Research article

Bifurcation analysis in a singular Beddington-DeAngelis predator-prey model with two delays and nonlinear predator harvesting

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Abstract: In this paper, a differential algebraic predator-prey model including two delays, Beddington-DeAngelis functional response and nonlinear predator harvesting is proposed. Without considering time delay, the existence of singularity induced bifurcation is analyzed by regarding economic interest as bifurcation parameter. In order to remove singularity induced bifurcation and stabilize the proposed system, state feedback controllers are designed in the case of zero and positive economic interest respectively. By the corresponding characteristic transcendental equation, the local stability of interior equilibrium and existence of Hopf bifurcation are discussed in the different case of two delays. By using normal form theory and center manifold theorem, properties of Hopf bifurcation are investigated. Numerical simulations are given to demonstrate our theoretical results.

Keywords: bioeconomic system; predator-prey model; nonlinear predator harvesting; Beddington-DeAngelis functional response; singularity induced bifurcation; Hopf bifurcation

1. Introduction

In recent decades, an increasing number of scholars have paid attention to population dynamics of prey-predator ecosystem [1–3]. During investigating such biological phenomena, dynamical behavior of biological and mathematical models are affected by many factors, such as the function response. All kinds of predator-prey models with Holling type, Crowley-Martin type and Leslie-Gower type, etc. have been investigated extensively by the researchers [4–9]. However, a few of literatures have studied the predator-prey systems with Beddington-DeAngelis type functional response [10–17].

The Beddington et al. [18, 19] gave the follows functional response

$$g(x_1, x_2) = \frac{l_1 x_1}{l x_1 + l_2 x_2 + d}. \tag{1.1}$$

Here, l_1 represents that per predator population per time can eat the maximum number of prey population, l is a positive constant and denotes the effect of handling time for predators, and l_2 is

positive and measures the magnitude of interference among predators, d represents the prey density where the attack rate is half-saturated. It is obvious that two cases are possible as following. One case is that, if $l = 1$, $l_2 = 0$ and $d > 0$, then it reduces to a Holling II functional response (or Michaelis-Menten functional response) [20]. The other case is that, if $l = 0$, $l_2 = 0$ and $d > 0$, then it reduces to a linear mass-action functional response (or Holling I functional response) [21].

Conforto et al. [16] considered a three-dimensional reaction-diffusion system incorporating the dynamics of handling and searching predators, and showed that its solutions converge when a small parameter tends to 0 towards the solutions of a reaction-cross diffusion system of predator-prey type involving a Holling-type II or Beddington-DeAngelis functional response. Employing the upper and lower solution method and comparison theory, Li et al. [12] got the sufficient conditions of the upper ultimate boundedness and permanence of this system which implies that impulse always changes the situation of survival for species, and obtained the conditions for the existence of unique globally stable positive periodic solution.

Generally speaking, the introduction of time lag in the mathematical models [22–32] tends to reflect the interaction and coexistence mechanism of population in the past. System with two delays is discussed by using the equivalent system with a single time delay in reference [30]. Liu et al. [33] used a Markovian switching process to model the telephone noise in the environment, proposed a stochastic regime-switching predator-prey model with harvesting and distributed delays, obtained the sufficient and necessary conditions for the existence of an optimal harvesting policy, and gave the explicit forms of the optimal harvesting effort and the maximum of sustainable yield.

Biological resources in the predator-prey system tend to be harvested and sold in order to obtain economic profit. In general, three types of harvesting function have been studied in the literatures [32, 34–37].

(I) Constant harvesting function is

$$H(x, E) = C,$$

where C is a suitable constant.
(II) Proportionate harvesting function is

$$H(x, E) = qEx,$$

where q is the catchability coefficient.
(III) Nonlinear harvesting function is

$$H(x, E) = \frac{qEx}{m_1 E + m_2 x},$$

where m_1, m_2 are suitable positive constants.

It is easy to find that there are several unrealistic features in the proportional harvesting, such as stochastic search for prey and unbounded linear harvesting. However, the nonlinear harvesting function (III) eliminates the above unrealistic features and satisfies

$$\lim_{E \to \infty} H(x, E) = \frac{qx}{m_1}, \quad \lim_{x \to \infty} H(x, E) = \frac{qE}{m_2},$$

that is to say, the nonlinear harvesting function shows saturation effects in terms of harvest effort and inventory abundance [38].

As we known that the harvested prey-predator systems have been focused by both theoretical and mathematical biologists [15, 39, 40]. Chakraborty et al. [40] studied the global dynamics and bifurcation of the predator-prey with constant harvesting. Liu et al. [15] investigated Hopf bifurcation and center stability for a predator-prey biological economic model with linear prey harvesting. Liu et al. [39] discussed bifurcation in a prey-predator model with nonlinear predator harvesting, and obtained the conditions for Turing and Hopf bifurcation. However, little work has been done on dynamic effects of economic interest on prey-predator system with nonlinear predator harvesting and Beddington-DeAngelis functional response.

In 1954, the common-property resource economic theory was proposed by Gordon in [41], which investigated the dynamic effects of harvesting efforts on the ecosystem from an economic point of view. Thus, in order to study the economic interest of commercial harvesting, an equation is proposed:

$$\text{Net Economic Revenue(NER)} = \text{Total Revenue (TR)} - \text{Total Cost (TC).} \qquad (1.2)$$

Recently, a number of delay differential algebraic systems were proposed to investigate the impact of commercial harvesting on prey-predator model [42, 43]. Liu et al. [44] investigated a delayed differential-algebraic system with double time delays, Holling type II functional response and linear commercial harvesting effort on predator population. By jointly using the normal form of differential algebraic system and the bifurcation theory, Li et al. [45] discussed the stability and bifurcations, and obtained richer dynamics of the bioeconomic differential algebraic predator-prey model with nonlinear prey harvesting.

Due to the above analysis and discussion, we will extend the work in [44, 45] by considering nonlinear predator harvesting, double delays and Beddington-DeAngelis functional response function into our bioeconomic differential algebraic predator-prey system in this paper. The aim of our work is to reveal the dynamical behavior of Beddington-DeAngelis predator-prey model with two delays and nonlinear predator harvesting, to obtain a reasonable profit and provide some guidance for the harvest of biological economy system.

The rest of the paper is organized as follows. In Section 2, a differential algebraic predator-prey model including two delays, Beddington-DeAngelis functional response and nonlinear predator harvesting is proposed. In Section 3, positive solutions are analyzed when $m = 0$ and $m > 0$, respectively. In Section 4, in the absence of time delay, the singularity induced bifurcation is discussed by using differential-algebraic system theory, what's more, state feedback controllers are designed to remove singular induced bifurcation. In the presence of time delay, by analyzing the corresponding characteristic transcendental equation, the local stability around the interior equilibrium are studied. Furthermore, the existence of Hopf bifurcation is investigated. Directions of Hopf bifurcation and stability of the bifurcating periodic solutions are also discussed by using normal form theory and center manifold theorem. Numerical simulations illustrate the effectiveness of the mathematical conclusions in Section 5. Discussions and conclusions are included in the last section.

2. Model formation

Leslie and Gower [46] proposed the classical predator-prey system, which is as follows

$$
\begin{cases}
\dot{X}(T) = s_1 X(T)(1 - \dfrac{X(T)}{K}) - \dfrac{l_1 X(T)Y(T)}{X(T) + l_2 Y(T)}, \\
\dot{Y}(T) = s_2 Y(T)(1 - \dfrac{l_3 Y(T)}{X(T)}).
\end{cases}
\tag{2.1}
$$

Here, $X(T)$, $Y(T)$ indicate prey and predator population at time T respectively, s_1, s_2 and K are intrinsic growth rate of prey, intrinsic growth rate of predator and carrying capacity. l_1 is the maximum number of prey each predator can eat each time and l_2 is a semi saturation constant, and the predation rate is $\frac{l_2}{2}$. The carrying capacity of predator is proportional to the size of prey population. l_3 is the amount of prey needed to support a predator population in equilibrium.

Based on the fact that Beddington-DeAngelis functional response is more authentic [47], we introduce it in system (2.1). Thus, the new system is

$$
\begin{cases}
\dot{X}(T) = s_1 X(T)(1 - \dfrac{X(T)}{K}) - \dfrac{l_1 X(T)Y(T)}{lX(T) + l_2 Y(T) + d}, \\
\dot{Y}(T) = s_2 Y(T)(1 - \dfrac{l_3 Y(T)}{X(T)}).
\end{cases}
\tag{2.2}
$$

Considering nonlinear predator harvesting, system (2.2) is constructed as follows

$$
\begin{cases}
\dot{X}(T) = s_1 X(T)(1 - \dfrac{X(T)}{K}) - \dfrac{l_1 X(T)Y(T)}{lX(T) + l_2 Y(T) + d}, \\
\dot{Y}(T) = s_2 Y(T)(1 - \dfrac{l_3 Y(T)}{X(T)}) - \dfrac{qE(T)Y(T)}{m_1 E(T) + m_2 Y(T)}.
\end{cases}
\tag{2.3}
$$

By using following transformations

$$
t = s_1 T, x = \frac{X}{K}, y = \frac{l_1 Y}{s_1 K}, \alpha = \frac{K m_2 s_1}{m_1 l_1}, \beta = \frac{s_2 l_3}{l_1}, r = \frac{l_1}{s_1 l_3}, d = K, q = s_1 m_1, l = a, b = \frac{l_2 s_1}{l_1},
$$

system (2.3) is non-dimensionalized as follows

$$
\begin{cases}
\dot{x}(t) = x(t)(1 - x(t)) - \dfrac{x(t)y(t)}{1 + ax(t) + by(t)}, \\
\dot{y}(t) = \beta y(t)\left(r - \dfrac{y(t)}{x(t)}\right) - \dfrac{E(t)y(t)}{E(t) + \alpha y(t)},
\end{cases}
\tag{2.4}
$$

where $a, b, \alpha, \beta, r, q, p, c, m_1$ and m_2 are all positive constants.

Simultaneously, an algebraic equation is also included due to the economic profit of harvesting. Based on Eq (1.2), we have

$$
m = \text{TR} - \text{TC} = \frac{qE(t)(py(t) - c)}{m_1 E(t) + m_2 y(t)},
\tag{2.5}
$$

here p is the price per unit harvested biomass, c is the cost per unit harvest, while m means the net economic revenue.

In general, the delay differential equation model can produce more efficient and accurate dynamics than the ordinary differential equation model as capturing oscillation dynamics.

Therefore, by combining the above biological economic algebraic equation and time delay, a differential-algebraic system with two time delays is given as follows

$$
\begin{cases}
\dot{x}(t) = x(t - \tau_1)(1 - x(t - \tau_1)) - \dfrac{x(t - \tau_1)y(t)}{1 + ax(t - \tau_1) + by(t)}, \\[2mm]
\dot{y}(t) = \beta y(t - \tau_2)\left(r - \dfrac{y(t - \tau_2)}{x(t - \tau_2)}\right) - \dfrac{E(t)y(t)}{E(t) + \alpha y(t)}, \\[2mm]
0 = \dfrac{qE(t)}{m_1 E(t) + m_2 y(t)}(py(t) - c) - m,
\end{cases}
\tag{2.6}
$$

where τ_1 is maturation time for prey population, τ_2 is gestation delay for predator population. The initial conditions of system (2.6) take the following form:

$$
x(\theta) > 0, y(\theta) > 0, E(0) > 0, \theta \in [-\tau, 0], \tau = \max\{\tau_1, \tau_2\}.
$$

System (2.6) can be transformed into matrix form as follows

$$
\bar{A}\dot{U}(t) = f(U),
\tag{2.7}
$$

where

$$
U(t) = \begin{pmatrix} x(t) \\ y(t) \\ E(t) \end{pmatrix}, \quad
\bar{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
f(U) = \begin{pmatrix} f_1(U) \\ f_2(U) \\ f_3(U) \end{pmatrix} =
\begin{pmatrix}
x(t - \tau_1)(1 - x(t - \tau_1)) - \dfrac{x(t-\tau_1)y(t)}{1+ax(t-\tau_1)+by(t)} \\[2mm]
\beta y(t - \tau_2)\left(r - \dfrac{y(t-\tau_2)}{x(t-\tau_2)}\right) \\[2mm]
\dfrac{qE(t)(py(t)-c)}{m_1 E(t)+m_2 y(t)} - m
\end{pmatrix}.
$$

Remark 1. *Compared with system proposed in [44], nonlinear predator harvesting and Beddington-DeAngelis functional response are considered in system (2.6).*

Remark 2. *Compared with system proposed in [45], system (2.6) contains two delays and Beddington-DeAngelis functional response, and focuses on economic interest of commercial harvest effort on predator.*

3. The existence of equilibria

For system (2.6), there is a bioeconomic equilibrium state when $m = 0$. First, we give the following assumptions:

(H1): $c > rx_0$ and $px_0(\beta r - 1) > \beta c$,

(H2): $c < rx_0$ and $px_0(\beta r - 1) < \beta c$,

(H3): $\dfrac{bc}{p} + 1 - \dfrac{c}{p} > 0$.

