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Research article

A two-state neuronal model with alternating exponential excitation

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Abstract: We develop a stochastic neural model based on point excitatory inputs. The nerve cell depolarisation is determined by a two-state point process corresponding the *two states of the cell.* The model presumes *state-dependent* excitatory stimuli amplitudes and decay rates of membrane potential. The state switches at each stimulus time.
We analyse the neural firing time distribution and the mean firing time. The limit of the firing time at a definitive scaling condition is also obtained.
The results are based on an analysis of the first crossing time of the depolarisation process through the firing threshold. The Laplace transform technique is widely used.

Keywords: jump-telegraph process; first passage time; neural activity; firing probability; asymptotical behaviour

1. The model of neural activity with alternating states

A neuron is surrounded by a membrane with selective conductivity depending on its current state. The membrane potential $V = V(t)$ undergoes a sudden change, which is called the "spike" potential. The spike (impulse) is generated in response to an external influence and only when it exceeds a certain threshold. Since most cells of central nervous system are characterised by spontaneously emitted train of impulses, stochastic modelling is of primary interest in the field. It is assumed that the excitatory stimuli that occur in random time are random and depend on the current state of the neuron. Each stimulus is followed by a refractory period, which corresponds to an exponential decay of the potential. In view of this, the first-passage-time problem for the underlying stochastic processes is important for the description of the neuronal firing.

A large number of models of single neurons were developed: from simple threshold models to biologically plausible "portrait" models. Best results were achieved when complicated experimental features were combined with a rather simple mathematical model, see, for example, the mathematical model of the Nobel Prize winner Hodgkin's of the squid giant axon by [1]. The first simple threshold neuron model is considered to be the model proposed by Louis Lapicque in 1907 [2] which is usually called the "leaky integrator" or the "forgetful integrate-and-fire" model. In modern terms, the simplest version of this model with an external forcing term described by a Brownian motion $W = W(t)$ yields the following stochastic differential equation

$$\mathrm{d}V(t) = \left(\frac{I}{C} - \frac{V(t)}{CR}\right)\mathrm{d}t + \sigma \mathrm{d}W(t), \qquad V(0) = V_0.$$

A spike is generated, once the process $V(t)$ hits the firing threshold H.

Since excitatory stimuli are intermittent, synaptic input should be modelled by means of point processes. This approach was proposed by R. Stein [3], see also [4]. In [5], Stein's model is briefly presented in a rigorous manner. See also recent paper [6] on this subject.

Stein's model presumes a time evolution of the potential described by a stochastic equation based on two independent Poisson processes N_+ and N_-

$$\mathrm{d}V(t) = -\frac{1}{\tau}V(t-)\mathrm{d}t + a_+\mathrm{d}N_+(t) - a_-\mathrm{d}N_-(t),$$

where $a_\pm$ denote the amplitudes of excitatory/inhibitory currents.

An excitatory model with an exponentially decaying membrane potential and a non-homogeneous Poisson process driving the consecutive neuronal stimuli is studied by [7]. Methods for assessing how well this model describes neural spikes are based on the time-rescaling technique, [8].

The detailed review of the existing models can be found in [9].

The model proposed in this paper suggests that neurons take one of two states, alternating at each stimulus time. Similar ideas are widely used in neural modelling. For example, in the recent monograph [10], the three-phase Stein model was presented: the standard Stein model is supplemented by an additional 0-phase, which starts at the end of the refractory period and lasts until depolarisation occurs. In [11], a two-phase model is studied, based on neuronal oscillations interrupted by stochastic behaviour. The authors claim that this can be explained by a bistability in the ensemble dynamics of coupled integrate and fire neurons. See also the paper [12], where some practical observations are presented that can serve as the basis for such an approach.

To describe the model, consider the right-continuous process $\varepsilon = \varepsilon(t)$, $t \geq 0$, which has the state space $\{0,\ 1\}$, with independent consecutive (random) holding times $\{T_n\}_{n\geq 1}$. Let $N = N(t)$ be the process counting the switching of ε till time t,

$$N(t) = \max\left\{n : \sum_{k\leq n} T_k \leq t\right\}, \qquad t > 0.$$

To construct a model of neural activity, we will use a well-studied class of jump-telegraph stochastic processes, see, for example, the review in [13] and [14]. Recent paper [15] intensively developed methods for studying the distributions of first passage times for such processes.

The jump-telegraph process $X = X(t)$, $t \geq 0$, with additive jumps, is defined by the stochastic equation

$$\mathrm{d}X(t) = c_{\varepsilon(t-)}\mathrm{d}t + Y_{N(t-)}\mathrm{d}N(t), \quad t > 0, \qquad X(0) = 0, \tag{1.1}$$

which by integration yields

$$X(t)=\int_0^t c_{\varepsilon(\tau)}\mathrm{d}\tau+\sum_{n=1}^{N(t)}Y_n. \tag{1.2}$$

Here velocities $c_0,\ c_1$ are two real constants, $Y_n,\ n\geq 1$ are independent random variables independent of ε, corresponding to jumps which accompany each velocity switching. By $X_i(t),\ t\geq 0$, we denote the solution of (1.1), provided with the additional initial condition: $\varepsilon(0)=i,\ i\in\{0,1\}$.

We propose a neural potential model that includes state-dependent decay rates of membrane potential, along with multiplicative state-dependent stimuli. The model suggests a change in state at the each stimulus time.

This model presumes the nerve cell depolarisation $V=V(t), t\geq 0$, to be determined by the stochastic exponential of X, see (1.1)-(1.2). In other words, process $V=V(t)$ is the solution of the stochastic equation

$$\begin{aligned}\mathrm{d}V(t)=&V(t-)\left(c_{\varepsilon(t-)}\mathrm{d}t+Y_{N(t)}\mathrm{d}N(t)\right),\qquad t>0,\\ V|_{t\downarrow 0}=&V_0.\end{aligned} \tag{1.3}$$

The counting process $N=N(t)$ corresponds to the number of excitatory stimuli received by the neuron till time t, Y_n is the voltage level at the stimulus time, and the *negative* constants $c_0,\ c_1$ correspond to the rates of exponential decay of the membrane potential being in the state 0 and 1 respectively.

The solution $V=V(t)$ of (1.3) is given by stochastic exponential of X,

$$\begin{aligned}V(t)=V_0\mathscr{E}_t(X)=&V_0\exp\left(\int_0^t c_{\varepsilon(s)}\mathrm{d}s\right)\prod_{n=1}^{N(t)}(1+Y_n)\\ =&V_0\exp\left(\int_0^t c_{\varepsilon(s)}\mathrm{d}s+\sum_{n=1}^{N(t)}\log(1+Y_n)\right).\end{aligned} \tag{1.4}$$

Let the holding times $T_n,\ n\geq 1$, have alternating distributions π_0 and π_1, that is $\pi_i(\mathrm{d}t)=\mathbb{P}\{T_1\in \mathrm{d}t\mid \varepsilon(0)=i\},\ i\in\{0,1\}$, and

$$\begin{aligned}\pi_0(\mathrm{d}t):=&\mathbb{P}\{T_n\in\mathrm{d}t\mid\varepsilon(T_1+\ldots+T_{n-1})=0\},\\ \pi_1(\mathrm{d}t):=&\mathbb{P}\{T_n\in\mathrm{d}t\mid\varepsilon(T_1+\ldots+T_{n-1})=1\},\end{aligned}\qquad n\geq 2.$$

Jumps Y_n have the distributions g_0 and g_1, alternately, together with alternating distributions π_0 and π_1 of holding times.

For each $t,\ t>0$, the distribution of $V(t)$ can be expressed by means of the given distributions $\pi_0,\pi_1,\ g_0,g_1$, see e. g. [15].

The main interest of the neural modelling lies in the properties of the first passage time of the depolarisation process $V=V(t)$ through the firing threshold $H,\ H>0$. We are interested to study the distribution of of the stopping time $\mathscr{T}_x,\ x=\log H/V_0$, of the jump-telegraph process,

$$\begin{aligned}\mathscr{T}_x=&\inf\left\{t>0\mid\int_0^t c_{\varepsilon(s)}\mathrm{d}s+\sum_{n=1}^{N(t)}\log(1+Y_n)>x\right\}\\ =&\inf\{t>0\mid V(t)>H\}.\end{aligned}$$

For a continuous version of such processes, $Y_n = 0\ \forall n$, this mathematical problem is well known and studied, see e.g. [16, 17, 18]. See also the recent paper [19], where arbitrary sequences of velocities and jumps intensities were applied. The number of level-crossings for the telegraph process has been analysed in [20].

Properties of $\mathscr{T}_x$ with nontrivial jumps are less known, see [7, 21, 22], where a single state model with independent exponentially distributed jumps, $\log(1+Y_n) \sim \mathrm{Exp}(b)$, is being studied. Some solutions for a jump-diffusion process are presented in [23, 24, 25], the martingale methods are developed by [26] (see the review in [27]). For applications of jump-diffusion processes to models of neuronal activity see [28].

In this paper, we generalise the results of [7] on a two-state model with positive independent random jumps $Z_n = \log(1+Y_n)$, $n \geq 1$, having the alternating exponential distributions, $\mathrm{Exp}(b_0)$ and $\mathrm{Exp}(b_1)$, $b_0, b_1 > 0$, which corresponds to the (alternating) Pareto distributions of the second kind (Lomax distributions) of Y_n, used in economics and actuarial science, see [29, 30],

$$\mathbb{P}\{Y_n \geq y\} = (1+y)^{-b_i}, \qquad y > 0, \qquad i \in \{0,1\}.$$

In Section 2, we treat the problem in a general setting. For the case of exponentially distributed jumps, explicit formulae for the moment generating function of $\mathscr{T}_x$, $x > 0$, are obtained. The firing probabilities and the mean values of the firing time are also studied.

