Mathematical Biosciences and Engineering

http://www.aimspress.com/journal/MBE

MBE, 16(5): 3595–3622
DOI: 10.3934/mbe.2019179
Received: 21 February 2019
Accepted: 14 April 2019
Published: 23 April 2019

Research article

Modelling and analysis of an alcoholism model with treatment and effect of Twitter

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Abstract: A new alcoholism model with treatment and effect of Twitter is introduced. The stability of all equilibria which is determined by the basic reproductive number R_0 is obtained. The occurrence of backward and forward bifurcation for a certain defined range of R_0 are established by the center manifold theory. Numerical results and sensitivity analysis on several parameters are conducted. Our results show that Twitter may be a good indicator of alcoholism model and affect the emergence and spread of drinking behavior.

Keywords: Twitter; treatment; stability; backward and forward bifurcation

1. Introduction

The harmful use of alcohol causes a large disease, social and economic burden in societies. In 2012, about 3.3 million deaths, or 5.9% of all global deaths, were attributable to alcohol consumption. Alcohol consumption can have an impact not only on the incidence of diseases, injuries and other health conditions, but also on the course of disorders and their outcomes in individuals [1]. According to a research report by the Shanghai Institute of Environmental Economics, the number of patients due to alcoholism has increased by 28.5 times, and the number of deaths has increased by 30.6 times in the past seven years [2]. Thus, it is very important to study drinking behavior.

Recently, many authors have studied mathematical models of drinking [3, 4, 5, 6, 7, 8, 9]. Bani et al. [3] studied the influence of environmental factors on college alcohol drinking patterns. Mulone et al. [4] developed a two-stage (four compartments) model for youths with serious drinking problems and their treatment, and the stability of all the equilibria was obtained. Mushayabasa et al. [5] formulated a deterministic model for evaluating the impact of heavy alcohol drinking on the reemerging gonorrhea epidemic. Lee et al. [6] studied the optimal control intervention strategies in low- and high-risk problem drinking populations. Mubayi et al. [7] studied the impact of relative residence times on the distribution of heavy drinkers in highly distinct environments and found that alcohol consumption is a function of social dynamics, environmental contexts, individuals' preferences and family history. Huo, Chen and Xiang [8] introduced a more realistic binge drinking model with time delay, in which time delay is used to represent the time lag of the immunity against drinking. Xiang, Liu and Huo [9] proposed a new SAIRS alcoholism model with birth and death on complex heterogeneous networks.

Media coverage is one of the effective ways to control alcoholism or infectious diseases, many authors have studied alcoholism or epidemic models with media coverage [10, 11, 12, 13, 14]. Cui et al. [10] developed a three dimensional compartmental model to investigate the impact of media coverage to the spread and control of infectious diseases. Pawelek et al. [11] studied the impact of twitter on influenza epidemics. Huo and Zhang [12] introduced a more realistic mathematical influenza model including dynamics of Twitter, which might reduce and increase the spread of influenza. Huo and Zhang [13] formulated a novel alcoholism model which involved impact of Twitter and investigated the occurrence of backward, forward bifurcation and Hopf bifurcation. Huo and Yang [14] introduced a novel SEIS epidemic model with the impact of media. Above results show that media coverage can regard as a good indicator in controlling the emergence and spread of the epidemic disease or alcoholism. Many scholars have done a lot of researches on drinking or infectious diseases with or without media coverage [15, 16, 17, 18, 19, 20, 21, 22].

Alcoholism can be defined as a pattern of alcohol use that compromises the health and safety of oneself and others. There are a variety of treatment methods currently available, such as behavioral treatments, medications and mutual-support Groups [23]. The goal of a person pursuing treatment is to abstain from alcohol or to cut back on drinking. Many people have studied the epidemic or alcoholism models with treatment [24, 25, 26].

Motivated by the above [13, 14], we set up a new alcoholism model with treatment and effect of Twitter in this paper. We derive the basic reproductive number of the model and study the stability of our model. Furthermore, we investigate the occurrence of backward and forward bifurcation.

The organization of this paper is as follows: In Section 2, we present a new alcoholism model with treatment and effect of Twitter. In Section 3, we derive the basic reproductive number and study the stability of all equilibria. We also study the occurrence of backward and forward bifurcation. In Section 4, we perform some numerical simulations to illustrate and extend our main results. Sensitivity analysis and some discussion are given in last section.

2. The model formulation

2.1. System description

The total population in this model is divided into four compartments: $S(t)$, $L(t)$, $H(t)$, $R(t)$. $S(t)$ represents the number of moderate drinkers, that is, the people who do not drink or drink within daily and weekly limits [13]. $L(t)$ represents the number of light problem drinkers, that is, the drinkers who drink beyond daily or weekly ceiling [13]. $H(t)$ represents the number of heavy problem drinkers, that is, the drinkers who drink far more than daily and weekly limits [13]. $R(t)$ represents the number of quitting drinkers , that is, the people who quit problem drinking by treatment permanently. $T(t)$ represents the number of messages that Twitter provide about alcoholism at time t. The total number of population at time t is given by

$$N(t) = S(t) + L(t) + H(t) + R(t).$$

The population flowing among those compartments is shown in the following diagram (Figure 1).

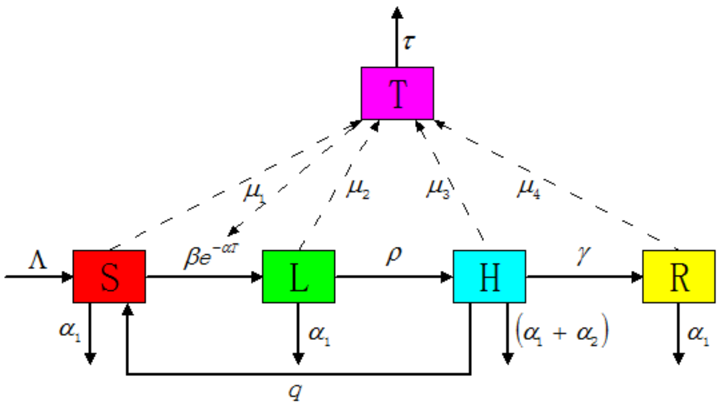


Figure 1. Flowchart of the alcoholism model with the influence of Twitter.

The diagram leads to the following system of ordinary differential equations:

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = \Lambda + qH - \beta SHe^{-\alpha T} - \alpha_1 S, \\ \dfrac{\mathrm{d}L}{\mathrm{d}t} = \beta SHe^{-\alpha T} - \rho L - \alpha_1 L, \\ \dfrac{\mathrm{d}H}{\mathrm{d}t} = \rho L - \gamma H - qH - (\alpha_1 + \alpha_2)H, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \gamma H - \alpha_1 R, \\ \dfrac{\mathrm{d}T}{\mathrm{d}t} = \mu_1 S + \mu_2 L + \mu_3 H + \mu_4 R - \tau T. \end{cases} \tag{2.1}$$

Where all the parameters are positive constants and Λ is the recruitment rate of the population. α_1 is the natural death rate. α_2 is the alcoholism-related death rate. β is the rate of transmission between moderate drinkers and heavy problem drinkers, and it is reduced by a factor $e^{-\alpha T}$ due to the behavior change of the public after reading information about alcoholism. α is the coefficient that determines how effective the drinking information can reduce the transmission rate. τ is outdated-rate of tweets. ρ is the transmission rate from the light problem drinkers to the heavy problem drinkers. After treatment, the transfer rate of the heavy problem drinkers to the moderate drinkers is q, the transfer rate of the heavy problem drinkers to the quitting drinkers is γ. $\mu_i (i = 1, 2, 3, 4)$ are the rates that moderate drinkers, light problem drinkers, heavy problem drinkers and quitting drinkers may tweet about alcoholism during an alcoholism occasion, respectively.

Adding the first four equations of system (2.1), we have

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{\mathrm{d}S}{\mathrm{d}t} + \frac{\mathrm{d}L}{\mathrm{d}t} + \frac{\mathrm{d}H}{\mathrm{d}t} + \frac{\mathrm{d}R}{\mathrm{d}t} = \Lambda - \alpha_1 N - \alpha_2 H \leq \Lambda - \alpha_1 N.$$

Then it follows that $\limsup_{t\to\infty} N(t) \leq \frac{\Lambda}{\alpha_1}$.

According to the fifth equation of system (2.1), we obtain

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \mu_1 S + \mu_2 L + \mu_3 H + \mu_4 R - \tau T \leq \frac{\Lambda(\mu_1 + \mu_2 + \mu_3 + \mu_4)}{\alpha_1} - \tau T,$$

then it follows that $\limsup_{t\to\infty} T(t) \leq \frac{\Lambda(\mu_1+\mu_2+\mu_3+\mu_4)}{\alpha_1 \tau}$, so the set is

$$\Omega = \Big\{(S, L, H, R, T) \in R_+^5 : 0 \leq S, L, H, R \leq N \leq \frac{\Lambda}{\alpha_1}, 0 \leq T \leq \frac{\Lambda(\mu_1 + \mu_2 + \mu_3 + \mu_4)}{\alpha_1 \tau}\Big\}. \tag{2.2}$$

Therefore, we will consider the global stability of system (2.1) on the set Ω.

3. Analysis of the model

3.1. Alcohol free equilibrium and the basic reproductive number

It is easy to see that system (2.1) always has a alcohol free equilibrium $P_0 = (S_0, L_0, H_0, R_0, T_0)$, where

$$S_0 = \frac{\Lambda}{\alpha_1}, L_0 = 0, H_0 = 0, R_0 = 0, T_0 = \frac{\Lambda\mu_1}{\alpha_1\tau}.$$

By applying the method of the next generation matrix in [27], we obtain the basic reproduction number R_0. System (2.1) can be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(x) - V(x),$$

where $x = (L, S, H, R, T)^T$,

$$F(x) = \begin{pmatrix} \beta S H e^{-\alpha T} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{and } V(x) = \begin{pmatrix} \rho L + \alpha_1 L \\ -\Lambda - qH + \beta S H e^{-\alpha T} + \alpha_1 S \\ -\rho L + qH + \gamma H + (\alpha_1 + \alpha_2)H \\ -\gamma H + \alpha_1 R \\ -\mu_1 S - \mu_2 L - \mu_3 H - \mu_4 R + \tau T \end{pmatrix}.$$

The Jacobian matrices of $F(x)$ and $V(x)$ at the alcohol free equilibrium P_0 are

$$DF(P_0) = \begin{pmatrix} 0 & 0 & \frac{\Lambda\beta}{\alpha_1} e^{-\frac{\alpha\mu_1\Lambda}{\alpha_1\tau}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$DV(P_0) = \begin{pmatrix} \rho + \alpha_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \frac{\Lambda\beta}{\alpha_1} e^{-\frac{\alpha\mu_1\Lambda}{\alpha_1\tau}} - q & 0 & 0 \\ -\rho & 0 & \alpha_1 + \alpha_2 + q + \gamma & 0 & 0 \\ 0 & 0 & -\gamma & \alpha_1 & 0 \\ -\mu_2 & -\mu_1 & -\mu_3 & -\mu_4 & \tau \end{pmatrix}.$$

Therefore, the basic reproduction number R_0 is

$$R_0 = \frac{\Lambda\beta\rho e^{\frac{-\alpha\mu_1\Lambda}{\alpha_1\tau}}}{\alpha_1(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma)}. \tag{3.1}$$

3.2. Stability of alcohol free equilibrium

Theorem 1. When $R_0 < 1$ and $T(t) \geq \frac{\Lambda\mu_1}{\alpha_1\tau}$, the alcohol free equilibrium P_0 of system (2.1) is globally asymptotically stable; When $R_0 < 1$ and $T(t) < \frac{\Lambda\mu_1}{\alpha_1\tau}$, the alcohol free equilibrium P_0 of system (2.1) is locally asymptotically stable; When $R_0 > 1$, the alcohol free equilibrium P_0 of system (2.1) is unstable.
Proof. The characteristic equation of the system (2.1) at the alcohol free equilibrium P_0 is

$$\begin{vmatrix} \lambda+\alpha_1 & 0 & \frac{\Lambda\beta}{\alpha_1}e^{\frac{-\alpha\mu_1\Lambda}{\alpha_1\tau}}-q & 0 & 0 \\ 0 & \lambda+(\alpha_1+\rho) & -\frac{\Lambda\beta}{\alpha_1}e^{\frac{-\alpha\mu_1\Lambda}{\alpha_1\tau}} & 0 & 0 \\ 0 & -\rho & \lambda+(\alpha_1+\alpha_2+q+\gamma) & 0 & 0 \\ 0 & 0 & -\gamma & \lambda+\alpha_1 & 0 \\ -\mu_1 & -\mu_2 & -\mu_3 & -\mu_4 & \lambda+\tau \end{vmatrix} = 0. \tag{3.2}$$

Therefore, Eq.(3.2) can be written as

$$(\lambda+\tau)(\lambda+\alpha_1)^2\Big[(\lambda+(\alpha_1+\rho))(\lambda+(\alpha_1+\alpha_2+q+\gamma)) - \frac{\Lambda\beta\rho}{\alpha_1}e^{\frac{-\alpha\mu_1\Lambda}{\alpha_1\tau}}\Big] = 0. \tag{3.3}$$

Therefore, the three eigenvalues of the Eq.(3.2) are $\lambda_1 = -\tau$, $\lambda_2 = -\alpha_1$, $\lambda_3 = -\alpha_1$, and the other eigenvalues are determined by the equation

$$(\lambda+(\alpha_1+\rho))(\lambda+(\alpha_1+\alpha_2+q+\gamma)) - \frac{\Lambda\beta\rho}{\alpha_1}e^{\frac{-\alpha\mu_1\Lambda}{\alpha_1\tau}} = 0. \tag{3.4}$$

Therefore, the Eq.(3.4) can be written as

$$\lambda^2 + \lambda(2\alpha_1+\alpha_2+q+\gamma+\rho) + (\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma)(1-R_0) = 0. \tag{3.5}$$

By Viete theorem, we have

$$\lambda_4+\lambda_5 = -(2\alpha_1+\alpha_2+q+\gamma+\rho) < 0,$$

and

$$\lambda_4\lambda_5 = (\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma)(1-R_0).$$

Thus, when $R_0 < 1$, the alcohol free equilibrium P_0 is locally asymptotically stable; when $R_0 > 1$, the alcohol free equilibrium P_0 is unstable.

Define the Lyapunov function

$$M(S, L, H, R, T) = \rho L(t) + (\alpha_1+\rho)H(t).$$

It is clear that $M(t) \geq 0$ and the equality holds if and only if $L(t) = H(t) = 0$. Differentiating $M(S, L, H, R, T)$ and from the Eq.(2.2), we obtain $S(t) \leq \frac{\Lambda}{\alpha_1}$. Therefore, when $T(t) \geq \frac{\Lambda\mu_1}{\alpha_1\tau}$, we have

$$\begin{aligned}
\frac{\mathrm{d}M(S, L, H, R, T)}{\mathrm{d}t} &= \rho\frac{dL(t)}{dt} + (\alpha_1 + \rho)\frac{dH(t)}{dt} \\
&= \rho(\beta S H e^{-\alpha T} - (\alpha_1 + \rho)L) + (\alpha_1 + \rho)(\rho L - (\alpha_1 + \alpha_2 + \gamma + q)H) \\
&= [\rho\beta S e^{-\alpha T} - (\alpha_1 + \rho)(\alpha_1 + \alpha_2 + \gamma + q)]H \\
&\leq \Big[\frac{\Lambda\beta\rho}{\alpha_1} e^{\frac{-\alpha\mu_1\Lambda}{\alpha_1\tau}} - (\alpha_1 + \rho)(\alpha_1 + \alpha_2 + \gamma + q)\Big]H \\
&= (\alpha_1 + \rho)(\alpha_1 + \alpha_2 + \gamma + q)H\Big[\frac{\Lambda\beta\rho e^{\frac{-\alpha\mu_1\Lambda}{\alpha_1\tau}}}{\alpha_1(\alpha_1 + \rho)(\alpha_1 + \alpha_2 + \gamma + q)} - 1\Big] \\
&= (\alpha_1 + \rho)(\alpha_1 + \alpha_2 + \gamma + q)H[R_0 - 1].
\end{aligned} \tag{3.6}$$

It follows that $M(S, L, H, R, T)$ is bounded and non-increasing. Thus, $\lim\limits_{t\to\infty} M(S, L, H, R, T)$ exists. Note that $R_0 < 1$ guarantees that $\frac{\mathrm{d}M(S,L,H,R,T)}{\mathrm{d}t} \leq 0$ for all $t \geq 0$. Consequently, for system (2.1) there holds

$$\lim_{t\to\infty} L(t) = 0, \quad \lim_{t\to\infty} H(t) = 0.$$

Hence, by LaSalle's Invariance Principle [28], the alcohol free equilibrium is globally attractive. We show that the alcohol free equilibrium P_0 is globally asymptotic stability when $R_0 < 1$.

3.3. Existence of alcoholism equilibrium

Theorem 2. (I) When $\theta = 0$ and $R_0 > 1$, the system (2.1) has a unique positive alcoholism equilibrium P_0^*;
(II)When $\theta \neq 0$ and $R_0 > \max\{R_{01}, 1\}$, the system (2.1) has a unique positive alcoholism equilibrium P_1^*;
(III)When $R_{02} = R_0 < \min\{R_{01}, 1\}$, the system (2.1) has a unique positive alcoholism equilibrium P_2^*;
(IV)When $R_{02} < R_0 < \min\{R_{01}, 1\}$, the system (2.1) has two different positive alcoholism equilibria P_3^* and P_4^*.

Proof. Assuming the right-hand sides of system (2.1) is 0, we have

$$\begin{cases}
\Lambda + qH - \beta S H e^{-\alpha T} - \alpha_1 S = 0, \\
\beta S H e^{-\alpha T} - \rho L - \alpha_1 L = 0, \\
\rho L - \gamma H - qH - (\alpha_1 + \alpha_2)H = 0, \\
\gamma H - \alpha_1 R = 0, \\
\mu_1 S + \mu_2 L + \mu_3 H + \mu_4 R - \tau T = 0.
\end{cases} \tag{3.7}$$

Let $(S, L, H, R, T) = (S^*, L^*, H^*, R^*, T^*)$ be the solution of Eq.(3.7), we have

$$\begin{cases}
\Lambda + qH^* - \beta S^* H^* e^{-\alpha T^*} - \alpha_1 S^* = 0, \\
\beta S^* H^* e^{-\alpha T^*} - \rho L^* - \alpha_1 L^* = 0, \\
\rho L^* - \gamma H^* - qH^* - (\alpha_1 + \alpha_2)H^* = 0, \\
\gamma H^* - \alpha_1 R^* = 0, \\
\mu_1 S^* + \mu_2 L^* + \mu_3 H^* + \mu_4 R^* - \tau T^* = 0.
\end{cases} \tag{3.8}$$

By Eq.(3.8), we obtain

$$S^* = \frac{\Lambda}{\alpha_1} + \frac{[\rho q - (\alpha_1+\rho)(\alpha_1+\alpha_2+\gamma+q)]H^*}{\alpha_1\rho}, \tag{3.9}$$

$$L^* = \frac{(\alpha_1+\alpha_2+\gamma+q)H^*}{\rho}, \tag{3.10}$$

$$R^* = \frac{\gamma H^*}{\alpha_1}, \tag{3.11}$$

$$T^* = \frac{\Lambda\mu_1}{\alpha_1\tau} + \frac{H^*}{\alpha_1\rho\tau}[\mu_1 q\rho - \mu_1(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma) + \alpha_1\mu_2(\alpha_1+\alpha_2+q+\gamma) + \alpha_1\mu_3\rho + \gamma\mu_4\rho]. \tag{3.12}$$

Combine the above Eqs.(3.9)-(3.12) and the first equation of Eq.(3.8), we have

$$\Big[1 - \frac{\theta H^*}{R_{01}}\Big]R_0 = e^{-\theta H^*}, \tag{3.13}$$

where

$$R_{01} = \frac{\Lambda\rho\theta}{(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma) - \rho q}, \tag{3.14}$$

and

$$\theta = \frac{-\alpha[\mu_1 q\rho - \mu_1(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma) + \alpha_1\mu_2(\alpha_1+\alpha_2+q+\gamma) + \alpha_1\mu_3\rho + \gamma\mu_4\rho]}{\alpha_1\rho\tau}. \tag{3.15}$$

For the sake of simplicity, we define

$$R_{02} = R_{01}e^{1-R_{01}}. \tag{3.16}$$

In what follows, we assume

$$F(H^*) = R_0 - \frac{R_0}{R_{01}}\theta H^* - e^{-\theta H^*}. \tag{3.17}$$

Thus

$$F'(H^*) = \theta e^{-\theta H^*} - \frac{R_0}{R_{01}}\theta, \tag{3.18}$$

$$F''(H^*) = -\theta^2 e^{-\theta H^*}. \tag{3.19}$$

The following work is to discuss the properties of Eq.(3.17).