Therefore, if one of the assumptions (H_1) and (H_2) holds, then the interior equilibrium $P_0 = (x_0, y_0, E_0) = (x_0, \frac{c}{p}, \frac{pac\beta(c - rx_0)}{px_0(\beta r - 1) - \beta c})$ exists. Here x_0 satisfies the following equation

$$ax_0{}^2 + \left(\frac{bc}{p} + 1 - a\right)x_0 - \frac{bc}{p} - 1 + \frac{c}{p} = 0. \tag{3.1}$$

Obviously, a simple sufficient condition that Eq (3.1) has at least one positive root is that (H_1) and (H_3) or (H_2) and (H_3) hold.

In the case of $m > 0$, we suppose

(H4): $pq(1 + ax^* - x^* - ax^{*2}) > (qc + mm_1)(1 + bx^* - b) > 0$,

(H5): $pq(1 + ax^* - x^* - ax^{*2}) < (1 + bx^* - b)(qc + mm_1) < 0$.

If one of the assumptions (H_4) and (H_5) holds, then the interior equilibrium $P^* = (x^*, y^*, E^*)$ exists, here $E^* = \frac{mm_2(1 + ax^* - x^* - ax^{*2})}{pq(1 + ax^* - x^* - ax^{*2}) - (qc + mm_1)(1 + bx^* - b)}, y^* = \frac{1 + ax^* - x^* - ax^{*2}}{1 + bx^* - b}$ and x^* satisfies the following equation

$$A_1 x^6 + A_2 x^5 + A_3 x^4 + A_4 x^3 + A_5 x^2 + A_6 x + A_7 = 0, \tag{3.2}$$

here

$$A_1 = pq\,\alpha^2\beta\,a^2,$$
$$A_2 = \alpha\beta\,rpqa + \alpha\beta\,rab^2(mm_1 + qc) - mm_2\,ab - pq\,\alpha^2\beta\,r(2\,a^3 - a^2 - a),$$
$$A_3 = -mm_2\,ab(\beta\,r - 1) + \alpha\beta\,rpq(-2\,a^2 + 2\,a) - \alpha\beta\,r(mm_1 + qc)$$
$$\left(3\,ab^2 - b^2 - 2\,ab\right) - mm_2\left(-2\,a^2b + a + ab\right) - pq\,\alpha^2\beta\,r\left(-3\,a^3 + 8\,a^2 - 2\,a\right),$$
$$A_4 = -mm_2\left(3\,a^2b - 2\,a^2 + 2\,a - 6\,ab + b\right) - pq\,\alpha^2\beta\,r\left(a^3 - 9\,a^2 + 9\,a - 1\right)$$
$$- \alpha\beta\,r\,(mm_1 + qc)\left(-3\,ab^2 + 3\,b^2 - a + 4\,ab - 2\,b\right)$$
$$+ mm_2\,(\beta\,r - 1)(2\,ab - a - b) + \alpha\beta\,rpq\left(a^2 - 4\,a + 1\right),$$
$$A_5 = -mm_2\left(-a^2b + a^2 - 4\,a + 6\,ab - 3\,b + 1\right) - pq\,\alpha^2\beta\,r\left(3\,a^2 - 9\,a + 3\right)$$
$$- \alpha\beta\,r\,(mm_1 + qc)\left(ab^2 - 2\,ab - 3\,b^2 + a + 4\,b - 1\right)$$
$$+ mm_2\,(\beta\,r - 1)(2\,b - ab - 1 + a) + \alpha\beta\,rpq\,(2\,a - 2),$$
$$A_6 = mm_2\,(\beta\,r - 1)(1 - b) + \alpha\beta\,rpq - \alpha\beta\,r\,(mm_1 + qc)\,(1 - b)^2$$
$$- mm_2\,(2\,a - 2\,ab + 3\,b - 2) - pq\,\alpha^2\beta\,r\,(3\,a - 3),$$
$$A_7 = -mm_2\,(1 - b) - pq\,\alpha^2\beta\,r.$$

Based on Routh-Hurwitz criterion [1], Eq (3.2) has at least one positive root when $1 < b < 1 + \frac{pq\alpha^2\beta r}{mm_2}$. Suppose $(\tau_1, \tau_2, m) \in \Omega_1$, here

$$\Omega_1 = \left\{(\tau_1, \tau_2, m)\middle|\tau_1 \geq 0, \tau_2 \geq 0, 0 < m < \frac{pq\alpha^2\beta r}{(b - 1)m_2}\right\}.$$

4. Bifurcation analysis of positive equilibria

4.1. System without delays

Here, we are interested in the stability of system (2.6) at the interior equilibrium P_0.

When $\tau_1 = 0, \tau_2 = 0$, system (2.6) takes the following form

$$
\begin{cases}
\dot{x}(t) = x(t)(1 - x(t)) - \dfrac{x(t)\,y(t)}{1 + ax(t) + by(t)}, \\[3mm]
\dot{y}(t) = \beta\, y(t)\left(r - \dfrac{y(t)}{x(t)}\right) - \dfrac{E(t)y(t)}{E(t) + \alpha\, y(t)}, \\[3mm]
0 = \dfrac{qE(t)}{m_1 E(t) + m_2 y(t)}(py(t) - c) - m.
\end{cases}
\tag{4.1}
$$

Lemma 4.1. *(Singularity induced bifurcation theorem [49, 50]). If the differential-algebraic equations satisfy the following conditions at the singular equilibrium:*

(I) $D_y f_3$ has a simple zero eigenvalue and

$$
trace\left(\left(\begin{array}{c} D_E f_1 \\ D_E f_2 \end{array}\right) adj(D_E f_3)\left(\begin{array}{cc} D_x f_3 & D_y f_3 \end{array}\right)\right)_{p_0} > 0,
$$

(II)

$$
\left(\begin{array}{ccc}
D_x f_1 & D_y f_1 & D_E f_1 \\
D_x f_2 & D_y f_2 & D_E f_2 \\
D_x f_3 & D_y f_3 & D_E f_3
\end{array}\right)
$$

is nonsingular,
(III)

$$
\left(\begin{array}{cccc}
D_x f_1 & D_y f_1 & D_E f_1 & D_v f_1 \\
D_x f_2 & D_y f_2 & D_E f_2 & D_v f_2 \\
D_x f_3 & D_y f_3 & D_E f_3 & D_v f_3 \\
D_x \Delta & D_y \Delta & D_E \Delta & D_v \Delta
\end{array}\right)
$$

is also nonsingular, here $\Delta = D_E f_3$, then the singularity induced bifurcation occurs at the singular equilibrium.

Theorem 4.2. *A singularity induced bifurcation takes place at the interior equilibrium P_0 of the differential algebraic system (4.1). When the bifurcation parameter m increases through zero, system (4.1) is unstable at P_0.*

Proof. We can obtain that the Jacobin matrix of system (4.1) around P_0 is

$$
J_{P_0} = \left(\begin{array}{ccc}
J_{11} & J_{12} & 0 \\
\dfrac{\beta\, y_0^2}{x_0^2} & J_{22} & -\dfrac{\alpha\, y_0^2}{(\alpha\, y_0 + E_0)^2} \\
0 & \dfrac{pq\, E_0}{(E_0 m_1 + m_2 y_0)^2} & 0
\end{array}\right),
$$

where, $J_{11} = 1 - 2x_0 - (b + \frac{x_0}{y_0})(1 - x_0)^2$, $J_{12} = \frac{-x_0(1-x_0)^2(ax_0+1)}{y_0^2}$, $J_{22} = -\frac{E_0^2}{(\alpha\, y_0 + E_0)^2} + \beta\, r - 2\frac{\beta\, y_0}{x_0}$.

Let m be bifurcation parameter, D be differential operator. We can obtain the following results.

(I)

$$\text{trace}\left(\begin{pmatrix} D_E f_1 \\ D_E f_2 \end{pmatrix} \text{adj}(D_E f_3) \begin{pmatrix} D_x f_3 & D_y f_3 \end{pmatrix}\right)_{P_0} = \frac{\alpha y_0{}^2 p E_0 (p m_1 E_0 + c m_2)}{(\alpha y_0 + E_0)^2 (m_1 E_0 + m_2 y_0)^2} > 0.$$

(II) If $J_{11} \neq 0$ holds, then it can be obtained that

$$\det\begin{pmatrix} D_x f_1 & D_y f_1 & D_E f_1 \\ D_x f_2 & D_y f_2 & D_E f_2 \\ D_x f_3 & D_y f_3 & D_E f_3 \end{pmatrix}_{P_0} = \frac{\alpha y^2 q E_0 (p E_0 m_1 + c m_2) J_{11}}{(\alpha y_0 + E_0)^2 (m_1 E_0 + m_2 y_0)^2} \neq 0.$$

(III) Defining $\Delta = D_E f_3 = \frac{q m_2 y (p y - c)}{(m_1 E + m_2 y)^2}$, it follows from simple computations that

$$\det\begin{pmatrix} D_x f_1 & D_y f_1 & D_E f_1 & D_v f_1 \\ D_x f_2 & D_y f_2 & D_E f_2 & D_v f_2 \\ D_x f_3 & D_y f_3 & D_E f_3 & D_v f_3 \\ D_x \Delta & D_y \Delta & D_E \Delta & D_v \Delta \end{pmatrix}_{P_0} = -\frac{\alpha y_0{}^3 p q m_2 J_{11}}{(\alpha y_0 + E_0)^2 (m_1 E_0 + m_2 y_0)^3} \neq 0.$$

Hence a singularity induced bifurcation occurs around P_0 of system (4.1) and the bifurcation value is $m = 0$.

On the other hand,

$$C_1 = \text{trace}\left(\begin{pmatrix} D_E f_1 \\ D_E f_2 \end{pmatrix} \text{adj}(D_E f_3) \begin{pmatrix} D_x f_3 & D_y f_3 \end{pmatrix}\right)_{P_0} = \frac{\alpha y_0{}^2 p E_0 (p m_1 E_0 + c m_2)}{(\alpha y_0 + E_0)^2 (m_1 E_0 + m_2 y_0)^2} > 0,$$

$$C_2 = \left(D_v \Delta - \begin{pmatrix} D_x \Delta & D_y \Delta & D_E \Delta \end{pmatrix} P_1^{-1} \begin{pmatrix} D_v f_1 \\ D_v f_2 \\ D_v f_3 \end{pmatrix}\right)_{P_0} = \frac{m_2 y_0}{E_0 (m_1 E_0 + m_2 y_0)^2} > 0,$$

where $P_1 = \begin{pmatrix} D_x f_1 & D_y f_1 & D_E f_1 \\ D_x f_2 & D_y f_2 & D_E f_2 \\ D_x f_3 & D_y f_3 & D_E f_3 \end{pmatrix}$.

According to Lemma 4.1 and Theorem 3 in [49], when m increases through 0, one eigenvalue of system (4.1) moves from $\mathbb{C}^-$ to $\mathbb{C}^+$ along real axis by diverging through ∞. Hence, the Theorem 4.2 is proved. □

By using the similar proof of Theorem 4.2, it is easy to show system (4.1) is unstable around P^*, what's more, state feedback controllers are designed to remove singularity induced bifurcation and stabilize system (4.1) around P_0 and P^*, respectively.

In the case of $m = 0$, according to the leading matrix $\bar{A}$ in (4.1) and J_{P_0}, we can obtain rank $(J_{P_0}, \bar{A} J_{P_0}, \bar{A}^2 J_{P_0}) = 3$. It is not hard to find that system (4.1) is locally controllable around P_0 based on Theorem 2-2.1 in [48]. Therefore, a feedback controller can be used to stabilize system (4.1) around P_0. By using Theorem 3-1.2 in [48], a feedback controller is as follows

$$v(t) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & \hat{k} \end{pmatrix} \begin{pmatrix} x(t) - x_0 \\ y(t) - y_0 \\ E(t) - E_0 \end{pmatrix},$$

where $\hat{k}$ is a feedback gain.

Applying the controller into system (4.1), we can obtain that a controlled system is

$$
\begin{cases}
\dot{x}(t) = x(t)(1 - x(t)) - \dfrac{x(t)y(t)}{1 + ax(t) + by(t)}, \\[2mm]
\dot{y}(t) = \beta y(t)\left(r - \dfrac{y(t)}{x(t)}\right) - \dfrac{E(t)y(t)}{E(t) + \alpha y(t)}, \\[2mm]
0 = \dfrac{qE(t)(py(t) - c)}{m_1 E(t) + m_2 y(t)} + \hat{k}(E(t) - E_0).
\end{cases}
\tag{4.2}
$$

Theorem 4.3. *If the feedback gain $\hat{k} > \max\{Q_1, Q_2, Q_3\}$, here*

$$
Q_1 = \frac{\alpha y_0{}^2 pq\, E_0}{(\alpha y_0 + E_0)(E_0\, m_1 + m_2\, y_0)^2},
$$

$$
Q_2 = \frac{-\alpha\, y_0{}^2 pq\, E_0}{(\alpha y_0 + E_0)(E_0\, m_1 + m_2\, y_0)^2\, J_{11} J_{22}},
$$

$$
Q_3 = \frac{-\alpha\, y_0{}^2 pq\, E_0 J_{11}}{(\alpha y_0 + E_0)(E_0\, m_1 + m_2\, y_0)^2\left(-J_{11} - J_{22} + \frac{\beta(1 - x_0)^2 (ax_0 + 1)}{x_0}\right)},
$$

then system (4.2) is stable around P_0.