Section 3 concerns the limit behaviour of $\mathscr{T}_x$ under small frequent stimuli. In Section 4 we give an overview of the single-state case, including the limit behaviour under the parameters' scaling similar to that described in Section 3.

2. Firing probabilities and the first passage time of the jump-telegraph process

Let $\mathscr{T}_x^{(i)}$ be the the first passage time of $X_i(t)$, (1.1)-(1.2), through the positive threshold x,

$$\mathscr{T}_x^{(i)} = \inf\{t > 0 :\ X_i(t) > x\}, \qquad i \in \{0,1\}.$$

By definition, we set $\mathscr{T}_x^{(i)}|_{x<0} \equiv 0,\ i \in \{0,1\}$.

Since $c_0, c_1 \leq 0$, the process $\mathbb{X}$ exceeds the threshold xx, $x > 0$, just by jumping. Conditioning on the first switching we have the following identities in law:

$$\mathscr{T}_x^{(0)} \overset{D}{=} T^{(0)} + \mathscr{T}^{(1)}_{x-c_0T^{(0)}-Y^{(0)}}, \qquad \mathscr{T}_x^{(1)} \overset{D}{=} T^{(1)} + \mathscr{T}^{(0)}_{x-c_1T^{(1)}-Y^{(1)}}, \tag{2.1}$$

see the definition of $X(t)$, (1.2). Here $T^{(0)}/\ Y^{(0)}$ and $T^{(1)}/\ Y^{(1)}$ are the first holding time / the first stimulus amplitude at the states $\varepsilon(0) = 0$ and $\varepsilon(0) = 1$ respectively.

Denote by $\phi_i(x) = \phi_i(x;q)$ the Laplace transform of $\mathscr{T}_x^{(i)}$,

$$\phi_i(x) := \mathbb{E}\left[\exp(-q\mathscr{T}_x^{(i)})\right], \qquad q > 0. \tag{2.2}$$

By definition, $0 \leq \phi_i(x) \leq 1\ \forall x$, and $\phi_0(x) \equiv 1,\ \phi_1(x) \equiv 1$, if $x < 0$. Integrating by parts in (2.2), we have

$$\begin{aligned}\phi_i(x) =&\mathbb{E}\left[\exp\left(-q\mathscr{T}_x^{(i)}\right)\right] = \int_0^\infty \mathrm{e}^{-qt}\mathrm{d}\mathbb{P}\{\mathscr{T}_x^{(i)} < t\}\\ =&\int_0^\infty q\mathrm{e}^{-qt}\mathbb{P}\{\mathscr{T}_x^{(i)} < t\}\mathrm{d}t\\ =&\mathbb{P}\{\mathscr{T}_x^{(i)} < \mathrm{e}_q\} = \mathbb{P}\{\sup_{0<t<\mathrm{e}_q} X_i(t) > x\},\end{aligned}$$

where e_q is an exponentially distributed random variable, $\mathrm{Exp}(q)$, independent of ε and $\{Y_n\}_{n\geq 1}$.

Assuming that the sequential jumps Y_n have the distributions g_0 and g_1, alternating together with the alternating distributions π_0 and π_1 of holding times, identity (2.1) can be written as

$$\begin{aligned}\phi_0(x) =&\mathbb{E}\left[\exp(-q\mathscr{T}_x^{(0)})\right] = \mathbb{E}\left[\mathrm{e}^{-qT}\times\exp(-q\mathscr{T}^{(1)}_{x-c_0T-Y}) \mid T\sim\pi_0,\, Y\sim g_0\right],\\ \phi_1(x) =&\mathbb{E}\left[\exp(-q\mathscr{T}_x^{(1)})\right] = \mathbb{E}\left[\mathrm{e}^{-qT}\times\exp(-q\mathscr{T}^{(0)}_{x-c_1T-Y}) \mid T\sim\pi_1,\, Y\sim g_1\right].\end{aligned}\tag{2.3}$$

Equations (2.3) are equivalent to

$$\begin{cases}\phi_0(x) = \displaystyle\int_0^\infty \pi_0(\tau)\mathrm{e}^{-q\tau}\left[\overline{G}_0(x-c_0\tau) + \int_{-\infty}^{x-c_0\tau}\phi_1(x-c_0\tau-y)g_0(\mathrm{d}y)\right]\mathrm{d}\tau,\\ \phi_1(x) = \displaystyle\int_0^\infty \pi_1(\tau)\mathrm{e}^{-q\tau}\left[\overline{G}_1(x-c_1\tau) + \int_{-\infty}^{x-c_1\tau}\phi_0(x-c_1\tau-y)g_1(\mathrm{d}y)\right]\mathrm{d}\tau.\end{cases}\tag{2.4}$$

Here

$$\overline{G}_i(y) = \mathbb{P}\{Y\geq y \mid \varepsilon = i\} = \int_y^\infty g_i(\mathrm{d}y)$$

denotes the conditional survivor function of the stimulus amplitude under the state $\varepsilon = i,\ i\in\{0,1\}$.

In what follows, assume excitatory inputs to be positive and exponentially distributed, that is

$$\overline{G}_0(y) = \mathrm{e}^{-b_0y}\wedge 1, \qquad \overline{G}_1(y) = \mathrm{e}^{-b_1y}\wedge 1,\tag{2.5}$$

with the corresponding density functions

$$g_0(y) = b_0\exp(-b_0y)\mathbb{1}_{y>0}, \qquad g_1(y) = b_1\exp(-b_1y)\mathbb{1}_{y>0},\tag{2.6}$$

$$b_0, b_1 > 0.$$

We try to find the solution $\vec{\phi} = (\phi_0,\ \phi_1)'$ of (2.4) in the form

$$\vec{\phi}(x) = \sum_{k=1}^N \mathrm{e}^{-\xi_k x}\mathbf{A}_k, \qquad x>0,\tag{2.7}$$

with the indefinite coefficients $\mathbf{A}_k = (A_{0k},\ A_{1k})'$, $\mathbf{A}_k\neq\mathbf{0}$, and $\xi_k,\ \xi_k\neq b_0, b_1$, $k = 1,\ldots,N$. Substituting

(2.7) and (2.5)-(2.6) into (2.4), we obtain the following algebraic system:

$$\begin{cases} \sum_{k=1}^{N} A_{0k}\mathrm{e}^{-\xi_k x} = \mathrm{e}^{-b_0 x}\widehat{\pi}_0(q-c_0 b_0) \\ \qquad\qquad + b_0 \sum_{k=1}^{N} \frac{A_{1k}}{b_0-\xi_k}\left[\mathrm{e}^{-\xi_k x}\widehat{\pi}_0(q-c_0\xi_k) - \mathrm{e}^{-b_0 x}\widehat{\pi}_0(q-c_0 b_0)\right], \\ \sum_{k=1}^{N} A_{1k}\mathrm{e}^{-\xi_k x} = \mathrm{e}^{-b_1 x}\widehat{\pi}_1(q-c_1 b_1) \\ \qquad\qquad + b_1 \sum_{k=1}^{N} \frac{A_{0k}}{b_1-\xi_k}\left[\mathrm{e}^{-\xi_k x}\widehat{\pi}_1(q-c_1\xi_k) - \mathrm{e}^{-b_1 x}\widehat{\pi}_1(q-c_1 b_1)\right], \end{cases} \tag{2.8}$$

where

$$\widehat{\pi}_i(p) = \int_0^\infty \mathrm{e}^{-pt}\pi_i(t)\mathrm{d}t, \qquad i \in \{0,1\},$$

is the Laplace transform of the distribution of the holding time.

From (2.8), we get the following linear equations for the indefinite coefficients ξ_k and A_{ik}, $i \in \{0,1\}$, $k = 1,\ldots,N$:

$$b_0 \sum_{k=1}^{N} \frac{A_{1k}}{b_0-\xi_k} = 1, \qquad b_1 \sum_{k=1}^{N} \frac{A_{0k}}{b_1-\xi_k} = 1 \tag{2.9}$$

and

$$A_{1k} = \frac{b_0-\xi_k}{b_0}\cdot\frac{A_{0k}}{\widehat{\pi}_0(q-c_0\xi_k)}, \qquad A_{0k} = \frac{b_1-\xi_k}{b_1}\cdot\frac{A_{1k}}{\widehat{\pi}_1(q-c_1\xi_k)}, \tag{2.10}$$
$$k = 1,\ldots,N.$$

The k-th system of (2.10) has a nontrivial ($A_{ik} \neq 0$) solution, if and only if $\xi_k = \xi_k(q)$ is the root of the equation

$$\widehat{\pi}_0(q-c_0\xi)\cdot\widehat{\pi}_1(q-c_1\xi) = \left(1-\frac{\xi}{b_0}\right)\cdot\left(1-\frac{\xi}{b_1}\right). \tag{2.11$_q$}$$

Since the mappings $\xi \to \widehat{\pi}_0(q-c_0\xi)$ and $\xi \to \widehat{\pi}_1(q-c_1\xi)$, $\xi > 0$, (with negative c_0 and c_1) are positive decreasing functions and $\widehat{\pi}_0(q)\cdot\widehat{\pi}_1(q) < 1\ \forall q > 0$, equation $(2.11)_q$ has *exactly two real positive roots* $\xi_1 = \xi_1(q)$, $\xi_1 < b_0 \wedge b_1$, and $\xi_2 = \xi_2(q)$, $\xi_2 > b_0 \vee b_1$, see Fig. 1.

