there exist positive constant C_3 and small $\mu = \mu^+ > 0$ such that

$$\|V_i(t,\cdot)\|_{H^1_\omega} \leq C_3 e^{-\mu^+ t} (\sum_{i=1}^{3} \|V_i(0,\cdot)\|^2_{H^1_\omega})^{1/2},\ \forall t > 0,\ i = 1,2,3. \tag{4.44}$$

Since $\omega(\xi) \geq 1$, we have $H^1_\omega(\mathbb{R}) \hookrightarrow H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$. Thus,

$$\sup_{x\in\mathbb{R}} |V_i(t,\xi)| \leq C_4 \|V_i(t,\cdot)\|^2_{H^1} \leq C_4 \|V_i(t,\cdot)\|^2_{H^1_\omega},\ i = 1,2,3,$$

for some $C_4 > 0$. Hence, it follows from (4.44) that there exists a positive constant C^+ such that

$$\sup_{x\in\mathbb{R}} \|U^+(t,x) - \Phi(x+ct)\| \le C^+ e^{-\mu^+ t}, \text{ for } t > 0.$$

Similar to the previous discussions, there exist positive constant C^- and small $\mu = \mu^- > 0$ such that

$$\sup_{x\in\mathbb{R}} \|U^-(t,x) - \Phi(x+ct)\| \le C^- e^{-\mu^- t}, \text{ for } t > 0.$$

Hence, we can conclude that

$$\sup_{x\in\mathbb{R}} \|u(t,x) - \Phi(x+ct)\| \le Ce^{-\mu t}, \ \forall t > 0,$$

for some positive constants C and μ. The proof of Theorem 4.1 is complete.

5. Stability of traveling wavefronts for (1.7) with large c

In this section, we will also apply the weighted energy method to study the stability of traveling wavefronts obtained in Theorem 3.1. However, due to the lattice structure of system (1.7), we should adopt different weighted spaces to derive the weighted energy estimates. Therefore, we first introduce the following notations.

Definition 5.1. *Let $\omega(\cdot) \in C(\mathbb{R})$ be a given weighted function, for any fixed $t \ge 0$ and $c > c_1^*$, we denote the spaces ℓ^2 and weighted spaces ℓ^2_ω by*

$$\ell^2 := \{v = \{v_i\}_{i\in\mathbb{Z}} | \ v_i \in \mathbb{R} \text{ and } \sum_{i\in\mathbb{Z}} v_i^2 < \infty\}$$

and

$$\ell^2_\omega(t) := \{v = \{v_i\}_{i\in\mathbb{Z}} | \ v_i \in \mathbb{R} \text{ and } \sum_{i\in\mathbb{Z}} \omega(i+ct) v_i^2 < \infty\},$$

which are endowed with the following norms:

$$\|v\|_{\ell^2} := \Big(\sum_{i\in\mathbb{Z}} v_i^2\Big)^{1/2} \text{ for } v \in \ell^2 \text{ and } \|v\|_{\ell^2_\omega(t)} := \Big(\sum_{i\in\mathbb{Z}} \omega(i+ct) v_i^2\Big)^{1/2} \text{ for } v \in \ell^2_\omega(t).$$

According to Definition 5.1, let us consider the initial value problem of (1.7) with initial data and $\{u_i(0)\}_{i\in\mathbb{Z}}, \ \{v_i(0)\}_{i\in\mathbb{Z}}, \ \{w_i(0)\}_{i\in\mathbb{Z}}$ satisfying the assumption

(L1) $(u_i(0), v_i(0), w_i(0)) \in [\mathbf{E}_1, \mathbf{E}_2]$ for all $i \in \mathbb{Z}$ and
$$\{u_i(0) - \phi_1(i)\}_{i\in\mathbb{Z}}, \ \{v_i(0) - \phi_2(i)\}_{i\in\mathbb{Z}}, \ \{w_i(0) - \phi_3(i)\}_{i\in\mathbb{Z}} \in \ell^2_\omega(0).$$

Then we can obtain the following stability result.

Theorem 5.1. *Assume that* (S1)*,* (S2) *and* (L1) *hold. Let $\Phi(i+ct)$ be a traveling wavefront of* (1.7) *satisfying* (1.9) *and with speed $c > \max\{c_1^*, c_1, c_2, c_3\}$. Then the initial value problem of* (1.7) *admits a unique solution $\{u_i(t)\}_{i\in\mathbb{Z}}, \ \{v_i(t)\}_{i\in\mathbb{Z}}, \ \{w_i(t)\}_{i\in\mathbb{Z}}$ satisfying $(u_i(t), v_i(t), w_i(t)) \in [\mathbf{E}_1, \mathbf{E}_2]$ for all $t > 0, \ i \in \mathbb{Z}$. In addition, for $t > 0$, we have*

$$\{u_i(i) - \phi_1(i+ct)\}_{i\in\mathbb{Z}} \in \ell^2_\omega(t), \ \sup_{i\in\mathbb{Z}} |u_i(i) - \phi_1(i+ct)| \le Ce^{-\mu t};$$

$$\{v_i(t) - \phi_2(i+ct)\}_{i\in\mathbb{Z}} \in \ell^2_\omega(t),\ \sup_{i\in\mathbb{Z}} |v_i(i) - \phi_2(i+ct)| \le Ce^{-\mu t};$$
$$\{w_i(t) - \phi_3(i+ct)\}_{i\in\mathbb{Z}} \in \ell^2_\omega(t),\ \sup_{i\in\mathbb{Z}} |w_i(i) - \phi_3(i+ct)| \le Ce^{-\mu t},$$

for some positive constants C and μ.

Proof. The proof is similar to that of Theorem 4.1 by replacing the weighted spaces L^2 and L^2_ω as ℓ^2 and ℓ^2_ω respectively, we sketch it in the sequel.