Proof. At first, the Jacobin matrix of system (4.2) around P_0 is

$$
\hat{J}_{P_0} =
\begin{pmatrix}
J_{11} & J_{12} & 0 \\[2mm]
\dfrac{\beta y_0{}^2}{x_0{}^2} & J_{22} & -\dfrac{\alpha\, y_0{}^2}{(\alpha\, y_0 + E_0)^2} \\[2mm]
0 & \dfrac{pq\, E_0}{(E_0\, m_1 + m_2\, y_0)^2} & \hat{k}
\end{pmatrix}.
$$

The characteristic equation of system (4.2) around P_0 is $\det(\lambda \bar{A} - \hat{J}_{P_0}) = 0$ based on $\bar{A}$ in system (4.2) and $\hat{J}_{P_0}$, which can be written as follows

$$
\lambda^2 + \Delta_1 \lambda + \Delta_2 = 0,
$$

where

$$
\Delta_1 = -J_{11} - J_{22} - \frac{1}{\hat{k}} B_1, \qquad\qquad \Delta_2 = -J_{11} J_{22} + B_2 + \frac{1}{\hat{k}} J_{11} B_1,
$$

$$
B_1 = \frac{\alpha\, y_0{}^2 pq\, E_0}{(\alpha y_0 + E_0)^2 (E_0\, m_1 + m_2\, y_0)}, \qquad B_2 = \frac{\beta(1 - x_0)^2 (ax_0 + 1)}{x_0}.
$$

By simple computation, if $\hat{k} > \max\{Q_1, Q_2, Q_3\}$, then system (4.2) is stable around P_0. $\qquad\square$

Similarly, in the case of $m > 0$, a controlled system is

$$
\begin{cases}
\dot{x}(t) = x(t)(1 - x(t)) - \dfrac{x(t)y(t)}{1 + ax(t) + by(t)}, \\[2mm]
\dot{y}(t) = \beta y(t)\left(r - \dfrac{y(t)}{x(t)}\right) - \dfrac{E(t)y(t)}{E(t) + \alpha y(t)}, \\[2mm]
0 = \dfrac{qE(t)(py(t) - c)}{m_1 E(t) + m_2 y(t)} - m + \tilde{k}(E(t) - E^*),
\end{cases}
\tag{4.3}
$$

where $\tilde{k}$ is a feedback gain.

By using the similar analysis in Theorem 4.3, we can obtain the following result.

Theorem 4.4. *If controller feedback gain $\tilde{k} > max\{\tilde{Q}_1, \tilde{Q}_2\}$, where*

$$\tilde{Q}_1 = \frac{\alpha y^* q E^* (m_2 c + m_1 p E^*) - q m_2 y^* (p y^* - c)(\alpha y^* + E^*)^2 \xi_5}{(\alpha y^* + E^*)^2 (m_1 E^* + m_2 y^*)^2 (\xi_1 + \xi_3)},$$

$$\tilde{Q}_2 = \frac{\alpha y^{*2} q E^* (m_2 c + m_1 p E^*) \xi_1}{(\alpha y^* + E^*)^2 (m_1 E^* + m_2 y^*)^2 [\xi_1 + \xi_3 + \xi_4]} - \frac{q m_2 y^* (p y^* - c)}{(m_1 E^* + m_2 y^*)^2},$$

$$\xi_0 = \frac{E^{*2}}{(E + \alpha y^*)^2} - \frac{\alpha y^* E^* (p m_1 E^* + c m_2)}{m_2 (p y^* - c)(E^* + \alpha y^*)^2}, \quad \xi_2 = \xi_0 \xi_1,$$

$$\xi_1 = 2 x^* + \frac{x^* (1 - x^*)^2}{y^*} + b (1 - x^*)^2 - 1, \quad \xi_5 = \xi_1 \xi_3,$$

$$\xi_3 = \frac{E^{*2}}{(E + \alpha y^*)^2} + \frac{2 \beta y^*}{x^*} - \beta r, \quad \xi_4 = \frac{\beta (1 - x^*)^2 (1 + a x^*)}{x^*},$$

then system (4.3) is stable around P^*.

Remark 3. *According to design of feedback controller, we can make interior equilibria be stable, which shows that prey-predator ecosystem can be kept sustainable and economic interest can be kept ideal by controlling commercial harvest effort on predator.*

4.2. System with delays

4.2.1. Case I: $\tau_1 > 0, \tau_2 = 0$

By analyzing system (2.6), a characteristic equation of around P^* is

$$\lambda^2 + \xi_0 \lambda + (\xi_1 \lambda + \xi_2) e^{-\lambda \tau_1} + (\xi_3 \lambda + \xi_4) e^{-\lambda \tau_2} + \xi_5 e^{-\lambda (\tau_1 + \tau_2)} = 0. \tag{4.4}$$

When $\tau_1 > 0$, and $\tau_2 = 0$, Eq (4.4) takes the following form

$$\lambda^2 + (\xi_0 + \xi_3) \lambda + (\xi_1 \lambda + \xi_2 + \xi_5) e^{-\lambda \tau_1} + \xi_4 = 0. \tag{4.5}$$

By simple computation, we can obtain that $\lambda = 0$ is not a root of Eq (4.5). We suppose that $\lambda = i \beta_1$ (β_1 is a positive real number) is a root of Eq (4.5). By separating real and imaginary parts, we can obtain the following two transcendental equations

$$\begin{cases} \xi_1 \beta_1 \sin(\beta_1 \tau_1) + (\xi_2 + \xi_5) \cos(\beta_1 \tau_1) = \beta_1^2 - \xi_4, \\ \xi_1 \beta_1 \cos(\beta_1 \tau_1) - (\xi_2 + \xi_5) \sin(\beta_1 \tau_1) = -\beta_1 (\xi_0 + \xi_3). \end{cases} \tag{4.6}$$

By computing two equations in Eq (4.6), it gives that

$$\beta_1^4 + \left[(\xi_0 + \xi_3)^2 - 2\xi_4 - \xi_1^2 \right] \beta_1^2 + \xi_4^2 - (\xi_2 + \xi_5)^2 = 0. \tag{4.7}$$

Theorem 4.5. *If $\xi_4^2 - (\xi_2 + \xi_5)^2 < 0$ holds,*
(i) system (2.6) is locally asymptotically stable around P^* when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_2$;
(ii) system (2.6) undergoes Hopf bifurcation around P^* when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_3$. Here, Ω_2 and Ω_3
are defined as follows

$$\Omega_2 = \left\{ (\tau_1, \tau_2, m) \middle| 0 < \tau_1 < \tau_{10}, \tau_2 = 0, m > 0 \right\},$$

$$\Omega_3 = \left\{ (\tau_1, \tau_2, m) \middle| \tau_1 = \tau_{10}, \tau_2 = 0, 0 < m < \frac{-\alpha y^* pq E^*}{m_2(E^* + \alpha y^*)^2 N_1 - m_2 E^{*2}} \right\},$$

where $N_1 = (\xi_1^2 + 2\xi_4)^{\frac{1}{2}} + \frac{E^* \alpha y^*}{(E + \alpha y^*)^2} - \frac{\beta y^*}{x^*}$ and $(E^* + \alpha y^*)^2 N_1 < E^{*2}$.

Proof. If $\xi_4^2 - (\xi_2 + \xi_5)^2 < 0$, based on Routh-Hurwitz criterion [1], we can guarantee that Eq (4.7) has at least one positive root. Consequently, Eq (4.5) has a pair of purely imaginary roots $\lambda = \pm i\beta_1^*$.

Based on Eq (4.6), we can calculate τ_{1k} as follows

$$\tau_{1k} = \frac{1}{\beta_1^*} \arccos \frac{[(\xi_2 + \xi_5) - \xi_1(\xi_0 + \xi_3)]\beta_1^{*2} - \xi_4(\xi_2 + \xi_5)}{\xi_1^2 \beta_1^{*2} + (\xi_2 + \xi_5)^2} + \frac{2k\pi}{\beta_1^*}. \tag{4.8}$$

By using Butlers lemma [55], system (2.6) is locally asymptotically stable around P^* when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_2$.

Next, we will determine

$$\Theta = \operatorname{sign} \left\{ \frac{\mathrm{d}(\mathrm{Re}\lambda)}{\mathrm{d}\tau_1} \right\}_{\lambda = i\beta_1^*} = \operatorname{sign} \left\{ \mathrm{Re}(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1})^{-1} \right\}_{\lambda = i\beta_1^*}.$$

By differentiating Eq (4.5) with respect to τ_1, we have

$$(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1})^{-1} = \frac{2\lambda + (\xi_0 + \xi_3)}{-\lambda(\lambda^2 + (\xi_0 + \xi_3)\lambda + \xi_4)} + \frac{\xi_1}{\xi_1 \lambda^2 + (\xi_2 + \xi_5)\lambda} - \frac{\tau_1}{\lambda}.$$

By virtue of Eq (4.6), we have

$$\Theta = \operatorname{sign} \left\{ \mathrm{Re}(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau_1})^{-1} \right\}_{\lambda = i\beta_1^*} = \operatorname{sign} \left\{ \frac{2\beta_1^{*2} + (\xi_0 + \xi_3)^2 - 2\xi_4 - \xi_1^2}{(\beta_1^{*2} - \xi_4)^2 + (\xi_0 + \xi_3)^2 \beta_1^{*2}} \right\}.$$

If $0 < m < \frac{-\alpha y^* pq E^*}{m_2(E^* + \alpha y^*)^2 N_1 - m_2 E^{*2}}$ and $(E^* + \alpha y^*)^2 N_1 < E^{*2}$, then $(\xi_0 + \xi_3)^2 - 2\xi_4 - \xi_1^2 > 0$ holds, which implies that $\Theta > 0$. Hence, when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_3$, transversality condition holds and Hopf bifurcation occurs. □

4.2.2. Case II: $\tau_1 = 0, \tau_2 > 0$

When $\tau_1 = 0$ and $\tau_2 > 0$, Eq (4.4) can be written the following form

$$\lambda^2 + (\xi_0 + \xi_1)\lambda + \xi_2 + (\xi_3 \lambda + \xi_4 + \xi_5)e^{-\lambda \tau_2} = 0. \tag{4.9}$$

Similarly, it shows that $\lambda = 0$ is not a root of Eq (4.9). We suppose that $\lambda = i\beta_2$ ($\beta_2 > 0$) is a root of Eq (4.9). We can obtain the following two transcendental equations

$$\begin{cases} \xi_3 \beta_2 \sin(\beta_2 \tau_2) + (\xi_4 + \xi_5) \cos(\beta_2 \tau_2) = \beta_2 - \xi_2, \\ (\xi_4 + \xi_5) \sin(\beta_2 \tau_2) - \xi_3 \beta_2 \cos(\beta_2 \tau_2) = (\xi_0 + \xi_1)\beta_2, \end{cases} \tag{4.10}$$

which gives that

$$\beta_2^4 + \left[(\xi_0 + \xi_1)^2 - 2\xi_2 - \xi_3^2\right]\beta_2^2 + \xi_2^2 - (\xi_4 + \xi_5)^2 = 0. \tag{4.11}$$

Theorem 4.6. *If $\xi_2^2 - (\xi_4 + \xi_5)^2 < 0$ holds,*
(i) system (2.6) is locally asymptotically stable around P^* when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_4$;
(ii) system (2.6) undergoes Hopf bifurcation around P^* when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_5$, where Ω_4 and Ω_5 are defined as follows

$$\Omega_4 = \{(\tau_1, \tau_2, m) | \tau_1 = 0, 0 < \tau_2 < \tau_{20}, m > 0\},$$

$$\Omega_5 = \left\{(\tau_1, \tau_2, m) \middle| \tau_1 = 0, \ \tau_2 = \tau_{20}, 0 < m < \frac{\alpha p q y^* E^*}{\sqrt{m_2\left(-\xi_2{}^2 + \xi_3{}^2\right)}(\alpha y^* + E^*)^2 - m_2 E^*}\right\},$$

here, $\sqrt{m_2\left(-\xi_2{}^2 + \xi_3{}^2\right)}(\alpha y^* + E^*)^2 > m_2 E^*$.

By computing Eq (4.10), we can obtain τ_{2k} is

$$\tau_{2k} = \frac{1}{\beta_2^*} \arccos \frac{[(\xi_4 + \xi_5) - \xi_3(\xi_0 + \xi_1)]\beta_2^{*2} - \xi_2(\xi_4 + \xi_5)}{\xi_3^2 \beta_2^{*2} + (\xi_4 + \xi_5)^2} + \frac{2k\pi}{\beta_2^*}. \tag{4.12}$$

The proof of Theorem 4.6 is similar to that of Theorem 4.5. So, we omit it.