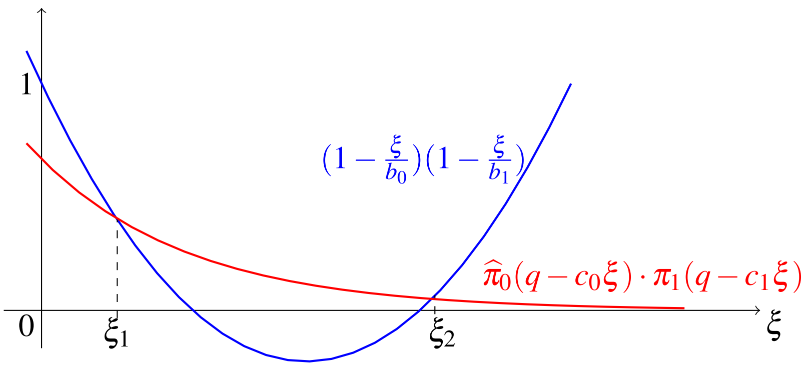

Figure 1. Positive roots ξ_1 and ξ_2 of equation $(2.11)_q$.

Hence, the moment generating function $\vec{\phi}(x;q)$ is defined by (2.7) with $N=2$,

$$\vec{\phi}(x;q)=\mathrm{e}^{-\xi_1(q)x}\mathbf{A}_1+\mathrm{e}^{-\xi_2(q)x}\mathbf{A}_2. \tag{2.12}$$

The corresponding coefficients $\mathbf{A}_1=(A_{01},\,A_{11})'$ and $\mathbf{A}_2=(A_{02},\,A_{12})'$ are determined by system (2.9)-(2.10), which splits onto two dual linear systems

$$\begin{cases} b_1\cdot\left(\dfrac{A_{01}}{b_1-\xi_1}+\dfrac{A_{02}}{b_1-\xi_2}\right)=1,\\ \dfrac{A_{01}}{\widehat{\pi}_0(q-c_0\xi_1)}+\dfrac{A_{02}}{\widehat{\pi}_0(q-c_0\xi_2)}=1,\end{cases}$$

and

$$\begin{cases} b_0\cdot\left(\dfrac{A_{11}}{b_0-\xi_1}+\dfrac{A_{12}}{b_0-\xi_2}\right)=1,\\ \dfrac{A_{11}}{\widehat{\pi}_1(q-c_1\xi_1)}+\dfrac{A_{12}}{\widehat{\pi}_1(q-c_1\xi_2)}=1.\end{cases}$$

After easy algebra, one can obtain the following explicit formulae

$$A_{01}=\frac{b_1-\xi_1}{b_1}\cdot\frac{f_0(\xi_2)-b_1}{f_0(\xi_2)-f_0(\xi_1)}=\frac{b_1-\xi_1}{b_1}\cdot\left(1-\frac{b_1-f_0(\xi_1)}{f_0(\xi_2)-f_0(\xi_1)}\right), \tag{2.13}$$

$$A_{02}=\frac{b_1-\xi_2}{b_1}\cdot\frac{b_1-f_0(\xi_1)}{f_0(\xi_2)-f_0(\xi_1)}=\frac{b_1-\xi_2}{b_1}\cdot\left(1-\frac{f_0(\xi_2)-b_1}{f_0(\xi_2)-f_0(\xi_1)}\right), \tag{2.14}$$

$$A_{11}=\frac{b_0-\xi_1}{b_0}\cdot\frac{f_1(\xi_2)-b_0}{f_1(\xi_2)-f_1(\xi_1)}=\frac{b_0-\xi_1}{b_0}\cdot\left(1-\frac{b_0-f_1(\xi_1)}{f_1(\xi_2)-f_1(\xi_1)}\right), \tag{2.15}$$

$$A_{12}=\frac{b_0-\xi_2}{b_0}\cdot\frac{b_0-f_1(\xi_1)}{f_1(\xi_2)-f_1(\xi_1)}=\frac{b_0-\xi_2}{b_0}\cdot\left(1-\frac{f_1(\xi_2)-b_0}{f_1(\xi_2)-f_1(\xi_1)}\right), \tag{2.16}$$

where the following notations

$$f_0(\xi)=f_0(\xi;q)=\frac{b_1-\xi}{\widehat{\pi}_0(q-c_0\xi)},\qquad f_1(\xi)=f_1(\xi;q)=\frac{b_0-\xi}{\widehat{\pi}_1(q-c_1\xi)},\quad \xi\geq 0, \tag{2.17}$$

are used.

To study the firing probabilities, $\mathbb{P}\left\{\mathscr{T}_x^{(0)}<\infty\right\}$, $\mathbb{P}\left\{\mathscr{T}_x^{(1)}<\infty\right\}$, and the mean value of firing time, $\mathbb{E}\left[\mathscr{T}_x^{(0)}\right]$, $\mathbb{E}\left[\mathscr{T}_x^{(1)}\right]$, we are interested to analyse the limits of the moment generating function and its derivative under $q\downarrow 0$,

$$\lim_{q\downarrow 0}\vec{\phi}(x;q),\qquad \lim_{q\downarrow 0}\frac{\mathrm{d}\vec{\phi}(x;q)}{\mathrm{d}q}.$$

To do this, keeping in mind (2.12), we need $\xi_k(q)|_{q\downarrow 0}$ and $\dfrac{\mathrm{d}\xi_k(q)}{\mathrm{d}q}|_{q\downarrow 0}$, $k=1,2$, where $\xi_1=\xi_1(q)$, $\xi_1(q)<b_0\wedge b_1$ and $\xi_2=\xi_2(q)$, $\xi_2(q)>b_1\vee b_1$, $q>0$, are the two branches of (positive) roots of $(2.11)_q$.

The firing probabilities $\mathbb{P}\left\{\mathscr{T}_x^{(i)}<\infty\right\}=\lim_{q\downarrow 0}\phi_i(x;q)$, $i\in\{0,1\}$, are presented by the following proposition.

Proposition 2.1. *Let the mean value of the holding time intervals* T_n, $n \geq 1$, *exists*, *that is*

$$\begin{aligned}\mathbb{E}_0[T] &:= \mathbb{E}[T \mid \varepsilon = 0] = -\frac{\mathrm{d}\widehat{\pi}_0(q)}{\mathrm{d}q}|_{q=0} < \infty,\\ \mathbb{E}_1[T] &:= \mathbb{E}[T \mid \varepsilon = 1] = -\frac{\mathrm{d}\widehat{\pi}_1(q)}{\mathrm{d}q}|_{q=0} < \infty.\end{aligned} \tag{2.18}$$

- *If*

$$c_0\mathbb{E}_0[T] + c_1\mathbb{E}_1[T] + b_0^{-1} + b_1^{-1} < 0, \tag{2.19}$$

then the limits $\xi_* = \lim_{q\downarrow 0}\xi_1(q)$ *and* $\xi^* = \lim_{q\downarrow 0}\xi_2(q)$ *exist and are positive.*
The firing probabilities are given by

$$\begin{aligned}\mathbb{P}\left\{\mathscr{T}_x^{(0)} < \infty\right\} &= A_{01}^*\mathrm{e}^{-\xi_* x} + A_{02}^*\mathrm{e}^{-\xi^* x},\\ \mathbb{P}\left\{\mathscr{T}_x^{(1)} < \infty\right\} &= A_{11}^*\mathrm{e}^{-\xi_* x} + A_{12}^*\mathrm{e}^{-\xi^* x},\end{aligned} \tag{2.20}$$

where $A_{ik}^* = A_{ik}(\xi_*, \xi^*)$, $i \in \{0,1\}, k = 1,2$, *are defined by* (2.13)-(2.16) *with* $\xi_1 = \xi_*$, $\xi_2 = \xi^*$.
- *Otherwise*, *if*

$$c_0\mathbb{E}_0[T] + c_1\mathbb{E}_1[T] + b_0^{-1} + b_1^{-1} \geq 0, \tag{2.21}$$

then the firing occurs a. s.

$$\mathbb{P}\left\{\mathscr{T}_x^{(0)} < \infty\right\} = \mathbb{P}\left\{\mathscr{T}_x^{(1)} < \infty\right\} = 1. \tag{2.22}$$

Proof. Note that $\widehat{\pi}_0(q - c_0\xi)$, $\widehat{\pi}_1(q - c_1\xi)$, $c_0, c_1 < 0$, are positive decreasing convex functions of $q > 0$ and of $\xi > 0$.

Since $\widehat{\pi}_0(0) = \widehat{\pi}_1(0) = 1$, then $\xi = 0$ is the root of equation $(2.11)_0$. The other roots of equation $(2.11)_0$ depend on the relation between the values of the derivative on ξ at point $\xi = 0$ of the sides of this equation. The derivatives of the RHS and of the LHS of $(2.11)_q$ are given by

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(1 - \frac{\xi}{b_0}\right)\cdot\left(1 - \frac{\xi}{b_1}\right)\right]|_{\xi\downarrow 0} = -\left(\frac{1}{b_0} + \frac{1}{b_1}\right), \tag{2.23}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\widehat{\pi}_0(q - c_0\xi)\widehat{\pi}_1(q - c_1\xi)\right]|_{\xi\downarrow 0,\, q\downarrow 0} = c_0\mathbb{E}_0[T] + c_1\mathbb{E}_1[T], \tag{2.24}$$

respectively, see (2.18).

We have two distinct situations.

- Let (2.19) holds, that is $-\left(\dfrac{1}{b_0} + \dfrac{1}{b_1}\right) > c_0\mathbb{E}_0[T] + c_1\mathbb{E}_1[T]$.
Since $\widehat{\pi}_0(-c_0\xi)\cdot\widehat{\pi}_1(-c_1\xi)$, $\xi > 0$, is the positive decreasing convex function, $\xi \to (1 - \xi/b_0)\cdot(1 - \xi/b_1)$ is convex, and

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left[\widehat{\pi}_0(-c_0\xi)\widehat{\pi}_1(-c_1\xi)\right]|_{\xi\downarrow 0} < \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(1 - \frac{\xi}{b_0}\right)\cdot\left(1 - \frac{\xi}{b_1}\right)\right]|_{\xi\downarrow 0}$$

equation $(2.11)_0$ has explicitly two positive roots: $\xi_* < b_0 \wedge b_1$, $\xi^* > b_0 \vee b_1$, see Fig. 2 (left). Functions $q \to \widehat{\pi}_0(q - c_0\xi)$ and $q \to \widehat{\pi}_1(q - c_1\xi)$ are monotone decreasing. Hence, the roots $\xi_1(q)$, $\xi_2(q)$ of $(2.11)_q$, $q > 0$, are monotone functions. Further, for any positive q we have

$$\xi_* < \xi_1(q) < b_0 \wedge b_1 < b_0 \vee b_1 < \xi_2(q) < \xi^*,$$

and

$$\lim_{q\downarrow 0} \xi_1(q) = \xi_*, \qquad \lim_{q\downarrow 0} \xi_2(q) = \xi^*.$$

Equalities (2.20) follow by passing to limit in (2.12)-(2.17).