(I) When $\theta = 0$ and $R_0 > 1$, the existence of the unique alcoholism equilibrium P_0^* of system (2.1) can be obtained by Eq.(3.13), as shown in line L_4 of Figure 2, where

$$H_0^* = \frac{\Lambda\rho}{(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma) - \rho q}\left(1 - \frac{1}{R_0}\right),$$

$$S_0^* = \frac{\Lambda}{\alpha_1} - \frac{\Lambda}{\alpha_1}\left(1 - \frac{1}{R_0}\right),$$

$$L_0^* = \frac{\Lambda(\alpha_1+\alpha_2+q+\gamma)}{(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma) - \rho q}\left(1 - \frac{1}{R_0}\right),$$

$$R_0^* = \frac{\Lambda\gamma\rho}{\alpha_1[(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma) - \rho q]}\left(1 - \frac{1}{R_0}\right),$$

$$T_0^* = \frac{\Lambda\mu_1}{\alpha_1\tau} + \frac{\Lambda}{\alpha_1\tau[(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma)-\rho q]}[\mu_1 q\rho + \alpha_1\mu_3\rho + \gamma\mu_4\rho$$
$$+\alpha_1\mu_2(\alpha_1+\alpha_2+q+\gamma) - \mu_1(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma)](1-\frac{1}{R_0}).$$

(II) When $\theta \neq 0$ and $R_0 > 1$, we have $F(0) = R_0 - 1 > 0$ and $F(\infty) = -\infty < 0$. Assume that $F^{'}(H^*) < \theta(1-\frac{R_0}{R_{01}})$. Thus, when $\theta > 0$ and $R_0 > R_{01}$ or $\theta < 0$, we obtain $F^{'}(H^*) < 0$. Therefore, there is a unique positive solution for Eq.(3.17). Thus, the alcoholism equilibrium $P_1^* = (S_1^*, L_1^*, H_1^*, R_1^*, T_1^*)$ can be obtained, as shown in regions Ω_A and Ω_B of Figure 2.
(III) When $\theta > 0$ and $R_0 < 1$, we have $F(0) = R_0 - 1 < 0$, $F(\infty) = -\infty < 0$ and $F^{''}(H^*) < 0$. Assume that $F^{'}(H^*) = \theta e^{-\theta H^*} - \frac{R_0}{R_{01}}\theta = \theta(e^{-\theta H^*} - \frac{R_0}{R_{01}})$. If $F^{'}(H^*) = 0$, Eq.(3.17) has the unique positive solution $H_2^* = \frac{1}{\theta}\ln(\frac{R_{01}}{R_0})$ when $R_0 < R_{01}$. Meanwhile, we also have

$$F(H_2^*) = R_0 - \frac{R_0}{R_{01}}\theta H_2^* - e^{-\theta H_2^*} = 0.$$

Therefore, we obtain $R_0 = R_{02} = R_{01}e^{(1-R_{01})}$. Thus, the alcoholism equilibrium $P_2^* = (S_2^*, L_2^*, H_2^*, R_2^*, T_2^*)$ can be obtained, as shown in line L_2 of Figure 2.
(IV) When $R_{02} < R_0 < 1$, we have $F(H_2^*) > 0$. Eq.(3.17) has two different positive solutions H_3^* and H_4^*, where H_3^* and H_4^* satisfy the following condition $H_3^* < H_2^* < H_4^*$. Thus, the alcoholism equilibria $P_3^* = (S_3^*, L_3^*, H_3^*, R_3^*, T_3^*)$ and $P_4^* = (S_4^*, L_4^*, H_4^*, R_4^*, T_4^*)$ can be obtained, as shown in region Ω_E of Figure 2.

Remark 1. For simplicity, the six curves ($L_i, i = 1,2,3,4,5,6$) divide the space in which R_0 and θ are located into seven regions as shown in Figure 2.

$$\begin{aligned}
L_1: &\quad R_0 = R_{01}(\theta), \text{with } R_0 > 1,\\
L_2: &\quad R_0 = R_{01}(\theta)e^{1-R_{01}(\theta)}, \text{with } R_0 < \min\{R_{01}(\theta), 1\},\\
L_3: &\quad \theta = 0, \text{with } R_0 < 1,\\
L_4: &\quad \theta = 0, \text{with } R_0 > 1,\\
L_5: &\quad R_0 = R_{01}(\theta)e^{1-R_{01}(\theta)}, \text{with } R_{01}(\theta) < R_0 < 1,\\
L_6: &\quad R_0 = 1.
\end{aligned}$$

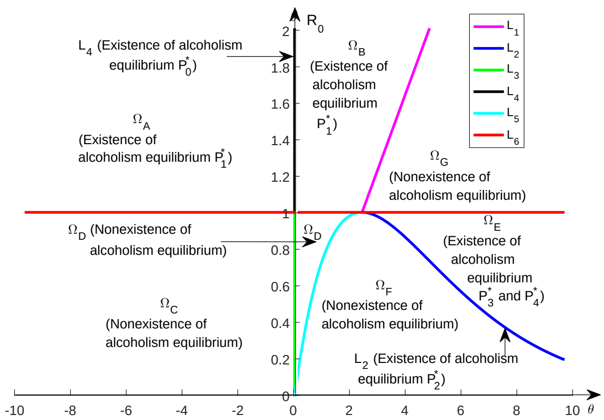


Figure 2. The regions for the existence of alcoholism equilibrium of system (2.1).

3.4. Stability of the alcoholism equilibrium

In this section, we study the local stability of the alcoholism equilibria $P_i^*(i = 0,1,2,3,4)$. First we obtain the characteristic matrix of system (2.1) at the alcoholism equilibria $P_i^*(i = 0,1,2,3,4)$, as follows

$$\begin{vmatrix} \lambda+\beta H_i^* e^{-\alpha T_i^*}+\alpha_1 & 0 & \beta S_i^* e^{-\alpha T_i^*}-q & 0 & -\alpha\beta S_i^* H_i^* e^{-\alpha T_i^*} \\ -\beta H_i^* e^{-\alpha T_i^*} & \lambda+(\alpha_1+\rho) & -\beta S_i^* e^{-\alpha T_i^*} & 0 & \alpha\beta S_i^* H_i^* e^{-\alpha T_i^*} \\ 0 & -\rho & \lambda+(\alpha_1+\alpha_2+q+\gamma) & 0 & 0 \\ 0 & 0 & -\gamma & \lambda+\alpha_1 & 0 \\ -\mu_1 & -\mu_2 & -\mu_3 & -\mu_4 & \lambda+\tau \end{vmatrix} = 0. \quad (3.20)$$

In order to simplify Eq.(3.20), we have

$$\begin{aligned} \Phi = \beta e^{-\alpha T_i^*} &= \frac{\alpha_1(\alpha_1+\rho)(\alpha_1+\alpha_2+\gamma+q)e^{-\alpha T_i^*}}{\Lambda\rho e^{-\frac{\Lambda\alpha\mu_1}{\alpha_1\tau}}} R_0 \\ &= \frac{\alpha_1(\alpha_1+\rho)(\alpha_1+\alpha_2+\gamma+q)e^{\theta H_i^*}}{\Lambda\rho} R_0, \end{aligned}$$

$$\Phi S_i^* = \frac{(\alpha_1+\rho)(\alpha_1+\alpha_2+\gamma+q)}{\rho} = \frac{\Lambda\theta}{R_{01}} + q.$$

Then the characteristic equation can be rewritten as:

$$(\lambda+\alpha_1)G(\lambda) = 0, \quad (3.21)$$

$$G(\lambda) = \lambda^4 + a_1(H_i^*)\lambda^3 + a_2(H_i^*)\lambda^2 + a_3(H_i^*)\lambda + a_4(H_i^*) = 0, \quad (3.22)$$

where

$$\begin{aligned} a_1(H_i^*) &= 3\alpha_1+\alpha_2+q+\gamma+\rho+\tau+H_i^*\Phi, \quad (3.23) \\ a_2(H_i^*) &= (2\alpha_1+\alpha_2+q+\gamma+\rho+\tau)(\alpha_1+H_i^*\Phi)+(2\alpha_1+\alpha_2+q+\gamma+\rho)\tau \\ &\quad +\alpha H_i^*(\frac{\Lambda\theta}{R_{01}}+q)(\mu_2-\mu_1), \quad (3.24) \\ a_3(H_i^*) &= (2\alpha_1+\alpha_2+q+\gamma+\rho)(\alpha_1+H_i^*\Phi)\tau \\ &\quad +(\alpha_1+\alpha_2+q+\gamma)\alpha_1 H_i^*\Phi+(\alpha_1+\alpha_2+\gamma)\rho H_i^*\Phi \\ &\quad +\alpha H_i^*(\frac{\Lambda\theta}{R_{01}}+q)[(2\alpha_1+\alpha_2+q+\gamma)(\mu_2-\mu_1)+\rho(\mu_3-\mu_1)], \quad (3.25) \\ a_4(H_i^*) &= \tau(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma)[(\alpha_1+H_i^*\Phi)-\alpha_1(1+H_i^*\theta)]. \quad (3.26) \end{aligned}$$

Theorem 3. For system (2.1), we assume that $\mu_1 = \mu_2 = \mu_3 = \mu_4$.
(I) When $\theta = 0, \alpha_2 = 0$ and $R_0 > 1(i.e., L_4)$, the alcoholism equilibrium P_0^* is locally asymptotically stable;
(II)When $\theta \neq 0$, $R_0 > \max\{1, R_{01}\}(i.e., \Omega_A$, and $\Omega_B)$, $\Phi > \alpha_1\theta$ and $\tau > \rho$, the alcoholism equilibrium P_1^* is locally asymptotically stable;
(III)When $R_{02} = R_0 < \min\{1, R_{01}\}(i.e., L_2)$, the alcoholism equilibrium P_2^* may be locally stable or not;

(IV)(i)When $R_{02} < R_0 < \min\{1, R_{01}\}(i.e., \Omega_E)$, the alcoholism equilibrium P_3^* is unstable,
(ii)When $R_{02} < R_0 < \min\{1, R_{01}\}(i.e., \Omega_E)$ and $\tau > \rho$, the alcoholism equilibrium P_4^* is locally asymptotically stable.

Proof. (I)When $\theta = 0$, applying to the proof of (I) of Theorem 2. We linearize the system (2.1) and evaluate the characteristic equation at the alcoholism equilibrium P_0^*, and get

$$\begin{vmatrix} \lambda + \beta H_0^* e^{-\alpha T_0^*} + \alpha_1 & 0 & \beta S_0^* e^{-\alpha T_0^*} - q & 0 & -\alpha\beta S_0^* H_0^* e^{-\alpha T_0^*} \\ -\beta H_0^* e^{-\alpha T_0^*} & \lambda + (\alpha_1 + \rho) & -\beta S_0^* e^{-\alpha T_0^*} & 0 & \alpha\beta S_0^* H_0^* e^{-\alpha T_0^*} \\ 0 & -\rho & \lambda + (\alpha_1 + q + \gamma) & 0 & 0 \\ 0 & 0 & -\gamma & \lambda + \alpha_1 & 0 \\ -\mu_1 & -\mu_1 & -\mu_1 & -\mu_1 & \lambda + \tau \end{vmatrix} = 0.$$

Thus, the characteristic equation can be rewritten as:

$$(\lambda + \alpha_1)(\lambda + \tau)G_1(\lambda) = 0,$$

$$G_1(\lambda) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0,$$

where

$$\begin{aligned} b_1 &= 3\alpha_1 + q + \gamma + \rho + H_0^*\Phi, \\ b_2 &= (2\alpha_1 + q + \gamma + \rho)(\alpha_1 + H_0^*\Phi), \\ b_3 &= [(\alpha_1 + \rho)(\alpha_1 + \gamma) + \alpha_1 q]H_0^*\Phi. \end{aligned}$$

It is clear that $b_1 > 0$, $b_2 > 0$ and $b_3 > 0$. Applying Routh−Hurwitz [13], by assuming that $B = b_1b_2 - b_3$. Then, we obtain

$$\begin{aligned} B &= (2\alpha_1 + q + \gamma + \rho)(H_0^*\Phi)^2 \\ &\quad + \left(7\alpha_1{}^2 + 5\alpha_1 q + 5\alpha_1\gamma + 5\alpha_1\rho + q^2 + 2q\gamma + 2q\rho + \gamma^2 + \gamma\rho + \rho^2\right)(H_0^*\Phi) \\ &\quad + 6\alpha_1{}^3 + 5\,\alpha_1{}^2 q + 5\alpha_1{}^2\gamma + 5\alpha_1{}^2\rho + \alpha_1 q^2 + 2\alpha_1 q\gamma \\ &\quad + 2\alpha_1 q\rho + \alpha_1\gamma^2 + 2\alpha_1\gamma\rho + \alpha_1\rho^2 > 0 \end{aligned}$$

Hence, the alcoholism equilibrium P_0^* is locally asymptotically stable.
(II)When $\mu_1 = \mu_2 = \mu_3 = \mu_4$ and $\Phi > \alpha_1\theta$, by Eqs.(3.23)-(3.26), we have

$$\begin{aligned} a_1(H_1^*) &= 3\alpha_1 + \alpha_2 + q + \gamma + \rho + \tau + H_1^*\Phi > 0, \\ a_2(H_1^*) &= (2\alpha_1 + \alpha_2 + q + \gamma + \rho + \tau)(\alpha_1 + H_1^*\Phi) \\ &\quad + (2\alpha_1 + \alpha_2 + q + \gamma + \rho)\tau > 0, \\ a_3(H_1^*) &= (2\alpha_1 + \alpha_2 + q + \gamma + \rho)(\alpha_1 + H_1^*\Phi)\tau \\ &\quad + (\alpha_1 + \alpha_2 + q + \gamma)\alpha_1 H_1^*\Phi + (\alpha_1 + \alpha_2 + \gamma)\rho H_1^*\Phi > 0, \\ a_4(H_1^*) &= \tau(\alpha_1 + \rho)(\alpha_1 + \alpha_2 + q + \gamma)\big[(\alpha_1 + H_1^*\Phi) - \alpha_1(1 + H_1^*\theta)\big] > 0. \end{aligned}$$

Applying Routh−Hurwitz [13], let $C = a_1a_2 - a_3$. Then, we obtain

$$C = c_1 H_1^{*2} + c_2 H_1^* + c_3 > 0,$$

where

$$\begin{aligned}
c_1 &= 2\,\Phi^2\,\alpha_1 + \Phi^2\,\alpha_2 + \Phi^2\,\gamma + \Phi^2\,q + \Phi^2\,\rho + \Phi^2\,\tau > 0,\\
c_2 &= 7\,\Phi\,{\alpha_1}^2 + \Phi\,{\alpha_2}^2 + \Phi\,\gamma^2 + \Phi\,q^2 + \Phi\,\rho^2 + \Phi\,\tau^2 + 5\,\Phi\,\alpha_1\,\alpha_2 + 5\,\Phi\,\alpha_1\,\gamma\\
&\quad +2\,\Phi\,\alpha_2\,\gamma + 5\,\Phi\,\alpha_1\,q + 2\,\Phi\,\alpha_2\,q + 5\,\Phi\,\alpha_1\,\rho + \Phi\,\alpha_2\,\rho + 6\,\Phi\,\alpha_1\,\tau\\
&\quad +2\,\Phi\,q\,\rho + 2\,\Phi\,q\,\tau + 2\,\Phi\,\rho\,\tau + \Phi\,\gamma\,\rho + 2\,\Phi\,\gamma\,q + 2\,\Phi\,\alpha_2\,\tau + 2\,\Phi\,\gamma\,\tau > 0,\\
c_3 &= \alpha_1{\alpha_2}^2 + 5{\alpha_1}^2\alpha_2 + \alpha_1\gamma^2 + 5{\alpha_1}^2\gamma + \alpha_1 q^2 + 5{\alpha_1}^2 q + \alpha_1\rho^2 + 5{\alpha_1}^2\rho\\
&\quad +9\,{\alpha_1}^2\,\tau + \alpha_2\,\tau^2 + {\alpha_2}^2\,\tau + \gamma\,\tau^2 + \gamma^2\,\tau + q\,\tau^2 + q^2\,\tau + \rho\,\tau^2 + \rho^2\,\tau\\
&\quad +2\,\alpha_1\,\alpha_2\,q + 2\,\alpha_1\,\alpha_2\,\rho + 6\,\alpha_1\,\alpha_2\,\tau + 2\,\alpha_1\,\gamma\,q + 2\,\alpha_1\,\gamma\,\rho + 6\,\alpha_1\,\gamma\,\tau\\
&\quad +6\,\alpha_1\,q\,\tau + 2\,\alpha_2\,q\,\tau + 6\,\alpha_1\,\rho\,\tau + 2\,\alpha_2\,\rho\,\tau + 2\,\gamma\,q\,\tau + 2\,\gamma\,\rho\,\tau + 2\,q\,\rho\,\tau\\
&\quad +2\,\alpha_1\,q\,\rho + 3\alpha_1\tau^2 + 6\,{\alpha_1}^3 + 2\,\alpha_1\,\alpha_2\,\gamma + 2\,\alpha_2\,\gamma\,\tau > 0.
\end{aligned}$$

Then, let $D = a_3[a_1a_2 - a_3] - a_1^2 a_4$, and get

$$D = d_1 H_1^{*2} + d_2 H_1^* + d_3,$$

It is clear that $D > 0$ and $d_i > 0 (i = 1, 2, 3)$, when $\tau > \rho$. Because the expression of $d_i (i = 1, 2, 3)$ are too long, we list them in Appendix. Hence, the alcoholism equilibrium P_1^* is locally asymptotically stable.

(III)Applying to the proof of (III) of Theorem 2, when $H_2^* = \frac{1}{\theta}\ln(\frac{R_{01}}{R_0})$, we obtain $\Phi = \alpha_1\theta$. Thus, by Eq.(3.26), we have

$$a_4(H_2^*) = \tau(\alpha_1 + \rho)(\alpha_1 + \alpha_2 + q + \gamma)\big[(\alpha_1 + H_2^*\Phi) - \alpha_1(1 + H_2^*\theta)\big] = 0,$$

and by Eq.(3.23), we have

$$a_1(H_2^*) = 3\alpha_1 + \alpha_2 + q + \gamma + \rho + \tau + H_2^*\Phi > 0.$$

Thus, we know that Eq.(3.21) has negative and zero eigenvalues. Therefore, the alcoholism equilibrium P_2^* may be locally stable or not.