4.2.3. Case III: $\tau_1 > 0, \tau_2 = \hat{\tau}_2 \in (0, \tau_{20}), \tau_1 \neq \tau_2$

In this part, τ_1 is considered as a parameter while τ_2 is regarded as a fixed value $\hat{\tau}_2 \in (0, \tau_{20})$, which is a stable interval calculated in Subsection 4.2.2. Here Ω_6 is defined as follows

$$\Omega_6 = \{(\tau_1, \tau_2, m) | \tau_1 > 0, \tau_2 = \hat{\tau}_2 \in (0, \tau_{20}), m > 0\}.$$

Let $\lambda = i\alpha_1$ (α_1 is a positive real number) represent a purely imaginary root of Eq (4.4). By separating real and imaginary parts, we have the following two transcendental equations

$$\begin{cases} \alpha_1^2 - \xi_3\alpha_1 \sin(\alpha_1\hat{\tau}_2) - \xi_4 \cos(\alpha_1\hat{\tau}_2) = \xi_1\alpha_1 \sin(\alpha_1\tau_1) + \xi_2 \cos(\alpha_1\tau_1) + \xi_5 \cos[\alpha_1(\tau_1 + \hat{\tau}_2)], \\ \xi_0\alpha_1 + \xi_3\alpha_1 \cos(\alpha_1\hat{\tau}_2) - \xi_4 \sin(\alpha_1\hat{\tau}_2) = \xi_2 \sin(\alpha_1\tau_1) - \xi_1\alpha_1 \cos(\alpha_1\tau_1) + \xi_5 \sin[\alpha_1(\tau_1 + \hat{\tau}_2)]. \end{cases} \tag{4.13}$$

Based on Eq (4.13), it derives that

$$cos(\alpha_1\tau_1) = \frac{l_{10}(\alpha_1, \hat{\tau}_2)}{l_{12}(\alpha_1, \hat{\tau}_2)}, \quad sin(\alpha_1\tau_1) = \frac{l_{11}(\alpha_1, \hat{\tau}_2)}{l_{12}(\alpha_1, \hat{\tau}_2)}, \tag{4.14}$$

where

$$l_{10}(\alpha_1, \hat{\tau}_2) = (\xi_2 - \xi_0\xi_1)\alpha_1^2 - \xi_4\xi_5 + [(\xi_5 - \xi_1\xi_3)\alpha_1^2 - \xi_2\xi_4] \cos(\alpha_1\hat{\tau}_2)$$
$$+ (\xi_0\xi_5 - \xi_2\xi_3 + \xi_1\xi_4)\alpha_1 \sin(\alpha_1\hat{\tau}_2),$$
$$l_{11}(\alpha_1, \hat{\tau}_2) = \xi_1\alpha_1^3 + (\xi_0\xi_2 + \xi_3\xi_5)\alpha_1 - [(\xi_5 + \xi_1\xi_3)\alpha_1^2 + \xi_2\xi_4] \sin(\alpha_1\hat{\tau}_2)$$
$$+ (\xi_0\xi_5 + \xi_2\xi_3 - \xi_1\xi_4)\alpha_1 \cos(\alpha_1\hat{\tau}_2),$$

$$l_{12}(\alpha_1, \hat{\tau}_2) = [\xi_2 + \xi_5 \cos(\alpha_1 \hat{\tau}_2)]^2 + [\xi_5 \sin(\alpha_1 \hat{\tau}_2) - \xi_1 \alpha_1]^2.$$

Let

$$L_1(\alpha_1, \hat{\tau}_2) = l_{10}^2(\alpha_1, \hat{\tau}_2) + l_{11}^2(\alpha_1, \hat{\tau}_2) - l_{12}^2(\alpha_1, \hat{\tau}_2). \tag{4.15}$$

Due to its complicated form, it is not easy for us to analyze properties of roots of transcendental equation (4.15). Based on dynamical system theory [1], we know that if and only if every eigenvalue has negative real part, system (2.6) is locally asymptotically stable around P^*. What's more, by analyzing existence of Hopf bifurcation around the corresponding interior equilibrium P^*, the periodic oscillation of system (2.6) is investigated. Hale [51] proposed that when the corresponding eigenvalue has a pair of purely imaginary roots, system usually exhibits Hopf bifurcation. Obviously, if Eq (4.15) has finite positive and simple roots $0 < \alpha_{10} < \alpha_{11} < \cdots < \alpha_{1n}$, Eq (4.4) has a pair of purely imaginary roots.

Without loss of generality, we denote $\alpha_{1c} = \max\{\alpha_{1k}\}, k = 0, 1, 2, \cdots n$ and regard τ_1 as the bifurcation parameter, while we have the following corresponding critical value τ_{1c}

$$\tau_{1c} = \min\left\{\frac{\omega_{1c} + 2k\pi}{\alpha_{1c}}\right\}, \tag{4.16}$$

here, $\omega_{1c} \in [0, 2\pi]$ satisfies the following equations

$$\cos \omega_{1c} = \frac{l_{10}(\alpha_{1c}, \hat{\tau}_2)}{l_{12}(\alpha_{1c}, \hat{\tau}_2)}, \qquad \sin \omega_{1c} = \frac{l_{11}(\alpha_{1c}, \hat{\tau}_2)}{l_{12}(\alpha_{1c}, \hat{\tau}_2)}. \tag{4.17}$$

Furthermore, it is important to check transversal condition. The necessary condition for the existence of Hopf branches is that the eigenvalues passes through the imaginary axis with non-zero speed [51].

By differentiating λ with respect to τ_1 in Eq (4.4), it derives that

$$\frac{d\lambda}{d\tau_1} = \frac{\lambda[(\xi_1\lambda + \xi_2)e^{-\lambda\tau_1} + \xi_5 e^{-\lambda(\tau_1 + \hat{\tau}_2)}]}{h_1(\lambda, \hat{\tau}_2, \tau_1)}, \tag{4.18}$$

where

$$h_1(\lambda, \hat{\tau}_2, \tau_1) = 2\lambda + \xi_0 + [\xi_1 - (\xi_1\lambda + \xi_2)\tau_1]e^{-\lambda\tau_1}$$
$$+ [\xi_3 - (\xi_3\lambda + \xi_4)\hat{\tau}_2]e^{-\lambda\hat{\tau}_2} - \xi_5(\tau_1 + \hat{\tau}_2)e^{-\lambda(\tau_1 + \hat{\tau}_2)}.$$

By virtue of Eq (4.18), it can be obtained that

$$\hat{\Theta} = \text{sign}\left\{\text{Re}(\frac{d\lambda}{d\tau_1})^{-1}\right\}_{\tau_1 = \tau_{1c}} = \text{sign}\left\{\frac{B_{11}B_{13} + B_{12}B_{14}}{B_{13}^2 + B_{14}^2}\right\}, \tag{4.19}$$

where

$$B_{11} = [(\xi_2\tau_{1c} - \xi_1)\alpha_{1c} - \xi_5(\tau_{1c} + \hat{\tau}_2)\alpha_{1c}\sin(\alpha_{1c}\hat{\tau}_2)]\sin(\alpha_{1c}\tau_{1c})$$
$$+ [\xi_5(\tau_{1c} + \hat{\tau}_2)\alpha_{1c}\cos(\alpha_{1c}\hat{\tau}_2) - \xi_1\tau_{1c}\alpha_{1c}^2]\cos(\alpha_{1c}\tau_{1c}),$$
$$+ 2\alpha_{1c}^2 - \xi_3\hat{\tau}_2\alpha_{1c}^2\cos(\alpha_{1c}\hat{\tau}_2) + (\xi_4\hat{\tau}_2 - \xi_3)\alpha_{1c}\sin(\alpha_{1c}\hat{\tau}_2)$$

$$B_{12} = -\xi_0\alpha_{1c} + \xi_3\hat{\tau}_2\alpha_{1c}\sin(\alpha_{1c}\hat{\tau}_2) + (\xi_4\hat{\tau}_2 - \xi_3)\alpha_{1c}\cos(\alpha_{1c}\hat{\tau}_2)$$
$$+ [(\xi_2\tau_{1c} - \xi_1)\alpha_{1c} + \xi_5(\tau_{1c} + \hat{\tau}_2)\alpha_{1c}\cos(\alpha_{1c}\hat{\tau}_2)]\cos(\alpha_{1c}\tau_{1c})$$
$$+ [\xi_1\tau_{1c}\alpha_{1c}^2 - \xi_5(\tau_{1c} + \hat{\tau}_2)\alpha_{1c}\sin(\alpha_{1c}\hat{\tau}_2)]\sin(\alpha_{1c}\tau_{1c}),$$
$$B_{13} = \alpha_{1c}^4 - [\xi_3\alpha_{1c}\sin(\alpha_{1c}\hat{\tau}_2) + \xi_4\cos(\alpha_{1c}\hat{\tau}_2)]\alpha_{1c}^2,$$
$$B_{14} = -\xi_0\alpha_{1c}^3 - [\xi_3\alpha_{1c}\cos(\alpha_{1c}\hat{\tau}_2) - \xi_4\sin(\alpha_{1c}\hat{\tau}_2)]\alpha_{1c}^2.$$

It is obvious to show $\hat{\Theta} > 0$ when $B_{11}B_{13} + B_{12}B_{14} > 0$. Therefore, we have the following results on stability and bifurcation in system (2.6).

Theorem 4.7. *For system (2.6), suppose that $B_{11}B_{13} + B_{12}B_{14} > 0$ holds and $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_6$.*
(i) If Eq (4.15) has no positive root, then system (2.6) is locally asymptotically stable around P^* when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_6$.
(ii) If Eq (4.15) has at least one positive and simple root α_1^*, there exists a critical delay $\tau_1^* = min\left\{\frac{\omega_1^* + 2k\pi}{\alpha_1^*}\right\} > 0$ such that system (2.6) is locally asymptotically stable around P^* when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_6 \cap \Omega_7$, here

$$\Omega_7 = \{(\tau_1, \tau_2, m) \mid 0 < \tau_1 < \tau_1^*, \tau_2 = \hat{\tau}_2, m > 0\}.$$

(iii) If Eq (4.15) has finite positive and simple roots $0 < \alpha_{10} < \alpha_{11} < \cdots < \alpha_{1n}$, there exists a critical delay τ_{1c} defined in (4.16) such that system (2.6) is locally asymptotically stable around P^* when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_6 \cap \Omega_8$, here

$$\Omega_8 = \{(\tau_1, \tau_2, m) \mid 0 < \tau_1 < \tau_{1c}, \tau_2 = \hat{\tau}_2, m > 0\}.$$

If $B_{11}B_{13} + B_{12}B_{14} > 0$, then system (2.6) undergoes a Hopf bifurcation around P^* when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_6 \cap \Omega_9$, here

$$\Omega_9 = \{(\tau_1, \tau_2, m) \mid \tau_1 = \tau_{1c}, \tau_2 = \hat{\tau}_2, m > 0\}.$$

4.2.4. Case IV: $\tau_1 \in (0, \tau_{10}), \tau_2 > 0, \tau_1 \neq \tau_2$

In this part, τ_2 is seen as a parameter, while τ_1 is regarded as a fixed value $\hat{\tau}_1 \in (0, \tau_{10})$ that is a stable interval calculated in Subsection 4.2.1. Ω_{10} is defined as follows:

$$\Omega_{10} = \{(\tau_1, \tau_2, m) \mid \tau_1 = \hat{\tau}_1 \in (0, \tau_{10}), \tau_2 > 0, m > 0\}.$$

Let $\lambda = i\alpha_2$ (α_2 is a positive real number) represent a purely imaginary root of Eq (4.4). We define

$$L_2(\alpha_2, \hat{\tau}_1) = l_{20}^2(\alpha_2, \hat{\tau}_1) + l_{21}^2(\alpha_2, \hat{\tau}_1) - l_{22}^2(\alpha_2, \hat{\tau}_1) = 0, \tag{4.20}$$

where,

$$l_{20}(\alpha_2, \hat{\tau}_1) = (\xi_4 - \xi_0\xi_3)\alpha_2^2 - \xi_2\xi_5 + [(\xi_5 - \xi_1\xi_3)\alpha_2^2 - \xi_2\xi_4]\cos(\alpha_2\hat{\tau}_1)$$
$$+ (\xi_0\xi_5 + \xi_2\xi_3 - \xi_1\xi_4)\alpha_2\sin(\alpha_2\hat{\tau}_1),$$
$$l_{21}(\alpha_2, \hat{\tau}_1) = \xi_3\alpha_2^3 + (\xi_0\xi_4 + \xi_1\xi_5)\alpha_2 - [(\xi_5 + \xi_1\xi_3)\alpha_2^2 + \xi_2\xi_4]\sin(\alpha_2\hat{\tau}_1)$$
$$+ (\xi_0\xi_5 - \xi_2\xi_3 + \xi_1\xi_4)\alpha_2\cos(\alpha_2\hat{\tau}_1),$$
$$l_{22}(\alpha_1, \hat{\tau}_2) = [\xi_4 + \xi_5\cos(\alpha_2\hat{\tau}_1)]^2 + [\xi_5\sin(\alpha_2\hat{\tau}_1) - \xi_3\alpha_2]^2.$$