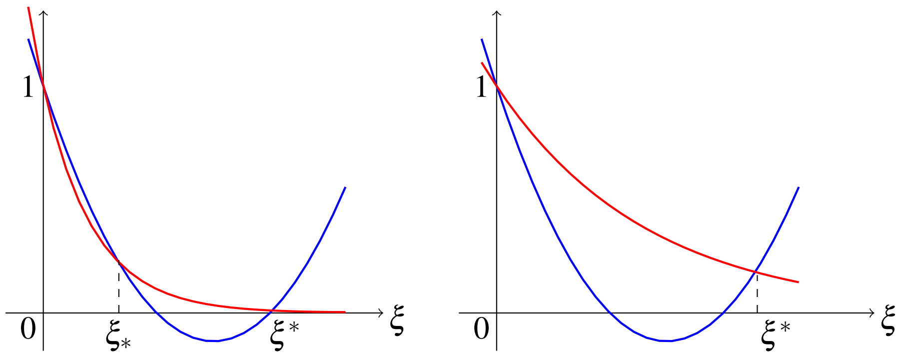

Figure 2. Two positive roots ξ_* and ξ^* of equation $(2.11)_0$ in the case (2.19), (left); one positive root ξ^* of $(2.11)_0$ in the case (2.21), (right).

- On the contrary, let (2.21) holds, that is $-\left(\frac{1}{b_0} + \frac{1}{b_1}\right) \le c_0\mathbb{E}_0[T] + c_1\mathbb{E}_1[T]$.

  In the case of the strict inequality,

$$\frac{\mathrm{d}}{\mathrm{d}\xi}[\widehat{\pi}_0(-c_0\xi)\widehat{\pi}_1(-c_1\xi)]|_{\xi\downarrow 0}$$
$$> \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\left(1 - \frac{\xi}{b_0}\right) \cdot \left(1 - \frac{\xi}{b_1}\right)\right]|_{\xi\downarrow 0},$$

equation $(2.11)_0$ has only one positive root
ξ^*, $\xi^* > b_0 \vee b_1$, see Fig. 2 (right), and

$$\lim_{q\downarrow 0} \xi_1(q) = 0, \qquad \lim_{q\downarrow 0} \xi_2(q) = \xi^*. \tag{2.25}$$

In this case, see (2.17),

$$\lim_{q\downarrow 0} f_0(\xi_1(q); q) = b_1, \ \lim_{q\downarrow 0} f_1(\xi_1(q); q) = b_0.$$

Hence, by (2.13)-(2.16) we have

$$\lim_{q\downarrow 0} A_{01} = \lim_{q\downarrow 0} A_{11} = 1, \qquad \lim_{q\downarrow 0} A_{02} = \lim_{q\downarrow 0} A_{12} = 0. \tag{2.26}$$

Therefore,

$$\lim_{q\downarrow 0}\phi_0(x;q)=\lim_{q\downarrow 0}\phi_1(x;q)=1$$

and

$$\mathbb{P}\left\{\mathscr{T}_x^{(0)}<\infty\right\}=\mathbb{P}\left\{\mathscr{T}_x^{(1)}<\infty\right\}=1.$$

If the equality holds, $-\left(\frac{1}{b_0}+\frac{1}{b_1}\right)=c_0\mathbb{E}_0[T]+c_1\mathbb{E}_1[T]$, then we have

$$2c_0c_1\mathbb{E}_0[T]\mathbb{E}_1[T]=\left(b_0^{-1}+b_1^{-1}\right)^2-c_0^2(\mathbb{E}_0[T])^2-c_1^2(\mathbb{E}_1[T])^2,$$

and

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[\widehat{\pi}_0(-c_0\xi)\widehat{\pi}_1(-c_1\xi)\right]|_{\xi\downarrow 0}\\&=c_0^2\mathbb{E}_0T^2+c_1^2\mathbb{E}_1T^2+2c_0c_1\mathbb{E}_0[T]\mathbb{E}_1[T]\\&=\left(b_0^{-1}+b_1^{-1}\right)^2+c_0^2\mathrm{Var}_0[T]+c_1^2\mathrm{Var}_1[T]\\&>\frac{2}{b_0b_1}=\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[\left(1-\frac{\xi}{b_0}\right)\cdot\left(1-\frac{\xi}{b_1}\right)\right]|_{\xi\downarrow 0},\end{aligned}$$

Therefore, since $\widehat{\pi}_0(-c_0\xi)\widehat{\pi}_1(-c_1\xi)$ and $\left(1-\frac{\xi}{b_0}\right)\left(1-\frac{\xi}{b_1}\right)$ are decreasing convex functions, the same result occurs: (2.25)-(2.26), and then (2.22).

□

The mean value of the firing time can be obtained in a similar way. We need some auxiliary results.

Let $\xi_1(q),\ \xi_2(q)$ be the two branches of positive roots of $(2.11)_q$, $q>0$, and condition (2.21) holds. By proposition 2.1, $0<\xi_1(q)<b_0\vee b_1<\xi_2(q)$ and

$$\lim_{q\downarrow 0}\xi_1(q)=0,\qquad \lim_{q\downarrow 0}\xi_2(q)=\xi^*.$$

Let coefficients $A_{ik}=A_{ik}(\xi_1,\xi_2;\ q)$ be defined by (2.13)-(2.17).

Lemma 2.2. *Let* (2.21) *be satisfied. The following limit relations hold*:

$$\frac{\mathrm{d}\xi_1(q)}{\mathrm{d}q}|_{q\downarrow 0}=\frac{\mathbb{E}_0[T]+\mathbb{E}_1[T]}{b_0^{-1}+b_1^{-1}+c_0\mathbb{E}_0[T]+c_1\mathbb{E}_1[T]}=:\sigma>0 \tag{2.27}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}q}\Big[A_{01}(\xi_1(q),\ \xi_2(q);\ q)\Big]|_{q\downarrow 0}=B_0(\xi^*)-\frac{\sigma}{b_1}, \tag{2.28}$$

$$\frac{\mathrm{d}}{\mathrm{d}q}\Big[A_{02}(\xi_1(q),\ \xi_2(q);\ q)\Big]|_{q\downarrow 0}=\left(\frac{\xi^*}{b_1}-1\right)B_0(\xi^*), \tag{2.29}$$

$$\frac{\mathrm{d}}{\mathrm{d}q}\Big[A_{11}(\xi_1(q),\ \xi_2(q);\ q)\Big]|_{q\downarrow 0}=B_1(\xi^*)-\frac{\sigma}{b_0}, \tag{2.30}$$

$$\frac{\mathrm{d}}{\mathrm{d}q}\Big[A_{12}(\xi_1(q),\ \xi_2(q);\ q)\Big]|_{q\downarrow 0} = \left(\frac{\xi^*}{b_0}-1\right)B_1(\xi^*), \tag{2.31}$$

where

$$\begin{aligned} B_0(\xi^*) =& \frac{b_1\mathbb{E}_0[T]-\sigma(1+b_1c_0\mathbb{E}_0[T])}{f_0(\xi^*;0)-b_1}, \\ B_1(\xi^*) =& \frac{b_0\mathbb{E}_1[T]-\sigma(1+b_0c_1\mathbb{E}_1[T])}{f_1(\xi^*;0)-b_0}. \end{aligned} \tag{2.32}$$

Proof. Substitute $\xi_1 = \xi_1(q)$ into $(2.11)_q$. Formula (2.27) follows from (2.23)-(2.24) by differentiating in $(2.11)_q$. The derivative in q at $q \downarrow 0$ gives

$$\begin{aligned} -(\mathbb{E}_0[T]+\mathbb{E}_1[T]) + \frac{\mathrm{d}\xi_1(q)}{\mathrm{d}q}|_{q\downarrow 0}\cdot\Big(c_0\mathbb{E}_0[T]+c_1\mathbb{E}_1[T]\Big) \\ = -\frac{\mathrm{d}\xi_1(q)}{\mathrm{d}q}|_{q\downarrow 0}\cdot\Big(\frac{1}{b_0}+\frac{1}{b_1}\Big). \end{aligned}$$

Under condition (2.21), $\lim_{q\downarrow 0}\xi_1(q) = 0$. By definition (2.17), it follows that

$$\lim_{q\downarrow 0} f_0(\xi_1(q),\ q) = b_1, \qquad \lim_{q\downarrow 0} f_1(\xi_1(q),\ q) = b_0,$$

and

$$\begin{aligned} \lim_{q\downarrow 0}\left[\frac{\partial f_0(\xi;q)}{\partial q}|_{\xi=\xi_1(q)}\right] =& b_1\mathbb{E}_0[T], \\ \lim_{q\downarrow 0}\left[\frac{\partial f_1(\xi;q)}{\partial q}|_{\xi=\xi_1(q)}\right] =& b_0\mathbb{E}_1[T], \\ \lim_{q\downarrow 0}\left[\frac{\partial f_0(\xi;q)}{\partial \xi}|_{\xi=\xi_1(q)}\right] =& -1-b_1c_0\mathbb{E}_0[T], \\ \lim_{q\downarrow 0}\left[\frac{\partial f_1(\xi;q)}{\partial \xi}|_{\xi=\xi_1(q)}\right] =& -1-b_0c_1\mathbb{E}_1[T]. \end{aligned}$$