Step 1. Let $\{U_i(t)\}_{i\in\mathbb{Z}} = \{(u_i(t), v_i(t), w_i(t))\}_{i\in\mathbb{Z}}$ be the solution of system (1.7) with initial data $\{U_i(0)\}_{i\in\mathbb{Z}} = \{(u_i(0), v_i(0), w_i(0))\}_{i\in\mathbb{Z}}$ satisfying (L1). Then, $\forall i \in \mathbb{Z}$, we set

$$\begin{aligned} U_i^-(0) :=& (\min\{u_i(0), \phi_1(i)\}, \min\{v_i(0), \phi_2(i)\}, \min\{w_i(0), \phi_3(i)\}), \\ U_i^+(0) :=& (\max\{u_i(0), \phi_1(i)\}, \max\{v_i(0), \phi_2(i)\}, \max\{w_i(0), \phi_3(i)\}). \end{aligned}$$

Based on assumption (A2), it is clear that $U_i^\pm(0)$ satisfy

$$\mathbf{E}_1 \le U_i^-(0) \le U_i(0), \Phi(i) \le U_i^+(0) \le \mathbf{E}_2,\ \forall i \in \mathbb{Z}. \tag{5.1}$$

Let $\{U_i^\pm(t)\}_{i\in\mathbb{Z}}$ be the solutions of (1.7) with initial data $\{U_i^\pm(0)\}_{i\in\mathbb{Z}}$, then we have

$$\mathbf{E}_1 \le U_i^-(t) \le U_i(t), \Phi(i+ct) \le U_i^+(t) \le \mathbf{E}_2,\ \forall (t,i) \in \mathbb{R}^+ \times \mathbb{Z}. \tag{5.2}$$

Then it follows from (4.7) that

$$\|U_i(t) - \Phi(i+ct)\| \le \max\{\|U_i^+(t) - \Phi(i+ct)\|, \|U_i^-(t) - \Phi(i+ct)\|\}, \tag{5.3}$$

for any $(t,i) \in \mathbb{R}^+ \times \mathbb{Z}$. Therefore, to derive a priori estimate of $U_i(t) - \Phi(i+ct)$, it suffices to estimate the functions $U_i^\pm(t) - \Phi(i+ct)$. For convenience, let's denote

$$V_i^\pm(t) := U_i^\pm(t) - \Phi(i+ct) \quad \text{and} \quad V_i^\pm(0) := U_i^\pm(0) - \Phi(i),\ \forall (t,i) \in \mathbb{R}^+ \times \mathbb{Z}.$$

Then it follows that

$$\mathbf{E}_1 \le V_i^\pm(0) \le \mathbf{E}_2 \quad \text{and} \quad \mathbf{E}_1 \le V_i^\pm(t) \le \mathbf{E}_2,\ \forall (t,i) \in \mathbb{R}^+ \times \mathbb{Z}.$$

Hence, we only need to estimate $\{V_i^+(t)\}_{i\in\mathbb{Z}}$, since $\{V_i^-(t)\}_{i\in\mathbb{Z}}$ can also be discussed in the same way. For convenience, we drop the sign " + " for $\{V_i^+(t)\}_{i\in\mathbb{Z}}$, $\{U_i^+(t)\}_{i\in\mathbb{Z}}$ and set

$$V_i(t) = (X_i(t), Y_i(t), Z_i(t)) := U_i(t) - \Phi(i+ct),\ \forall (t,i) \in \mathbb{R}^+ \times \mathbb{Z}.$$

Step 2. Similar to (4.8)–(4.10), $V_i(t)$ satisfies

$$\begin{aligned} X_{it} =& d_1 D[X_i] + [r_1 - c_{12} r_2 - 2\phi_1 + c_{12}(Y_i + \phi_2) + c_{13}(Z_i + \phi_3)]X_i + \\ & c_{12}\phi_1 Y_i + c_{13}\phi_1 Z_i - X_i^2, \end{aligned} \tag{5.4}$$
$$\begin{aligned} Y_{it} =& d_2 D[Y_i] + [-r_2 + 2\phi_2 - c_{21}(X_i + \phi_1) - c_{23}(Z_i + \phi_3)]Y_i + \\ & c_{21}(r_2 - \phi_2)X_i + c_{23}(r_2 - \phi_2)Z_i + Y_i^2, \end{aligned} \tag{5.5}$$
$$Z_{it} = d_3 D[Z_i] + [r_3 - c_{32} r_2 - 2\phi_3 + c_{31}(X_i + \phi_1) + c_{32}(Y_i + \phi_2)]Z_i +$$

$$c_{31}\phi_3 X_i + c_{32}\phi_3 Y_i - Z_i^2. \tag{5.6}$$

Step 3. Multiplying both sides of (5.4), (5.5) and (5.6) by $e^{2\mu t}\omega(\xi)X_i(t)$, $e^{2\mu t}\omega(\xi)Y_i(t)$ and $e^{2\mu t}\omega(\xi)Z_i(t)$ respectively, we can obtain

$$\begin{aligned}
&(\frac{1}{2}e^{2\mu t}\omega X_i^2)_t - d_1 e^{2\mu t}\omega X_i[X_{i+1} + X_{i-1}] \\
=&e^{2\mu t}\omega X_i^2 \hat{Q}_i(t) + e^{2\mu t}\omega X_i[c_{12}\phi_1 Y_i + c_{13}\phi_1 Z_i - X_i^2], \qquad (5.7)\\
&(\frac{1}{2}e^{2\mu t}\omega Y_i^2)_t - d_2 e^{2\mu t}\omega Y_i[Y_{i+1} + Y_{i-1}] \\
=&e^{2\mu t}\omega Y_i^2 \bar{Q}_i(t) + e^{2\mu t}\omega Y_i[c_{21}(r_2 - \phi_2)X_i + c_{23}(r_2 - \phi_2)Z_i + Y_i^2], \qquad (5.8)\\
&(\frac{1}{2}e^{2\mu t}\omega Z_i^2)_t - d_3 e^{2\mu t}\omega Z_i[Z_{i+1} + Z_{i-1}] \\
=&e^{2\mu t}\omega Z_i^2 \tilde{Q}_i(t) + e^{2\mu t}\omega Z_i[c_{31}\phi_3 X_i + c_{32}\phi_3 Y_i - Z_i^2], \qquad (5.9)
\end{aligned}$$