(IV)(i)Applying to the proof of (IV) of Theorem 2, when $H_3^* < H_2^* = \frac{1}{\theta}\ln(\frac{R_{01}}{R_0})$, we obtain $\Phi < \alpha_1\theta$. Thus, by Eq.(3.26), we have

$$a_4(H_3^*) = \tau(\alpha_1 + \rho)(\alpha_1 + \alpha_2 + q + \gamma)\big[(\alpha_1 + H_3^*\Phi) - \alpha_1(1 + H_3^*\theta)\big] < 0,$$

and by Eq.(3.23), we have

$$a_1(H_3^*) = 3\alpha_1 + \alpha_2 + q + \gamma + \rho + \tau + H_3^*\Phi > 0.$$

Assuming $g_1(H_3^*), g_2(H_3^*), g_3(H_3^*), g_4(H_3^*)$ is the root of Eq.(3.22), and we assume that the real parts satisfying $Re(g_1(H_3^*)) \le Re(g_2(H_3^*)) \le Re(g_3(H_3^*)) \le Re(g_4(H_3^*))$, so we obtain

$$g_1(H_3^*) + g_2(H_3^*) + g_3(H_3^*) + g_4(H_3^*) = -a_1(H_3^*) < 0,$$

and

$$g_1(H_3^*)g_2(H_3^*)g_3(H_3^*)g_4(H_3^*) = a_4(H_3^*) < 0.$$

There are $Re(g_1(H_3^*)) < 0$ and $Re(g_4(H_3^*)) > 0$, thus, the alcoholism equilibrium P_3^* is unstable.
(ii)When $H_4^* > H_2^* = \frac{1}{\theta}\ln(\frac{R_{01}}{R_0})$, we obtain $\Phi > \alpha_1\theta$. Thus, by Eq.(3.26), we have

$$a_4(H_4^*) = \tau(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma)\big[(\alpha_1+H_4^*\Phi)-\alpha_1(1+H_4^*\theta)\big] > 0.$$

By $\mu_1 = \mu_2 = \mu_3 = \mu_4$, we have

$$\begin{aligned}
a_1(H_4^*) &= 3\alpha_1+\alpha_2+q+\gamma+\rho+\tau+H_4^*\Phi > 0,\\
a_2(H_4^*) &= (2\alpha_1+\alpha_2+q+\gamma+\rho+\tau)(\alpha_1+H_4^*\Phi)+(2\alpha_1+\alpha_2+q+\gamma+\rho)\tau > 0,\\
a_3(H_4^*) &= (2\alpha_1+\alpha_2+q+\gamma+\rho)(\alpha_1+H_4^*\Phi)\tau\\
&\quad +(\alpha_1+\alpha_2+q+\gamma)\alpha_1H_4^*\Phi+(\alpha_1+\alpha_2+\gamma)\rho H_4^*\Phi > 0.
\end{aligned}$$

Applying Routh−Hurwitz [13], by assuming that $E = a_1a_2 - a_3$. Then, we obtain

$$E = e_1H_4^{*2} + e_2H_4^* + e_3 > 0,$$

where

$$\begin{aligned}
e_1 &= 2\,\Phi^2\,\alpha_1+\Phi^2\,\alpha_2+\Phi^2\,\gamma+\Phi^2\,q+\Phi^2\,\rho+\Phi^2\,\tau > 0,\\
e_2 &= 7\,\Phi\,\alpha_1{}^2+\Phi\,\alpha_2{}^2+\Phi\,\gamma^2+\Phi\,q^2+\Phi\,\rho^2+\Phi\,\tau^2+5\,\Phi\,\alpha_1\,\alpha_2+5\,\Phi\,\alpha_1\,\gamma+2\,\Phi\,\gamma\,q+\Phi\,\gamma\,\rho\\
&\quad +5\,\Phi\,\alpha_1\,q+2\,\Phi\,\alpha_2\,\gamma+5\,\Phi\,\alpha_1\,\rho+2\,\Phi\,\alpha_2\,q+\Phi\,\alpha_2\,\rho+6\,\Phi\,\alpha_1\,\tau+2\,\Phi\,\alpha_2\,\tau+2\,\Phi\,\gamma\,\tau\\
&\quad +2\,\Phi\,q\,\rho+2\,\Phi\,q\,\tau+2\,\Phi\,\rho\,\tau > 0,\\
e_3 &= 5\,\alpha_1{}^2\,\alpha_2+\alpha_1\,\alpha_2{}^2+\alpha_1\,\gamma^2+5\,\alpha_1{}^2\,\gamma+\alpha_1\,q^2+5\,\alpha_1{}^2\,q+\alpha_1\,\rho^2+5\,\alpha_1{}^2\,\rho+3\,\alpha_1\,\tau^2\\
&\quad +9\,\alpha_1{}^2\,\tau+\alpha_2\,\tau^2+\alpha_2{}^2\,\tau+\gamma\,\tau^2+\gamma^2\,\tau+q\,\tau^2+q^2\,\tau+\rho\,\tau^2+\rho^2\,\tau+6\,\alpha_1{}^3+2\,\alpha_1\,\alpha_2\,\gamma\\
&\quad +2\,\alpha_1\,\alpha_2\,q+2\,\alpha_1\,\alpha_2\,\rho+6\,\alpha_1\,\alpha_2\,\tau+2\,\alpha_1\,\gamma\,q+2\,\alpha_1\,\gamma\,\rho+6\,\alpha_1\,\gamma\,\tau+2\,\alpha_2\,\gamma\,\tau+2\,\alpha_1\,q\,\rho\\
&\quad +2\,\alpha_2\,q\,\tau+6\,\alpha_1\,q\,\tau+6\,\alpha_1\,\rho\,\tau+2\,\alpha_2\,\rho\,\tau+2\,\gamma\,\rho\,\tau+2\,\gamma\,q\,\tau+2\,q\,\rho\,\tau > 0.
\end{aligned}$$

Then, by assuming that $F = a_3[a_1a_2 - a_3] - a_1^2a_4$, and get

$$F = f_1H_4^{*2} + f_2H_4^* + f_3,$$

It is clear that $D > 0$, when $f_i > 0 (i = 1, 2, 3)$ and $\tau > \rho$. Because the expression of $f_i (i = 1, 2, 3)$ are too long, we do not list it here, and it is placed in Appendix. Hence, the alcoholism equilibrium P_4^* is locally asymptotically stable.

3.5. Forward and Backward Bifurcation

In this section, we study the change of the parameter β causing a forward or a backward bifurcation to occur.

Theorem 4. (I) If $R_{01} > 1$, when $R_0 = 1$, the system (2.1) appears a backward bifurcation.
(II) If $R_{01} < 1$, when $R_0 = 1$, the system (2.1) appears a forward bifurcation.

Proof. Using the central manifold theory described in [29]. Introducing $x_1 = S$, $x_2 = L$, $x_3 = H$, $x_4 = R$, $x_5 = T$, the system (2.1) becomes

$$\begin{cases} \dfrac{dx_1}{dt} = \Lambda + qx_3 - \beta x_1 x_3 e^{-\alpha x_5} - \alpha_1 x_1 := f_1, \\ \dfrac{dx_2}{dt} = \beta x_1 x_3 e^{-\alpha x_5} - \rho x_2 - \alpha_1 x_2 := f_2, \\ \dfrac{dx_3}{dt} = \rho x_2 - \gamma x_3 - qx_3 - (\alpha_1 + \alpha_2)x_3 := f_3, \\ \dfrac{dx_4}{dt} = \gamma x_3 - \alpha_1 x_4 := f_4, \\ \dfrac{dx_5}{dt} = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3 + \mu_4 x_4 - \tau x_5 := f_5. \end{cases} \tag{3.27}$$

Thus, the alcohol free equilibrium P_0 is

$$P_0 = X_0 = (\frac{\Lambda}{\alpha_1}, 0, 0, 0, \frac{\Lambda\mu_1}{\alpha_1\tau}),$$

In view of Theorem 4.1 [29], the Jacobian matrix $J(P_0)$ of the system (3.27) in the alcohol free equilibrium is

$$J(X_0) = \begin{pmatrix} -\alpha_1 & 0 & q - \frac{\Lambda\beta}{\alpha_1} e^{\frac{-\alpha\mu_1\Lambda}{\alpha_1\tau}} & 0 & 0 \\ 0 & -(\alpha_1+\rho) & \frac{\Lambda\beta}{\alpha_1} e^{\frac{-\alpha\mu_1\Lambda}{\alpha_1\tau}} & 0 & 0 \\ 0 & \rho & -(\alpha_1+\alpha_2+q+\gamma) & 0 & 0 \\ 0 & 0 & \gamma & -\alpha_1 & 0 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & -\tau \end{pmatrix}.$$

We establish the local stability of alcohol free equilibrium taking β as bifurcation parameter, when $R_0 = 1$ corresponding to $\beta = \beta^* = \frac{\alpha_1(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma)e^{\frac{\Lambda\alpha\mu_1}{\alpha_1\tau}}}{\Lambda\rho}$, thus, we obtain

$$J(X_0) = \begin{pmatrix} -\alpha_1 & 0 & q - \frac{\alpha_1(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma)}{\rho} & 0 & 0 \\ 0 & -(\alpha_1+\rho) & \frac{\alpha_1(\alpha_1+\rho)(\alpha_1+\alpha_2+q+\gamma)}{\rho} & 0 & 0 \\ 0 & \rho & -(\alpha_1+\alpha_2+q+\gamma) & 0 & 0 \\ 0 & 0 & \gamma & -\alpha_1 & 0 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & -\tau \end{pmatrix}.$$

It is clear that 0 is a simple eigenvalue of $J(P_0)$. Therefore, there is a right eigenvector associated with 0 eigenvalues that is $R = (r_1, r_2, r_3, r_4, r_5)^T$, there is a left eigenvector associated with 0 eigenvalues is $L = (l_1, l_2, l_3, l_4, l_5)$, and it is required to satisfy $LR = 1$.

Therefore, the right eigenvector is

$$R = \begin{pmatrix} \frac{-\alpha_1(\alpha_1+\alpha_2+q+\gamma)-\rho(\alpha_1+\alpha_2+\gamma)}{\alpha_1\rho} \\ \frac{\alpha_1+\alpha_2+q+\gamma}{\rho} \\ 1 \\ \frac{\gamma}{\alpha_1} \\ \frac{-(\mu_1-\mu_2)(\alpha_1+\alpha_2+q+\gamma)}{\rho\tau} - \frac{(\alpha_1\mu_1-\alpha_1\mu_3+\alpha_2\mu_1+\gamma\mu_1-\gamma\mu_4)}{\alpha_1\tau} \end{pmatrix},$$

the left eigenvector is

$$L = \Big(0, \frac{\rho}{2\alpha_1 + \alpha_2 + q + \gamma + \rho}, \frac{\alpha_1 + \rho}{2\alpha_1 + \alpha_2 + q + \gamma + \rho}, 0, 0\Big).$$

In view of Theorem 4.1 [29], we know that

$$a = \sum_{k,i,j=1}^{5} l_k r_i r_j \frac{\partial^2 f_k(X_0)}{\partial x_i \partial x_j},$$
$$b = \sum_{k,i=1}^{5} l_k r_i \frac{\partial^2 f_k(X_0)}{\partial x_i \partial \beta}.$$

Therefore, we obtain

$$\begin{aligned} a =& l_2 r_1 r_3 \frac{\partial^2 f_2(X_0)}{\partial x_1 \partial x_3} + l_2 r_3 r_1 \frac{\partial^2 f_2(X_0)}{\partial x_3 \partial x_1} + l_2 r_3 r_5 \frac{\partial^2 f_2(X_0)}{\partial x_3 \partial x_5} + l_2 r_5 r_3 \frac{\partial^2 f_2(X_0)}{\partial x_5 \partial x_3} \\ =& 2 l_2 \Big(r_1 r_3 \frac{\partial^2 f_2(X_0)}{\partial x_1 \partial x_3} + r_3 r_5 \frac{\partial^2 f_2(X_0)}{\partial x_3 \partial x_5}\Big) \\ =& \frac{-2\rho\Lambda\alpha\beta e^{-\frac{\Lambda\alpha\mu_1}{\alpha_1\tau}}}{\alpha_1(2\alpha_1 + \alpha_2 + q + \gamma + \rho)} \Big[\frac{\alpha_1(\mu_2 - \mu_1)(\alpha_1 + \alpha_2 + q + \gamma) - \rho(\alpha_1\mu_1 - \alpha_1\mu_3 + \alpha_2\mu_1 + \gamma\mu_1 - \gamma u_4)}{\alpha_1\rho\tau}\Big] \\ &+ \frac{2\rho\beta e^{-\frac{\Lambda\alpha\mu_1}{\alpha_1\tau}}}{2\alpha_1 + \alpha_2 + q + \gamma + \rho} \Big[\frac{-(\alpha_1 + \rho)(\alpha_1 + \alpha_2 + \gamma) - \alpha_1 q}{\alpha_1\rho}\Big] \\ =& 2\Big[\frac{\alpha_1(\alpha_1 + \rho)(\alpha_1 + \alpha_2 + q + \gamma)^2 + \rho(\alpha_1 + \rho)(\alpha_1 + \alpha_2 + \gamma)(\alpha_1 + \alpha_2 + q + \gamma)}{\Lambda\rho(2\alpha_1 + \alpha_2 + q + \gamma + \rho)}\Big](R_{01} - 1). \\ b =& l_2 r_3 \frac{\partial^2 f_2(X_0)}{\partial x_3 \partial \beta} = \frac{\Lambda\rho e^{-\frac{\Lambda\alpha\mu_1}{\alpha_1\tau}}}{\alpha_1(2\alpha_1 + \alpha_2 + q + \gamma + \rho)} > 0. \end{aligned}$$

According to Theorem 4.1 of [29], we notice that the coefficient b is always positive. The coefficient a is positive when $R_{01} > 1$. In this case, the direction of the bifurcation of the system (2.1) at $R_0 = 1$ is backward, as shown in the Figure 9(a). The coefficient a is negative when $R_{01} < 1$. Under this circumstance, the direction of the bifurcation of the system (2.1) at $R_0 = 1$ is forward, as shown in the Figure 9(b).

4. Numerical results

The goal of this section is to present some numerical simulations which complement the theoretical results in the previous sections. We choose some parameters based on the Table 1.

Table 1. The parameters description of the alcoholism model.

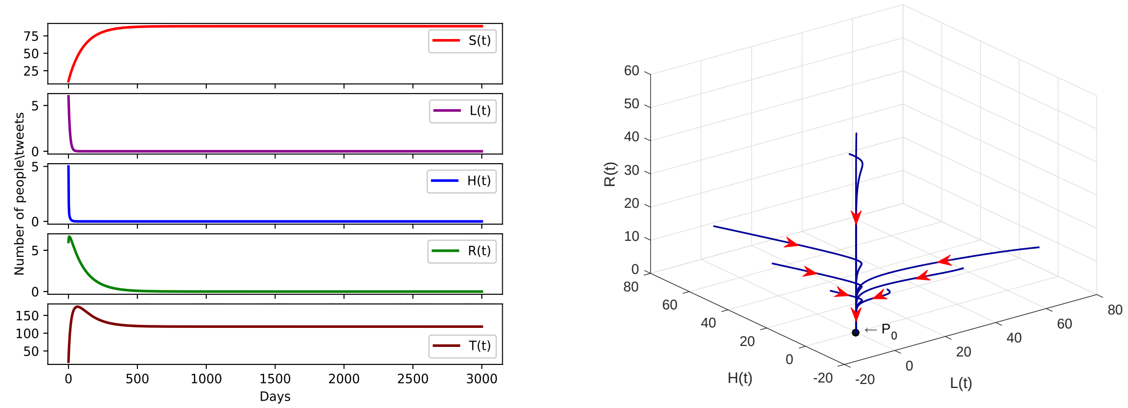
Parameter	Description	Value	Source
Λ	The constant recruitment rate of the population	$0.7-0.8day^{-1}$	[30]
β	Transmission coefficient from the moderate drinkers compartment to the light problem drinkers compartment	$0.0099-0.9person^{-1}$	Estimate
α	The coefficient that determines how effective the positive drinking information can reduce the transmission rate	$0.00091-0.8tweet^{-1}$	Estimate
ρ	Transmission coefficient from the light problem drinkers compartment to the heavy problem drinkers compartment	$0.04-0.99day^{-1}$	Estimate
μ_1	The rates that the moderate drinkers may tweet about alcoholism during an alcoholism occasion	$0-1day^{-1}$	[11]
μ_2	The rates that the light problem drinkers may tweet about alcoholism during an alcoholism occasion	$0-1day^{-1}$	[11]
μ_3	The rates that the heavy problem drinkers may tweet about alcoholism during an alcoholism occasion	$0-1day^{-1}$	[11]
μ_4	The rates that quitting drinkers may tweet about alcoholism during an alcoholism occasion	$0-1day^{-1}$	[13]
α_1	The natural death rate of the population	$0.009-0.6year^{-1}$	[4, 5]
α_2	The death rate due to heavy alcoholism	$0.02-0.5day^{-1}$	Estimate
q	Transmission coefficient from the heavy problem drinkers compartment to the moderate drinkers compartment	$0.006-0.99day^{-1}$	Estimate
γ	Transmission coefficient from the heavy problem drinkers compartment to quitting drinkers compartment	$0.006-0.99day^{-1}$	Estimate
τ	The rate that message become outdated	$0.03-0.6year^{-1}$	[11]

As an example, we choose a set of the following parameters, the parameter values are $\Lambda = 0.8, \alpha = 0.007, \alpha_1 = 0.009, \alpha_2 = 0.5, \mu_1 = 0.04, \mu_2 = 0.8, \mu_3 = 0.8, \mu_4 = 0.8, \gamma = 0.1, q = 0.07, \rho = 0.09, \tau = 0.03$ and $\beta = 0.001$. It follows from Theorem 1 that the alcohol free equilibrium $P_0 = (88.89, 0, 0, 0, 118.52)$ of system (2.1) is globally asymptotically stable for any value of time t when $R_0 = 0.0519 < 1$ (see Figure 3 (a) and (b)). Furthermore, we can also observe that the value of the equilibrium $P^*(t)$ changes as t increasing and eventually tends to $P_0 = (88.89, 0, 0, 0, 118.52)$ from Figure 3 (a) and (b).

In order to verify the local stability of the alcoholism equilibrium P_1^*, we choose a set of the following parameters, the parameter values are $\Lambda = 0.8, \alpha = 0.007, \alpha_1 = 0.009, \alpha_2 = 0.5, \mu_1 = 0.04, \mu_2 = 0.04, \mu_3 = 0.04, \mu_4 = 0.04, \gamma = 0.1, q = 0.07, \rho = 0.09, \tau = 0.03$ and $\beta = 0.004$. It follows from Theorem 3 that the alcoholism equilibrium $P_1^* = (28.16, 6.09, 0.81, 8.97, 58.71)$ of system (2.1) is locally asymptotically stable for any value of time t when $R_0 = 2.0765 > \max\{1, R_{01}\}$, where $R_{01} = 0.6128$ (see Figure 4 (a) and (b)). Furthermore, we can also observe that the value of the equilibrium $P^*(t)$ changes with t increasing and eventually tends to $P_1^* = (28.16, 6.09, 0.81, 8.97, 58.71)$ from Figure 4 (a) and (b).