With this in mind, by (2.13)-(2.14) you can get

$$\begin{aligned} &\lim_{q\downarrow 0}\left[\frac{\partial A_{01}(\xi_1,\xi_2;\ q)}{\partial \xi_1}|_{\xi_1=\xi_1(q),\ \xi_2=\xi_2(q)}\right] \\ &= -\frac{1}{b_1} + \frac{\lim_{q\downarrow 0}\left[\frac{\partial f_0(\xi;\ q)}{\partial \xi}|_{\xi=\xi_1(q)}\right]}{f_0(\xi^*;0)-b_1} = -\frac{1}{b_1} - \frac{1+b_1c_0\mathbb{E}_0[T]}{f_0(\xi^*;0)-b_1}, \end{aligned} \tag{2.33}$$

$$\begin{aligned} &\lim_{q\downarrow 0}\left[\frac{\partial A_{02}(\xi_1,\xi_2;\ q)}{\partial \xi_1}|_{\xi_1=\xi_1(q),\ \xi_2=\xi_2(q)}\right] \\ =& \frac{b_1-\xi^*}{b_1}\cdot\frac{-\lim_{q\downarrow 0}\left[\frac{\partial f_0(\xi)}{\partial \xi}|_{\xi=\xi_1(q)}\right]}{f_0(\xi^*;0)-b_1}, = \frac{b_1-\xi^*}{b_1}\cdot\frac{1+b_1c_0\mathbb{E}_0[T]}{f_0(\xi^*;0)-b_1}, \end{aligned} \tag{2.34}$$

$$
\begin{aligned}
&\lim_{q\downarrow 0}\left[\frac{\partial A_{01}(\xi_1,\xi_2;\,q)}{\partial q}|_{\xi_1=\xi_1(q),\,\xi_2=\xi_2(q)}\right]\\
&=\frac{\lim_{q\downarrow 0}\frac{\partial f_0(\xi_1(q))}{\partial q}|_{\xi_1=\xi_1(q)}}{f_0(\xi^*;0)-b_1}=\frac{b_1\mathbb{E}_0[T]}{f_0(\xi^*;0)-b_1},
\end{aligned}
\tag{2.35}
$$

$$
\begin{aligned}
&\lim_{q\downarrow 0}\left[\frac{\partial A_{02}(\xi_1,\xi_2;\,q)}{\partial q}|_{\xi_1=\xi_1(q),\,\xi_2=\xi_2(q)}\right]\\
&=\frac{b_1-\xi^*}{b_1}\cdot\frac{-\lim_{q\downarrow 0}\frac{\partial f_0(\xi_1(q))}{\partial q}|_{\xi_1=\xi_1(q)}}{f_0(\xi^*;0)-b_1}=\frac{b_1-\xi^*}{b_1}\cdot\frac{b_1\mathbb{E}_0[T]}{f_0(\xi^*;0)-b_1}.
\end{aligned}
\tag{2.36}
$$

and

$$
\begin{aligned}
&\lim_{q\downarrow 0}\left[\frac{\partial A_{01}(\xi_1,\xi_2;\,q)}{\partial \xi_2}|_{\xi_1=\xi_1(q),\,\xi_2=\xi_2(q)}\right]=0,\\
&\lim_{q\downarrow 0}\left[\frac{\partial A_{02}(\xi_1,\xi_2;\,q)}{\partial \xi_2}|_{\xi_1=\xi_1(q),\,\xi_2=\xi_2(q)}\right]=0.
\end{aligned}
\tag{2.37}
$$

Substituting (2.33)-(2.37) into

$$
\begin{aligned}
\frac{\mathrm{d}A_{01}}{\mathrm{d}q}|_{q\downarrow 0}&=\frac{\partial A_{01}}{\partial \xi_1}|_{q\downarrow 0}\cdot\xi_1'(0)+\frac{\partial A_{01}}{\partial \xi_2}|_{q\downarrow 0}\cdot\xi_2'(0)+\frac{\partial A_{01}}{\partial q}|_{q\downarrow 0},\\
\frac{\mathrm{d}A_{02}}{\mathrm{d}q}|_{q\downarrow 0}&=\frac{\partial A_{02}}{\partial \xi_1}|_{q\downarrow 0}\cdot\xi_1'(0)+\frac{\partial A_{02}}{\partial \xi_2}|_{q\downarrow 0}\cdot\xi_2'(0)+\frac{\partial A_{02}}{\partial q}|_{q\downarrow 0},
\end{aligned}
$$

where $\xi_k'(0):=\mathrm{d}\xi_k(q)/\mathrm{d}q|_{q\downarrow 0}$, $k=1,2$, one can get Eqns (2.28)-(2.29). Eqns (2.30)-(2.31) follow similarly. □

Proposition 2.3. *If* (2.21) *holds, then the mean firing time* $\mathbf{M}(x)=\left(\mathbb{E}\left[\mathscr{T}_x^{(0)}\right],\,\mathbb{E}\left[\mathscr{T}_x^{(1)}\right]\right)$ *is finite and is given by the entries*

$$
\mathbb{E}\left[\mathscr{T}_x^{(0)}\right]=\sigma\left(x+\frac{1}{b_1}\right)-B_0(\xi^*)\left[1+\left(\frac{\xi^*}{b_1}-1\right)\exp(-\xi^*x)\right],
\tag{2.38}
$$

$$
\mathbb{E}\left[\mathscr{T}_x^{(1)}\right]=\sigma\left(x+\frac{1}{b_0}\right)-B_1(\xi^*)\left[1+\left(\frac{\xi^*}{b_0}-1\right)\exp(-\xi^*x)\right],
\tag{2.39}
$$

where σ, $B_0(\xi^*)$, $B_1(\xi^*)$ *and* $\xi^*=\lim_{q\downarrow 0}\xi_2(q)$ *are defined in Lemma 2.2.*

A numerical example with various values of jump amplitudes is depicted in Fig.3: the average firing time increases when up jumps vary from large to small.

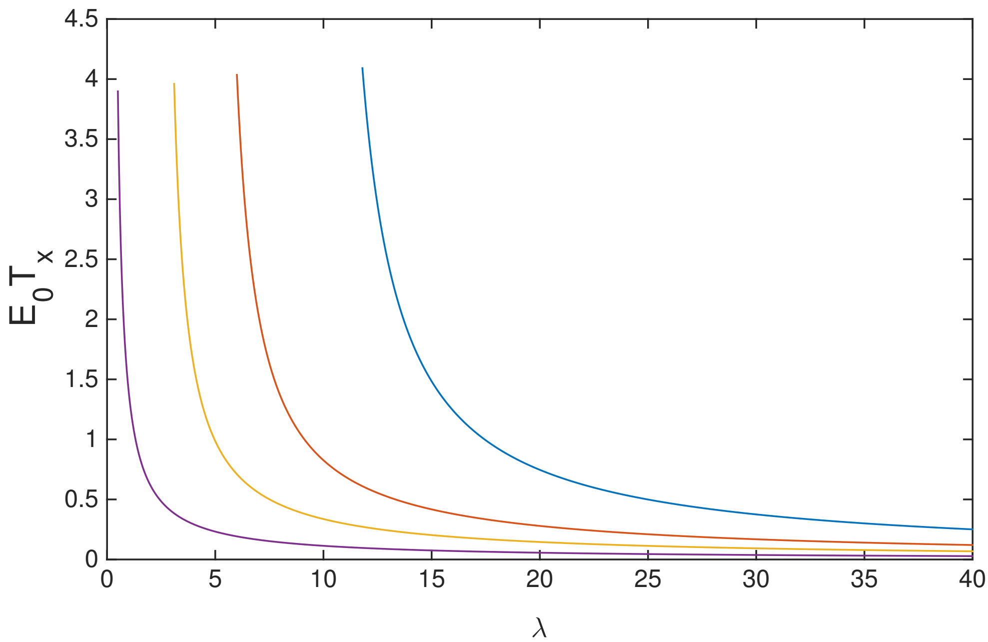

Figure 3. Mean values of firing times $\mathbb{E}\mathscr{T}_x^{(0)}$, $x = 1$, $c_0 = -1, c_1 = -2$, (2.38), with exponentially distributed holding times, $\mathrm{Exp}(\lambda)$, depending on $\lambda = \lambda_0 = \lambda_1$. From left to right: $b_0 = 0.1, b_1 = 0.5$; $\quad b_0 = 1, b_1 = 5$; $\quad b_0 = 2, b_1 = 10$; $\quad b_0 = 4, b_1 = 20$.

Proof. Since $\mathbf{M}(x) = -\lim_{q\downarrow 0} \dfrac{\mathrm{d}\vec{\phi}(x;q)}{\mathrm{d}q}$, by (2.12)

$$\begin{aligned}\mathbf{M}(x) =& -\exp\left(-x\xi_1|_{q\downarrow 0}\right)\frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}q}|_{q\downarrow 0} - \exp\left(-x\xi_2|_{q\downarrow 0}\right)\frac{\mathrm{d}\mathbf{A}_2}{\mathrm{d}q}|_{q\downarrow 0}\\ &+x\exp\left(-x\xi_1|_{q\downarrow 0}\right)\left(\xi_1'\mathbf{A}_1\right)|_{q\downarrow 0} + x\exp\left(-x\xi_2|_{q\downarrow 0}\right)\left(\xi_2'\mathbf{A}_2\right)|_{q\downarrow 0}.\end{aligned}$$

By (2.25) and (2.26) we have $\mathbf{A}_1|_{q\downarrow 0} = \mathbf{1}$, $\mathbf{A}_2|_{q\downarrow 0} = \mathbf{0}$, $\xi_1|_{q\downarrow 0} = 0$, $\xi_2|_{q\downarrow 0} = \xi^*$ and $\xi_1'|_{q\downarrow 0} = \sigma$. Therefore,

$$\mathbf{M}(x) = -\frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}q}|_{q\downarrow 0} - \exp\left(-\xi^* x\right)\frac{\mathrm{d}\mathbf{A}_2}{\mathrm{d}q}|_{q\downarrow 0} + x\sigma\mathbf{1},$$

which by Lemma 2.2 gives (2.38)-(2.39). □

Remark 2.1. **Firing time distribution in the markovian case.** Let the holding times be exponentially distributed with alternating mean values λ_0^{-1}, λ_1^{-1}. In this case, the pair $\langle X(t), \varepsilon(t)\rangle$, $t \geq 0$, is the Markov process.

