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Research article

Sliding mode of compulsory treatment in infectious disease controlling

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Abstract: Preventing the infectious disease from breakout and maintaining public health have always been placed at the first place when making public healthy policy. When the epidemic trend of infectious disease arises, compulsory treatment is an efficient pattern to control the rapid spreading. A sliding mode is carried out to evaluate the effect of compulsory treatment in the infectious disease controlling. When the number of infected persons reach a certain level I_c, the policy of compulsory treatment will be carried out at rate f. We analyze the influence of the compulsory treatment rate f and threshold value I_c to commence the control. Finally we investigate the theorems and the existence of the optimality combination.

Keywords: sliding mode; compulsory treatment; infectious disease; ratio dependent transmission; final infected size

1. Introduction

Despite the development of medical technology, infectious diseases still threaten the health of human and economics of our society. The outbreaks of SARS in 2003 and the H1N1 influenzas pandemic in 2009 force people to rethink the danger of infectious disease to public health. It has become a reminder of keeping vigilant to the threats of infectious diseases [1]. The UK's Health Protection Agency publicized a special case in 2005, a man in his forties refused treatment of his tuberculosis and went on infecting at least 12 others. That incident had aroused scholars' new cogitation in the public health policy [2]. Not alone, the emerging of influenza A(H1N1) in the United States and Mexico which causes more than 2% death rate in Mexico also led to cross infection since the patients had not accepted treatment [3]. Although the disease makes their bodies face more threaten even death, to the patients, they hold their own opinions about the treatment for the unbearable treatment procedure, side effects of the drugs or subjective abandonment. To the public, each untreated patient acts as a source of the infection and healthy people expose to the danger of

being infected. Following the improper activities of the patients, the infectious disease expands and becomes popular. The most horrible thing is that the variation of the virus during transmission will make the disease out of control. So the patients should take the treatment responsibly and politely till he or she is not contagious any longer.

Basing on those facts, some hidden problems in the field of medical ethics have aroused scholars' attention. During the controlling of the infectious disease, the contradiction between individual privilege and group health and the conflict between personal privacy and information public become the focus of controversy. Some discussion about compulsory treatment and patient responsibility were carried out [4–6]. Among those discussion, R.M. Anderson and R.M. May talked about the control of infectious diseases [7], O. O'Neill studied the confliction between the informed consent and public health [8], and G. Richard and T. Cassam even investigated the role of the law in controlling the infectious disease [9]. N. Kondo inspected the case of Japan and analyzed the effect of the public strategy [10]. Those opinions only discussed the problems in moral philosophy, however people concern more about practical measures to control the infectious disease. In the state of emergency, some coercive means may be carried out considering most people's safety. In this paper we focus on the effect of compulsory treatment and its optical policy.

From 1970's to the end of last century, the number of study of complex dynamic systems increased in the ascendant [11]. Among those achievement, the research of sliding mode is an outstanding accomplishment. Vadim I.Utkin dedicated a lot to that field and issued his opinion. In [12], he stated the basic theory and corollaries of sliding mode and in [13] he made further efforts to the control and optimization. Furthermore, Edwards C. and Spurgeon S.K. also dedicated to the theory and applications of sliding mode in his work published in 1998 [14]. With sliding mode, some scholars got a lots of good results in the control field. W. J. Chang and F. L. Hsu investigated the multiple performance constrained sliding mode fuzzy control problem with multiplicative noises [15]. M. Cui, W. Liu, H. Liu et al. designed an adaptive sliding mode to control the trajectory tracking of the differential-driving mobile robots [16]. Some further application of sliding mode as well as discrete-time sliding mode also got a lot of development [17–22].