Theorem 4.8. *For system (2.6), suppose that $B_{21}B_{23} + B_{22}B_{24} > 0$ holds and $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_{10}$.*
(i) If Eq (4.20) has no positive root, then system (2.6) is locally asymptotically stable around P^* when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_{10}$.
(ii) If Eq (4.20) has at least one positive and simple root α_2^*, there exists a critical delay $\tau_2^* = min\left\{\frac{\omega_2^* + 2k\pi}{\alpha_2^*}\right\} > 0$ such that system (2.6) is locally asymptotically stable around P^* when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_{10} \cap \Omega_{11}$, here

$$\Omega_{11} = \left\{(\tau_1, \tau_2, m) \mid \tau_1 = \hat{\tau}_1, 0 < \tau_2 < \tau_2^*, m > 0\right\}.$$

(iii) If Eq (4.20) has finite positive and simple roots $0 < \alpha_{20} < \alpha_{21} < \cdots \alpha_{2n}$, there exists a critical delay τ_{2c} defined in (4.19) such that system (2.6) is locally asymptotically stable around P^* when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_{10} \cap \Omega_{12}$, here

$$\Omega_{12} = \left\{(\tau_1, \tau_2, m) \mid \tau_1 = \hat{\tau}_1, 0 < \tau_2 < \tau_{2c}, m > 0\right\}.$$

If $B_{21}B_{23} + B_{22}B_{24} > 0$ holds, then system (2.6) undergoes a Hopf bifurcation around P^* when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_{10} \cap \Omega_{13}$, here

$$\Omega_{13} = \left\{(\tau_1, \tau_2, m) \mid \tau_1 = \hat{\tau}_1, \tau_2 = \tau_{2c}, m > 0\right\}.$$

Where the corresponding critical value τ_{2c} satisfies

$$\tau_{2c} = min\left\{\frac{\omega_{2c} + 2k\pi}{\alpha_{2c}}\right\}, \tag{4.21}$$

here $\omega_{2c} \in [0, 2\pi]$ satisfies

$$\cos \omega_{2c} = \frac{l_{20}(\alpha_{2c}, \hat{\tau}_1)}{l_{22}(\alpha_{2c}, \hat{\tau}_1)}, \quad \sin \omega_{2c} = \frac{l_{21}(\alpha_{2c}, \hat{\tau}_1)}{l_{22}(\alpha_{2c}, \hat{\tau}_1)}.$$

Similarly, by differentiating λ with respect to τ_2 in Eq (4.4) and computing Eq (4.13), we can obtain

$$\bar{\Theta} = \text{sign}\left\{\text{Re}(\frac{d\lambda}{d\tau_2})^{-1}\right\}_{\tau_2 = \tau_{2c}} = \text{sign}\left\{\frac{B_{21}B_{23} + B_{22}B_{24}}{B_{23}^2 + B_{24}^2}\right\},$$

here

$$
\begin{aligned}
B_{21} &= 2\alpha_{2c}^2 + (\xi_2\hat{\tau}_1 - \xi_1)\alpha_{2c}\sin(\alpha_{2c}\hat{\tau}_1) - \xi_1\alpha_{2c}^2\cos(\alpha_{2c}\hat{\tau}_1) \\
&\quad + [(\xi_4\tau_{2c} - \xi_3)\alpha_{2c} + \xi_5(\tau_{2c} + \hat{\tau}_1)\alpha_{2c}\cos(\alpha_{2c}\hat{\tau}_1)]\sin(\alpha_{2c}\tau_{2c}) \\
&\quad + [\xi_5(\tau_{2c} + \hat{\tau}_1)\alpha_{2c}\sin(\alpha_{2c}\hat{\tau}_1) - \xi_3\tau_{2c}\alpha_{2c}^2]\cos(\alpha_{2c}\tau_{2c}), \\
B_{22} &= -\xi_0\alpha_{2c} + \xi_1\alpha_{1c}^2\sin(\alpha_{2c}\hat{\tau}_1) + (\xi_2\hat{\tau}_1 - \xi_1)\alpha_{2c}\cos(\alpha_{2c}\hat{\tau}_1) \\
&\quad + [(\xi_4\tau_{2c} - \xi_3)\alpha_{2c} + \xi_5(\tau_{2c} + \hat{\tau}_1)\alpha_{2c}\cos(\alpha_{2c}\hat{\tau}_1)]\sin(\alpha_{2c}\tau_{2c}) \\
&\quad + [\xi_5(\tau_{2c} + \hat{\tau}_1)\alpha_{2c}\sin(\alpha_{2c}\hat{\tau}_1) - \xi_2\tau_{2c}\alpha_{2c}^2]\cos(\alpha_{2c}\tau_{2c}), \\
B_{23} &= \alpha_{2c}^4 - [\xi_1\alpha_{2c}\sin(\alpha_{2c}\hat{\tau}_1) + \xi_2\cos(\alpha_{2c}\hat{\tau}_1)]\alpha_{2c}^2, \\
B_{24} &= [-\xi_0\alpha_{2c}^3 - (\xi_1\alpha_{2c}\cos(\alpha_{2c}\hat{\tau}_1) + \xi_2\sin(\alpha_{2c}\hat{\tau}_1)]\alpha_{2c}^2.
\end{aligned}
$$

4.3. Properties of Hopf bifurcation

In this part, we shall study the direction of the Hopf bifurcation and the stability of bifurcating periodic solution of system (2.6) when τ_1 is regarded as a parameter, $\tau_2 = \hat{\tau}_2$ is a fixed value. Similarly, we can discuss other cases. The approach employed here is the normal form method and center manifold theorem introduced by Hassard et al. [52] and Guckenheimer et al. [53]. It follows from implicit function theorem [53] and the third equation of system (2.6) that $E(t) = \frac{mm_2y(t)}{q(py(t)-c)-mm_1}$. Hence, system (2.6) can be transformed as follows

$$
\begin{cases}
\dot{x}(t) = x(t - \tau_1)(1 - x(t - \tau_1)) - \dfrac{x(t - \tau_1)y(t)}{1 + ax(t - \tau_1) + by(t)}, \\
\dot{y}(t) = \beta y(t - \tau_2)\left(r - \dfrac{y(t - \tau_2)}{x(t - \tau_2)}\right) - \dfrac{mm_2y(t)}{vm_2 + \alpha[q(py(t) - c) - mm_1]}.
\end{cases}
\tag{4.22}
$$

Some transformations associated with $P^*(x^*, y^*)$ are given as follows:

$$
u_1(t) = x(t) - x^*, u_2(t) = y(t) - y^*.
$$

Here, we define $\bar{\tau}_1 = \mu + \tau_{1c}$, then $\mu = 0$ is the Hopf bifurcation value of system (2.6).

By simplifying, system (2.6) can be transformed to the following functional differential equation that is in the Banach space of continuous functions mapping $C = C([-\tilde{\tau}, 0], \mathbb{R}^2)$ ($\tilde{\tau} = \max\{\tau_{1c}, \hat{\tau}_2\}$), here $\tau_{1c}, \hat{\tau}_2$ are defined in (4.16) and (4.21), respectively,

$$
\begin{cases}
\dot{u}_1(t) = a_{11}u_1(t - \tau_{1c} - \mu) + a_{12}u_2(t) + a_{13}u_1^2(t - \tau_{1c} - \mu) \\
\qquad + a_{14}u_1(t - \tau_{1c} - \mu)u_2(t) + a_{15}u_2^2(t), \\
\dot{u}_2(t) = a_{21}u_2(t) + a_{22}u_2(t - \hat{\tau}_2) + a_{23}u_1(t - \hat{\tau}_2) + a_{24}u_1^2(t - \hat{\tau}_2) \\
\qquad + a_{25}u_2^2(t) + a_{26}u_2^2(t - \hat{\tau}_2) + a_{27}u_1(t - \hat{\tau}_2)u_2(t - \hat{\tau}_2),
\end{cases}
\tag{4.23}
$$

here,

$$
a_{12} = -\frac{x^*\beta(1 - x^*)^2(1 + ax^*)}{y^{*2}}, \qquad a_{13} = \frac{2ay^*(1 + by^*)}{(1 + ax^* + by^*)^3} - 2,
$$

$$
a_{14} = \frac{2by^*(by^* - 1) - 2ax^* - 1}{(1 + ax^* + by^*)^3}, \qquad a_{15} = \frac{2b}{(1 + ax^* + by^*)^3},
$$

$$
a_{21} = \frac{vm_2(vm_2 - \alpha vm_1 - \alpha qc)}{(vm_2 + \alpha q(py^* - c) - \alpha vm_1)^3}, \qquad a_{23} = \frac{\beta y^{*2}}{x^{*2}}, \quad a_{24} = -\frac{2\beta y^{*2}}{x^{*3}},
$$

$$
a_{25} = \frac{-2\alpha pqvm_2(vm_2 - \alpha vm_1 - \alpha qc)}{(vm_2 + \alpha q(py^* - c) - \alpha vm_1)^3}, \qquad a_{26} = \frac{-2\beta}{x^*}, \quad a_{27} = \frac{2\beta y^*}{x^{*2}},
$$

$$
a_{11} = 1 - 2x^* - \frac{x^*(1 - x^*)^2}{y^*} - b(1 - x^*)^2, \quad a_{22} = \beta r - \frac{2\beta y^*}{x^*}.
$$

Based to Riesz representation theorem [54], there is a 2×2 matrix function $\eta(\theta, \mu)$, here, $\theta \in [-\tilde{\tau}, 0]$ and

$$
L_\mu(\phi) = \int_{-\tilde{\tau}}^0 d\eta(\theta, \mu)\phi(\theta),
\tag{4.24}
$$

where $\phi(\theta) = (\phi_1(\theta), \phi_2(\theta) \in C([-\tilde{\tau}, 0], \mathbb{R}^2)$, and

$$\eta(\theta, \mu) = \begin{pmatrix} 0 & a_{12} \\ 0 & a_{21} \end{pmatrix} \delta(\theta) + \begin{pmatrix} 0 & 0 \\ a_{23} & a_{22} \end{pmatrix} \delta(\theta + \hat{\tau}_2) - \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix} \delta(\theta + \tau_{1c} + \mu), \qquad (4.25)$$

here, δ denotes the Dirac delta function.

Next, for $\phi \in C_1([-\tilde{\tau}, 0], \mathbb{R}^2)$, we define the operator $A(\mu)$ as

$$A(\mu)\phi = \begin{cases} \dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \theta \in [-\tilde{\tau}, 0), \\ \displaystyle\int_{-\tilde{\tau}}^{0} \mathrm{d}\eta(\mu, s)\phi(s), & \theta = 0, \end{cases}$$

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-\tilde{\tau}, 0), \\ F(\mu, \phi), & \theta = 0, \end{cases}$$

where $F(\mu, \phi) = (F_1(\mu, \phi), F_2(\mu, \phi))^T$ and

$$\begin{cases} F_1(\mu, \phi) = a_{13}\phi_1^2(-\tau_{1c} - \mu) + a_{14}\phi_1(-\tau_{1c})\phi_2(0) + a_{15}\phi_2^2(0), \\ F_2(\mu, \phi) = a_{24}\phi_1^2(-\hat{\tau}_2) + a_{25}\phi_2^2(0) + a_{26}\phi_2^2(-\hat{\tau}_2) + a_{27}\phi_1(-\hat{\tau}_2)\phi_2(-\hat{\tau}_2), \end{cases} \qquad (4.26)$$

then system (4.23) is equivalent to

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t. \qquad (4.27)$$

For $\psi \in C^1([-\tilde{\tau}, 0], (\mathbb{R}^2)^*)$, we suppose that

$$A^*\phi(s) = \begin{cases} -\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & s \in (0, \tilde{\tau}], \\ \displaystyle\int_{-\tilde{\tau}}^{0} \mathrm{d}\eta^T(t, 0)\psi(-t), & s = 0, \end{cases}$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{-\tilde{\tau}}^{0} \int_{\rho=0}^{\theta} \bar{\psi}(\rho - \theta) \mathrm{d}\eta(\theta)\phi(\rho)\mathrm{d}\rho, \qquad (4.28)$$

where $\eta(\theta) = \eta(\theta, 0)$.

From the above analysis, we know that A and A^* are adjoint operators. According to discussion in Subsection 4.2, $\pm i\alpha_{1c}$ are eigenvalues of both A and A^*.