Since,

$$\widehat{\pi}_0(p) = \frac{\lambda_0}{\lambda_0 + p}, \qquad \widehat{\pi}_1(p) = \frac{\lambda_1}{\lambda_1 + p},$$

functions f_0 and f_1, see (2.17), are defined by

$$\begin{aligned}f_0(\xi) =& \lambda_0^{-1}(b_1 - \xi)(\lambda_0 + q - c_0\xi),\\ f_1(\xi) =& \lambda_1^{-1}(b_0 - \xi)(\lambda_1 + q - c_1\xi),\end{aligned}$$

and equation $(2.11)_q$ becomes

$$\frac{\lambda_0\lambda_1}{(\lambda_0+q-c_0\xi)(\lambda_1+q-c_1\xi)}=\left(1-\frac{\xi}{b_0}\right)\cdot\left(1-\frac{\xi}{b_1}\right).$$

In this case formulae (2.38)-(2.39) for mean firing times hold with

$$B_0(\xi^*)=\frac{b_1-\sigma(\lambda_0+b_1c_0)}{\xi^*(c_0\xi^*-\lambda_0-b_1c_0)},$$
$$B_1(\xi^*)=\frac{b_0-\sigma(\lambda_1+b_0c_1)}{\xi^*(c_1\xi^*-\lambda_1-b_0c_1)},$$

where $\xi^*=\lim_{q\downarrow 0}\xi_2(q)$.

Formulae (2.38)-(2.39) for mean firing times can be simplified to an explicit form also in the case an alternating compound Poisson process, that is if $c_0=c_1=0$, which gives a nice additional result.

Proposition 2.4. *Let the telegraph component vanish,* $c_0=c_1=0$.
In this case, condition (2.21) *always holds. The mean firing times of* $\mathscr{T}_x$ *are given by*

$$\mathbb{E}\left[\mathscr{T}_x^{(0)}\right]=\sigma\left(x+\frac{1}{b_1}\right)+\frac{b_1\mathbb{E}_0[T]-\sigma}{2b}\left[1+\frac{b_0}{b_1}\exp(-2bx)\right], \tag{2.40}$$

and

$$\mathbb{E}\left[\mathscr{T}_x^{(1)}\right]=\sigma\left(x+\frac{1}{b_0}\right)+\frac{b_0\mathbb{E}_1[T]-\sigma}{2b}\left[1+\frac{b_1}{b_0}\exp(-2bx)\right], \tag{2.41}$$

where σ, *see* (2.27)*, is simplified to*

$$\sigma=\frac{b_0b_1}{2b}\left(\mathbb{E}_0[T]+\mathbb{E}_1[T]\right),\qquad 2b=b_0+b_1.$$

Proof. The moment generating function ϕ is given by (2.12), where $\xi_1(q),\ \xi_2(q)$ are the two (positive) roots of the equation $(2.11)_q$ with $c_0=c_1=0$:

$$(b_0-\xi)(b_1-\xi)=C_q,\qquad C_q=b_0b_1\widehat{\pi}_0(q)\widehat{\pi}_1(q),\qquad q\geq 0.$$

Explicitly,

$$\xi_1=b-\frac{1}{2}D,\quad \xi_2=b+\frac{1}{2}D,\qquad \text{where }\ D=\sqrt{(b_0-b_1)^2+4C_q}. \tag{2.42}$$

Formulae for the coefficients $A_{ik},\ i\in\{0,1\},k=1,2$, can be simplified. First, by (2.17) we have

$$f_0(\xi_1)=\frac{b_1-b_0+D}{2\widehat{\pi}_0(q)},\qquad f_0(\xi_2)=\frac{b_1-b_0-D}{2\widehat{\pi}_0(q)}. \tag{2.43}$$

Further, by (2.13) and (2.14)

$$A_{01}=\frac{b_1-b_0+D}{2b_1}\cdot\frac{(b_1-b_0-D)-2b_1\widehat{\pi}_0(q)}{-2D}=\frac{2C_q+b_1\widehat{\pi}_0(q)(b_1-b_0+D)}{2b_1D}$$

$$= \frac{\widehat{\pi}_0(q)\,(1+\Delta_0)}{2}$$

and

$$A_{02} = \frac{b_1 - b_0 - D}{2b_1} \cdot \frac{2b_1\widehat{\pi}_0(q) - (b_1 - b_0 + D)}{-2D} = \frac{(b_1 - b_0 - D)b_1\widehat{\pi}_0(q) + 2C_q}{-2b_1D}$$
$$= \frac{\widehat{\pi}_0(q)\,(1-\Delta_0)}{2},$$

where $\Delta_0 = \dfrac{b_1 - b_0 + 2b_0\widehat{\pi}_1(q)}{D}$.

Similarly, by (2.15) and (2.16)

$$A_{11} = \frac{\widehat{\pi}_1(q)\,(1+\Delta_1)}{2}, \qquad A_{12} = \frac{\widehat{\pi}_1(q)\,(1-\Delta_1)}{2},$$

where $\Delta_1 = \dfrac{b_0 - b_1 + 2b_1\widehat{\pi}_0(q)}{D}$.

After easy algebra one can obtain the explicit formulae for the moment generating functions of $\mathscr{T}_x$:

$$\phi_0(x;q) = \widehat{\pi}_0(q)\exp(-bx)\left[\cosh(Dx/2) + \Delta_0 \cdot \sinh(Dx/2)\right],$$
$$\phi_1(x;q) = \widehat{\pi}_1(q)\exp(-bx)\left[\cosh(Dx/2) + \Delta_1 \cdot \sinh(Dx/2)\right].$$

Further, by (2.42) $\xi^* = \lim_{q\downarrow 0}\xi_2(q) = 2b$; by (2.43) $f_0(\xi^*;0) = -b_0$. Similarly, $f_1(\xi^*;0) = -b_1$ and by (2.32)

$$B_0(\xi^*) = \frac{b_1\mathbb{E}_0[T] - \sigma}{-2b}, \qquad B_1(\xi^*) = \frac{b_0\mathbb{E}_1[T] - \sigma}{-2b}.$$

Under these simplifications, formulae (2.38)-(2.39), Proposition 2.3, become (2.40)-(2.41). □

3. Firing time under small frequent stimuli

Consider the case of small frequent stimuli. We assume that the mean values of jumps, b_0^{-1}, b_1^{-1}, and the holding times, T_n, consistently tend to zero. We are interested to study the asymptotical behaviour of the firing time under these circumstances. Let's set the exact assertion.

Let the parameters of stochastic stimulation of a neuron be scaled as follows: first, the mean stimuli amplitudes consistently tend to zero,

$$\mathbb{E}_0[Y] = b_0^{-1} \to 0, \qquad \mathbb{E}_1[Y] = b_1^{-1} \to 0; \tag{3.1}$$

second, the holding time intervals tend to zero. More precisely, assume that for any fixed positive q

$$\widehat{\pi}_0(q) \to 1, \qquad \widehat{\pi}_1(q) \to 1, \tag{3.2}$$

such that the derivatives of the moment generating functions $\widehat{\pi}_0$ and $\widehat{\pi}_1$ exist and vanish:

$$m_0(q) := -\widehat{\pi}_0'(q) = \mathbb{E}_0\left[T e^{-qT}\right] \to 0, \qquad m_1(q) := -\widehat{\pi}_1'(q) = \mathbb{E}_1\left[T e^{-qT}\right] \to 0, \tag{3.3}$$
$$\widehat{\pi}_0''(q) = \mathbb{E}_0\left[T^2 e^{-qT}\right] \to 0, \qquad \widehat{\pi}_1''(q) = \mathbb{E}_1\left[T^2 e^{-qT}\right] \to 0. \tag{3.4}$$

Assume the convergence rates at (3.1) to be comparable,

$$\frac{b_0}{b_1}=\frac{\mathbb{E}_1[T]}{\mathbb{E}_0[T]}\to\beta; \tag{3.5}$$

convergence rates at (3.1), (3.3)-(3.4) to be consistent as follows:

$$\frac{\mathbb{E}_0[Y]}{-\widehat{\pi}_0'(q)}=\frac{1}{b_0m_0(q)}\to v_0,\qquad \frac{\mathbb{E}_1[Y]}{-\widehat{\pi}_1'(q)}=\frac{1}{b_1m_1(q)}\to v_1,\qquad v_0,\ v_1\geq 0, \tag{3.6}$$

and

$$b_0\mathbb{E}_0\left[T^2\mathrm{e}^{-qT}\right]\to 0,\qquad b_1\mathbb{E}_1\left[T^2\mathrm{e}^{-qT}\right]\to 0. \tag{3.7}$$

Coefficients $v_0,\ v_1$ describe an additional positive trend arising in the jump-telegraph process due to small frequent positive jumps.