where

$$\begin{aligned}
\hat{Q}_i(t) :=&\mu - 2d_1 + [r_1 - 2\phi_1 + c_{12}(Y_i + \phi_2 - r_2) + c_{13}(Z_i + \phi_3)],\\
\bar{Q}_i(t) :=&\mu - 2d_2 + [-r_2 + 2\phi_2 - c_{21}(X_i + \phi_1) - c_{23}(Z_i + \phi_3)],\\
\tilde{Q}_i(t) :=&\mu - 2d_3 + [r_3 - 2\phi_3 + c_{31}(X_i + \phi_1) + c_{32}(Y_i + \phi_2 - r_2)].
\end{aligned}$$

Step 4. Let us set $X(t) = \{X_i(t)\}_{i\in\mathbb{Z}}, Y(t) = \{Y_i(t)\}_{i\in\mathbb{Z}}$ and $Z(t) = \{Z_i(t)\}_{i\in\mathbb{Z}}$. Summing over all $i \in \mathbb{Z}$ for (5.7)–(5.9), integrating them over $[0, t]$ and applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
e^{2\mu t}\|X(t)\|^2_{\ell^2_\omega} \leq&\|X(0)\|^2_{\ell^2_\omega} + d_1\int_0^t\sum_{i\in\mathbb{Z}} e^{2\mu s}\omega[2 + \frac{\omega(\xi+1)}{\omega} + \frac{\omega(\xi-1)}{\omega}]X_i^2 ds+\\
&2\int_0^t\sum_{i\in\mathbb{Z}} e^{2\mu s}\omega\hat{Q}_i(s)X_i^2 ds + \int_0^t\sum_{i\in\mathbb{Z}} e^{2\mu s}\omega c_{12}\phi_1(X_i^2 + Y_i^2)ds+\\
&\int_0^t\sum_{i\in\mathbb{Z}} e^{2\mu s}\omega c_{13}\phi_1(X_i^2 + Z_i^2)ds, \qquad (5.10)
\end{aligned}$$

$$\begin{aligned}
e^{2\mu t}\|Y(t)\|^2_{\ell^2_\omega} \leq&\|Y(0)\|^2_{\ell^2_\omega} + d_2\int_0^t\sum_{i\in\mathbb{Z}} e^{2\mu s}\omega[2 + \frac{\omega(\xi+1)}{\omega} + \frac{\omega(\xi-1)}{\omega}]Y_i^2 ds+\\
&2\int_0^t\sum_{i\in\mathbb{Z}} e^{2\mu s}\omega\bar{Q}_i(s)Y_i^2 ds + \int_0^t\sum_{i\in\mathbb{Z}} e^{2\mu s}\omega c_{21}(r_2 - \phi_2)(X_i^2 + Y_i^2)ds+\\
&\int_0^t\sum_{i\in\mathbb{Z}} e^{2\mu s}\omega c_{23}(r_2 - \phi_2)(Y_i^2 + Z_i^2)ds + 2\int_0^t\sum_{i\in\mathbb{Z}} e^{2\mu s}\omega Y_i^3 ds, \qquad (5.11)\\
e^{2\mu t}\|Z(t)\|^2_{\ell^2_\omega} \leq&\|Z(0)\|^2_{\ell^2_\omega} + d_3\int_0^t\sum_{i\in\mathbb{Z}} e^{2\mu s}\omega[2 + \frac{\omega(\xi+1)}{\omega} + \frac{\omega(\xi-1)}{\omega}]Z_i^2 ds+\\
&2\int_0^t\sum_{i\in\mathbb{Z}} e^{2\mu s}\omega\tilde{Q}_i(s)Z_i^2 ds + \int_0^t\sum_{i\in\mathbb{Z}} e^{2\mu s}\omega c_{31}\phi_3(X_i^2 + Z_i^2)ds+
\end{aligned}$$

$$\int_0^t \sum_{i\in\mathbb{Z}} e^{2\mu s}\omega c_{32}\phi_3(Y_i^2+Z_i^2)ds. \tag{5.12}$$

Summing up the inequalities (5.10)–(5.12), we can derive

$$\begin{aligned}&e^{2\mu t}(\|X(t)\|^2_{\ell^2_\omega}+\|Y(t)\|^2_{\ell^2_\omega}+\|Z(t)\|^2_{\ell^2_\omega})+\int_0^t\sum_{i\in\mathbb{Z}}e^{2\mu s}\omega(\hat{R}^\mu_i(s)X_i^2+\bar{R}^\mu_i(s)Y_i^2+\tilde{R}^\mu_i(s)Z_i^2)ds\\&\leq(\|X(0)\|^2_{\ell^2_\omega}+\|Y(0)\|^2_{\ell^2_\omega}+\|Z(0)\|^2_{\ell^2_\omega}),\end{aligned} \tag{5.13}$$

where

$$\begin{aligned}\hat{R}^\mu_i(t):=&-d_1[2+\frac{\omega(\xi+1)}{\omega(\xi)}+\frac{\omega(\xi-1)}{\omega(\xi)}]-2\hat{Q}_i(t)-(c_{12}+c_{13})\phi_1-c_{21}(r_2-\phi_2)-c_{31}\phi_3,\\ \bar{R}^\mu_i(t):=&-d_2[2+\frac{\omega(\xi+1)}{\omega(\xi)}+\frac{\omega(\xi-1)}{\omega(\xi)}]-2\bar{Q}_i(t)-c_{12}\phi_1-(c_{21}+c_{23})(r_2-\phi_2)-c_{32}\phi_3-2Y_i,\\ \tilde{R}^\mu_i(t):=&-d_3[2+\frac{\omega(\xi+1)}{\omega(\xi)}+\frac{\omega(\xi-1)}{\omega(\xi)}]-2\tilde{Q}_i(t)-(c_{31}+c_{32})\phi_3-c_{23}(r_2-\phi_2)-c_{13}\phi_1.\end{aligned}$$