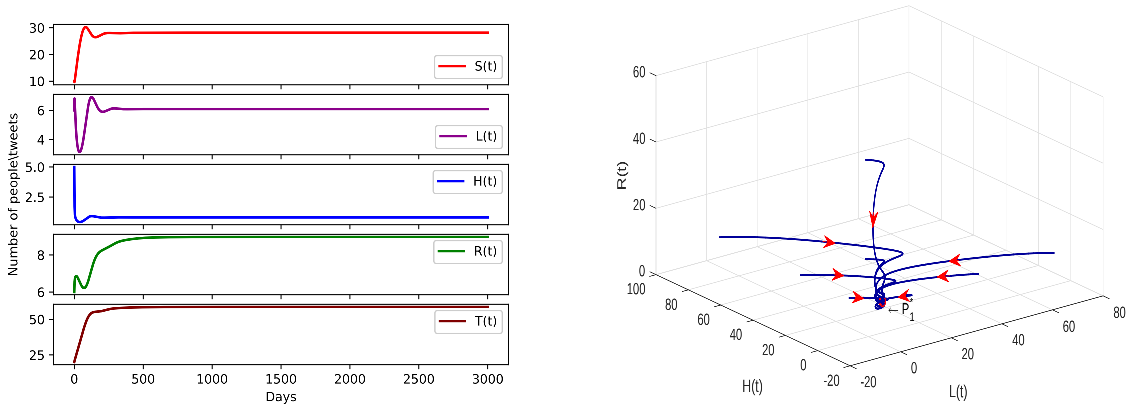
In order to verify the local stability of the alcoholism equilibrium P_4^*, we choose a set of the following parameters, the parameter values are $\Lambda = 8, \alpha = 0.07, \alpha_1 = 0.003, \alpha_2 = 0.005, \mu_1 = 0.025, \mu_2 = 0.025, \mu_3 = 0.025, \mu_4 = 0.025, \gamma = 0.01, q = 0.07, \rho = 0.4, \tau = 0.45$ and $\beta = 0.9$. It follows from Theorem 3 that the alcoholism equilibrium $P_4^* = (254.1677, 85.2877, 387.8429, 1292.7081, 112.2223)$

of system (2.1) is locally asymptotically stable for any value of time t when $R_{02} < R_0 < \max\{1, R_{01}\}$, where $R_{01} = 2.7788$, $R_0 = 0.8486$ and $R_{02} = 0.4692$ (see Figure 5 (a) and (b)). Furthermore, we can also observe that the value of the equilibrium $P^*(t)$ changes with t increasing and eventually tends to $P_4^* = (254.1677, 85.2877, 387.8429, 1292.7081, 112.2223)$ from Figure 5 (a) and (b).



(a) (b)

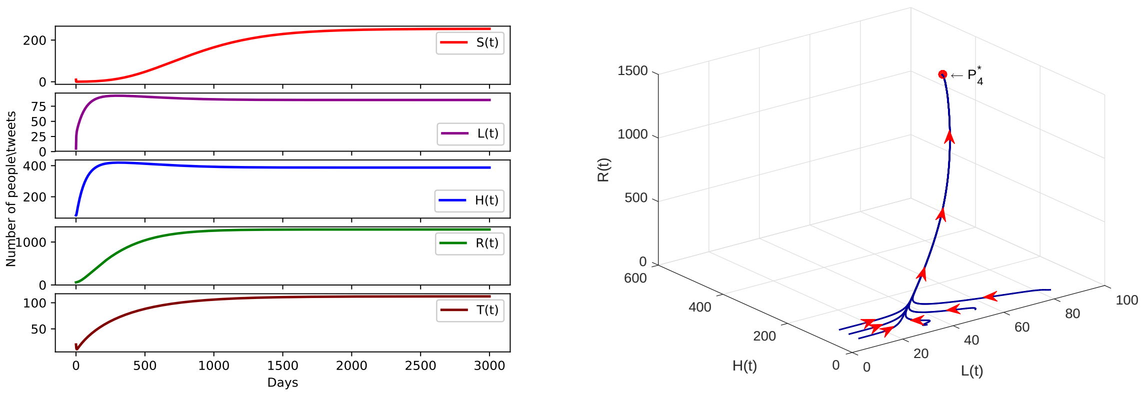
Figure 3. Alcohol free equilibrium P_0 of system (2.1) is globally asymptotically stable.



(a) (b)

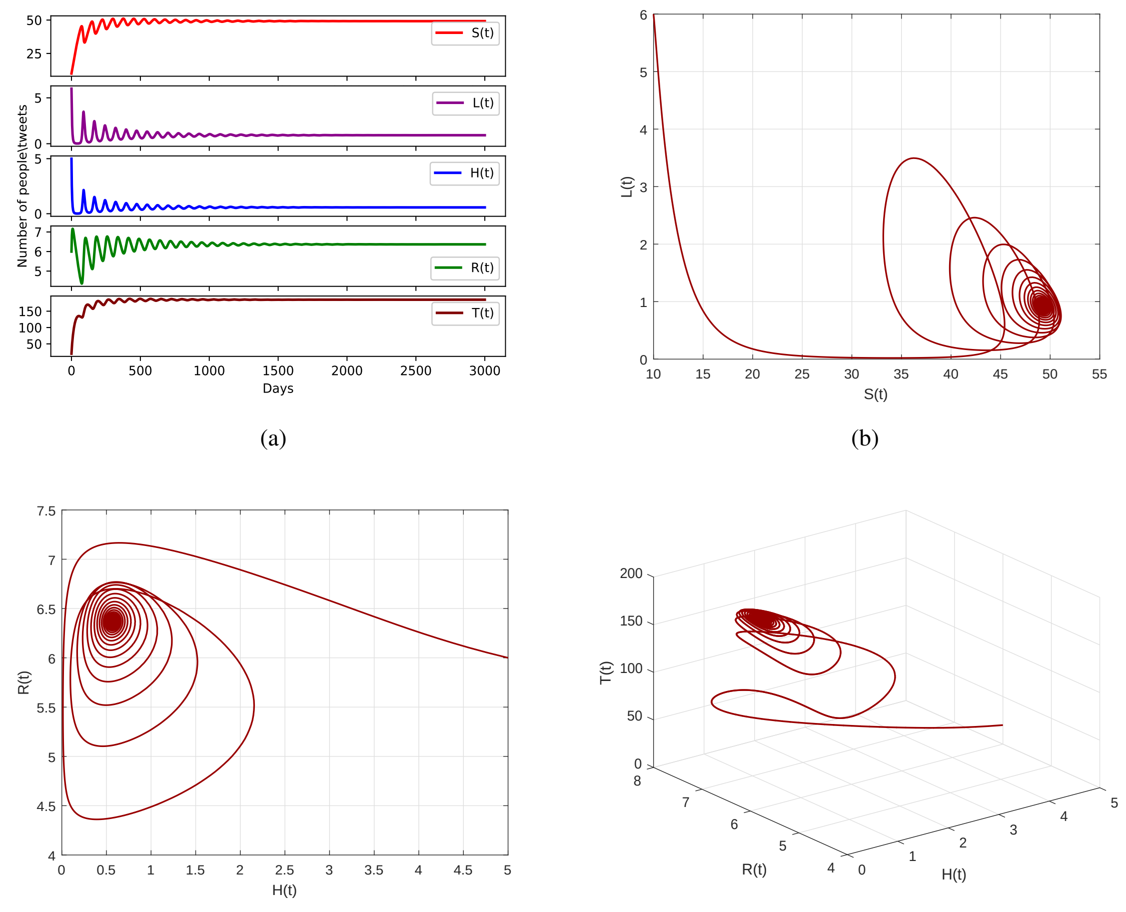
Figure 4. Alcoholism equilibrium P_1^* of system (2.1) is locally asymptotically stable.

Then, we choose a set of the following parameters, the parameter values are $\Lambda = 0.8, \alpha = 0.007, \alpha_1 = 0.009, \alpha_2 = 0.5, \mu_1 = 0, \mu_2 = 0.008, \mu_3 = 0.8, \mu_4 = 0.8, \gamma = 0.1, q = 0.99, \rho = 0.99, \tau = 0.03$ and $\beta = 0.0204$. It follows from Theorem 3 that the alcoholism equilibrium $P_1^* = (49.19, 0.92, 0.57, 6.37, 185.30)$ of system (2.1) is locally asymptotically stable for any value of time t when $R_0 = 1.1238 > \max\{1, R_{01}\}$ and $\beta < \beta^*$, where $R_{01} = -2.9044$ and $\beta^* = 0.021$ (see Figure 6 (a) and (b)).



(a) (b)

Figure 5. Alcoholism equilibrium P_4^* of system (2.1) is locally asymptotically stable.



(a) (b)

(c) (d)

Figure 6. Alcoholism equilibrium P_1^* of system (2.1) is locally asymptotically stable when $\beta < \beta^*$.

If we choose β as 0.076 (see Figure 7 (a) and (b)), we have more intricate dynamic behaviors on system (2.1). As an example, we choose a set of the following parameters, the parameter values are $\Lambda = 0.8, \alpha = 0.00626, \alpha_1 = 0.009, \alpha_2 = 0.4, \mu_1 = 0.009, \mu_2 = 0.004, \mu_3 = 0.8, \mu_4 = 0.8, \gamma = 0.1, q = 0.06, \rho = 0.9, \tau = 0.03$ and $\beta = 0.076$. The alcoholism equilibrium P_1^* of system (2.1) occurs a Hopf bifurcation when $R_0 = 9.9478 > 1$ and $\beta > \beta^*$, where $\beta^* = 0.011$ (see Figure 7 (a-d)). In Figure 7 (a-d), we can readily see that the solution curves of system (2.1) perform a sustained periodic oscillation and phase trajectories approaches limit cycle.

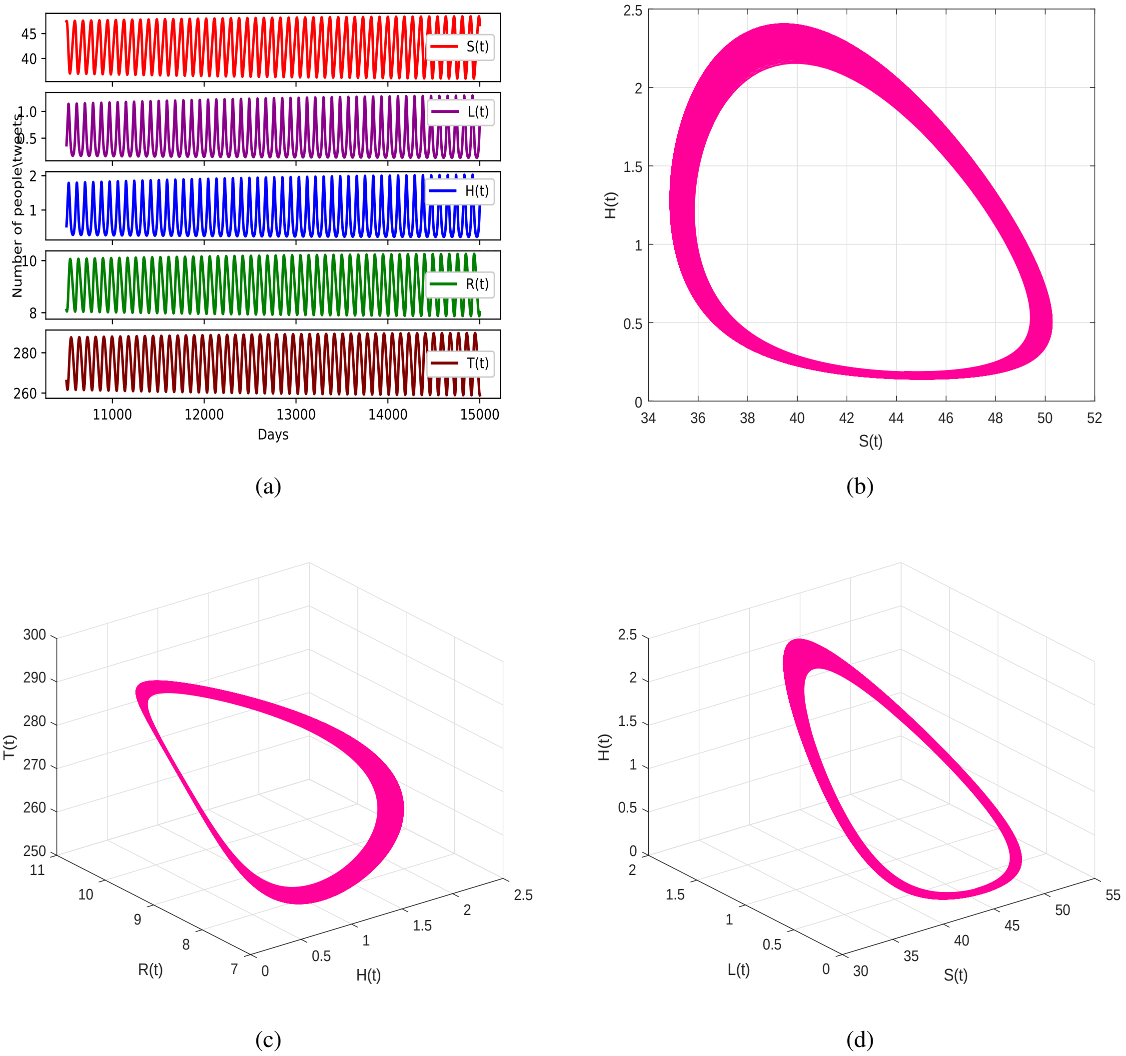


Figure 7. Alcoholism equilibrium P_1^* of system (2.1) occurs a Hopf bifurcation when $\beta > \beta^*$.

In order to demonstrate some results about Hopf bifurcation, we consider β as bifurcation parameter. We know that the alcoholism equilibrium P_1^* is feasible for $\beta \in [0.0099, 0.8]$. Thus, system (2.1) is stable when $0.0099 \leq \beta < 0.011$, and Hopf bifurcation occurs at the alcoholism equilibrium P_1^* when $0.011 \leq \beta < 0.08$, and system (2.1) becomes stable again when $0.08 \leq \beta \leq 0.2$, as depicted in Figure 8(a-e).

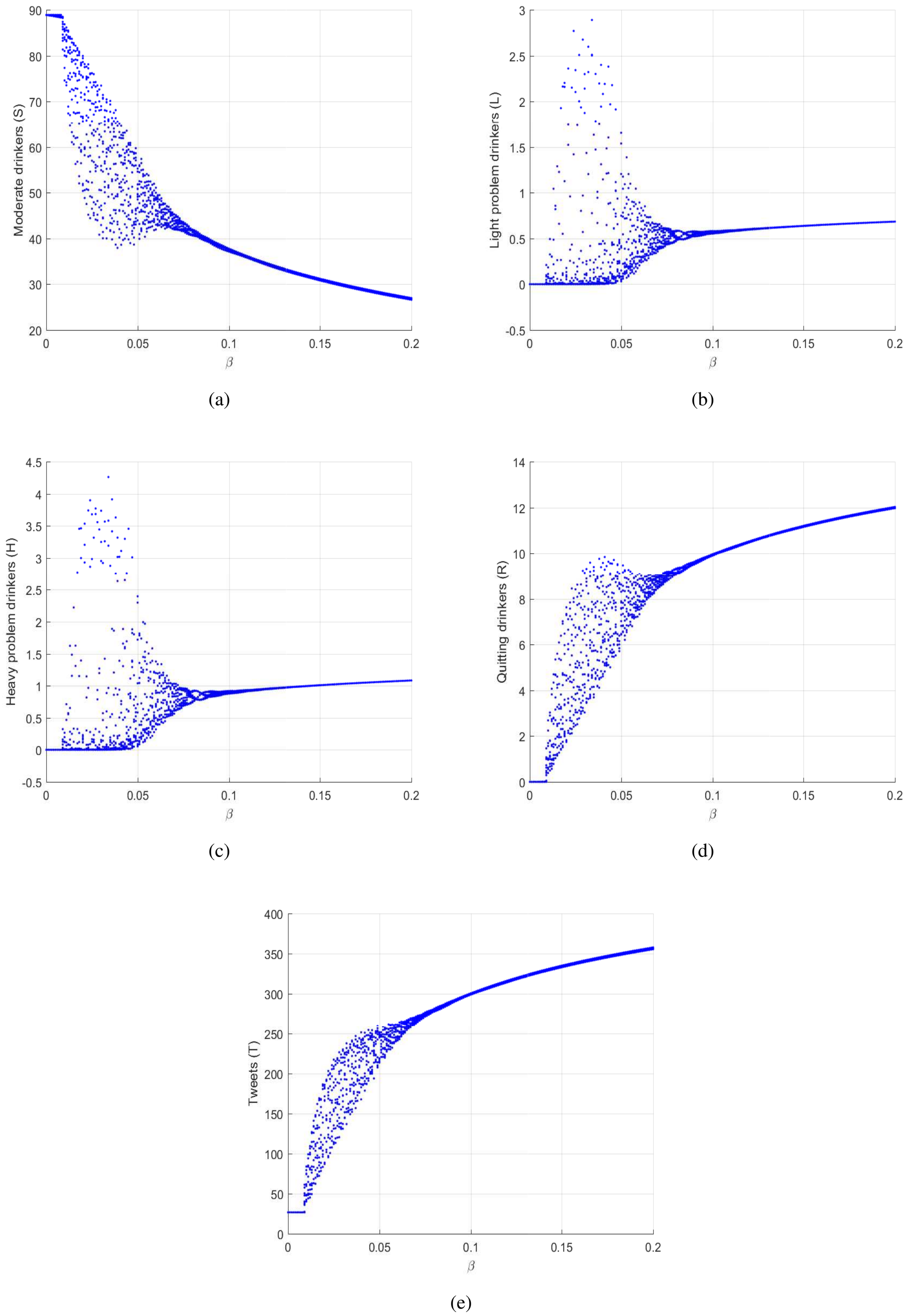
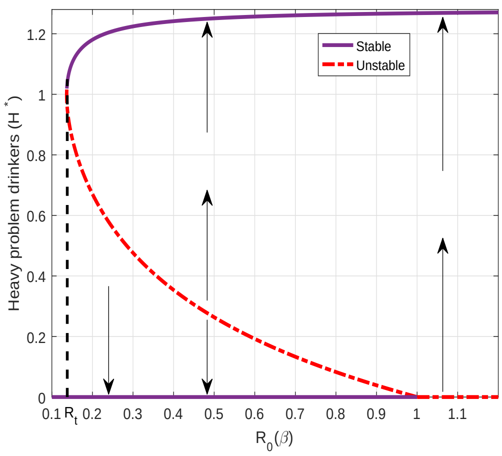
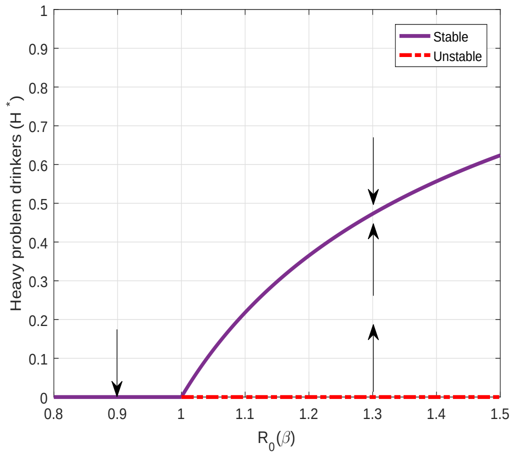


Figure 8. Hopf bifurcation occurs at the alcoholism equilibrium P_1^* when $0.0099 \leq \beta \leq 0.2$.