Meanwhile, the control of infectious disease has always been a central issue of public health, and some mathematical techniques have been applied to the draft of public health policy [23–28]. Yanni Xiao discussed the control of breakout with sliding mode considering the sufficient awareness of the infectious disease can help cut down the transition rate [24]. Inspired by that work, we carry out a simple SI model to evaluate the effect of compulsory treatment, and the compulsory treatment acts as a sliding item in the model. To prevent the great damage of the strong explosion to human, compulsory treatment is implemented to prevent most people from enrolling in the infectious disease. Here we include the demographic process to explore the longer-term persistence and assume Λ as the constant recruitment rate and d as the natural death rate of the population. Let S and I be the susceptible and infected population separately and the model equations are

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \beta\dfrac{SI}{S+I} - dS + (\gamma + \varepsilon f)I, \\ \dfrac{dI(t)}{dt} = \beta\dfrac{SI}{S+I} - dI - \alpha I - (\gamma + \varepsilon f)I, \end{cases} \tag{1.1}$$

with

$$\varepsilon = \begin{cases} 0, & \sigma(I) = I - I_c < 0, \\ 1, & \sigma(I) = I - I_c > 0. \end{cases} \tag{1.2}$$

In the model β represents the transmission coefficient, α is the extra death rate caused by the disease, γ describes the recovery rate from infection, and f is the recovery rate of compulsory treatment. All the parameters in the model are positive. Here we suppose the transmission rate of the disease depends on the ratio of infection over the population size and the compulsory treatment is effective to all the infectious with the disease. Model (1.1) with (1.2) describe a control policy which is referred to as an on-off control or as a special and simple case of variable structure control literature. In the control policy described by model (1.1) with (1.2) the infectious disease spreads freely in the initial stage, and the amount of infectious reaching the threshold value I_c will trigger the compulsory treatment at the rate of f (the amount of infectious taking compulsory treatment over the amount of total infectious is f). The compulsory treatment separates the dynamic change of the disease into two parts by the threshold value I_c. In each side of the threshold value I_c the model expresses in the style with compulsory control or not. The compulsory treatment which is required by the public health policy can not commence together with the appearance of the infectious disease, while it can only be carried out some time later. This circumstance can not be depict with a common ODE model, so the sliding model is set up here to improve the accuracy of the research.

This paper discusses the dynamic structures of the infectious disease with compulsory treatment implemented at threshold value I_c and considers the influence of the value I_c to the explosion of the disease. The paper is organized in the following. Section 2 discusses the preliminaries of the basic model and section 3 considers the sliding domain relaying on the compulsory treatment rate f and threshold value I_c. The most important part of the paper, i.e. the global behavior of the sliding mode is presented in section 4 and we draw our conclusion and discuss the result in section 5.

2. Preliminaries to the model

It is obvious that the solutions of model (1.1) are ultimately uniform bounded by $\frac{\Lambda}{d}$, then

$$D = \left\{ (S, I) \in R_+^2 \middle| S(t) + I(t) \leq \frac{\Lambda}{d} \right\} \tag{2.1}$$

is the attraction region for system (1.1). In the initial stage of the disease the dynamic structure follows the model without control ($\varepsilon = 0$) and the model (1.1) with (1.2) can be simplified into

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \beta \dfrac{SI}{S+I} - dS + \gamma I, \\ \dfrac{dI(t)}{dt} = \beta \dfrac{SI}{S+I} - dI - \alpha I - \gamma I. \end{cases} \tag{2.2}$$

While the amount of infectious over the threshold value I_c the implement of compulsory treatment($\varepsilon = 1$) makes the model changed into

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \beta \dfrac{SI}{S+I} - dS + (\gamma + f)I, \\ \dfrac{dI(t)}{dt} = \beta \dfrac{SI}{S+I} - dI - \alpha I - (\gamma + f)I. \end{cases} \tag{2.3}$$

We are quite familiar with model (2.2) and model (2.3) for their popular in the research of infectious disease. However the dynamic of trajectories in the transition area when $I = I_c$ and where the trajectories heads for when they hit the transition area are still mystics. For convenience we define the hyperplane

$$\Sigma_0 = \left\{(S, I) \in R_+^2 \middle| \sigma(I) = 0\right\}$$

which divides R_+^2 into two regions

$$\Sigma_1 = \left\{(S, I) \in R_+^2 \middle| \sigma(I) < 0\right\} \text{ and } \Sigma_2 = \left\{(S, I) \in R_+^2 \middle| \sigma(I) > 0\right\}.$$

Then Σ_0 is a discontinuous surface between two different structures of the system, and model (2.2) is valid in Σ_1 and Σ_2 is the region for model (2.3).