Step 5. Similar to Lemma 4.3, there exists $\tilde{C}_0>0$ such that

$$\hat{R}^\mu_i(t),\bar{R}^\mu_i(t),\tilde{R}^\mu_i(t)>\tilde{C}_0,\ \forall i\in\mathbb{Z}\text{ and }t>0.$$

Then, for $t\geq 0$, (5.13) implies that there exists a positive constant $\tilde{C}_1$ such that

$$\begin{aligned}&(\|X(t)\|^2_{\ell^2_\omega}+\|Y(t)\|^2_{\ell^2_\omega}+\|Z(t)\|^2_{\ell^2_\omega})+\int_0^t e^{-2\mu(t-s)}(\|X(s)\|^2_{\ell^2_\omega}+\|Y(s)\|^2_{\ell^2_\omega}+\|Z(s)\|^2_{\ell^2_\omega})ds\\&\leq\tilde{C}_1e^{-2\mu t}(\|X(0)\|^2_{\ell^2_\omega}+\|Y(0)\|^2_{\ell^2_\omega}+\|Z(0)\|^2_{\ell^2_\omega}).\end{aligned} \tag{5.14}$$

Step 6. Since $\omega(\xi)\geq 1$, we have $\|\cdot\|_{\ell^2}\leq\|\cdot\|_{\ell^2_\omega}$. By the Sobolev's embedding inequality $\ell^2\hookrightarrow\ell^\infty$, we have

$$\begin{aligned}\sup_{i\in\mathbb{Z}}|X_i(t)|\leq C\|X(t)\|_{\ell^2}\leq C\|X(t)\|_{\ell^2_\omega},\\ \sup_{i\in\mathbb{Z}}|Y_i(t)|\leq C\|Y(t)\|_{\ell^2}\leq C\|Y(t)\|_{\ell^2_\omega},\\ \sup_{i\in\mathbb{Z}}|Z_i(t)|\leq C\|Z(t)\|_{\ell^2}\leq C\|Z(t)\|_{\ell^2_\omega},\end{aligned}$$

for some constant $C>0$. Then it follows from (5.14) that

$$\sup_{i\in\mathbb{Z}}\|U_i^+(t)-\Phi(i+ct)\|\leq C_1^+e^{-\mu t},$$

for some constant $C_1^+>0$. By (5.3) and similar arguments, we have

$$\|U_i(t)-\Phi(i+ct)\|\leq\sup_{i\in\mathbb{Z}}\max\{\|U_i^+(t)-\Phi(i+ct)\|,\|U_i^-(t)-\Phi(i+ct)\|\}\leq C_2e^{-\mu t},$$

$\forall(t,i)\in\mathbb{R}^+\times\mathbb{Z}$, for some constant $C_2>0$. The proof is complete. □

6. Stability of traveling wavefronts for $c > c_1^*$

In this section, we will improve the stability results of Theorem 4.1 and Theorem 5.1 to any $c > c_1^*$. Different to (4.1), we consider the weighted function

$$\omega^*(\xi) := e^{-\mu_1^* \xi}, \ \forall \xi \in \mathbb{R}. \tag{6.1}$$

Note that $\mu_1^* > 0$ is a constant given in Lemma 2.1 such that

$$c_1^* \mu_1^* = d_1(e^{\mu_1^*} + e^{-\mu_1^*} - 2) + r_1 - c_{12} r_2. \tag{6.2}$$

Furthermore, we impose the following assumption:

(S3) $r_1 > 2(c_{12} + c_{13} + c_{31})u_* + (3c_{12} + 2c_{21} + 4 + 2c_{23})r_2 + 2(c_{31} + c_{32})w_*$.
(S4) $\hat{\mu} := \min\{\frac{1}{2}[\min\{u_* - c_{12}r_2 - c_{13}w_*, w_* - c_{31}u_* - c_{32}r_2\} - \max\{c_{13}u_*, c_{31}w_*\}], -2r_2 + 2c_{21}u_* + 2c_{23}w_*\} > 0$.

Example 6.1. *Assume that*

$$r_1 = 6, \ r_2 = 0.1, \ r_3 = 6, \ c_{12} = c_{13} = c_{31} = c_{32} = 0.01, \ c_{21} = c_{23} = 1.$$

Then the parameters satisfy the assumptions (H1), (H2), (S1), (S3) *and* (S4). *In addition, we have*

$$\mathbf{E}_2 = (6.\overline{06}, 0.1, 6.\overline{06}) \text{ and } (\ell_1, \ell_2, \ell_3) \simeq (11.817, 12.734, 11.817).$$

Similar to (4.19), we can obtain the following estimation:

$$\sum_{i=1}^{3} e^{2\mu t} \|V_i(t,\xi)\|^2_{L^2_{\omega^*}} + \int_0^t \int_{\mathbb{R}} e^{2\mu s} \omega^* \sum_{i=1}^{3} \mathcal{R}_i^{\mu}(s,\xi) V_i^2 d\xi ds \le \sum_{i=1}^{3} \|V_i(0,\xi)\|^2_{L^2_{\omega^*}}, \tag{6.3}$$

where each $\mathcal{R}_i^{\mu}(t,\xi)$ has the same form as $R_i^{\mu}(t,\xi)$ but replacing $\omega(\cdot)$ as $\omega^*(\cdot)$. Similar and simpler than Lemma 4.3, we have the following result.