Let

$$\kappa:=1+c_0v_0^{-1}+\beta(1+c_1v_1^{-1}). \tag{3.8}$$

To analyse the comportment of the roots ξ_1 and ξ_2 of $(2.11)_q$ we will use the following decomposition of the moment generating functions of the holding times distributions:

$$\begin{aligned}\widehat{\pi}_0(q-c_0\xi)=&\widehat{\pi}_0(q)+c_0\xi m_0(q)+R_0(\xi;q),\\ \widehat{\pi}_1(q-c_1\xi)=&\widehat{\pi}_1(q)+c_1\xi m_1(q)+R_1(\xi;q),\end{aligned} \tag{3.9}$$

where by (3.4)

$$\begin{aligned}R_0(\xi;q)=&c_0^2\xi^2\mathbb{E}_0\left[T^2\mathrm{e}^{-qT}\sum_{n\geq 0}^{\infty}\frac{(c_0\xi T)^n}{(n+2)!}\right]\leq c_0^2\xi^2\mathbb{E}_0\left[T^2\mathrm{e}^{-qT}\right]\to 0,\\ R_1(\xi;q)=&c_1^2\xi^2\mathbb{E}_1\left[T^2\mathrm{e}^{-qT}\sum_{n\geq 0}^{\infty}\frac{(c_1\xi T)^n}{(n+2)!}\right]\leq c_1^2\xi^2\mathbb{E}_1\left[T^2\mathrm{e}^{-qT}\right]\to 0.\end{aligned}$$

Condition (3.7) provides the uniform in ξ convergence:

$$\begin{aligned}\frac{b_0R_0(\xi;q)}{\xi^2}&=b_0\frac{\widehat{\pi}_0(q-c_0\xi)-\widehat{\pi}_0(q)-c_0\xi m_0(q)}{\xi^2}\to 0,\\ \frac{b_1R_1(\xi;q)}{\xi^2}&=b_1\frac{\widehat{\pi}_1(q-c_1\xi)-\widehat{\pi}_1(q)-c_1\xi m_1(q)}{\xi^2}\to 0.\end{aligned} \tag{3.10}$$

Theorem 3.1. *Let $b_0,\ b_1\to\infty$, and the holding times be asymptotically zero, such that conditions* (3.1)-(3.6) *and* (3.7) *met.*

If $\kappa\in(0,\ +\infty]$, then for any $x,\ x>0$,

$$\mathscr{T}_x\to\gamma x,\qquad a.s.,$$

where

$$\gamma=\frac{v_0^{-1}+\beta v_1^{-1}}{\kappa}=\frac{v_0^{-1}+\beta v_1^{-1}}{1+c_0v_0^{-1}+\beta(1+c_1v_1^{-1})},\qquad \gamma>0.$$

If $\kappa\leq 0$, then

$$\mathscr{T}_x\to+\infty\qquad a.s.$$

Proof. To analyse the asymptotical behaviour of $\mathscr{T}_x$ we need to evaluate the comportment of the positive roots $\xi_1(q), \xi_2(q),\ \xi_1(q) < \xi_2(q)$, of equation $(2.11)_q$.

It turns out, the behaviour of the smaller root $\xi_1,\ \xi_1 < b_0 \wedge b_1$, depends on the sign of κ, (3.8).

Due to (3.9) equation $(2.11)_q$ takes the form

$$(\widehat{\pi}_0(q) + c_0\xi m_0(q) + R_0(\xi;q))\,(\widehat{\pi}_1(q) + c_1\xi m_1(q) + R_1(\xi;q)) = \left(1 - \frac{\xi}{b_0}\right)\left(1 - \frac{\xi}{b_1}\right),$$

which can be rewritten as

$$A_0 - A_1\xi + A_2\xi^2 = 0. \tag{3.11}$$

Here A_0 and A_1 are constants (depending only on q),

$$A_0 = 1 - \widehat{\pi}_0(q)\widehat{\pi}_1(q),$$

$$A_1 = \frac{1}{b_0} + \frac{1}{b_1} + c_0 m_0(q)\widehat{\pi}_1(q) + c_1 m_1(q)\widehat{\pi}_0(q),$$

and $A_2 = A_2(\xi)$ is given by

$$\begin{aligned} A_2 = A_2(\xi) =& \frac{1}{b_0 b_1} - c_0 c_1 m_0(q) m_1(q) \\ & - \xi^{-2}\left[R_0 \cdot (\widehat{\pi}_1(q) + c_1\xi m_1(q)) + R_1 \cdot (\widehat{\pi}_0(q) + c_0\xi m_0(q)) + R_0 R_1\right]. \end{aligned}$$

By (3.10) and (3.6)

$$\begin{aligned} \lim\,(b_0 A_0) =& \lim b_0 \cdot (1 - \widehat{\pi}_0(q)\widehat{\pi}_1(q)) = q \lim b_0 \cdot (\mathbb{E}_0[T] + \mathbb{E}_1[T]) \\ =& q \lim b_0 \cdot (m_0(0) + m_1(0)) = q\left(v_0^{-1} + \beta v_1^{-1}\right) \geq 0. \end{aligned} \tag{3.12}$$

By (3.6) and (3.1)-(3.2), $b_0 A_1$ converges to κ, (3.8),

$$\begin{aligned} \lim[b_0 A_1] =& \lim\left[1 + \frac{b_0}{b_1} + c_0 b_0 m_0(q)\widehat{\pi}_1(q) + c_1 b_0 m_1(q)\widehat{\pi}_0(q)\right] \\ =& 1 + \beta + c_0 v_0^{-1} + \beta c_1 v_1^{-1} = \kappa. \end{aligned} \tag{3.13}$$

Next,

$$\begin{aligned} b_0 A_2 =& b_1^{-1} - c_0 c_1 m_1(q) \cdot b_0 m_0(q) \\ & - \frac{b_0 R_0}{\xi^2}(\widehat{\pi}_1(q) + c_1\xi m_1(q)) - \frac{b_0 R_1}{\xi^2}(\widehat{\pi}_0(q) + c_0\xi m_0(q)) - \frac{b_0 R_0}{\xi^2} R_1. \end{aligned}$$

Since $\xi = \xi_1 < b_0 \wedge b_1$, the terms $\xi m_0(q)$ and $\xi m_1(q)$ by (3.6) are uniformly bounded. Therefore, by (3.10),

$$\lim[b_0 A_2(\xi_1)] = 0. \tag{3.14}$$

If $\kappa > 0$, the smaller root ξ_1 of (3.11) has the following limit:

$$\lim \xi_1(q) = \lim \frac{2A_0}{A_1 + \sqrt{A_1^2 - 4A_0 A_2}}. \tag{3.15}$$

By (3.12), (3.13) and (3.14) this limit is positive and finite:

$$\lim \xi_1(q) = \lim \frac{A_0}{A_1} = \frac{q(v_0^{-1} + \beta v_1^{-1})}{\kappa} = q\gamma \geq 0.$$

Further, under this scaling

$$\lim \frac{f_0(\xi_1)}{b_1} = \lim \frac{1 - \xi_1/b_1}{\widehat{\pi}_0(q - c_0\xi_1)} = 1, \qquad \lim \frac{f_1(\xi_1)}{b_0} = \lim \frac{1 - \xi_1/b_0}{\widehat{\pi}_1(q - c_1\xi_1)} = 1$$

Under the scaling (3.1), the greater root ξ_2, $\xi_2 > b_0 \vee b_1$, always goes to infinity, $\xi_2 \to \infty$; moreover, $f_0(\xi_2)/b_1$ and $f_1(\xi_2)/b_0$ are finite.

As a consequence, A_{01} and A_{11}, which are defined by (2.13) and (2.15), converge to 1 and A_{02} and A_{12} are bounded.

Summarising, we obtain

$$\lim \phi(x) = \exp(-q\gamma x),$$

which means

$$\mathscr{T}_x \to \frac{v_0 + \beta v_1}{\kappa} x = \gamma x \qquad a.s.$$

If the limit $\lim[b_0 A_1] = \kappa$ in (3.13) is not positive, $\kappa \leq 0$, then (see (3.15))

$$\lim \xi_1(q) = +\infty,$$

which corresponds to

$$\lim \vec{\phi}(x) = 0$$

and

$$\mathscr{T}_x \to +\infty \qquad a.s.$$

□

Remark 3.1. The result of Theorem 3.1 can be interpreted as the behaviour of two types of neurons: if $\kappa > 0$, then the scaled model corresponds to the so called a *tonically discharging cell,* a *phasic cell* appears when $\kappa \leq 0$, that is $\mathscr{T}_x \to \infty$, the firing rate drops to zero, see [31].

Remark 3.2. In the markovian case, that is if $\widehat{\pi}_0(q) = \lambda_0/(q + \lambda_0)$, $\widehat{\pi}_1(q) = \lambda_1/(q + \lambda_1)$, conditions (3.2)-(3.7) hold when

$$\lambda_0,\ \lambda_1 \to +\infty,$$
$$\lambda_0/b_0 \to v_0, \qquad \lambda_1/b_1 \to v_1.$$

The crucial parameter κ becomes

$$\kappa = 1 + c_0 v_0^{-1} + \beta\left(1 + c_1 v_1^{-1}\right) = 1 + \lim\left[b_0 \cdot \left(b_1^{-1} + c_0\lambda_0^{-1} + c_1\lambda_1^{-1}\right)\right].$$

4. Single-state homogeneous model

The model of neural activity based on a jump-telegraph process, see (1.1), (1.4), which is studied in Sections 2 and 3 can be simplified, restricting to the case with one state. Consider the particular case of the neural model (1.4) based on the single-state symmetric process

$$X(t) = ct + \sum_{n=1}^{N(t)} Y_n, \qquad c \le 0, \tag{4.1}$$

with independent positive exponentially distributed stimuli amplitudes Y_n, $Y_n \sim \mathrm{Exp}(b)$, $b > 0$; the independent inter-arrival times $\{T_n\}_{n\ge1}$, are identically distributed with the density function $\pi(t)$. When T_n are exponentially distributed, such a model has been studied in detail by [7] and [21].

The moment generating function $\phi(x) = \mathbb{E}[\exp(-q\mathscr{T}_x)]$ of the first passage time $\mathscr{T}_x$ is given by (2.12),

$$\phi(x) = \mathrm{e}^{-\xi_1 x}A_1 + \mathrm{e}^{-\xi_2 x}A_2,$$

where $A_1 = A_{01} = A_{11}$, $A_2 = A_{02} = A_{12}$ are defined by (2.13)-(2.17), $\xi_1 = \xi_1(q)$ and $\xi_2 = \xi_2(q)$ are the two branches of positive roots of $(2.11)_q$.