The backward and forward bifurcation diagram of system (2.1) is shown in Figure 9, and the direction of bifurcation depends upon the value of R_{01}. As seen in the backward bifurcation diagram of Figure 9(a) when $R_{01} = 4.4936 > 1$, there is a threshold quantity R_t which is the value of R_0. The alcohol free equilibrium is globally asymptotically stable when $R_0 < R_t$, where $R_t = 0.1350$. There are two alcoholism equilibria and a alcohol free equilibrium when $R_t < R_0 < 1$, the upper ones are stable, the middle ones are unstable and the lower ones is globally asymptotically stable. There are a stable alcoholism equilibria and an unstable alcohol free equilibrium when $R_0 > 1$. As seen in the forward bifurcation diagram of Figure 9(b) when $R_{01} = 0.5357 < 1$, the alcohol free equilibrium is globally asymptotically stable when $R_0 < 1$. There are a stable alcoholism equilibria and an unstable alcohol free equilibrium when $R_0 > 1$.



(a)



(b)

Figure 9. (a) Illustration of backward bifurcation when one parameter β in R_0 is varied. (b) Illustration of forward bifurcation when one parameter β in R_0 is varied.

5. Sensitivity analysis

In this section, we examine the effects of changes in some parameters on the number of heavy problem drinkers. Therefore, we carry out the sensitivity analysis of heavy problem drinkers H.

Figure 10 shows a comparison between the parameters of system (2.1) versus the heavy problem drinkers, we main consider the effect of μ_1, q, γ, τ on the dynamics of heavy problem drinkers. Firstly, we choose the effect of parameter μ_1 on the dynamics of heavy problem drinkers, the other parameter values are $\Lambda = 0.8, \alpha = 0.07, \alpha_1 = 0.009, \alpha_2 = 0.5, \mu_2 = 0.8, \mu_3 = 0.8, \mu_4 = 0.8, \gamma = 0.1, q = 0.09, \rho = 0.09, \tau = 0.03$ and $\beta = 0.04$, as depicted in Figure 10(a). We know that the number of heavy problem drinkers will decrease when μ_1 increase from Figure 10(a). The simulation shows that more Twitter messages can result in the lower alcoholism cases, and changing the number of Twitter messages posted per day does affect the time when the alcoholism reaches the peak. Secondly, we choose the effect of parameter q on the dynamics of heavy problem drinkers, the other parameter values are $\Lambda = 0.8, \alpha = 0.007, \alpha_1 = 0.009, \alpha_2 = 0.5, \mu_1 = 0.04, \mu_2 = 0.8, \mu_3 = 0.8, \mu_4 = 0.8, \gamma = 0.1, \rho = 0.09, \tau = 0.03$ and $\beta = 0.15$, as depicted in Figure 10(b). We know that the number of heavy problem drinkers will decrease when q increase from Figure 10(b). Thirdly, we choose the effect of parameter γ on the dynamics of heavy problem drinkers, the other parameter values are $\Lambda = 0.8, \alpha = 0.007, \alpha_1 = 0.009, \alpha_2 = 0.5, \mu_1 = 0.04, \mu_2 = 0.8, \mu_3 = 0.8, \mu_4 = 0.8, q = 0.07, \tau = 0.03, \rho = 0.01$ and $\beta = 0.15$, as depicted in Figure 10(c). We know that the number of heavy problem drinkers will decrease when γ increase from Figure 10(c). The simulation results in Figure 10(b) and 10(c) show that treatment significantly reduces the number of alcoholism cases. Finally, we choose the effect of parameter τ on the dynamics of heavy problem drinkers, the other parameter values are $\Lambda = 0.8, \alpha = 0.007, \alpha_1 = 0.009, \alpha_2 = 0.5, \mu_1 = 0.04, \mu_2 = 0.8, \mu_3 = 0.8, \mu_4 = 0.8, \gamma = 0.1, q = 0.07, \rho = 0.09$ and $\beta = 0.15$, as depicted in Figure 10(d). We know that the number of heavy problem drinkers will increase when τ increase from Figure 10(d). This indicates that the rate of upper outdated Twitter messages result in the upper alcoholism cases.

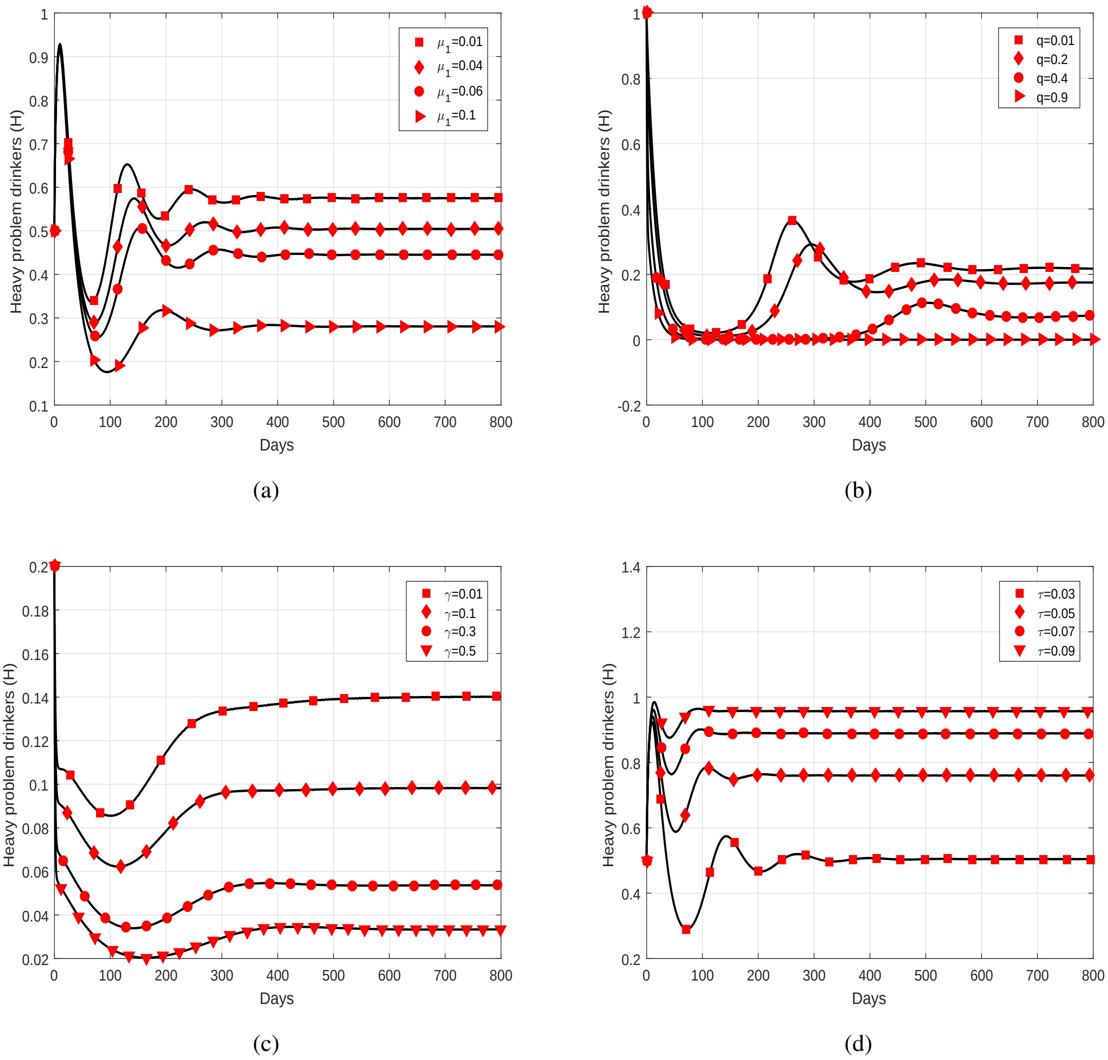


Figure 10. Sensitivity analysis of heavy problem drinkers H.

6. Discussion and conclusion

We construct a new alcoholism model with treatment and effect of Twitter in this paper. We study the stability of all equilibria and derive the basic reproductive number R_0. We also investigate the occurrence of backward and forward bifurcation for a certain defined range of R_0 by the center manifold theory. Furthermore, we give some numerical results and sensitivity analysis to extend and illustrate our results. Our results show that Twitter may be a good indicator of alcoholism model and affect the emergence and spread of drinking behavior. How to prove existence of Hopf bifurcation analytically is interesting and still open. We will leave this work in future.

Acknowledgments

We are grateful to the anonymous referees and the editors for their valuable comments and suggestions which improved the quality of the paper. This work is supported by the National Natural Science Foundation of China (11861044 and 11661050), and the HongLiu first-class disciplines Development Program of Lanzhou University of Technology.

Conflict of interest

The authors declare there is no conflict of interest.

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Appendix A

The formula of d_1, d_2, d_3 in the proof of (II) of Theorem 3.

$$\begin{aligned}
d_1 \;=\;& 7\,\Phi^2\alpha_1{}^4 + 12\,\Phi^2\alpha_1{}^3\alpha_2 + 12\,\Phi^2\alpha_1{}^3\gamma + 12\,\Phi^2\alpha_1{}^3 q + 12\,\Phi^2\alpha_1{}^3\rho + 18\,\Phi^2\alpha_1{}^3\tau + 6\,\Phi^2\alpha_1{}^2\alpha_2{}^2 \\
&+12\,\Phi^2\alpha_1{}^2\alpha_2\gamma + 12\,\Phi^2\alpha_1{}^2\alpha_2 q + 18\,\Phi^2\alpha_1{}^2\alpha_2\rho + 21\,\Phi^2\alpha_1{}^2\alpha_2\tau + 6\,\Phi^2\alpha_1{}^2\gamma^2 + 12\,\Phi^2\alpha_1{}^2\gamma q \\
&+18\,\Phi^2\alpha_1{}^2\gamma\rho + 21\,\Phi^2\alpha_1{}^2\gamma\tau + 6\,\Phi^2\alpha_1{}^2 q^2 + 12\,\Phi^2\alpha_1{}^2 q\rho + 21\,\Phi^2\alpha_1{}^2 q\tau + 6\,\Phi^2\alpha_1{}^2\rho^2 \\
&+21\,\Phi^2\alpha_1{}^2\rho\tau + 13\,\Phi^2\alpha_1{}^2\tau^2 + \Phi^2\alpha_1\alpha_2{}^3 + 3\,\Phi^2\alpha_1\alpha_2{}^2\gamma + 3\,\Phi^2\alpha_1\alpha_2{}^2 q + 7\,\Phi^2\alpha_1\alpha_2{}^2\rho \\
&+8\,\Phi^2\alpha_1\alpha_2{}^2\tau + 3\,\Phi^2\alpha_1\alpha_2\gamma^2 + 6\,\Phi^2\alpha_1\alpha_2\gamma q + 14\,\Phi^2\alpha_1\alpha_2\gamma\rho + 16\,\Phi^2\alpha_1\alpha_2\gamma\tau + 3\,\Phi^2\alpha_1\alpha_2 q^2 \\
&+10\,\Phi^2\alpha_1\alpha_2 q\rho + 16\,\Phi^2\alpha_1\alpha_2 q\tau + 7\,\Phi^2\alpha_1\alpha_2\rho^2 + 14\,\Phi^2\alpha_1\alpha_2\rho\tau + 10\,\Phi^2\alpha_1\alpha_2\tau^2 + \Phi^2\alpha_1\gamma^3 \\
&+3\,\Phi^2\alpha_1\gamma^2 q + 7\,\Phi^2\alpha_1\gamma^2\rho + 8\,\Phi^2\alpha_1\gamma^2\tau + 3\,\Phi^2\alpha_1\gamma q^2 + 10\,\Phi^2\alpha_1\gamma q\rho + 16\,\Phi^2\alpha_1\gamma q\tau \\
&+14\,\Phi^2\alpha_1\gamma\rho\tau + 10\,\Phi^2\alpha_1\gamma\tau^2 + \Phi^2\alpha_1 q^3 + 3\,\Phi^2\alpha_1 q^2\rho + 8\,\Phi^2\alpha_1 q^2\tau + 3\,\Phi^2\alpha_1 q\rho^2 \\
&+10\,\Phi^2\alpha_1 q\tau^2 + \Phi^2\alpha_1\rho^3 + 8\,\Phi^2\alpha_1\rho^2\tau + 10\,\Phi^2\alpha_1\rho\tau^2 + 2\,\Phi^2\alpha_1\tau^3 + \Phi^2\alpha_2{}^3\rho + \Phi^2\alpha_2{}^3\tau \\
&+3\,\Phi^2\alpha_2{}^2\gamma\tau + 2\,\Phi^2\alpha_2{}^2 q\rho + 3\,\Phi^2\alpha_2{}^2 q\tau + \Phi^2\alpha_2{}^2\rho^2 + 2\,\Phi^2\alpha_2{}^2\rho\tau + 2\,\Phi^2\alpha_2{}^2\tau^2 + 3\,\Phi^2\alpha_2\gamma^2\rho \\
&+3\,\Phi^2\alpha_2\gamma^2\tau + 4\,\Phi^2\alpha_2\gamma q\rho + 6\,\Phi^2\alpha_2\gamma q\tau + 2\,\Phi^2\alpha_2\gamma\rho^2 + 4\,\Phi^2\alpha_2\gamma\rho\tau + 4\,\Phi^2\alpha_2\gamma\tau^2 \\
&+3\,\Phi^2\alpha_2 q^2\tau + 2\,\Phi^2\alpha_2 q\rho^2 + 3\,\Phi^2\alpha_2 q\rho\tau + 4\,\Phi^2\alpha_2 q\tau^2 + \Phi^2\alpha_2\rho^3 + 2\,\Phi^2\alpha_2\rho^2\tau + 3\,\Phi^2\alpha_2\rho\tau^2 \\
&+\Phi^2\alpha_2\tau^3 + \Phi^2\gamma^3\rho + \Phi^2\gamma^3\tau + 2\,\Phi^2\gamma^2 q\rho + 3\,\Phi^2\gamma^2 q\tau + \Phi^2\gamma^2\rho^2 + 2\,\Phi^2\gamma^2\rho\tau + 2\,\Phi^2\gamma^2\tau^2 \\
&+\Phi^2\gamma q^2\rho + 3\,\Phi^2\gamma q^2\tau + 2\,\Phi^2\gamma q\rho^2 + 3\,\Phi^2\gamma q\rho\tau + 4\,\Phi^2\gamma q\tau^2 + \Phi^2\gamma\rho^3 + 2\,\Phi^2\gamma\rho^2\tau \\
&+3\,\Phi^2\gamma\rho\tau^2 + \Phi^2\gamma\tau^3 + \Phi^2 q^3\tau + \Phi^2 q^2\rho\tau + 2\,\Phi^2 q^2\tau^2 + \Phi^2 q\rho^2\tau + 2\,\Phi^2 q\rho\tau^2 + \Phi^2 q\tau^3 \\
&+\Phi^2\rho^3\tau + 2\,\Phi^2\rho^2\tau^2 + \Phi^2\rho\tau^3 + 6\,\alpha\,\Phi\,\alpha_1{}^3\alpha_2\rho + 8\,\alpha\,\Phi\,\alpha_1{}^2\alpha_2{}^2\rho + 8\,\alpha\,\Phi\,\alpha_1{}^2\alpha_2\gamma\rho \\
&+8\,\alpha\,\Phi\,\alpha_1{}^2\alpha_2 q\rho + 8\,\alpha\,\Phi\,\alpha_1{}^2\alpha_2\rho^2 + 2\,\alpha\,\Phi\,\alpha_1{}^2\alpha_2\rho\tau + 2\,\alpha\,\Phi\,\alpha_1\alpha_2{}^3\rho + 4\,\alpha\,\Phi\,\alpha_1\alpha_2{}^2\gamma\rho \\
&+4\,\alpha\,\Phi\,\alpha_1\alpha_2{}^2 q\rho + 10\,\alpha\,\Phi\,\alpha_1\alpha_2{}^2\rho^2 + 2\,\alpha\,\Phi\,\alpha_1\alpha_2{}^2\rho\tau + 2\,\alpha\,\Phi\,\alpha_1\alpha_2\gamma^2\rho + 4\,\alpha\,\Phi\,\alpha_1\alpha_2\gamma q\rho \\
&+10\,\alpha\,\Phi\,\alpha_1\alpha_2\gamma\rho^2 + 2\,\alpha\,\Phi\,\alpha_1\alpha_2\gamma\rho\tau + 2\,\alpha\,\Phi\,\alpha_1\alpha_2 q^2\rho + 10\,\alpha\,\Phi\,\alpha_1\alpha_2 q\rho^2 + 2\,\alpha\,\Phi\,\alpha_1\alpha_2 q\rho\tau \\
&+2\,\alpha\,\Phi\,\alpha_1\alpha_2\rho^3 + 2\,\alpha\,\Phi\,\alpha_1\alpha_2\rho^2\tau + 2\,\alpha\,\Phi\,\alpha_2{}^3\rho^2 + 4\,\alpha\,\Phi\,\alpha_2{}^2\gamma\rho^2 + 4\,\alpha\,\Phi\,\alpha_2{}^2 q\rho^2 + 2\,\alpha\,\Phi\,\alpha_2{}^2\rho^3 \\
&+2\,\alpha\,\Phi\,\alpha_2{}^2\rho^2\tau + 2\,\alpha\,\Phi\,\alpha_2\gamma^2\rho^2 + 4\,\alpha\,\Phi\,\alpha_2\gamma q\rho^2 + 2\,\alpha\,\Phi\,\alpha_2\gamma\rho^3 + 2\,\alpha\,\Phi\,\alpha_2\gamma\rho^2\tau + \Phi^2\alpha_2 q^2\rho \\
&+2\,\alpha\,\Phi\,\alpha_2 q^2\rho^2 + 2\,\alpha\,\Phi\,\alpha_2 q\rho^3 + 2\,\alpha\,\Phi\,\alpha_2 q\rho^2\tau + 3\,\Phi^2\alpha_2{}^2\gamma\rho + 7\,\Phi^2\alpha_1\gamma\rho^2 + 10\,\Phi^2\alpha_1 q\rho\tau, \\
d_2 \;=\;& 6\,\Phi\,\alpha_1{}^5 + 9\,\alpha\,\alpha_1{}^4\alpha_2\rho + 11\,\Phi\,\alpha_1{}^4\alpha_2 + 11\,\Phi\,\alpha_1{}^4\gamma + 11\,\Phi\,\alpha_1{}^4 q + 11\,\Phi\,\alpha_1{}^4\rho + 26\,\Phi\,\alpha_1{}^4\tau \\
&+15\,\alpha\,\alpha_1{}^3\alpha_2{}^2\rho + 6\,\Phi\,\alpha_1{}^3\alpha_2{}^2 + 15\,\alpha\,\alpha_1{}^3\alpha_2\gamma\rho + 12\,\Phi\,\alpha_1{}^3\alpha_2\gamma + 15\,\alpha\,\alpha_1{}^3\alpha_2 q\rho \\
&+15\,\alpha\,\alpha_1{}^3\alpha_2\rho^2 + 6\,\alpha\,\alpha_1{}^3\alpha_2\rho\tau + 18\,\Phi\,\alpha_1{}^3\alpha_2\rho + 33\,\Phi\,\alpha_1{}^3\alpha_2\tau + 6\,\Phi\,\alpha_1{}^3\gamma^2 + 12\,\Phi\,\alpha_1{}^3\gamma q \\
&+18\,\Phi\,\alpha_1{}^3\gamma\rho + 33\,\Phi\,\alpha_1{}^3\gamma\tau + 6\,\Phi\,\alpha_1{}^3 q^2 + 12\,\Phi\,\alpha_1{}^3 q\rho + 33\,\Phi\,\alpha_1{}^3 q\tau + 6\,\Phi\,\alpha_1{}^3\rho^2 \\
&+27\,\Phi\,\alpha_1{}^3\tau^2 + 7\,\alpha\,\alpha_1{}^2\alpha_2{}^3\rho + \Phi\,\alpha_1{}^2\alpha_2{}^3 + 14\,\alpha\,\alpha_1{}^2\alpha_2{}^2\gamma\rho + 3\,\Phi\,\alpha_1{}^2\alpha_2{}^2\gamma + 14\,\alpha\,\alpha_1{}^2\alpha_2{}^2 q\rho
\end{aligned}$$