Lemma 6.1. *Assume that* (S3) *holds and* $c > c_1^*$. *Then there exists a small* $\mu > 0$ *such that the following statements hold:*

(1) *There exists a positive constant* C_0 *such that*

$$\sum_{i=1}^{3} \mathcal{R}_i^{\mu}(t,\xi) \ge C_0, \ \forall (t,\xi) \in \mathbb{R}^+ \times \mathbb{R}, i = 1,2,3. \tag{6.4}$$

(2) *There exists a positive constant* C_1 *such that*

$$\sum_{i=1}^{3} \|V_i(\cdot,t)\|^2_{L^2_{\omega^*}} + \int_0^t e^{-2\mu(t-s)} \sum_{i=1}^{3} \|V_i(\cdot,s)\|^2_{L^2_{\omega^*}} ds \le C_1 e^{-2\mu t} \sum_{i=1}^{3} \|V_i(\cdot,0)\|^2_{L^2_{\omega^*}}. \tag{6.5}$$

Proof. (1) Noting that $(0,0,0) < (V_1 + \phi_1, V_2 + \phi_2, V_3 + \phi_3) < (u_*, r_2, w_*)$. Since $d_1 \ge d_2, d_3$, it follows from (6.2) that

$$c\mu_1^* \ge d_i(e^{\mu_1^*} + e^{-\mu_1^*} - 2) + r_1 - c_{12} r_2, \ \text{for } i = 1,2,3. \tag{6.6}$$

By (6.6) and elementary computations, we have

$$
\begin{aligned}
\mathcal{R}_1^0(t,\xi) =& -D_1(\mu_1^*) + c\mu_1^* - 2[r_1 - c_{12}r_2 - 2\phi_1 + c_{12}(V_2+\phi_2) + c_{13}(V_3+\phi_3)] \\
& - (c_{12}+c_{13})\phi_1 - c_{21}(r_2-\phi_2) - c_{31}\phi_3 \\
>& -(r_1 - c_{12}r_2) - 2c_{12}r_2 - 2c_{13}w_* - (c_{12}+c_{13})u_* - c_{21}r_2 - c_{31}w_*, \\
\mathcal{R}_2^0(t,\xi) =& -D_2(\mu_1^*) + c\mu_1^* - 2[-r_2 + 2\phi_2 - c_{21}(V_1+\phi_1) - c_{23}(V_3+\phi_3)] \\
& - c_{12}\phi_1 - (c_{21}+c_{23})(r_2-\phi_2) - c_{32}\phi_3 - 2V_2 \\
>& r_1 - c_{12}r_2 - 4r_2 - c_{12}u_* - (c_{21}+c_{23})r_2 - c_{32}w_*, \\
\mathcal{R}_3^0(t,\xi) >& r_1 - c_{12}r_2 - 2r_3 - 2c_{31}u_* - (c_{31}+c_{32})w_* - c_{23}r_2 - c_{13}u_*.
\end{aligned}
$$

Then it follows from (S3) that

$$
\sum_{i=1}^{3} \mathcal{R}_i^0(t,\xi) > r_1 - 2(c_{12}+c_{13}+c_{31})u_* - (3c_{12}+2c_{21}+4+2c_{23})r_2 - 2(c_{31}+c_{32})w_* > 0.
$$

Therefore, we may choose a small $\mu > 0$ such that (6.4) holds for some $C_0 > 0$.

(2) The proof of this part is the same as Lemma 4.3 and skipped. □

Similar to Lemmas 4.4 and, we have

Lemma 6.2. *Assume* (S3) *and $c > c_1^*$. There exists a small $\mu > 0$ such that the following statements hold:*

(1) *There exists a positive constant $\widehat{C}_0$ such that*

$$
\widehat{\mathcal{R}}_i^\mu(t,\xi) \geq \widehat{C}_0, \ \forall (t,\xi) \in \mathbb{R}^+ \times \mathbb{R}, \ i = 1,2,3. \tag{6.7}
$$

(2) *There exists a positive constant $\widehat{C}_1$ such that*

$$
\sum_{i=1}^{3} \|V_{i\xi}(t,\cdot)\|^2_{L^2_{\omega^*}} + \int_0^t e^{-2\mu(t-s)} \sum_{i=1}^{3} \|V_{i\xi}(s,\cdot)\|^2_{L^2_{\omega^*}} ds \leq \widehat{C}_1 e^{-2\mu t} \sum_{i=1}^{3} \|V_{i\xi}(0,\cdot)\|^2_{L^2_{\omega^*}}. \tag{6.8}
$$

Note that each $\widehat{\mathcal{R}}_i^\mu(t,\xi)$ has the same form as $\widehat{R}_i^\mu(t,\xi)$ but replacing $\omega(\cdot)$ as $\omega^*(\cdot)$. as a consequence Lemmas 6.1 and 6.2, we know that there exist positive constant $\tilde{C}$ and small $\mu = \tilde{\mu} > 0$ such that

$$
\|V_i(t,\cdot)\|_{H^1_{\omega^*}} \leq \tilde{C} e^{-\tilde{\mu}t} (\sum_{i=1}^{3} \|V_i(0,\cdot)\|^2_{H^1_{\omega^*}})^{1/2}, \ \forall t > 0, \ i = 1,2,3. \tag{6.9}
$$

Since $\omega^*(\xi) \to 0$ as $\xi \to \infty$, it is not true that $H^1_{\omega^*}(\mathbb{R}) \hookrightarrow C(\mathbb{R})$. However, for any $I = (-\infty, \bar{\xi}]$ for some large $\bar{\xi} \gg 1$, we can obtain $H^1_{\omega^*}(I) \hookrightarrow C(I)$. Thus, (6.9) implies the following lemma.

Lemma 6.3. *For all $t > 0$, $i = 1,2,3$, it holds that*

$$
\sup_{\xi \in I} |V_i(\xi,t)| \leq \widehat{C}_1 e^{-\tilde{\mu}t} (\sum_{i=1}^{3} \|V_{i0}(0)\|^2_{H^1_{\omega^*}})^{\frac{1}{2}}, \ \forall \xi \in I = (-\infty, \bar{\xi}], \tag{6.10}
$$

for some $\tilde{\mu} > 0$ and large $\bar{\xi} \gg 1$.

To extend the result of Lemma 6.3 to the whole space $(-\infty, \infty)$, we have to prove the convergence of $V_i(\xi, t)$ as $\xi \to \infty$.