Suppose $q(\theta) = (1, \chi)^T e^{i\alpha_{1c}\tau_{1c}\theta}$, θ is eigenvector of A corresponding to $i\alpha_{1c}$, which gives that $Aq(\theta) = i\alpha_{1c}q(\theta)$. Based on definition of A, (4.24), (4.25) and (4.26), we have

$$\int_{-\tilde{\tau}}^{0} \mathrm{d}\eta(\theta)q(\theta) = \begin{pmatrix} 0 & a_{12} \\ 0 & a_{21} \end{pmatrix} q(0) + \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix} q(-\tau_{1c}) = Aq(0) = i\alpha_{1c}q(0).$$

Hence, we can obtain $\chi = \frac{i\chi_{1c} - a_{11}e^{-i\alpha_{1c}\tau_{1c}}}{a_{12}}$.

Similarly, By simple computation, we can obtain eigenvector of A^* corresponding to $i\alpha_{1c}$, i.e., $q^*(s) = D(1, \chi^*)^T e^{i\alpha_{1c}\tau_{1c}s}$, here $\chi^* = \frac{i\alpha_{1c} - a_{11}e^{-i\alpha_{1c}\tau_{1c}}}{a_{23}}$.

In order to assume $\langle q^*(s), q(\theta) \rangle = 1$, we can determine the value of D. Based on (4.28), we have

$$\langle q^*(s), q(\theta) \rangle = \bar{D}(1, \bar{\chi}^*)(1, \chi)^T - \int_{-\tilde{\tau}}^{0} \int_{\rho=0}^{\theta} \bar{D}(1, \bar{\chi}^*) e^{-i\alpha_{1c}\tau_{1c}(\rho-\theta)} \mathrm{d}\eta(\theta) q(\rho) \mathrm{d}\rho$$

$$= \bar{D}(1 + \chi\bar{\chi}^*) - \bar{D} \int_{-\tilde{\tau}}^{0} (1, \bar{\chi}^*) \theta e^{i\alpha_{1c}\tau_{1c}\theta} \mathrm{d}\eta(\theta)(1, \chi)^T$$

$$= \bar{D}(1 + \chi\bar{\chi}^* + a_{11}\tau_{1c}e^{-i\alpha_{1c}\tau_{1c}}).$$

Hence, we can choose $\bar{D} = \frac{1}{1 + \chi\bar{\chi}^* + a_{11}\tau_{1c}e^{-i\alpha_{1c}\tau_{1c}}}$.

Next, let u_t be the solution of Eq (4.27) when $\mu = 0$, and

$$z(t) = \langle q^*, u_t \rangle, \qquad W(t, \theta) = u_t(\theta) - \mathrm{Re}\{z(t)q(\theta)\}. \tag{4.29}$$

On the center manifold C_0, it gives that

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots, \tag{4.30}$$

z and $\bar{z}$ are local coordinates for C_0 in the direction of q^* and $\bar{q}^*$.

For solution $u_t \in C_0$ of Eq (4.27), when $\mu = 0$, we have

$$\dot{z}(t) = i\alpha_{1c}z(t) + g(z, \bar{z}).$$

Where,

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots. \tag{4.31}$$

It follows from Eqs (4.29) and (4.30) that

$$u_t(\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + (1, \chi)^T e^{i\alpha_{1c}\tau_{1c}\theta}z + (1, \bar{\chi}^*)^T e^{i\alpha_{1c}\tau_{1c}\theta}\bar{z} + \cdots. \tag{4.32}$$

By virtue of (4.30), (4.31) and (4.32), we have

$$\begin{aligned}
g(z, \bar{z}) &= \tau_{1c}\bar{D}(1, \bar{\chi}^*)\begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \end{pmatrix} \\
&= \tau_{1c}\bar{D}[a_{13} + a_{14} + a_{15} + \bar{\chi}^*(a_{24} + a_{25} + a_{26} + a_{27})][z^2 + 2z\bar{z} + \bar{z}^2] \\
&+ \tau_{1c}\bar{D}a_{13}[W_{20}^{(2)}(-\tau_{1c}) + 2W_{11}^{(2)}(-\tau_{1c})] + a_{15}[W_{20}^{(2)}(0) + 2W_{11}^{(2)}(0)]z^2\bar{z} \\
&+ \tau_{1c}\bar{D}a_{14}[W_{11}^{(1)}(-\tau_{1c}) + W_{11}^{(2)}(0) + \frac{W_{20}^{(1)}(-\tau_{1c}) + W_{20}^{(2)}(0)}{2}]z^2\bar{z} \\
&+ \tau_{1c}\bar{D}\bar{\chi}^*[a_{24}(W_{20}^{(1)}(-\hat{\tau}_2) + 2W_{11}^{(1)}(-\hat{\tau}_2)) + a_{25}(W_{20}^{(2)}(0) + 2W_{11}^{(2)}(0))]z^2\bar{z} \\
&+ \tau_{1c}\bar{D}\bar{\chi}^*a_{26}[W_{20}^{(2)}(-\hat{\tau}_2) + 2W_{11}^{(2)}(-\hat{\tau}_2)]z^2\bar{z} \\
&+ \tau_{1c}\bar{D}\bar{\chi}^*a_{27}[W_{11}^{(1)}(-\hat{\tau}_2) + W_{11}^{(2)}(-\hat{\tau}_2) + \frac{W_{20}^{(1)}(-\hat{\tau}_2) + W_{20}^{(2)}(-\hat{\tau}_2)}{2}]z^2\bar{z} + \cdots,
\end{aligned} \tag{4.33}$$

where

$$\hat{F}_1 = a_{13}u_1^2(-\tau_{1c}) + a_{14}u_1(\tau_{1c})u_2(0) + a_{15}u_2^2(0),$$
$$\hat{F}_2 = a_{24}u_1^2(-\hat{\tau}_2) + a_{25}u_1^2(0) + a_{26}u_2^2(-\hat{\tau}_2) + a_{27}u_1(-\hat{\tau}_2)u_2(-\hat{\tau}_2).$$

According to comparing (4.31) with (4.33), we can obtain

$$g_{20} = 2\tau_{1c}\bar{D}[a_{13} + a_{14} + a_{15} + \bar{\chi}^*(a_{24} + a_{25} + a_{26} + a_{27})],$$
$$g_{11} = \tau_{1c}\bar{D}[a_{13} + a_{14} + a_{15} + \bar{\chi}^*(a_{24} + a_{25} + a_{26} + a_{27})],$$
$$g_{02} = 2\tau_{1c}\bar{D}[a_{13} + a_{14} + a_{15} + \bar{\chi}^*(a_{24} + a_{25} + a_{26} + a_{27})],$$
$$g_{21} = \tau_{1c}\bar{D}a_{13}[W_{20}^{(2)}(-\tau_{1c}) + 2W_{11}^{(2)}(-\tau_{1c})] + a_{15}[W_{20}^{(2)}(0) + 2W_{11}^{(2)}(0)]$$
$$+ \tau_{1c}\bar{D}a_{14}[W_{11}^{(1)}(-\tau_{1c}) + W_{11}^{(2)}(0) + \frac{W_{20}^{(1)}(-\tau_{1c}) + W_{20}^{(2)}(0)}{2}]$$
$$+ \tau_{1c}\bar{D}\bar{\chi}^*[a_{24}(W_{20}^{(1)}(-\hat{\tau}_2) + 2W_{11}^{(1)}(0\hat{\tau}_2)) + a_{25}(W_{20}^{(2)}(0) + 2W_{11}^{(2)}(0))]$$
$$+ \tau_{1c}\bar{D}\bar{\chi}^*a_{26}[W_{20}^{(2)}(-\hat{\tau}_2) + 2W_{11}^{(2)}(-\hat{\tau}_2)]$$
$$+ \tau_{1c}\bar{D}\bar{\chi}^*a_{27}[W_{11}^{(1)}(-\hat{\tau}_2) + W_{11}^{(2)}(-\hat{\tau}_2) + \frac{W_{20}^{(1)(-\hat{\tau}_2)} + W_{20}^{(2)}(-\hat{\tau}_2)}{2}] + \cdots.$$

By virtue of (4.30) and (4.32), we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} = \begin{cases} AW - 2\mathrm{Re}\left\{\bar{q}^*(0)F_0q(\theta)\right\}, & \theta \in [-1, 0) \\ AW - 2\mathrm{Re}\left\{\bar{q}^*(0)F_0q(0)\right\}, & \theta = 0 \end{cases} \tag{4.34}$$
$$\triangleq AW + H(z, \bar{z}, \theta).$$

Here,

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots. \tag{4.35}$$

By comparing coefficient and computing, we can obtain

$$(A - 2i\alpha_{1c}\tau_{1c})W_{20}(\theta) = -H_{20}(\theta), \quad AW_{11}(\theta) = -H_{11}(\theta), \cdots. \tag{4.36}$$

We know that

$$H(z, \bar{z}, \theta) = -g(z, \bar{z})q(\theta) - \bar{g}(z, \bar{z})\bar{q}(\theta), \quad \theta \in [-\tilde{\tau}, 0). \tag{4.37}$$

Based on (4.35) and (4.37), it derives that

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \tag{4.38}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \tag{4.39}$$

Based on (4.36) and (4.38), we can obtain

$$\dot{W}_{20}(\theta) = 2i\alpha_{1c}\tau_{1c}W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta).$$

Based on $q(\theta) = (1, \chi)^T e^{i\alpha_{1c}\tau_{1c}\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\alpha_{1c}\tau_{1c}}q(0)e^{i\alpha_{1c}\tau_{1c}\theta} + \frac{i\bar{g}_{20}}{3\alpha_{1c}\tau_{1c}}\bar{q}(0)e^{-i\alpha_{1c}\tau_{1c}\theta} + E_1e^{2i\alpha_{1c}\tau_{1c}\theta},$$

where $E_1 = (E_1^1, E_1^2)$ is a constant vector.

Similarly, based on (4.36) and (4.39), we have

$$W_{11}(\theta) = -\frac{ig_{11}}{\alpha_{1c}\tau_{1c}}q(0)e^{i\alpha_{1c}\tau_{1c}\theta} + \frac{i\bar{g}_{11}}{\alpha_{1c}\tau_{1c}}\bar{q}(0)e^{-i\alpha_{1c}\tau_{1c}\theta} + E_2,$$

here, $E_2 = (E_2^1, E_2^2)$ is a constant vector.

By above definitions and conditions, we have

$$E_1^1 = \frac{1}{G_1} \begin{vmatrix} a_{13}e^{-2i\alpha_{1c}\tau_{1c}} + a_{14}e^{-i\alpha_{1c}\tau_{1c}} + a_{15} & -a_{12} \\ (a_{24} + a_{26} + a_{27})e^{-2i\alpha_{1c}\hat{\tau}_2} + a_{25} & 2i\alpha_{1c} - a_{21} - a_{22} \end{vmatrix},$$

$$E_1^2 = \frac{1}{G_1} \begin{vmatrix} 2i\alpha_{1c} & a_{13}e^{-2i\alpha_{1c}\tau_{1c}} + a_{14}e^{-i\alpha_{1c}\tau_{1c}} + a_{15} \\ -a_{23} & (a_{24} + a_{26} + a_{27})e^{-2i\alpha_{1c}\hat{\tau}_2} + a_{25} \end{vmatrix},$$

$$E_2^1 = \frac{1}{G_2} \begin{vmatrix} 2a_{13}\cos(2\alpha_{1c}\tau_{1c}) + 2a_{14}\cos(\alpha_{1c}\tau_{1c}) + 2a_{15} & a_{12} \\ 2(a_{24} + a_{26} + a_{27})\cos(2\alpha_{1c}\hat{\tau}_2) + 2a_{25} & a_{21} + a_{22} \end{vmatrix},$$

$$E_2^2 = \frac{1}{G_2} \begin{vmatrix} a_{11} & 2a_{13}\cos(2\alpha_{1c}\tau_{1c}) + 2a_{14}\cos(\alpha_{1c}\tau_{1c}) + 2a_{15} \\ a_{23} & 2(a_{24} + a_{26} + a_{27})\cos(2\alpha_{1c}\hat{\tau}_2) + 2a_{25} \end{vmatrix},$$

where $G_1 = 2i\alpha_{1c}(2i\alpha_{1c} - a_{21} - a_{22}) - a_{12}a_{23}$, $G_2 = a_{11}(a_{21} + a_{22}) - a_{12}a_{23}$.

By above analyses and the results of Kuang [54], the following results can be given:

$$\begin{cases} c_1(0) = \dfrac{i}{2\alpha_{1c}\tau_{1c}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \dfrac{|g_{02}|^2}{3}\right) + \dfrac{g_{21}}{2}, \\ \mu_2 = -\dfrac{\mathrm{Re}c_1(0)}{\mathrm{Re}\lambda'(\tau_{1c})}, \\ \varepsilon_2 = 2\mathrm{Re}c_1(0), \\ T_2 = -\dfrac{\mathrm{Im}c_1(0) + \varepsilon_2\mathrm{Im}\lambda'(\tau_{1c})}{\alpha_{1c}\tau_{1c}}. \end{cases} \qquad (4.40)$$

By computing Eq (4.40), we have the following theorem.