$$
\begin{aligned}
&+3\,\Phi\,\alpha_1{}^2\,\alpha_2{}^2\,q + 23\,\alpha\,\alpha_1{}^2\,\alpha_2{}^2\,\rho^2 + 8\,\alpha\,\alpha_1{}^2\,\alpha_2{}^2\,\rho\,\tau + 8\,\Phi\,\alpha_1{}^2\,\alpha_2{}^2\,\rho + 14\,\Phi\,\alpha_1{}^2\,\alpha_2{}^2\,\tau \\
&+3\,\Phi\,\alpha_1{}^2\,\alpha_2\,\gamma^2 + 14\,\alpha\,\alpha_1{}^2\,\alpha_2\,\gamma\,q\,\rho + 6\,\Phi\,\alpha_1{}^2\,\alpha_2\,\gamma\,q + 23\,\alpha\,\alpha_1{}^2\,\alpha_2\,\gamma\,\rho^2 + 8\,\alpha\,\alpha_1{}^2\,\alpha_2\,\gamma\,\rho\,\tau \\
&+16\,\Phi\,\alpha_1{}^2\,\alpha_2\,\gamma\,\rho + 28\,\Phi\,\alpha_1{}^2\,\alpha_2\,\gamma\,\tau + 7\,\alpha\,\alpha_1{}^2\,\alpha_2\,q^2\,\rho + 3\,\Phi\,\alpha_1{}^2\,\alpha_2\,q^2 + 23\,\alpha\,\alpha_1{}^2\,\alpha_2\,q\,\rho^2 \\
&+8\,\alpha\,\alpha_1{}^2\,\alpha_2\,q\,\rho\,\tau + 11\,\Phi\,\alpha_1{}^2\,\alpha_2\,q\,\rho + 28\,\Phi\,\alpha_1{}^2\,\alpha_2\,q\,\tau + 7\,\alpha\,\alpha_1{}^2\,\alpha_2\,\rho^3 + 8\,\alpha\,\alpha_1{}^2\,\alpha_2\,\rho^2\,\tau \\
&+8\,\Phi\,\alpha_1{}^2\,\alpha_2\,\rho^2 + \alpha\,\alpha_1{}^2\,\alpha_2\,\rho\,\tau^2 + 26\,\Phi\,\alpha_1{}^2\,\alpha_2\,\rho\,\tau + 27\,\Phi\,\alpha_1{}^2\,\alpha_2\,\tau^2 + \Phi\,\alpha_1{}^2\,\gamma^3 + 3\,\Phi\,\alpha_1{}^2\,\gamma^2\,q \\
&+8\,\Phi\,\alpha_1{}^2\,\gamma^2\,\rho + 14\,\Phi\,\alpha_1{}^2\,\gamma^2\,\tau + 3\,\Phi\,\alpha_1{}^2\,\gamma\,q^2 + 11\,\Phi\,\alpha_1{}^2\,\gamma\,q\,\rho + 28\,\Phi\,\alpha_1{}^2\,\gamma\,q\,\tau + 8\,\Phi\,\alpha_1{}^2\,\gamma\,\rho^2 \\
&+26\,\Phi\,\alpha_1{}^2\,\gamma\,\rho\,\tau + 27\,\Phi\,\alpha_1{}^2\,\gamma\,\tau^2 + \Phi\,\alpha_1{}^2\,q^3 + 3\,\Phi\,\alpha_1{}^2\,q^2\,\rho + 14\,\Phi\,\alpha_1{}^2\,q^2\,\tau + 3\,\Phi\,\alpha_1{}^2\,q\,\rho^2 \\
&+19\,\Phi\,\alpha_1{}^2\,q\,\rho\,\tau + 27\,\Phi\,\alpha_1{}^2\,q\,\tau^2 + \Phi\,\alpha_1{}^2\,\rho^3 + 14\,\Phi\,\alpha_1{}^2\,\rho^2\,\tau + 27\,\Phi\,\alpha_1{}^2\,\rho\,\tau^2 + 7\,\Phi\,\alpha_1{}^2\,\tau^3 \\
&+\alpha\,\alpha_1\,\alpha_2{}^4\,\rho + 3\,\alpha\,\alpha_1\,\alpha_2{}^3\,\gamma\,\rho + 3\,\alpha\,\alpha_1\,\alpha_2{}^3\,q\,\rho + 9\,\alpha\,\alpha_1\,\alpha_2{}^3\,\rho^2 + 2\,\alpha\,\alpha_1\,\alpha_2{}^3\,\rho\,\tau + \Phi\,\alpha_1\,\alpha_2{}^3\,\rho \\
&+2\,\Phi\,\alpha_1\,\alpha_2{}^3\,\tau + 3\,\alpha\,\alpha_1\,\alpha_2{}^2\,\gamma^2\,\rho + 6\,\alpha\,\alpha_1\,\alpha_2{}^2\,\gamma\,q\,\rho + 18\,\alpha\,\alpha_1\,\alpha_2{}^2\,\gamma\,\rho^2 + 4\,\alpha\,\alpha_1\,\alpha_2{}^2\,\gamma\,\rho\,\tau \\
&+6\,\Phi\,\alpha_1\,\alpha_2{}^2\,\gamma\,\tau + 3\,\alpha\,\alpha_1\,\alpha_2{}^2\,q^2\,\rho + 18\,\alpha\,\alpha_1\,\alpha_2{}^2\,q\,\rho^2 + 4\,\alpha\,\alpha_1\,\alpha_2{}^2\,q\,\rho\,\tau + 2\,\Phi\,\alpha_1\,\alpha_2{}^2\,q\,\rho \\
&+9\,\alpha\,\alpha_1\,\alpha_2{}^2\,\rho^3 + 10\,\alpha\,\alpha_1\,\alpha_2{}^2\,\rho^2\,\tau + 2\,\Phi\,\alpha_1\,\alpha_2{}^2\,\rho^2 + \alpha\,\alpha_1\,\alpha_2{}^2\,\rho\,\tau^2 + 5\,\Phi\,\alpha_1\,\alpha_2{}^2\,\rho\,\tau + 9\,\Phi\,\alpha_1\,\alpha_2{}^2\,\tau^2 \\
&+\alpha\,\alpha_1\,\alpha_2\,\gamma^3\,\rho + 3\,\alpha\,\alpha_1\,\alpha_2\,\gamma^2\,q\,\rho + 9\,\alpha\,\alpha_1\,\alpha_2\,\gamma^2\,\rho^2 + 2\,\alpha\,\alpha_1\,\alpha_2\,\gamma^2\,\rho\,\tau + 3\,\Phi\,\alpha_1\,\alpha_2\,\gamma^2\,\rho \\
&+3\,\alpha\,\alpha_1\,\alpha_2\,\gamma\,q^2\,\rho + 18\,\alpha\,\alpha_1\,\alpha_2\,\gamma\,q\,\rho^2 + 4\,\alpha\,\alpha_1\,\alpha_2\,\gamma\,q\,\rho\,\tau + 4\,\Phi\,\alpha_1\,\alpha_2\,\gamma\,q\,\rho + 12\,\Phi\,\alpha_1\,\alpha_2\,\gamma\,q\,\tau \\
&+9\,\alpha\,\alpha_1\,\alpha_2\,\gamma\,\rho^3 + 10\,\alpha\,\alpha_1\,\alpha_2\,\gamma\,\rho^2\,\tau + 4\,\Phi\,\alpha_1\,\alpha_2\,\gamma\,\rho^2 + \alpha\,\alpha_1\,\alpha_2\,\gamma\,\rho\,\tau^2 + 10\,\Phi\,\alpha_1\,\alpha_2\,\gamma\,\rho\,\tau \\
&+18\,\Phi\,\alpha_1\,\alpha_2\,\gamma\,\tau^2 + \alpha\,\alpha_1\,\alpha_2\,q^3\,\rho + 9\,\alpha\,\alpha_1\,\alpha_2\,q^2\,\rho^2 + 2\,\alpha\,\alpha_1\,\alpha_2\,q^2\,\rho\,\tau + \Phi\,\alpha_1\,\alpha_2\,q^2\,\rho + 6\,\Phi\,\alpha_1\,\alpha_2\,q^2\,\tau \\
&+9\,\alpha\,\alpha_1\,\alpha_2\,q\,\rho^3 + 10\,\alpha\,\alpha_1\,\alpha_2\,q\,\rho^2\,\tau + 2\,\Phi\,\alpha_1\,\alpha_2\,q\,\rho^2 + \alpha\,\alpha_1\,\alpha_2\,q\,\rho\,\tau^2 + 5\,\Phi\,\alpha_1\,\alpha_2\,q\,\rho\,\tau \\
&+\alpha\,\alpha_1\,\alpha_2\,\rho^4 + 2\,\alpha\,\alpha_1\,\alpha_2\,\rho^3\,\tau + \Phi\,\alpha_1\,\alpha_2\,\rho^3 + \alpha\,\alpha_1\,\alpha_2\,\rho^2\,\tau^2 + 5\,\Phi\,\alpha_1\,\alpha_2\,\rho^2\,\tau + 15\,\Phi\,\alpha_1\,\alpha_2\,\rho\,\tau^2 \\
&+5\,\Phi\,\alpha_1\,\alpha_2\,\tau^3 + \Phi\,\alpha_1\,\gamma^3\,\rho + 2\,\Phi\,\alpha_1\,\gamma^3\,\tau + 2\,\Phi\,\alpha_1\,\gamma^2\,q\,\rho + 6\,\Phi\,\alpha_1\,\gamma^2\,q\,\tau + 2\,\Phi\,\alpha_1\,\gamma^2\,\rho^2 \\
&+9\,\Phi\,\alpha_1\,\gamma^2\,\tau^2 + \Phi\,\alpha_1\,\gamma\,q^2\,\rho + 6\,\Phi\,\alpha_1\,\gamma\,q^2\,\tau + 2\,\Phi\,\alpha_1\,\gamma\,q\,\rho^2 + 5\,\Phi\,\alpha_1\,\gamma\,q\,\rho\,\tau + 18\,\Phi\,\alpha_1\,\gamma\,q\,\tau^2 \\
&+\Phi\,\alpha_1\,\gamma\,\rho^3 + 5\,\Phi\,\alpha_1\,\gamma\,\rho^2\,\tau + 15\,\Phi\,\alpha_1\,\gamma\,\rho\,\tau^2 + 5\,\Phi\,\alpha_1\,\gamma\,\tau^3 + 2\,\Phi\,\alpha_1\,q^3\,\tau + 9\,\Phi\,\alpha_1\,q^2\,\tau^2 \\
&+5\,\Phi\,\alpha_1\,q\,\tau^3 + 2\,\Phi\,\alpha_1\,\rho^3\,\tau + 9\,\Phi\,\alpha_1\,\rho^2\,\tau^2 + 5\,\Phi\,\alpha_1\,\rho\,\tau^3 + \alpha\,\alpha_2{}^4\,\rho^2 + 3\,\alpha\,\alpha_2{}^3\,\gamma\,\rho^2 + 3\,\alpha\,\alpha_2{}^3\,q\,\rho^2 \\
&+2\,\alpha\,\alpha_2{}^3\,\rho^3 + 2\,\alpha\,\alpha_2{}^3\,\rho^2\,\tau + \Phi\,\alpha_2{}^3\,\tau^2 + 3\,\alpha\,\alpha_2{}^2\,\gamma^2\,\rho^2 + 6\,\alpha\,\alpha_2{}^2\,\gamma\,q\,\rho^2 + 4\,\alpha\,\alpha_2{}^2\,\gamma\,\rho^3 \\
&+3\,\Phi\,\alpha_2{}^2\,\gamma\,\tau^2 + 3\,\alpha\,\alpha_2{}^2\,q^2\,\rho^2 + 4\,\alpha\,\alpha_2{}^2\,q\,\rho^3 + 4\alpha\,\alpha_2{}^2\,q\,\rho^2\,\tau + \Phi\,\alpha_2{}^2\,q\,\tau(3\tau-\rho) + \alpha\,\alpha_2{}^2\,\rho^4 \\
&+2\,\alpha\,\alpha_2{}^2\,\rho^3\,\tau + \alpha\,\alpha_2{}^2\,\rho^2\,\tau^2 + 2\,\Phi\,\alpha_2{}^2\,\rho\,\tau^2 + \Phi\,\alpha_2{}^2\,\tau^3 + \alpha\,\alpha_2\,\gamma^3\,\rho^2 + 3\,\alpha\,\alpha_2\,\gamma^2\,q\,\rho^2 + 2\,\alpha\,\alpha_2\,\gamma^2\,\rho^3 \\
&+2\,\alpha\,\alpha_2\,\gamma^2\,\rho^2\,\tau + 3\,\Phi\,\alpha_2\,\gamma^2\,\tau^2 + 3\,\alpha\,\alpha_2\,\gamma\,q^2\,\rho^2 + 4\,\alpha\,\alpha_2\,\gamma\,q\,\rho^3 + 4\,\alpha\,\alpha_2\,\gamma\,q\,\rho^2\,\tau \\
&+\alpha\,\alpha_2\,\gamma\,\rho^4 + 2\,\alpha\,\alpha_2\,\gamma\,\rho^3\,\tau + \alpha\,\alpha_2\,\gamma\,\rho^2\,\tau^2 + 4\,\Phi\,\alpha_2\,\gamma\,\rho\,\tau^2 + 2\,\Phi\,\alpha_2\,\gamma\,\tau^3 + \alpha\,\alpha_2\,q^3\,\rho^2 \\
&+2\,\alpha\,\alpha_2\,q^2\,\rho^3 + 2\,\alpha\,\alpha_2\,q^2\,\rho^2\,\tau + \Phi\,\alpha_2\,q^2\,\tau(3\tau-2\rho) + \alpha\,\alpha_2\,q\,\rho^4 + 2\,\alpha\,\alpha_2\,q\,\rho^3\,\tau + \alpha\,\alpha_2\,q\,\rho^2\,\tau^2 \\
&+\Phi\,\alpha_2\,q\,\rho\,\tau(3\tau-2\rho) + 2\,\Phi\,\alpha_2\,q\,\tau^3 + 2\,\Phi\,\alpha_2\,\rho^2\,\tau^2 + \Phi\,\alpha_2\,\rho\,\tau^3 + \Phi\,\gamma^3\,\tau^2 + \Phi\,\rho^2\,\tau^3 \\
&+\Phi\,\gamma^2\,q\,\tau(3\tau-\rho) + 2\,\Phi\,\gamma^2\,\rho\,\tau^2 + \Phi\,\gamma^2\,\tau^3 + \Phi\,\gamma\,q^2\,\tau(3\tau-2\rho) + \Phi\,\gamma\,q\,\rho\,\tau(3\tau-2\rho) \\
&+2\,\Phi\,\gamma\,q\,\tau^3 + 2\,\Phi\,\gamma\,\rho^2\,\tau^2 + \Phi\,\gamma\,\rho\,\tau^3 + \Phi\,q^3\,\tau(\tau-\rho) + \Phi\,q^2\,\rho\,\tau(\tau-\rho) + \Phi q^2\tau(\tau^2-\rho^2) \\
&+\Phi\,q\,\rho^2\tau(\tau-\rho) + \Phi\,q\,\rho\,\tau^3 + \Phi\rho^3\,\tau^2 + 7\,\alpha\,\alpha_1{}^2\,\alpha_2\,\gamma^2\,\rho + 3\,\Phi\,\alpha_1\,\alpha_2{}^2\,\gamma\,\rho + 6\,\Phi\,\alpha_1\,\alpha_2{}^2\,q\,\tau \\
&+6\,\Phi\,\alpha_1\,\alpha_2\,\gamma^2\,\tau + 18\,\Phi\,\alpha_1\,\alpha_2\,q\,\tau^2 + 5\,\Phi\,\alpha_1\,\gamma^2\,\rho\,\tau + 12\,\Phi\,\alpha_1\,q\,\rho\,\tau^2 + 4\,\alpha\,\alpha_2{}^2\,\gamma\,\rho^2\,\tau \\
&+\Phi\,\alpha_2\,\gamma\,q\,\tau(6\tau-2\rho) + 33\,\Phi\,\alpha_1{}^3\,\rho\,\tau + 12\,\Phi\,\alpha_1{}^3\,\alpha_2\,q,
\end{aligned}
$$

$$
\begin{aligned}
d_3 \;=\;& 12\,\alpha_1{}^5\,\tau + 16\,\alpha_1{}^4\,\alpha_2\,\tau + 16\,\alpha_1{}^4\,\gamma\,\tau + 16\,\alpha_1{}^4\,q\,\tau + 16\,\alpha_1{}^4\,\rho\,\tau + 18\,\alpha_1{}^4\,\tau^2 + 7\,\alpha_1{}^3\,\alpha_2{}^2\,\tau \\
&+14\,\alpha_1{}^3\,\alpha_2\,q\,\tau + 14\,\alpha_1{}^3\,\alpha_2\,\rho\,\tau + 21\,\alpha_1{}^3\,\alpha_2\,\tau^2 + 7\,\alpha_1{}^3\,\gamma^2\,\tau + 14\,\alpha_1{}^3\,\gamma\,q\,\tau + 14\,\alpha_1{}^3\,\gamma\,\rho\,\tau
\end{aligned}
$$