Lemma 6.4. *Assume that* (S4) *holds. There exists some constant $C > 0$ such that*

$$\lim_{\xi\to\infty} V_i(\xi, t) \le Ce^{-\hat{\mu}t}, \ i = 1, 2, 3. \tag{6.11}$$

Note that $\hat{\mu}$ is given in (S4).

Proof. It's easy to see that $V_{i\xi}(\infty, t) = 0$ and $d_i D[V_i](+\infty) = 0$ for $i = 1, 2, 3$. Based on (4.8)–(4.10) and the boundedness of $\mathcal{V}_i(t) := V_i(\infty, t)$ for all $\xi \in (-\infty, \infty)$, letting $\xi \to \infty$, one immediately obtains

$$\begin{aligned}
\mathcal{V}_{1t}(t) =& -[u_* + \mathcal{V}_1(t) - c_{12}\mathcal{V}_2(t) - c_{13}\mathcal{V}_3(t)]\mathcal{V}_1(t) + c_{12}u_*\mathcal{V}_2(t) + c_{13}u_*\mathcal{V}_3(t),\\
\le& -[u_* - c_{12}r_2 - c_{13}w_*]\mathcal{V}_1(t) + c_{12}u_*\mathcal{V}_2(t) + c_{13}u_*\mathcal{V}_3(t), \qquad (6.12)\\
\mathcal{V}_{2t}(t) =& -[-r_2 + c_{21}u_* + c_{23}w_* + c_{21}\mathcal{V}_1(t) + c_{23}\mathcal{V}_3(t)]\mathcal{V}_2(t) + \mathcal{V}_2^2(t),\\
\le& -[-2r_2 + 2c_{21}u_* + 2c_{23}w_*]\mathcal{V}_2(t), \qquad (6.13)\\
\mathcal{V}_{3t}(t) =& -[w_* + \mathcal{V}_3(t) - c_{31}\mathcal{V}_1(t) - c_{32}\mathcal{V}_2(t)]\mathcal{V}_3(t) + c_{31}\phi_3\mathcal{V}_1(t) + c_{32}\phi_3\mathcal{V}_2(t)\\
\le& -[w_* - c_{31}u_* - c_{32}r_2]\mathcal{V}_3(t) + c_{31}w_*\mathcal{V}_1(t) + c_{32}w_*\mathcal{V}_2(t). \qquad (6.14)
\end{aligned}$$

Let's set

$$A_1 := u_* - c_{12}r_2 - c_{13}w_*, A_2 := -2r_2 + 2c_{21}u_* + 2c_{23}w_* \text{ and } A_3 := w_* - c_{31}u_* - c_{32}r_2.$$

By the assumption (S4), we see that $A_2 > 0$. Integrating (6.13) over $[0, t]$, we have

$$\mathcal{V}_2(t) \le \mathcal{V}_2(0)e^{-A_2 t}, \ \forall t > 0.$$

Then it follows from (6.12) and (6.14) that

$$\mathcal{V}_{1t}(t) + \mathcal{V}_{3t}(t) \le -\mathcal{A}[\mathcal{V}_1(t) + \mathcal{V}_3(t)] + (c_{12}u_* + c_{32}w_*)\mathcal{V}_2(0)e^{-A_2 t}, \ \forall t > 0,$$

where $\mathcal{A} := \min\{A_1, A_3\} - \max\{c_{12}u_*, c_{32}w_*\}$. We claim that there exists some positive constant $\hat{C}$ such that

$$\mathcal{V}_1(t) + \mathcal{V}_3(t) \le \hat{C}e^{-\hat{\mu}t}, \ \forall t > 0.$$

Note that $\hat{\mu} = \min\{\mathcal{A}/2, A_2\}$. In fact, if $\mathcal{A} \neq A_2$, we then have

$$\begin{aligned}
\mathcal{V}_1(t) + \mathcal{V}_3(t) \le& [\mathcal{V}_1(0) + \mathcal{V}_3(0)]e^{-\mathcal{A}t} + e^{-\mathcal{A}t}\int_0^t (c_{12}u_* + c_{32}w_*)\mathcal{V}_2(0)e^{(\mathcal{A}-A_2)s}ds\\
=& [\mathcal{V}_1(0) + \mathcal{V}_3(0)]e^{-\mathcal{A}t} + (c_{12}u_* + c_{32}w_*)\mathcal{V}_2(0)\frac{e^{-A_2 t} - e^{-\mathcal{A}t}}{\mathcal{A} - A_2}\\
\le& [\mathcal{V}_1(0) + \mathcal{V}_3(0)]e^{-\mathcal{A}t} + (c_{12}u_* + c_{32}w_*)\mathcal{V}_2(0)\frac{e^{-\min\{\mathcal{A},A_2\}t}}{|\mathcal{A} - A_2|}\\
\le& \hat{C}_1 e^{-\min\{\mathcal{A},A_2\}t} \le \hat{C}_1 e^{-\hat{\mu}t}, \ \forall t > 0,
\end{aligned}$$

where

$$\hat{C}_1 := \mathcal{V}_1(0) + \mathcal{V}_3(0) + \frac{(c_{12}u_* + c_{32}w_*)\mathcal{V}_2(0)}{|\mathcal{A} - A_2|}.$$

If $\mathcal{A} = A_2$, then we obtain

$$\begin{aligned}\mathcal{V}_1(t) + \mathcal{V}_3(t) \leq & [\mathcal{V}_1(0) + \mathcal{V}_3(0)]e^{-\mathcal{A}t} + e^{-\mathcal{A}t}\int_0^t (c_{12}u_* + c_{32}w_*)\mathcal{V}_2(0)ds \\ \leq & [\mathcal{V}_1(0) + \mathcal{V}_3(0) + (c_{12}u_* + c_{32}w_*)\mathcal{V}_2(0)t]e^{-\mathcal{A}t} \\ \leq & \hat{C}_2 e^{-\frac{\mathcal{A}}{2}t} \leq \hat{C}_2 e^{-\hat{\mu}t}, \ \forall t > 0,\end{aligned}$$

for some $\hat{C}_2 > 0$. Thus, the claim holds. Therefore, we conclude that

$$\lim_{\xi\to\infty} V_i(\xi, t) \leq Ce^{-\hat{\mu}t}, \ i = 1, 2, 3,$$

for some positive constant C. This completes the proof. □

Based on the above lemmas, we can also obtain the following stability result.