For the single-state process $X(t)$, defined by (4.1), equation $(2.11)_q$ is simplified to

$$\widehat{\pi}(q - c\xi) = \left|1 - \frac{\xi}{b}\right|.$$

More precisely, ξ_1, $\xi_1 < b$, is the positive root of $\widehat{\pi}(q-c\xi) = 1 - \xi/b$, and ξ_2, $\xi_2 > b$, is the root of $\widehat{\pi}(q-c\xi) = \xi/b - 1$. In this case, functions f_0 and f_1, which are defined by (2.17), coincide,

$$f_0(\xi) \equiv f_1(\xi) = \frac{b-\xi}{\widehat{\pi}(q-c\xi)} =: f(\xi),$$

that is,

$$f(\xi_1) = \frac{b-\xi_1}{1-\xi_1/b} = b, \qquad f(\xi_2) = \frac{b-\xi_2}{\xi_2/b-1} = -b. \tag{4.2}$$

By (4.2) and (2.13)-(2.16)

$$A_1 = A_{01} = A_{11} = \frac{b-\xi_1}{b} = \widehat{\pi}(q-c\xi_1), \qquad A_2 = A_{02} = A_{12} = 0,$$

and the moment generating function ϕ (depending only on ξ_1) is given by

$$\phi(x;q) = \frac{b-\xi}{b}\mathrm{e}^{-\xi x}, \tag{4.3}$$

where $\xi = \xi_1(q)$, $0 < \xi_1(q) < b$, $q > 0$, is the root of

$$\widehat{\pi}(q-c\xi) = 1 - \xi/b. \tag{4.4}$$

Function $\widehat{\pi} = \widehat{\pi}(q)$, $q \ge 0$, is positive, convex and decreasing, hence the root of (4.4) exists and function $\xi_1(q)$, $q \ge 0$, increases,

$$0 < q_1 < q_2 \Rightarrow 0 < \xi_1(q_1) < \xi_1(q_2) < b.$$

Therefore, $\lim_{q\downarrow 0} \xi_1(q)$ exists and, by Proposition 2.1,

$$\lim_{q\downarrow 0} \xi_1(q) = \begin{cases} \xi_* > 0, & \text{if} \quad c\mathbb{E}[T] + b^{-1} < 0, \\ \\ 0, & \text{if} \quad c\mathbb{E}[T] + b^{-1} \geq 0. \end{cases}$$

Further, the firing probability $\mathbb{P}\{\mathscr{T}_x < \infty\} = \lim_{q\downarrow 0} \phi(x;q)$ is given by

$$\mathbb{P}\{\mathscr{T}_x < \infty\} = \begin{cases} \dfrac{b-\xi_*}{b} \exp(-\xi_* x), & \text{if} \quad c\mathbb{E}[T] + b^{-1} < 0, \\ \\ 1, & \text{if} \quad c\mathbb{E}[T] + b^{-1} \geq 0, \end{cases}$$

In the case $c\mathbb{E}[T] + b^{-1} > 0$, by differentiating in (4.4) (see also (2.27)) one can obtain

$$\lim_{q\downarrow 0} \frac{\mathrm{d}\xi_1(q)}{\mathrm{d}q} = \frac{\mathbb{E}[T]}{b^{-1} + c\mathbb{E}[T]} > 0. \tag{4.5}$$

The mean firing time is finite: by (4.3) and (4.5) one can obtain

$$\mathbb{E}[\mathscr{T}_x] = -\frac{\mathrm{d}\phi(q)}{\mathrm{d}q}|_{q\downarrow 0} = \left(x + b^{-1}\right) \cdot \lim_{q\downarrow 0} \frac{\mathrm{d}\xi_1(q)}{\mathrm{d}q} = \frac{(1+bx)\mathbb{E}[T]}{1 + bc\mathbb{E}[T]}. \tag{4.6}$$

This result is in concordance with Proposition 2.3 . In this case, by (2.27),

$$\sigma = \frac{\mathbb{E}[T]}{b^{-1} + c\mathbb{E}[T]},$$

and by (2.32),

$$B_0(\xi^*) = B_1(\xi^*) = \frac{b\mathbb{E}[T] - \sigma(1 + bc\mathbb{E}[T])}{f(\xi^*) - b} = 0,$$

which by (2.38)-(2.39) gives (4.6).

In particular, consider the model (4.1) defined by the compound Poisson process with the (negative) drift c, that is, let the inter-switching times of model (4.1) be exponentially distributed, $\pi(t) = \lambda \exp(-\lambda t)$. Now, formula (4.3) holds with ξ, $0 < \xi < b$, which is the unique positive root of the equation (4.4):

$$\frac{\lambda}{\lambda + q - c\xi} = 1 - \xi/b.$$

We have

$$0 < \xi = \xi(q) = \frac{\lambda + q + bc - \sqrt{(\lambda + q + bc)^2 - 4bcq}}{2c} = b + \frac{\widetilde{q} - \sqrt{\widetilde{q}^2 + 4bc\lambda}}{2c} < b, \tag{4.7}$$

where $\widetilde{q} := q + \lambda - bc$. Due to (4.3), the moment generating function ϕ is given by

$$\phi(x;q) = \mathbb{E}e^{-q\mathscr{T}_x} = e^{-\xi x} + b^{-1} \frac{\mathrm{d}}{\mathrm{d}x} e^{-\xi x} \tag{4.8}$$

$$=\frac{2\lambda}{\lambda+q-bc+\sqrt{q^2+(\lambda+bc)^2+2q(\lambda-bc)}}$$
$$\times\exp\left(-\frac{\lambda+q+bc-\sqrt{q^2+(\lambda+bc)^2+2q(\lambda-bc)}}{2c}x\right),$$

which coincide with [7, (39)]. By applying the inverse Laplace transform $\mathscr{L}^{-1}_{q\to t}$ to (4.8) one can obtain the firing density $f_{\mathscr{T}_x}(t)$. Formula [32, 2.2.5-18] applied to $\exp(-\xi(q)x)$ with $\xi(q)$ defined by (4.7) shows

$$\mathscr{L}^{-1}_{q\to t}\left[\exp(-\xi(q)x)\right]=\frac{az}{\sqrt{t^2+2at}}I_1\left(z\sqrt{t^2+2at}\right)\exp\left(-bx-(\lambda-bc)t\right),$$

where $a=x/(-2c)$, $z=2\sqrt{-bc\lambda}$. After easy algebra, from (4.8) we obtain

$$f_{\mathscr{T}_x}(t)=\frac{\lambda x}{x-ct}\left[I_0(w)-2ct\frac{I_1(w)}{w}\right]\exp\left(-bx-(\lambda-bc)t\right) \tag{4.9}$$

with $w:=2\sqrt{\lambda bt(x-ct)}$. Formula (4.9) was derived in [7, Theorem 3.1] using another technique.

Firing probability, $\mathbb{P}\{\mathscr{T}_x<\infty\}$, and moments of $\mathscr{T}_x$ can be also obtained: if $\lambda+bc>0$, then (2.21) holds and

$$\mathbb{E}\left[\mathscr{T}_x\right]=\frac{1+bx}{\lambda+bc}.$$

If $\lambda+bc\le 0$, then $\mathbb{E}\left[\mathscr{T}_x\right]=+\infty$. This coincides with the known result, see [7, Proposition 4.2].

The limit behaviour under small frequent stimuli, Section 3, in the case of the single-state model, also looks simple.

In this case, condition (3.1) corresponds to $b\to\infty$ and $\beta=1$, see (3.5); (3.6)-(3.7) follows, if $\lambda\to+\infty$ and $\dfrac{\lambda}{b}\to v$.

By (4.3)

$$\mathbb{E}\exp(-q\mathscr{T}_x)=\phi(x)=\frac{b-\xi_1}{b}\exp(-\xi_1 x),$$

where $\xi_1=\xi_1(q)$ is the positive root, $0<\xi_1(q)<b$, of

$$\frac{\lambda}{\lambda+q-c\xi}=1-\frac{\xi}{b}.$$

One can see that for any $q>0$ under this scaling

$$\xi_1(q)=\frac{2q}{v+c+q/b+\sqrt{(v+c+q/b)^2-4cq/b}}\to\begin{cases}\dfrac{q}{v+c}, & \text{if } v+c>0,\\[2ex] +\infty, & \text{if } v+c\le 0,\end{cases}$$

and for any $x>0$

$$\phi(x)\to\begin{cases}\exp(-qx/(v+c)), & \text{if } v+c>0,\\[2ex] 0, & \text{if } h+c\le 0.\end{cases}$$

Therefore,

$$\mathscr{T}_x \to \begin{cases} \dfrac{x}{v+c}, & \text{if } v+c > 0\ , \\ \\ +\infty, & \text{if } v+c \leq 0. \end{cases}$$

5. Concluding remarks

The main goal of this paper is to study the stochastic model based on two states/phases of the nerve cell, alternating at random times of exponential excitatory inputs. The corresponding single-phase Stein's model is well known and presented in detail, see [3, 5, 10]. Our model generalises and modifies the two-phase cell cycle model, presented by [10, 11]. In this paper, we have obtained the explicit formulae for firing probability and the mean firing time, under the certain necessary condition. The asymptotical behaviour of the firing time under small frequent stimuli has been also presented. The known results of the single-state homogeneous model, [7], follow as a special case.

Since the real activity of neurons depends on the current state/phase of the organism, the proposed model, based on two alternating patterns, fits well with a naive understanding of this issue. The structure of this model can serve as a guide for practitioners: it would be interesting to discover this two-phase phenomenon of the behaviour of neurons in an experiment.

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Conflict of interest

The author declares no conflicts of interest in this paper.

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