$$+7\,\alpha_1{}^3\,q^2\,\tau + 14\,\alpha_1{}^3\,q\,\rho\,\tau + 21\,\alpha_1{}^3\,q\,\tau^2 + 7\,\alpha_1{}^3\,\rho^2\,\tau + 21\,\alpha_1{}^3\,\rho\,\tau^2 + 6\,\alpha_1{}^3\,\tau^3 + \alpha_1{}^2\,\alpha_2{}^3\,\tau$$
$$+3\,\alpha_1{}^2\,\alpha_2{}^2\,q\,\tau + 3\,\alpha_1{}^2\,\alpha_2{}^2\,\rho\,\tau + 8\,\alpha_1{}^2\,\alpha_2{}^2\,\tau^2 + 3\,\alpha_1{}^2\,\alpha_2\,\gamma^2\,\tau + 6\,\alpha_1{}^2\,\alpha_2\,\gamma\,q\,\tau + 6\,\alpha_1{}^2\,\alpha_2\,\gamma\,\rho\,\tau$$
$$+16\,\alpha_1{}^2\,\alpha_2\,\gamma\,\tau^2 + 3\,\alpha_1{}^2\,\alpha_2\,q^2\,\tau + 6\,\alpha_1{}^2\,\alpha_2\,q\,\rho\,\tau + 16\,\alpha_1{}^2\,\alpha_2\,q\,\tau^2 + 3\,\alpha_1{}^2\,\alpha_2\,\rho^2\,\tau + 16\,\alpha_1{}^2\,\alpha_2\,\rho\,\tau^2$$
$$+5\,\alpha_1{}^2\,\alpha_2\,\tau^3 + \alpha_1{}^2\,\gamma^3\,\tau + 3\,\alpha_1{}^2\,\gamma^2\,q\,\tau + 3\,\alpha_1{}^2\,\gamma^2\,\rho\,\tau + 8\,\alpha_1{}^2\,\gamma^2\,\tau^2 + 3\,\alpha_1{}^2\,\gamma\,q^2\,\tau + 6\,\alpha_1{}^2\,\gamma\,q\,\rho\,\tau$$
$$+16\,\alpha_1{}^2\,\gamma\,q\,\tau^2 + 3\,\alpha_1{}^2\,\gamma\,\rho^2\,\tau + 16\,\alpha_1{}^2\,\gamma\,\rho\,\tau^2 + 5\,\alpha_1{}^2\,\gamma\,\tau^3 + \alpha_1{}^2\,q^3\,\tau + 3\,\alpha_1{}^2\,q^2\,\rho\,\tau + 8\,\alpha_1{}^2\,q^2\,\tau^2$$
$$+3\,\alpha_1{}^2\,q\,\rho^2\,\tau + 16\,\alpha_1{}^2\,q\,\rho\,\tau^2 + 5\,\alpha_1{}^2\,q\,\tau^3 + \alpha_1{}^2\,\rho^3\,\tau + 8\,\alpha_1{}^2\,\rho^2\,\tau^2 + 5\,\alpha_1{}^2\,\rho\,\tau^3 + \alpha_1\,\alpha_2{}^3\,\tau^2$$
$$+3\,\alpha_1\,\alpha_2{}^2\,q\,\tau^2 + 3\,\alpha_1\,\alpha_2{}^2\,\rho\,\tau^2 + \alpha_1\,\alpha_2{}^2\,\tau^3 + 3\,\alpha_1\,\alpha_2\,\gamma^2\,\tau^2 + 6\,\alpha_1\,\alpha_2\,\gamma\,q\,\tau^2 + 6\,\alpha_1\,\alpha_2\,\gamma\,\rho\,\tau^2$$
$$+3\,\alpha_1\,\alpha_2\,q^2\,\tau^2 + 6\,\alpha_1\,\alpha_2\,q\,\rho\,\tau^2 + 2\,\alpha_1\,\alpha_2\,q\,\tau^3 + 3\,\alpha_1\,\alpha_2\,\rho^2\,\tau^2 + 2\,\alpha_1\,\alpha_2\,\rho\,\tau^3 + \alpha_1\,\gamma^3\,\tau^2$$
$$+3\,\alpha_1\,\gamma^2\,\rho\,\tau^2 + \alpha_1\,\gamma^2\,\tau^3 + 3\,\alpha_1\,\gamma\,q^2\,\tau^2 + 6\,\alpha_1\,\gamma\,q\,\rho\,\tau^2 + 2\,\alpha_1\,\gamma\,q\,\tau^3 + 3\,\alpha_1\,\gamma\,\rho^2\,\tau^2 + 2\,\alpha_1\,\gamma\,\rho\,\tau^3$$
$$+\alpha_1\,q^3\,\tau^2 + 3\,\alpha_1\,q^2\,\rho\,\tau^2 + \alpha_1\,q^2\,\tau^3 + 3\,\alpha_1\,q\,\rho^2\,\tau^2 + 2\,\alpha_1\,q\,\rho\,\tau^3 + \alpha_1\,\rho^3\,\tau^2 + \alpha_1\,\rho^2\,\tau^3$$
$$+14\,\alpha_1{}^3\,\alpha_2\,\gamma\,\tau + 21\,\alpha_1{}^3\,\gamma\,\tau^2 + 3\,\alpha_1{}^2\,\alpha_2{}^2\,\gamma\,\tau + 3\,\alpha_1\,\alpha_2{}^2\,\gamma\,\tau^2 + 2\,\alpha_1\,\alpha_2\,\gamma\,\tau^3 + 3\,\alpha_1\,\gamma^2\,q\,\tau^2.$$

The formula of f_1, f_2, f_3.

$$f_1 = 7\,\Phi^2\,\alpha_1{}^4 + 12\,\Phi^2\,\alpha_1{}^3\,\alpha_2 + 12\,\Phi^2\,\alpha_1{}^3\,\gamma + 12\,\Phi^2\,\alpha_1{}^3\,q + 12\,\Phi^2\,\alpha_1{}^3\,\rho + 18\,\Phi^2\,\alpha_1{}^3\,\tau + 6\,\Phi^2\,\alpha_1{}^2\,\alpha_2{}^2$$
$$+12\,\Phi^2\,\alpha_1{}^2\,\alpha_2\,\gamma + 12\,\Phi^2\,\alpha_1{}^2\,\alpha_2\,q + 18\,\Phi^2\,\alpha_1{}^2\,\alpha_2\,\rho + 21\,\Phi^2\,\alpha_1{}^2\,\alpha_2\,\tau + 6\,\Phi^2\,\alpha_1{}^2\,\gamma^2 + 12\,\Phi^2\,\alpha_1{}^2\,\gamma\,q$$
$$+18\,\Phi^2\,\alpha_1{}^2\,\gamma\,\rho + 21\,\Phi^2\,\alpha_1{}^2\,\gamma\,\tau + 6\,\Phi^2\,\alpha_1{}^2\,q^2 + 12\,\Phi^2\,\alpha_1{}^2\,q\,\rho + 21\,\Phi^2\,\alpha_1{}^2\,q\,\tau + 6\,\Phi^2\,\alpha_1{}^2\,\rho^2$$
$$+21\,\Phi^2\,\alpha_1{}^2\,\rho\,\tau + 13\,\Phi^2\,\alpha_1{}^2\,\tau^2 + \Phi^2\,\alpha_1\,\alpha_2{}^3 + 3\,\Phi^2\,\alpha_1\,\alpha_2{}^2\,\gamma + 3\,\Phi^2\,\alpha_1\,\alpha_2{}^2\,q + 7\,\Phi^2\,\alpha_1\,\alpha_2{}^2\,\rho$$
$$+8\,\Phi^2\,\alpha_1\,\alpha_2{}^2\,\tau + 3\,\Phi^2\,\alpha_1\,\alpha_2\,\gamma^2 + 6\,\Phi^2\,\alpha_1\,\alpha_2\,\gamma\,q + 14\,\Phi^2\,\alpha_1\,\alpha_2\,\gamma\,\rho + 16\,\Phi^2\,\alpha_1\,\alpha_2\,\gamma\,\tau + 3\,\Phi^2\,\alpha_1\,\alpha_2\,q^2$$
$$+10\,\Phi^2\,\alpha_1\,\alpha_2\,q\,\rho + 16\,\Phi^2\,\alpha_1\,\alpha_2\,q\,\tau + 7\,\Phi^2\,\alpha_1\,\alpha_2\,\rho^2 + 14\,\Phi^2\,\alpha_1\,\alpha_2\,\rho\,\tau + 10\,\Phi^2\,\alpha_1\,\alpha_2\,\tau^2 + \Phi^2\,\alpha_1\,\gamma^3$$
$$+3\,\Phi^2\,\alpha_1\,\gamma^2\,q + 7\,\Phi^2\,\alpha_1\,\gamma^2\,\rho + 8\,\Phi^2\,\alpha_1\,\gamma^2\,\tau + 3\,\Phi^2\,\alpha_1\,\gamma\,q^2 + 10\,\Phi^2\,\alpha_1\,\gamma\,q\,\rho + 16\,\Phi^2\,\alpha_1\,\gamma\,q\,\tau$$
$$+14\,\Phi^2\,\alpha_1\,\gamma\,\rho\,\tau + 10\,\Phi^2\,\alpha_1\,\gamma\,\tau^2 + \Phi^2\,\alpha_1\,q^3 + 3\,\Phi^2\,\alpha_1\,q^2\,\rho + 8\,\Phi^2\,\alpha_1\,q^2\,\tau + 3\,\Phi^2\,\alpha_1\,q\,\rho^2$$
$$+10\,\Phi^2\,\alpha_1\,q\,\tau^2 + \Phi^2\,\alpha_1\,\rho^3 + 8\,\Phi^2\,\alpha_1\,\rho^2\,\tau + 10\,\Phi^2\,\alpha_1\,\rho\,\tau^2 + 2\,\Phi^2\,\alpha_1\,\tau^3 + \Phi^2\,\alpha_2{}^3\,\rho + \Phi^2\,\alpha_2{}^3\,\tau$$
$$+3\,\Phi^2\,\alpha_2{}^2\,\gamma\,\tau + 2\,\Phi^2\,\alpha_2{}^2\,q\,\rho + 3\,\Phi^2\,\alpha_2{}^2\,q\,\tau + \Phi^2\,\alpha_2{}^2\,\rho^2 + 2\,\Phi^2\,\alpha_2{}^2\,\rho\,\tau + 2\,\Phi^2\,\alpha_2{}^2\,\tau^2 + 3\,\Phi^2\,\alpha_2\,\gamma^2\,\rho$$
$$+3\,\Phi^2\,\alpha_2\,\gamma^2\,\tau + 4\,\Phi^2\,\alpha_2\,\gamma\,q\,\rho + 6\,\Phi^2\,\alpha_2\,\gamma\,q\,\tau + 2\,\Phi^2\,\alpha_2\,\gamma\,\rho^2 + 4\,\Phi^2\,\alpha_2\,\gamma\,\rho\,\tau + 4\,\Phi^2\,\alpha_2\,\gamma\,\tau^2$$
$$+3\,\Phi^2\,\alpha_2\,q^2\,\tau + 2\,\Phi^2\,\alpha_2\,q\,\rho^2 + 3\,\Phi^2\,\alpha_2\,q\,\rho\,\tau + 4\,\Phi^2\,\alpha_2\,q\,\tau^2 + \Phi^2\,\alpha_2\,\rho^3 + 2\,\Phi^2\,\alpha_2\,\rho^2\,\tau + 3\,\Phi^2\,\alpha_2\,\rho\,\tau^2$$
$$+\Phi^2\,\alpha_2\,\tau^3 + \Phi^2\,\gamma^3\,\rho + \Phi^2\,\gamma^3\,\tau + 2\,\Phi^2\,\gamma^2\,q\,\rho + 3\,\Phi^2\,\gamma^2\,q\,\tau + \Phi^2\,\gamma^2\,\rho^2 + 2\,\Phi^2\,\gamma^2\,\rho\,\tau + 2\,\Phi^2\,\gamma^2\,\tau^2$$
$$+\Phi^2\,\gamma\,q^2\,\rho + 3\,\Phi^2\,\gamma\,q^2\,\tau + 2\,\Phi^2\,\gamma\,q\,\rho^2 + 3\,\Phi^2\,\gamma\,q\,\rho\,\tau + 4\,\Phi^2\,\gamma\,q\,\tau^2 + \Phi^2\,\gamma\,\rho^3 + 2\,\Phi^2\,\gamma\,\rho^2\,\tau$$
$$+3\,\Phi^2\,\gamma\,\rho\,\tau^2 + \Phi^2\,\gamma\,\tau^3 + \Phi^2\,q^3\,\tau + \Phi^2\,q^2\,\rho\,\tau + 2\,\Phi^2\,q^2\,\tau^2 + \Phi^2\,q\,\rho^2\,\tau + 2\,\Phi^2\,q\,\rho\,\tau^2 + \Phi^2\,q\,\tau^3$$
$$+\Phi^2\,\rho^3\,\tau + 2\,\Phi^2\,\rho^2\,\tau^2 + \Phi^2\,\rho\,\tau^3 + 6\,\alpha\,\Phi\,\alpha_1{}^3\,\alpha_2\,\rho + 8\,\alpha\,\Phi\,\alpha_1{}^2\,\alpha_2{}^2\,\rho + 8\,\alpha\,\Phi\,\alpha_1{}^2\,\alpha_2\,\gamma\,\rho$$
$$+8\,\alpha\,\Phi\,\alpha_1{}^2\,\alpha_2\,q\,\rho + 8\,\alpha\,\Phi\,\alpha_1{}^2\,\alpha_2\,\rho^2 + 2\,\alpha\,\Phi\,\alpha_1{}^2\,\alpha_2\,\rho\,\tau + 2\,\alpha\,\Phi\,\alpha_1\,\alpha_2{}^3\,\rho + 4\,\alpha\,\Phi\,\alpha_1\,\alpha_2{}^2\,\gamma\,\rho$$
$$+4\,\alpha\,\Phi\,\alpha_1\,\alpha_2{}^2\,q\,\rho + 10\,\alpha\,\Phi\,\alpha_1\,\alpha_2{}^2\,\rho^2 + 2\,\alpha\,\Phi\,\alpha_1\,\alpha_2{}^2\,\rho\,\tau + 2\,\alpha\,\Phi\,\alpha_1\,\alpha_2\,\gamma^2\,\rho + 4\,\alpha\,\Phi\,\alpha_1\,\alpha_2\,\gamma\,q\,\rho$$
$$+10\,\alpha\,\Phi\,\alpha_1\,\alpha_2\,\gamma\,\rho^2 + 2\,\alpha\,\Phi\,\alpha_1\,\alpha_2\,\gamma\,\rho\,\tau + 2\,\alpha\,\Phi\,\alpha_1\,\alpha_2\,q^2\,\rho + 10\,\alpha\,\Phi\,\alpha_1\,\alpha_2\,q\,\rho^2 + 2\,\alpha\,\Phi\,\alpha_1\,\alpha_2\,q\,\rho\,\tau$$
$$+2\,\alpha\,\Phi\,\alpha_1\,\alpha_2\,\rho^3 + 2\,\alpha\,\Phi\,\alpha_1\,\alpha_2\,\rho^2\,\tau + 2\,\alpha\,\Phi\,\alpha_2{}^3\,\rho^2 + 4\,\alpha\,\Phi\,\alpha_2{}^2\,\gamma\,\rho^2 + 4\,\alpha\,\Phi\,\alpha_2{}^2\,q\,\rho^2 + 2\,\alpha\,\Phi\,\alpha_2{}^2\,\rho^3$$
$$+2\,\alpha\,\Phi\,\alpha_2{}^2\,\rho^2\,\tau + 2\,\alpha\,\Phi\,\alpha_2\,\gamma^2\,\rho^2 + 4\,\alpha\,\Phi\,\alpha_2\,\gamma\,q\,\rho^2 + 2\,\alpha\,\Phi\,\alpha_2\,\gamma\,\rho^3 + 2\,\alpha\,\Phi\,\alpha_2\,\gamma\,\rho^2\,\tau + \Phi^2\,\alpha_2\,q^2\,\rho$$
$$+2\,\alpha\,\Phi\,\alpha_2\,q^2\,\rho^2 + 2\,\alpha\,\Phi\,\alpha_2\,q\,\rho^3 + 2\,\alpha\,\Phi\,\alpha_2\,q\,\rho^2\,\tau + 3\,\Phi^2\,\alpha_2{}^2\,\gamma\,\rho + 7\,\Phi^2\,\alpha_1\,\gamma\,\rho^2 + 10\,\Phi^2\,\alpha_1\,q\,\rho\,\tau,$$

$$f_2 = 6\,\Phi\,\alpha_1{}^5 + 9\,\alpha\,\alpha_1{}^4\,\alpha_2\,\rho + 11\,\Phi\,\alpha_1{}^4\,\alpha_2 + 11\,\Phi\,\alpha_1{}^4\,\gamma + 11\,\Phi\,\alpha_1{}^4\,q + 11\,\Phi\,\alpha_1{}^4\,\rho + 26\,\Phi\,\alpha_1{}^4\,\tau$$
$$+15\,\alpha\,\alpha_1{}^3\,\alpha_2{}^2\,\rho + 6\,\Phi\,\alpha_1{}^3\,\alpha_2{}^2 + 15\,\alpha\,\alpha_1{}^3\,\alpha_2\,\gamma\,\rho + 12\,\Phi\,\alpha_1{}^3\,\alpha_2\,\gamma + 15\,\alpha\,\alpha_1{}^3\,\alpha_2\,q\,\rho$$