Theorem 6.1. *Assume that* (S3)–(S4) *hold. Let* $\Phi(x + ct)$ *be a traveling wavefront of* (1.6) *satisfying* (1.9) *and with speed* $c > c_1^*$. *Then the initial value problem* (1.6) *admits a unique solution* $U(t, x) = (u(t, x), v(t, x), w(t, x))$ *satisfying* $U(t, x) \in [\mathbf{E}_1, \mathbf{E}_2]$ *for all* $t > 0, \ x \in \mathbb{R}$. *In addition, we have*

$$U(t, x) - \Phi(x + ct) \in C([0, +\infty); H^1_{\omega^*}(\mathbb{R})) \cap L^2([0, +\infty); H^1_{\omega^*}(\mathbb{R})) \tag{6.15}$$

and

$$\sup_{x\in\mathbb{R}} \|U(t, x) - \Phi(x + ct)\| \leq Ce^{-\mu t}, \ \forall t > 0, \tag{6.16}$$

for some positive constants C *and* μ.

By the same way, we also have the following stability result for (1.7).

Theorem 6.2. *Assume that* (S3)–(S4) *hold. Let* $\Phi(i + ct)$ *be a traveling wavefront of* (1.7) *satisfying* (1.9) *and with speed* $c > c_1^*$. *Then the initial value problem of* (1.7) *admits a unique solution* $\{u_i(t)\}_{i\in\mathbb{Z}}, \ \{v_i(t)\}_{i\in\mathbb{Z}}, \ \{w_i(t)\}_{i\in\mathbb{Z}}$ *satisfying* $(u_i(t), v_i(t), w_i(t)) \in [\mathbf{E}_1, \mathbf{E}_2]$ *for all* $t > 0, \ i \in \mathbb{Z}$. *In addition, for* $t > 0$*, we have*

$$\begin{aligned}&\{u_i(t) - \phi_1(i + ct)\}_{i\in\mathbb{Z}} \in \ell^2_\omega(t), \ \sup_{i\in\mathbb{Z}} |u_i(t) - \phi_1(i + ct)| \leq Ce^{-\mu t};\\ &\{v_i(t) - \phi_2(i + ct)\}_{i\in\mathbb{Z}} \in \ell^2_\omega(t), \ \sup_{i\in\mathbb{Z}} |v_i(t) - \phi_2(i + ct)| \leq Ce^{-\mu t};\\ &\{w_i(t) - \phi_3(i + ct)\}_{i\in\mathbb{Z}} \in \ell^2_\omega(t), \ \sup_{i\in\mathbb{Z}} |w_i(t) - \phi_3(i + ct)| \leq Ce^{-\mu t},\end{aligned}$$

for some positive constants C *and* μ.

7. Discussion

In population dynamics, traveling wave solution can be used to describe the spatial spread or invasion of the species. In this article we consider the existence and stability of the traveling wavefronts of discrete diffusive systems which come from the competition and cooperations between three species.

In Theorem 3.1, we proved that both systems (1.6) and (1.7) admit traveling wavefronts connecting the extinct state $\mathbf{E_1}$ and co-existence state $\mathbf{E_2}$, provided the assumptions (H1)-(H2) hold and the propagation wave speed c is greater than the minimum speed c_1^*. Roughly speaking, to guarantee the assumptions (H1)-(H2) hold, it is required that d_2, r_2, c_{12}, c_{32} are small enough, and d_1, r_1 are large enough. Biologically, it means that the diffusion effect, growth rate for the species v and the competition relation between v and the other species are very weak. Since the species u and w cooperate with each other; the species u has strong diffusion effect and growth rate; and their competition from the species v are very weak, this gives us the reason why the minimal speed is determined by the linearization problem of the first u-equation of both systems. And also the existence of traveling wavefronts propagating from the extinct state to the co-existence state.

As mentioned in introduction, when the traveling wavefronts are disturbed under small perturbations, only stable such solutions can be visualized in the real world. However, since such solutions exist for all $c > c_1^*$, generically any one of them won't be globally asymptotic stable. Therefore, we introduce the weight functions to split the domain of attractions of traveling wavefronts with different speeds, and then obtain the stability results.

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Conflict of interest.

The authors declare that they have no competing interests.

References

1. R. A. Fisher, The wave of advance of advantageous genes, *Ann. Eugenics*, **7** (1937), 355–369.
2. C. C. Chen, L. C. Hung, M. Mimura, et al., Exact travelling wave solutions of three-species competition-diffusion systems, *Discrete Contin. Dyn. Syst. Ser. B*, **17** (2012), 2653–2669.
3. C. C. Chen, L, C, Hung, M. Mimura, et al., Semi-exact equilibrium solutions for three-species competition-diffusion systems, *Hiroshima Math. J.*, **43** (2013), 179–206.
4. M. Mimura and M. Tohma, Dynamic coexistence in a three-species competition-diffusion system, *Ecol. Complex.*, **21** (2015), 215–232.
5. H. Ikeda, Travelling wave solutions of three-component systems with competition and diffusion, *Toyama Math. J.*, **24** (2001), 37–66.
6. H. Ikeda, Dynamics of weakly interacting front and back waves in three-component systems, *Toyama Math. J.*, **30** (2007), 1–34.