Theorem 4.9. *In Eq (4.40), the following results are true.*
(i) The sign of μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0(\mu_2 < 0)$, then Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $\tau > \tau_{1c}(\tau < \tau_{1c})$.
(ii) The sign of ε_2 determines the stability of the bifurcating periodic solution: the bifurcating periodic solutions are stable (unstable) if $\varepsilon_2 < 0(\varepsilon_2 > 0)$.
(iii) The sign of T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0(T_2 < 0)$.

5. Numerical simulation

In this section, some numerical results of system (2.6) are presented for supporting the analytic results obtained above. We choose the values of some parameters of system (2.6) are: $a = 2, b = 1.02, c = 1.89, m_1 = 0.3, \alpha = 0.3, \beta = 0.5, m_2 = 1, p = 15.2, q = 0.05, r = 1.528$.

When economic interest $m = 0$, if one of the assumptions (H_1) and (H_2) holds, then Eq (3.1) has a positive root $x_0 = 1.04$. Thus, an interior equilibrium $(1.04, 0.12, 0.59)$ can be obtained. When economic interest $m > 0$, we can obtain $0 < m < 2.61$ from Ω_1. We choose $m = 0.9$, which is enough to support the theoretical analysis. If one of the assumptions (H_4) and (H_5) holds, then one positive root is $x^* = 0.74$ by solving Eq (3.2). When $b = 1.02 < 1 + \frac{pq\alpha^2\beta r}{mm_2} = 1.16$, the interior equilibrium is $(0.74, 0.83, 2.75)$. Next, we will give dynamical responses of system (2.6) around the interior equilibrium.
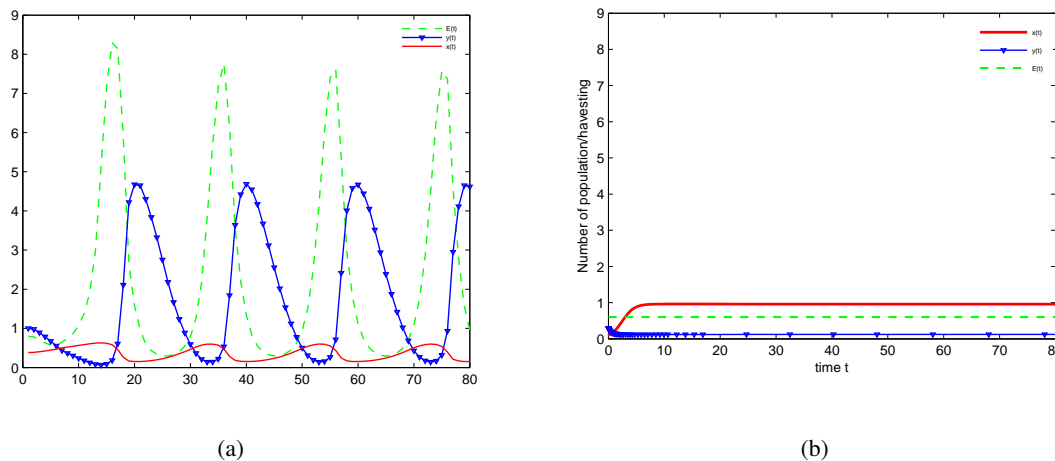
Figure 1. Dynamical responses around $(1.04, 0.12, 0.59)$. (a) system (4.1), (b) system (4.2).

When $\tau_1 = \tau_2 = 0$, according to Theorem 4.2, singularity induced bifurcation occur around $(1.04, 0.12, 0.59)$ as $m = 0$ (see Figure 1), and $(0.74, 0.83, 2.75)$ as $m = 0.9$ (see Figure 2). According to Theorem 4.3, we obtain that feedback gain satisfies $\hat{k} > 0.02$. Thus state feedback controller $u(t) = 10(E(t) - 0.59)$ is designed to stabilize system (4.1) around $(1.04, 0.12, 0.59)$ when $m = 0$. Dynamical responses are given in Figure 1b. According to Theorem 4.4, state feedback controller $\tilde{u}(t) = 3(E(t) - 2.75)$ is designed to stabilize system (4.1) around $(0.74, 0.83, 2.75)$ as $m = 0.9$. Dynamical responses are given in Figure 2b.

Based on Theorem 4.5, by computing $\xi_4^2 - (\xi_2 + \xi_5)^2 = -1.67 < 0$, we have that system (2.6) is locally asymptotically stable around $(0.74, 0.83, 2.75)$ when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_2 = \{(\tau_1, \tau_2, m) | 0 < \tau_1 < \tau_{10} = 3.6, \tau_2 = 0, 0 < m < 2.61\}$. However, system (2.6) undergoes Hopf bifurcation around $(0.74, 0.83, 2.75)$ when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_3 = \{(\tau_1, \tau_2, m) | \tau_1 = 3.8, \tau_2 = 0, 0 < m < 0.99\}$ (see Figures 3 and 4).

Similarly, system (2.6) is locally asymptotically stable around $(0.74, 0.83, 2.75)$ when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_4 = \{(\tau_1, \tau_2, m) | \tau_1 = 0, 0 < \tau_2 < \tau_{20} = 1.65, 0 < m < 2.61\}$. However, system (2.6) undergoes Hopf bifurcation around $(0.74, 0.83, 2.75)$ when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_5 = \{(\tau_1, \tau_2, m) | \tau_1 = 0, \tau_2 = 1.8, 0 < m < 1.32\}$ (see Figure 5).
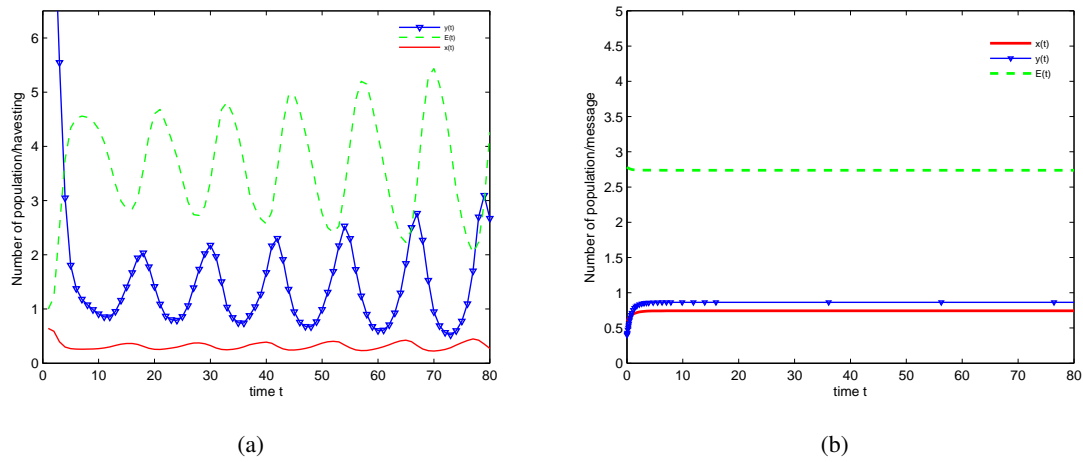
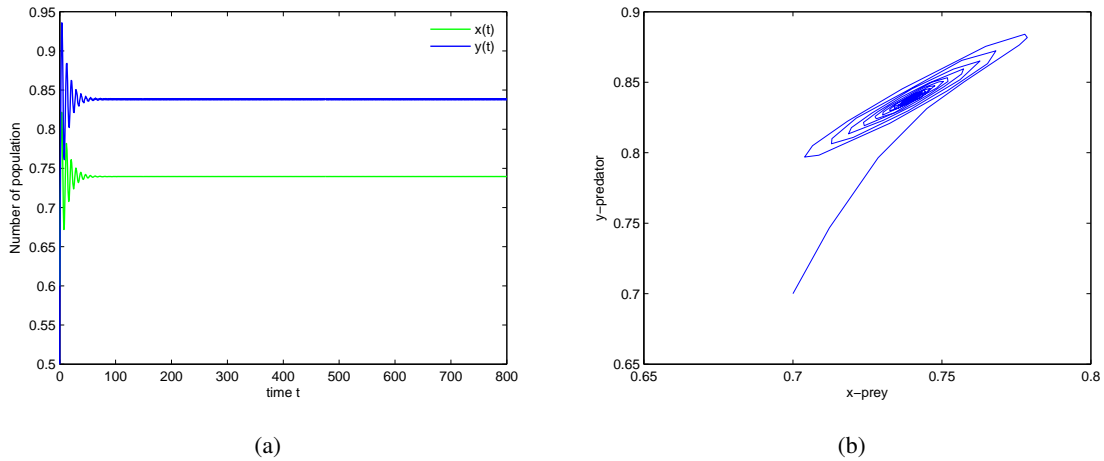**Figure 2.** Dynamical responses around $(0.74, 0.83, 2.75)$: (a) system (4.1),(b) system (4.3).

Figure 3. System (2.6) is locally asymptotically stable when $\tau_1 = 3 < \tau_{10}, \tau_2 = 0$, and $m = 0.9$. (a) dynamical responses of system (2.6), (b) phase diagram of system (2.6).
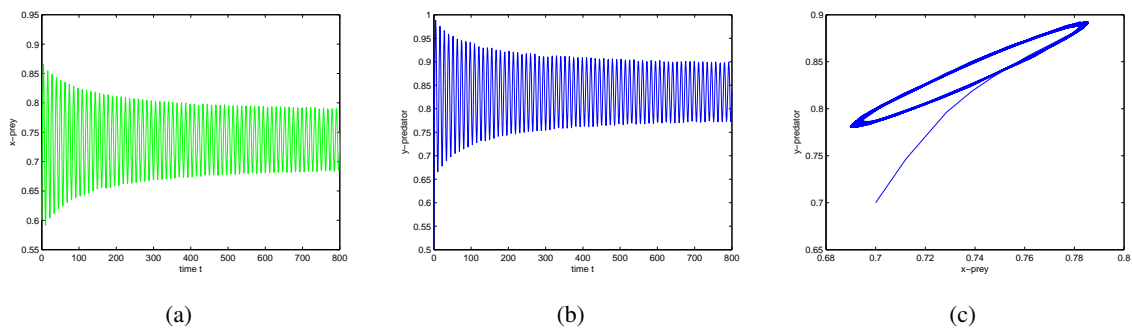

Figure 4. Hopf bifurcation occurs around $(0.74, 0.83, 2.75)$ when $\tau_1 = 3.8 > \tau_{10}, \tau_2 = 0$, and $m = 0.9$. (a) dynamical responses of the prey; (b) dynamical responses of the predator; (c) phase diagram.

(a) (b)
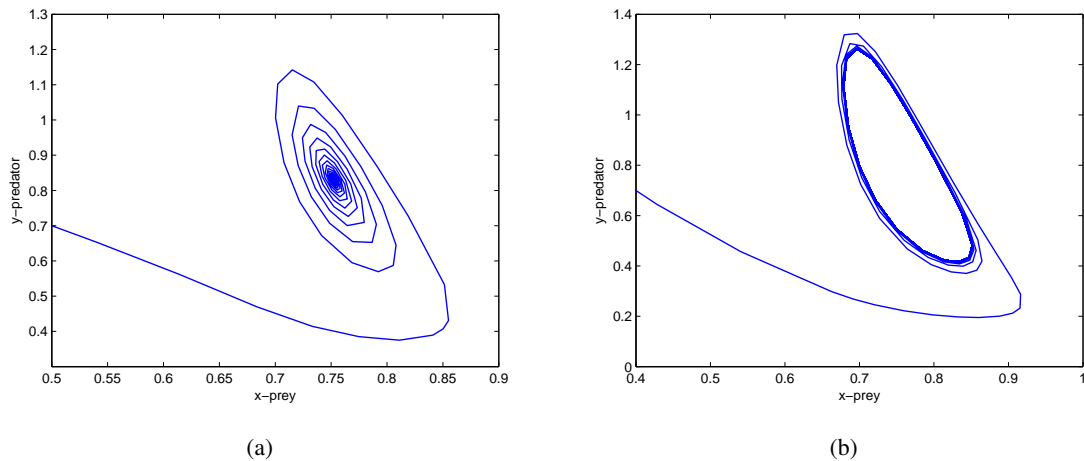
Figure 5. Dynamical responses of system (2.6). (a) system (2.6) is locally asymptotically stable when $\tau_2 = 1.6 < \tau_{20}$, (b) Hopf bifurcation occurs around $(0.74, 0.83, 2.75)$ when $\tau_2 = 1.8 > \tau_{20}$.