$$+15\,\alpha\,\alpha_1{}^3\,\alpha_2\,\rho^2+6\,\alpha\,\alpha_1{}^3\,\alpha_2\,\rho\,\tau+18\,\Phi\,\alpha_1{}^3\,\alpha_2\,\rho+33\,\Phi\,\alpha_1{}^3\,\alpha_2\,\tau+6\,\Phi\,\alpha_1{}^3\,\gamma^2+12\,\Phi\,\alpha_1{}^3\,\gamma\,q$$
$$+18\,\Phi\,\alpha_1{}^3\,\gamma\,\rho+33\,\Phi\,\alpha_1{}^3\,\gamma\,\tau+6\,\Phi\,\alpha_1{}^3\,q^2+12\,\Phi\,\alpha_1{}^3\,q\,\rho+33\,\Phi\,\alpha_1{}^3\,q\,\tau+6\,\Phi\,\alpha_1{}^3\,\rho^2$$
$$+27\,\Phi\,\alpha_1{}^3\,\tau^2+7\,\alpha\,\alpha_1{}^2\,\alpha_2{}^3\,\rho+\Phi\,\alpha_1{}^2\,\alpha_2{}^3+14\,\alpha\,\alpha_1{}^2\,\alpha_2{}^2\,\gamma\,\rho+3\,\Phi\,\alpha_1{}^2\,\alpha_2{}^2\,\gamma+14\,\alpha\,\alpha_1{}^2\,\alpha_2{}^2\,q\,\rho$$
$$+3\,\Phi\,\alpha_1{}^2\,\alpha_2{}^2\,q+23\,\alpha\,\alpha_1{}^2\,\alpha_2{}^2\,\rho^2+8\,\alpha\,\alpha_1{}^2\,\alpha_2{}^2\,\rho\,\tau+8\,\Phi\,\alpha_1{}^2\,\alpha_2{}^2\,\rho+14\,\Phi\,\alpha_1{}^2\,\alpha_2{}^2\,\tau$$
$$+3\,\Phi\,\alpha_1{}^2\,\alpha_2\,\gamma^2+14\,\alpha\,\alpha_1{}^2\,\alpha_2\,\gamma\,q\,\rho+6\,\Phi\,\alpha_1{}^2\,\alpha_2\,\gamma\,q+23\,\alpha\,\alpha_1{}^2\,\alpha_2\,\gamma\,\rho^2+8\,\alpha\,\alpha_1{}^2\,\alpha_2\,\gamma\,\rho\,\tau$$
$$+16\,\Phi\,\alpha_1{}^2\,\alpha_2\,\gamma\,\rho+28\,\Phi\,\alpha_1{}^2\,\alpha_2\,\gamma\,\tau+7\,\alpha\,\alpha_1{}^2\,\alpha_2\,q^2\,\rho+3\,\Phi\,\alpha_1{}^2\,\alpha_2\,q^2+23\,\alpha\,\alpha_1{}^2\,\alpha_2\,q\,\rho^2$$
$$+8\,\alpha\,\alpha_1{}^2\,\alpha_2\,q\,\rho\,\tau+11\,\Phi\,\alpha_1{}^2\,\alpha_2\,q\,\rho+28\,\Phi\,\alpha_1{}^2\,\alpha_2\,q\,\tau+7\,\alpha\,\alpha_1{}^2\,\alpha_2\,\rho^3+8\,\alpha\,\alpha_1{}^2\,\alpha_2\,\rho^2\,\tau$$
$$+8\,\Phi\,\alpha_1{}^2\,\alpha_2\,\rho^2+\alpha\,\alpha_1{}^2\,\alpha_2\,\rho\,\tau^2+26\,\Phi\,\alpha_1{}^2\,\alpha_2\,\rho\,\tau+27\,\Phi\,\alpha_1{}^2\,\alpha_2\,\tau^2+\Phi\,\alpha_1{}^2\,\gamma^3+3\,\Phi\,\alpha_1{}^2\,\gamma^2\,q$$
$$+8\,\Phi\,\alpha_1{}^2\,\gamma^2\,\rho+14\,\Phi\,\alpha_1{}^2\,\gamma^2\,\tau+3\,\Phi\,\alpha_1{}^2\,\gamma\,q^2+11\,\Phi\,\alpha_1{}^2\,\gamma\,q\,\rho+28\,\Phi\,\alpha_1{}^2\,\gamma\,q\,\tau+8\,\Phi\,\alpha_1{}^2\,\gamma\,\rho^2$$
$$+26\,\Phi\,\alpha_1{}^2\,\gamma\,\rho\,\tau+27\,\Phi\,\alpha_1{}^2\,\gamma\,\tau^2+\Phi\,\alpha_1{}^2\,q^3+3\,\Phi\,\alpha_1{}^2\,q^2\,\rho+14\,\Phi\,\alpha_1{}^2\,q^2\,\tau+3\,\Phi\,\alpha_1{}^2\,q\,\rho^2$$
$$+19\,\Phi\,\alpha_1{}^2\,q\,\rho\,\tau+27\,\Phi\,\alpha_1{}^2\,q\,\tau^2+\Phi\,\alpha_1{}^2\,\rho^3+14\,\Phi\,\alpha_1{}^2\,\rho^2\,\tau+27\,\Phi\,\alpha_1{}^2\,\rho\,\tau^2+7\,\Phi\,\alpha_1{}^2\,\tau^3$$
$$+\alpha\,\alpha_1\,\alpha_2{}^4\,\rho+3\,\alpha\,\alpha_1\,\alpha_2{}^3\,\gamma\,\rho+3\,\alpha\,\alpha_1\,\alpha_2{}^3\,q\,\rho+9\,\alpha\,\alpha_1\,\alpha_2{}^3\,\rho^2+2\,\alpha\,\alpha_1\,\alpha_2{}^3\,\rho\,\tau+\Phi\,\alpha_1\,\alpha_2{}^3\,\rho$$
$$+2\,\Phi\,\alpha_1\,\alpha_2{}^3\,\tau+3\,\alpha\,\alpha_1\,\alpha_2{}^2\,\gamma^2\,\rho+6\,\alpha\,\alpha_1\,\alpha_2{}^2\,\gamma\,q\,\rho+18\,\alpha\,\alpha_1\,\alpha_2{}^2\,\gamma\,\rho^2+4\,\alpha\,\alpha_1\,\alpha_2{}^2\,\gamma\,\rho\,\tau$$
$$+6\,\Phi\,\alpha_1\,\alpha_2{}^2\,\gamma\,\tau+3\,\alpha\,\alpha_1\,\alpha_2{}^2\,q^2\,\rho+18\,\alpha\,\alpha_1\,\alpha_2{}^2\,q\,\rho^2+4\,\alpha\,\alpha_1\,\alpha_2{}^2\,q\,\rho\,\tau+2\,\Phi\,\alpha_1\,\alpha_2{}^2\,q\,\rho$$
$$+9\,\alpha\,\alpha_1\,\alpha_2{}^2\,\rho^3+10\,\alpha\,\alpha_1\,\alpha_2{}^2\,\rho^2\,\tau+2\,\Phi\,\alpha_1\,\alpha_2{}^2\,\rho^2+\alpha\,\alpha_1\,\alpha_2{}^2\,\rho\,\tau^2+5\,\Phi\,\alpha_1\,\alpha_2{}^2\,\rho\,\tau+9\,\Phi\,\alpha_1\,\alpha_2{}^2\,\tau^2$$
$$+\alpha\,\alpha_1\,\alpha_2\,\gamma^3\,\rho+3\,\alpha\,\alpha_1\,\alpha_2\,\gamma^2\,q\,\rho+9\,\alpha\,\alpha_1\,\alpha_2\,\gamma^2\,\rho^2+2\,\alpha\,\alpha_1\,\alpha_2\,\gamma^2\,\rho\,\tau+3\,\Phi\,\alpha_1\,\alpha_2\,\gamma^2\,\rho$$
$$+3\,\alpha\,\alpha_1\,\alpha_2\,\gamma\,q^2\,\rho+18\,\alpha\,\alpha_1\,\alpha_2\,\gamma\,q\,\rho^2+4\,\alpha\,\alpha_1\,\alpha_2\,\gamma\,q\,\rho\,\tau+4\,\Phi\,\alpha_1\,\alpha_2\,\gamma\,q\,\rho+12\,\Phi\,\alpha_1\,\alpha_2\,\gamma\,q\,\tau$$
$$+9\,\alpha\,\alpha_1\,\alpha_2\,\gamma\,\rho^3+10\,\alpha\,\alpha_1\,\alpha_2\,\gamma\,\rho^2\,\tau+4\,\Phi\,\alpha_1\,\alpha_2\,\gamma\,\rho^2+\alpha\,\alpha_1\,\alpha_2\,\gamma\,\rho\,\tau^2+10\,\Phi\,\alpha_1\,\alpha_2\,\gamma\,\rho\,\tau$$
$$+18\,\Phi\,\alpha_1\,\alpha_2\,\gamma\,\tau^2+\alpha\,\alpha_1\,\alpha_2\,q^3\,\rho+9\,\alpha\,\alpha_1\,\alpha_2\,q^2\,\rho^2+2\,\alpha\,\alpha_1\,\alpha_2\,q^2\,\rho\,\tau+\Phi\,\alpha_1\,\alpha_2\,q^2\,\rho+6\,\Phi\,\alpha_1\,\alpha_2\,q^2\,\tau$$
$$+9\,\alpha\,\alpha_1\,\alpha_2\,q\,\rho^3+10\,\alpha\,\alpha_1\,\alpha_2\,q\,\rho^2\,\tau+2\,\Phi\,\alpha_1\,\alpha_2\,q\,\rho^2+\alpha\,\alpha_1\,\alpha_2\,q\,\rho\,\tau^2+5\,\Phi\,\alpha_1\,\alpha_2\,q\,\rho\,\tau$$
$$+\alpha\,\alpha_1\,\alpha_2\,\rho^4+2\,\alpha\,\alpha_1\,\alpha_2\,\rho^3\,\tau+\Phi\,\alpha_1\,\alpha_2\,\rho^3+\alpha\,\alpha_1\,\alpha_2\,\rho^2\,\tau^2+5\,\Phi\,\alpha_1\,\alpha_2\,\rho^2\,\tau+15\,\Phi\,\alpha_1\,\alpha_2\,\rho\,\tau^2$$
$$+5\,\Phi\,\alpha_1\,\alpha_2\,\tau^3+\Phi\,\alpha_1\,\gamma^3\,\rho+2\,\Phi\,\alpha_1\,\gamma^3\,\tau+2\,\Phi\,\alpha_1\,\gamma^2\,q\,\rho+6\,\Phi\,\alpha_1\,\gamma^2\,q\,\tau+2\,\Phi\,\alpha_1\,\gamma^2\,\rho^2$$
$$+9\,\Phi\,\alpha_1\,\gamma^2\,\tau^2+\Phi\,\alpha_1\,\gamma\,q^2\,\rho+6\,\Phi\,\alpha_1\,\gamma\,q^2\,\tau+2\,\Phi\,\alpha_1\,\gamma\,q\,\rho^2+5\,\Phi\,\alpha_1\,\gamma\,q\,\rho\,\tau+18\,\Phi\,\alpha_1\,\gamma\,q\,\tau^2$$
$$+\Phi\,\alpha_1\,\gamma\,\rho^3+5\,\Phi\,\alpha_1\,\gamma\,\rho^2\,\tau+15\,\Phi\,\alpha_1\,\gamma\,\rho\,\tau^2+5\,\Phi\,\alpha_1\,\gamma\,\tau^3+2\,\Phi\,\alpha_1\,q^3\,\tau+9\,\Phi\,\alpha_1\,q^2\,\tau^2$$
$$+5\,\Phi\,\alpha_1\,q\,\tau^3+2\,\Phi\,\alpha_1\,\rho^3\,\tau+9\,\Phi\,\alpha_1\,\rho^2\,\tau^2+5\,\Phi\,\alpha_1\,\rho\,\tau^3+\alpha\,\alpha_2{}^4\,\rho^2+3\,\alpha\,\alpha_2{}^3\,\gamma\,\rho^2+3\,\alpha\,\alpha_2{}^3\,q\,\rho^2$$
$$+2\,\alpha\,\alpha_2{}^3\,\rho^3+2\,\alpha\,\alpha_2{}^3\,\rho^2\,\tau+\Phi\,\alpha_2{}^3\,\tau^2+3\,\alpha\,\alpha_2{}^2\,\gamma^2\,\rho^2+6\,\alpha\,\alpha_2{}^2\,\gamma\,q\,\rho^2+4\,\alpha\,\alpha_2{}^2\,\gamma\,\rho^3$$
$$+3\,\Phi\,\alpha_2{}^2\,\gamma\,\tau^2+3\,\alpha\,\alpha_2{}^2\,q^2\,\rho^2+4\,\alpha\,\alpha_2{}^2\,q\,\rho^3+4\alpha\,\alpha_2{}^2\,q\,\rho^2\,\tau+\Phi\,\alpha_2{}^2\,q\,\tau(3\tau-\rho)+\alpha\,\alpha_2{}^2\,\rho^4$$
$$+2\,\alpha\,\alpha_2{}^2\,\rho^3\,\tau+\alpha\,\alpha_2{}^2\,\rho^2\,\tau^2+2\,\Phi\,\alpha_2{}^2\,\rho\,\tau^2+\Phi\,\alpha_2{}^2\,\tau^3+\alpha\,\alpha_2\,\gamma^3\,\rho^2+3\,\alpha\,\alpha_2\,\gamma^2\,q\,\rho^2+2\,\alpha\,\alpha_2\,\gamma^2\,\rho^3$$
$$+2\,\alpha\,\alpha_2\,\gamma^2\,\rho^2\,\tau+3\,\Phi\,\alpha_2\,\gamma^2\,\tau^2+3\,\alpha\,\alpha_2\,\gamma\,q^2\,\rho^2+4\,\alpha\,\alpha_2\,\gamma\,q\,\rho^3+4\,\alpha\,\alpha_2\,\gamma\,q\,\rho^2\,\tau$$
$$+\alpha\,\alpha_2\,\gamma\,\rho^4+2\,\alpha\,\alpha_2\,\gamma\,\rho^3\,\tau+\alpha\,\alpha_2\,\gamma\,\rho^2\,\tau^2+4\,\Phi\,\alpha_2\,\gamma\,\rho\,\tau^2+2\,\Phi\,\alpha_2\,\gamma\,\tau^3+\alpha\,\alpha_2\,q^3\,\rho^2$$
$$+2\,\alpha\,\alpha_2\,q^2\,\rho^3+2\,\alpha\,\alpha_2\,q^2\,\rho^2\,\tau+\Phi\,\alpha_2\,q^2\,\tau(3\tau-2\rho)+\alpha\,\alpha_2\,q\,\rho^4+2\,\alpha\,\alpha_2\,q\,\rho^3\,\tau+\alpha\,\alpha_2\,q\,\rho^2\,\tau^2$$
$$+\Phi\,\alpha_2\,q\,\rho\,\tau(3\tau-2\rho)+2\,\Phi\,\alpha_2\,q\,\tau^3+2\,\Phi\,\alpha_2\,\rho^2\,\tau^2+\Phi\,\alpha_2\,\rho\,\tau^3+\Phi\,\gamma^3\,\tau^2+\Phi\,\rho^2\,\tau^3$$
$$+\Phi\,\gamma^2\,q\,\tau(3\tau-\rho)+2\,\Phi\,\gamma^2\,\rho\,\tau^2+\Phi\,\gamma^2\,\tau^3+\Phi\,\gamma\,q^2\,\tau(3\tau-2\rho)+\Phi\,\gamma\,q\,\rho\,\tau(3\tau-2\rho)$$
$$+2\,\Phi\,\gamma\,q\,\tau^3+2\,\Phi\,\gamma\,\rho^2\,\tau^2+\Phi\,\gamma\,\rho\,\tau^3+\Phi\,q^3\,\tau(\tau-\rho)+\Phi\,q^2\,\rho\,\tau(\tau-\rho)+\Phi q^2\tau(\tau^2-\rho^2)$$
$$+\Phi\,q\,\rho^2\tau(\tau-\rho)+\Phi\,q\,\rho\,\tau^3+\Phi\,\rho^3\,\tau^2+7\,\alpha\,\alpha_1{}^2\,\alpha_2\,\gamma^2\,\rho+3\,\Phi\,\alpha_1\,\alpha_2{}^2\,\gamma\,\rho+6\,\Phi\,\alpha_1\,\alpha_2{}^2\,q\,\tau$$
$$+6\,\Phi\,\alpha_1\,\alpha_2\,\gamma^2\,\tau+18\,\Phi\,\alpha_1\,\alpha_2\,q\,\tau^2+5\,\Phi\,\alpha_1\,\gamma^2\,\rho\,\tau+12\,\Phi\,\alpha_1\,q\,\rho\,\tau^2+4\,\alpha\,\alpha_2{}^2\,\gamma\,\rho^2\,\tau$$

$$
\begin{aligned}
& +\Phi\,\alpha_2\,\gamma\,q\,\tau(6\tau-2\rho)+33\,\Phi\,{\alpha_1}^3\,\rho\,\tau+12\,\Phi\,{\alpha_1}^3\,\alpha_2\,q, \\
f_3 \;=\; & 12\,{\alpha_1}^5\,\tau+16\,{\alpha_1}^4\,\alpha_2\,\tau+16\,{\alpha_1}^4\,\gamma\,\tau+16\,{\alpha_1}^4\,q\,\tau+16\,{\alpha_1}^4\,\rho\,\tau+18\,{\alpha_1}^4\,\tau^2+7\,{\alpha_1}^3\,{\alpha_2}^2\,\tau \\
& +14\,{\alpha_1}^3\,\alpha_2\,q\,\tau+14\,{\alpha_1}^3\,\alpha_2\,\rho\,\tau+21\,{\alpha_1}^3\,\alpha_2\,\tau^2+7\,{\alpha_1}^3\,\gamma^2\,\tau+14\,{\alpha_1}^3\,\gamma\,q\,\tau+14\,{\alpha_1}^3\,\gamma\,\rho\,\tau \\
& +7\,{\alpha_1}^3\,q^2\,\tau+14\,{\alpha_1}^3\,q\,\rho\,\tau+21\,{\alpha_1}^3\,q\,\tau^2+7\,{\alpha_1}^3\,\rho^2\,\tau+21\,{\alpha_1}^3\,\rho\,\tau^2+6\,{\alpha_1}^3\,\tau^3+{\alpha_1}^2\,{\alpha_2}^3\,\tau \\
& +3\,{\alpha_1}^2\,{\alpha_2}^2\,q\,\tau+3\,{\alpha_1}^2\,{\alpha_2}^2\,\rho\,\tau+8\,{\alpha_1}^2\,{\alpha_2}^2\,\tau^2+3\,{\alpha_1}^2\,\alpha_2\,\gamma^2\,\tau+6\,{\alpha_1}^2\,\alpha_2\,\gamma\,q\,\tau+6\,{\alpha_1}^2\,\alpha_2\,\gamma\,\rho\,\tau \\
& +16\,{\alpha_1}^2\,\alpha_2\,\gamma\,\tau^2+3\,{\alpha_1}^2\,\alpha_2\,q^2\,\tau+6\,{\alpha_1}^2\,\alpha_2\,q\,\rho\,\tau+16\,{\alpha_1}^2\,\alpha_2\,q\,\tau^2+3\,{\alpha_1}^2\,\alpha_2\,\rho^2\,\tau+16\,{\alpha_1}^2\,\alpha_2\,\rho\,\tau^2 \\
& +5\,{\alpha_1}^2\,\alpha_2\,\tau^3+{\alpha_1}^2\,\gamma^3\,\tau+3\,{\alpha_1}^2\,\gamma^2\,q\,\tau+3\,{\alpha_1}^2\,\gamma^2\,\rho\,\tau+8\,{\alpha_1}^2\,\gamma^2\,\tau^2+3\,{\alpha_1}^2\,\gamma\,q^2\,\tau+6\,{\alpha_1}^2\,\gamma\,q\,\rho\,\tau \\
& +16\,{\alpha_1}^2\,\gamma\,q\,\tau^2+3\,{\alpha_1}^2\,\gamma\,\rho^2\,\tau+16\,{\alpha_1}^2\,\gamma\,\rho\,\tau^2+5\,{\alpha_1}^2\,\gamma\,\tau^3+{\alpha_1}^2\,q^3\,\tau+3\,{\alpha_1}^2\,q^2\,\rho\,\tau+8\,{\alpha_1}^2\,q^2\,\tau^2 \\
& +3\,{\alpha_1}^2\,q\,\rho^2\,\tau+16\,{\alpha_1}^2\,q\,\rho\,\tau^2+5\,{\alpha_1}^2\,q\,\tau^3+{\alpha_1}^2\,\rho^3\,\tau+8\,{\alpha_1}^2\,\rho^2\,\tau^2+5\,{\alpha_1}^2\,\rho\,\tau^3+\alpha_1\,{\alpha_2}^3\,\tau^2 \\
& +3\,\alpha_1\,{\alpha_2}^2\,q\,\tau^2+3\,\alpha_1\,{\alpha_2}^2\,\rho\,\tau^2+\alpha_1\,{\alpha_2}^2\,\tau^3+3\,\alpha_1\,\alpha_2\,\gamma^2\,\tau^2+6\,\alpha_1\,\alpha_2\,\gamma\,q\,\tau^2+6\,\alpha_1\,\alpha_2\,\gamma\,\rho\,\tau^2 \\
& +3\,\alpha_1\,\alpha_2\,q^2\,\tau^2+6\,\alpha_1\,\alpha_2\,q\,\rho\,\tau^2+2\,\alpha_1\,\alpha_2\,q\,\tau^3+3\,\alpha_1\,\alpha_2\,\rho^2\,\tau^2+2\,\alpha_1\,\alpha_2\,\rho\,\tau^3+\alpha_1\,\gamma^3\,\tau^2 \\
& +3\,\alpha_1\,\gamma^2\,\rho\,\tau^2+\alpha_1\,\gamma^2\,\tau^3+3\,\alpha_1\,\gamma\,q^2\,\tau^2+6\,\alpha_1\,\gamma\,q\,\rho\,\tau^2+2\,\alpha_1\,\gamma\,q\,\tau^3+3\,\alpha_1\,\gamma\,\rho^2\,\tau^2+2\,\alpha_1\,\gamma\,\rho\,\tau^3 \\
& +\alpha_1\,q^3\,\tau^2+3\,\alpha_1\,q^2\,\rho\,\tau^2+\alpha_1\,q^2\,\tau^3+3\,\alpha_1\,q\,\rho^2\,\tau^2+2\,\alpha_1\,q\,\rho\,\tau^3+\alpha_1\,\rho^3\,\tau^2+\alpha_1\,\rho^2\,\tau^3 \\
& +14\,{\alpha_1}^3\,\alpha_2\,\gamma\,\tau+21\,{\alpha_1}^3\,\gamma\,\tau^2+3\,{\alpha_1}^2\,{\alpha_2}^2\,\gamma\,\tau+3\,\alpha_1\,{\alpha_2}^2\,\gamma\,\tau^2+2\,\alpha_1\,\alpha_2\,\gamma\,\tau^3+3\,\alpha_1\,\gamma^2\,q\,\tau^2.
\end{aligned}
$$