7. Y. Kan-on and M. Mimura, Singular perturbation approach to a 3-component reaction-diffusion system arising in population dynamics, *SIAM J. Math. Anal.*, **29** (1998), 1519–1536.
8. P. D. Miller, Nonmonotone waves in a three species reaction-diffusion model, *Methods and Applications of Analysis,* **4** (1997), 261–282.
9. M. Mimura and P. C. Fife, A 3-component system of competition and diffusion, *Hiroshima Math. J.*, **16** (1986), 189–207.
10. J.-S. Guo, Y. Wang, C.-H. Wu, et al., The minimal speed of traveling wave solutions for a diffusive three species competition system, *Taiwan. J. Math.*, **19** (2015), 1805–1829.
11. L.-C. Hung, Traveling wave solutions of competitive-cooperative Lotka-Volterra systems of three species, *Nonlinear Anal. Real World Appl.*, **12** (2011), 3691–3700.
12. C.-H. Chang, Existence and stability of traveling wave solutions for a competitive-cooperative system of three species, preprint, (2018).
13. A. W. Leung, X. Hou and W. Feng, Traveling wave solutions for Lotka-Volterra system revisited, *Discrete Contin. Dyn. Syst.-B*, **15** (2011), 171–196.
14. M. Mei, C. Ou and X.-Q. Zhao, Global stability of monostable traveling waves for nonlocal time-delayed reaction-diffusion equations, *SIAM J. Math. Anal.*, **42** (2010), 2762–2790; Erratum, *SIAM J. Math. Anal.*, **44** (2012), 538–540.
15. D. Sattinger, On the stability of traveling waves, *Adv. Math.*, **22** (1976), 312–355.
16. M. Bramson, Convergence of solutions of the Kolmogorov equations to traveling waves, *Mem. Amer. Math. Soc.*, **44** (1983), 285.
17. A. I. Volpert, V. A. Volpert and V. A. Volpert, *Travelling wave solutions of parabolic systems, Translations of Mathematical Monographs*, **140**, Amer. Math. Soc., Providence, RI, 1994.
18. J. Xin, Front propagation in heterogeneous media, *SIAM Rev.*, **42** (2000), 161–230.
19. G.-S. Chen, S.-L. Wu and C.-H. Hsu, Stability of traveling wavefronts for a discrete diffusive competition system with three species, *J. Math. Anal. Appl.*, **474** (2019), 909–930.
20. C.-H. Hsu, T.-S. Yang and Z. X. Yu, Existence and exponential stability of traveling waves for delayed reaction-diffusion systems, *Nonlinearity*, **32** (2019), 1206–1236.
21. M. Mei, C. K. Lin, C. T. Lin, et al., Traveling wavefronts for time-delayed reaction-diffusion equation:(I) local nonlinearity, *J. Differ. Equations*, **247** (2009), 495–510.
22. M. Mei, C. K. Lin, C. T. Lin, et al., Traveling wavefronts for time-delayed reaction-diffusion equation:(II)local nonlinearity, *J. Differ. Equations*, **247** (2009), 511–529.
23. K. J. Brown and J. Carr, Deterministic epidemic waves of critical velocity, *Math. Proc. Cambridge Philos. Soc.*, **81** (1977), 431–433.
24. M. A. Lewis, B. Li and H. F. Weinberger, Spreading speed and linear determinacy for two-species competition models, *J. Math. Biol.*, **45** (2002), 219–233.
25. R. Martin and H. Smith, Abstract functional differential equations and reaction-diffusion systems, *Trans. Amer. Math. Soc.*, **321** (1990), 1–44.
26. M. Mei, J.W.-H. So, M. Y. Li, et al., Asymptotic stability of traveling waves for Nicholson's blowflies equation with diffusion, *Proc. Roy. Soc. Edinburgh*, **134A** (2004), 579–594.

27. W.-T. Li, L. Zhang and G.-B. Zhang, Invasion entire solutions in a competition system with nonlocal dispersal, *Discrete Contin. Dyn. Syst.*, **35** (2015), 1531–1560.

28. L.-C. Hung, Exact traveling wave solutions for diffusive Lotka-Volterra systems of two competing species, *Japan J. Indust. Appl. Math.*, **29** (2012), 237–251.

29. W. Huang, Problem on minimum wave speed for a Lotka-Volterra reaction-diffusion competition model, *J. Dynam. Differ. Equations*, **22** (2010), 285–297.

30. N. Fei and J. Carr, Existence of travelling waves with their minimal speed for a diffusing Lotka-Volterra system, *Nonlinear Anal. Real World Appl.*, **4** (2003) 503–524.

31. D. Henry, *G*eometric theory of semilinear parabolic equations, Lecture Notes in Mathematics, **840**, Springer-Verlag, New York Berlin, 1981.

32. C.-H. Hsu, J.-J. Lin and T.-S. Yang, Traveling wave solutions for delayed lattice reaction-diffusion systems, *IMA J. Appl. Math.*, **80** (2015), 302–323.

33. C.-H. Hsu and T.-S. Yang, Existence, uniqueness, monotonicity and asymptotic behavior of traveling waves for a epidemic model, *Nonlinearity*, **26** (2013), 121–139. Corrigendum: **26** (2013), 2925–2928.

34. Y. Kan-on, Note on propagation speed of travelling waves for a weakly coupled parabolic system, *Nonlinear Anal.-Theor.*, **44** (2001), 239–246.

35. Y. Kan-on, Fisher wave fronts for the Lotka-Volterra competition model with diffusion, *Nonlinear Anal.-Theor.*, **28** (1997), 145–164.

36. S. Ma, Traveling wavefronts for delayed reaction-diffusion systems via a fixed point theorem, *J. Differ. Equations*, **171** (2001), 294–314.

37. M. Rodrigo and M. Mimura, Exact solutions of a competition-diffusion system, *Hiroshima Math. J.*, **30** (2000), 257–270.

38. M. Rodrigo and M. Mimura, Exact solutions of reaction-diffusion systems and nonlinear wave equations, *Japan J. Indust. Appl. Math.*, **18** (2001), 657–696.

39. Q. Ye, Z. Li, M. X. Wang, et al., *Introduction to Reaction-Diffusion Equations*, 2^{nd} edition, Science Press, Beijing, 2011.

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