When we consider τ_1 as a parameter and $\tau_2 = 2.3 \in (0, 2.59)$ as a fixed value based on Theorem 4.7, we know that Eq (4.15) has finite positive roots, and obtain the critical value of delay is $\tau_{1c} = 2.12$. Therefore, system (2.6) is locally asymptotically stable around $(0.74, 0.83, 2.75)$ when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_6 \cap \Omega_8 = \{(\tau_1, \tau_2, m)|0 < \tau_1 < \tau_{1c} = 2.12, \tau_2 = \hat{\tau}_2 = 2.3, 0 < m < 2.61\}$. By simple computations, $B_{11}B_{13} + B_{12}B_{14} = 32.23 > 0$. When $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_6 \cap \Omega_9 = \{(\tau_1, \tau_2, m)|\tau_1 > \tau_{1c} = 2.12, \tau_2 = \hat{\tau}_2 = 2.3, 0 < m < 2.61\}$, system (2.6) undergoes Hopf bifurcation around $(0.74, 0.83, 2.75)$. Of course, from Figure 6, we can see that system (2.6) is locally asymptotically stable when $\tau_1 = 2 < \tau_{1c}, \tau_2 = 2.3, m = 0.9$. Hopf bifurcation occurs around $(0.74, 0.83, 2.75)$ of system (2.6) when $\tau_1 = 2.2 > \tau_{1c}, \tau_2 = 2.3, m = 0.9$, whose dynamical responses are plotted in Figure 7.
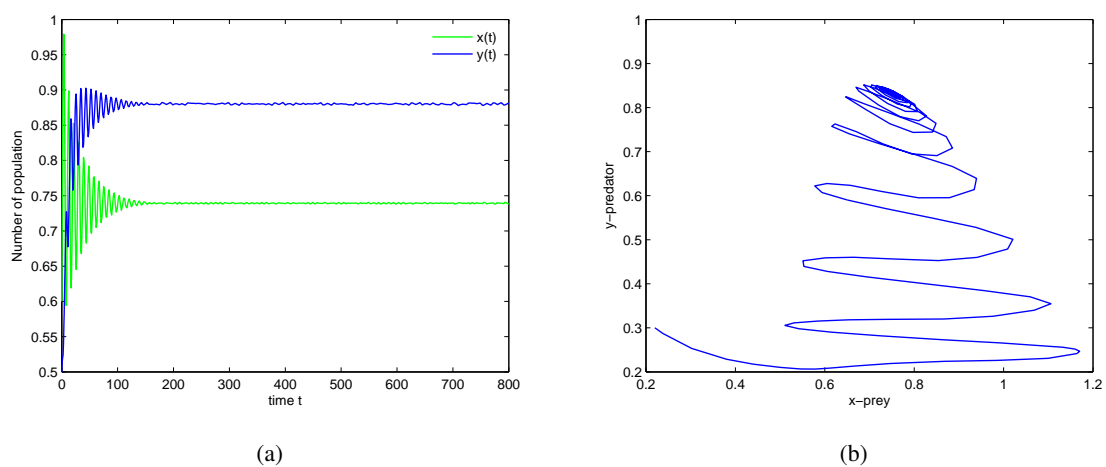

(a) (b)

Figure 6. System (2.6) is locally asymptotically stable when $\tau_1 = 2 < \tau_{1c}, \tau_2 = 2.3, m = 0.9$. (a) dynamical responses of system (2.6), (b) phase diagram of system (2.6).
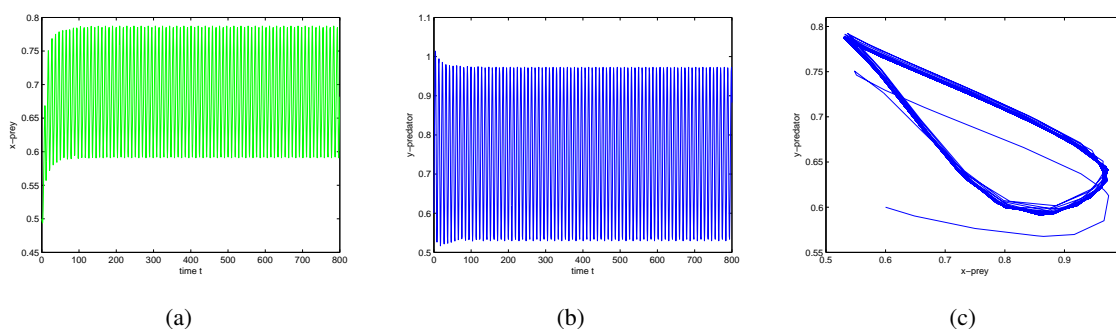
Figure 7. System (2.6) undergoes Hopf bifurcation around $(0.74, 0.83, 2.75)$ when $\tau_1 = 2.2 > \tau_{1c}, \tau_2 = 2.3$, and $m = 0.9$. (a) dynamical responses of the prey; (b) dynamical responses of the predator; (c) phase diagram.
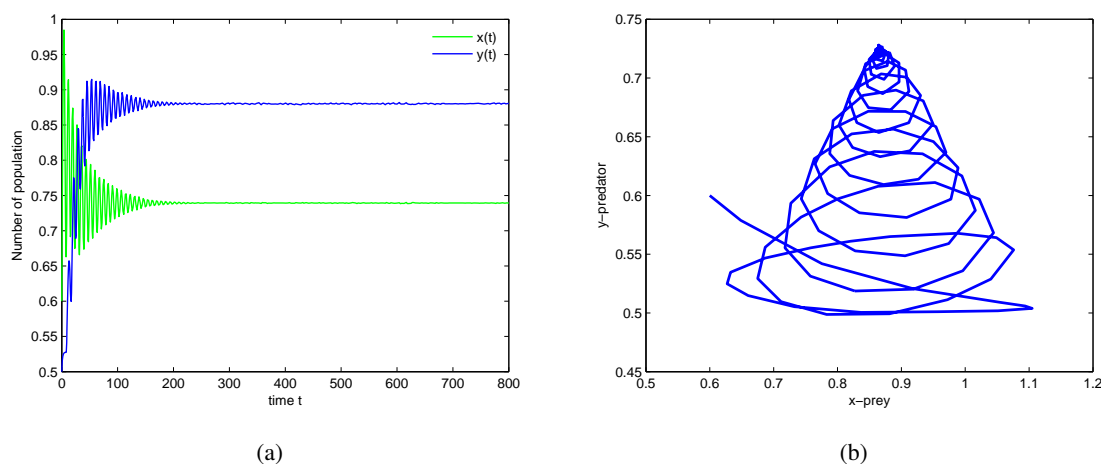

Figure 8. System (2.6) is locally asymptotically stable when $\tau_1 = 2, \tau_2 = 6 < \tau_{2c}$, and $m = 0.9$. (a) dynamical responses of system (2.6), (b) phase diagram of system (2.6).

Similarly, we regard τ_2 as a parameter and $\tau_1 = 2 \in (0, 6.10)$ based on Theorem 4.8. Furthermore, we know that Eq (4.20) has two positive roots, and obtain that the critical value of delay is $\tau_{2c} = 7$ by the given values. Thus, system (2.6) is locally asymptotically stable around $(0.74, 0.83, 2.75)$ when $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_{10} \cap \Omega_{12} = \{(\tau_1, \tau_2, m) | \tau_1 = \hat{\tau}_1 = 2, 0 < \tau_2 < \tau_{2c} = 7, 0 < m < 2.61\}$. By simple computations, we obtain that $B_{21}B_{23} + B_{22}B_{24} = 2.7796 > 0$. When $(\tau_1, \tau_2, m) \in \Omega_1 \cap \Omega_{10} \cap \Omega_{13} = \{(\tau_1, \tau_2, m) | \tau_1 = \hat{\tau}_1 = 2, \tau_2 = 7.5 > \tau_{2c} = 7, 0 < m < 2.61\}$, system (2.6) undergoes Hopf bifurcation. Of course, from Figure 8, system (2.6) with $\tau_2 = 6 < \tau_{2c}, \tau_1 = 2, m = 0.9$ is locally asymptotically stable around $(0.74, 0.83, 2.75)$. System (2.6) with $\tau_1 = 2, \tau_2 = 7.5 > \tau_{2c}, m = 0.9$ undergoes Hopf bifurcation, whose dynamical responses are plotted in Figure 9.

Based on Theorem 4.9, we can obtain $\mu_2 = -0.05 < 0, \varepsilon_2 = 0.48 > 0$ and $T_2 = -0.89 < 0$. Thus, Hopf bifurcation is subcritical and the bifurcating periodic solutions exist for $\tau_1 < \tau_{1c}$, bifurcating periodic solutions are unstable and the period decreases because of $\varepsilon_2 > 0, T_2 < 0$.
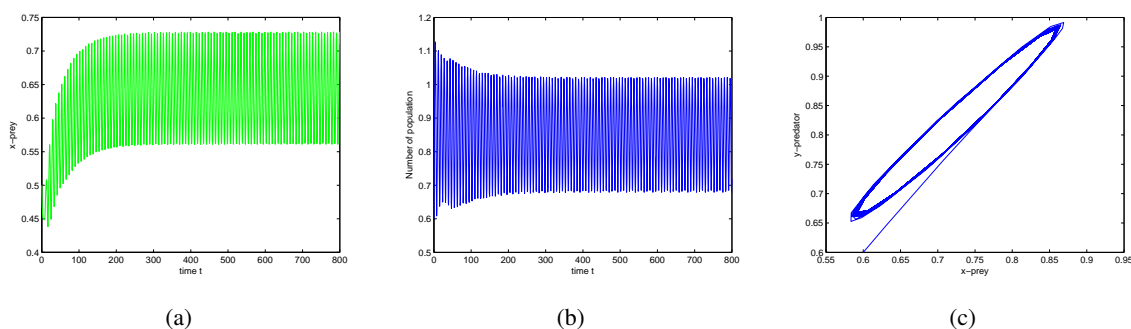
Figure 9. System (2.6) undergoes Hopf bifurcation around $(0.74, 0.83, 2.75)$ when $\tau_1 = 2, \tau_2 = 7.5 > \tau_{2c}$, and $m = 0.9$. (a) dynamical responses of the prey; (b) dynamical responses of the predator; (c) phase diagram.

6. Conclusions

It is well known that commercial harvesting and economic benefits have a strong impact on the dynamical behavior. Liu et al. [44] investigated a differential-algebraic prey-predator system with linear harvesting on predator and Holling-II. Li et al. [45] analyzed a differential-algebraic prey-predator system without time delay. However, nonlinear harvesting and Beddington-DeAngelis functional response more realistic. Therefore, this paper proposed a singular Beddington-DeAngelis predator-prey model with two delays and nonlinear predator harvesting by extending the work of references [44] and [45], and obtained some results. The existence of equilibria was analyzed. Without considering time delay, the existence of singularity induced bifurcation by regarding economic interest as bifurcation parameter was discussed. In order to remove singularity induced bifurcation and stabilize system (4.1), state feedback controllers $u(t) = 10(E(t) - 0.59)$ (see Figure 1b) and $\tilde{u}(t) = 3(E(t) - 2.75)$ (see Figure 2b) were designed, which shows that the system can be kept in a stable state with benefits by capturing predators. While considering time delay, stability of system were discussed by analyzing the corresponding characteristic transcendental equation. When $\tau_2 = 0$, the critical value of time delay is $\tau_{10} = 3.6$; when $\tau_1 = 0$, the critical value of time delay is $\tau_{20} = 1.65$; when τ_1 is regarded as a parameter and $\tau_2 = 2.3$ as a fixed value, the critical value of time delay is $\tau_{1c} = 2.12$. At the same time, when τ_2 is regarded as a parameter and $\tau_1 = 2$ as a fixed value, the critical value of time delay is $\tau_{2c} = 7$. It was obvious that system lost local stability around it's corresponding interior equilibrium when time delays crossed corresponding the critical values, and Hopf bifurcations occurred (see Figures 4, 5b, 7, and 9). Finally, by using normal form theory and center manifold theorem, Hopf bifurcation is subcritical and the bifurcating periodic solutions exist for $\tau_1 < \tau_{1c}$, bifurcating periodic solutions are unstable and the period decreases because of $\varepsilon_2 > 0, T_2 < 0$, which could be found in Theorem 4.9.

In fact, the prey and predator may be captured simultaneously in real world. Thus, in order to make this model more practical, nonlinear predator harvesting and nonlinear prey harvesting can be

introduced into our predator-prey system, which is

$$
\begin{cases}
\dot{x}(t) = \alpha x(t - \tau_1)\left(1 - \dfrac{x(t - \tau_1)}{k}\right) - \dfrac{x(t - \tau_1)y(t)}{1 + ax(t - \tau_1) + by(t)} - \dfrac{q_1 E(t)x(t)}{m_1 E(t) + m_2 x(t)}, \\[3mm]
\dot{y}(t) = \beta y(t - \tau_2)\left(1 - \dfrac{y(t - \tau_2)}{x(t - \tau_2)}\right) - \dfrac{q_2 E(t)y(t)}{m_3 E(t) + m_4 y(t)}, \\[3mm]
\quad 0 = \dfrac{q_1 E(t)}{m_1 E(t) + m_2 y(t)}(px(t) - c) + \dfrac{q_2 E(t)}{m_3 E(t) + m_4 y(t)}(p_1 y(t) - c_1) - m.
\end{cases}
$$

We leave this work in the future.

Acknowledgments

Conflict of interest

All authors declare no conflicts of interest in this paper.

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