First we consider the trajectories of (2.2) and (2.3) separately.

In the initial stage the disease spreads freely without control, and the case is described by model (2.2). It is easy to obtain the disease free equilibrium $E_0\left(\frac{\Lambda}{d}, 0\right)$ and endemic equilibrium $E_1(S_1, I_1)$, where $S_1 = \frac{\Lambda(d+\alpha+\gamma)}{\beta(d+\alpha)-\alpha(d+\alpha+\gamma)}$ and $I_1 = \Lambda\frac{\beta-(d+\alpha+\gamma)}{\beta(d+\alpha)-\alpha(d+\alpha+\gamma)}$. We can also calculate the basic reproduction number $R_{01} = \frac{\beta}{d+\alpha+\gamma}$. So E_0 is globally asymptotically stable when $R_{01} \leq 1$, which indicates that the disease will distinct naturally. While E_1 is globally asymptotically stable if $R_{01} > 1$. It follows from the Dulac Theorem that no limit cycle exists for model (2.2) in R_+^2 with function $B = \frac{1}{SI}$. Particularly, E_1 is either be a stable node if $\Delta_1 \geq 0$ or a stable focus if $\Delta_1 < 0$, where

$$\Delta_1 = \left(\frac{\beta}{R_{01}}(1 - R_{01}) + d\right)^2 - \frac{4\alpha\beta}{R_{01}^2}(1 - R_{01})^2.$$

Similarly, for the model (2.3), $E_0\left(\frac{\Lambda}{d}, 0\right)$ and $E_2(S_2, I_2)$ are disease free and endemic equilibrium respectively where $S_2 = \frac{\Lambda(d+\alpha+\gamma+f)}{\beta(d+\alpha)-\alpha(d+\alpha+\gamma+f)}$, $I_2 = \Lambda\frac{\beta-(d+\alpha+\gamma+f)}{\beta(d+\alpha)-\alpha(d+\alpha+\gamma+f)}$, and the basic reproduction number is given by $R_{02} = \frac{\beta}{d+\alpha+\gamma+f}$. We can also find that E_0 and E_2 are globally asymptotically stable when $R_{02} \leq 1$ and $R_{02} > 1$ respectively, and the limited circle does not exist in R_+^2. Particularly, E_2 is either a stable node if $\Delta_2 \geq 0$ or a stable focus if $\Delta_2 < 0$, where

$$\Delta_2 = \left(\frac{\beta}{R_{02}}(1 - R_{02}) + d\right)^2 - \frac{4\alpha\beta}{R_{02}^2}(1 - R_{02})^2.$$

Especially, when the compulsory treatment rate f satisfies $R_{01} > 1 > R_{02}$, i.e.

$$f > \beta - (d + \alpha + \gamma) = \beta(1 - \frac{1}{R_{01}}),$$

then the consistent treatment satisfying the above requirement makes the infectious disease eliminated.

Here both E_1 and E_2 located in D as well as $I_1 > I_2$ holds.

In model (1.1) with (1.2), we consider three different types of equilibria: sliding equilibrium, real equilibrium and virtual equilibrium. Those belonging to the sliding domain are called sliding equilibria. If the equilibria located in the valid area of the model they are called real equilibrium while they are named virtual ones when seated in the opposite regions. When the equilibrium becomes a virtual one the trajectories will not approach to it for the dynamics changed as soon as they cross the threshold value I_c.

3. The structure of sliding domain

Following the theorem proved by Utkin(Utkin 1992) a sliding mode comes into being while the vector fields of both structures in vicinity of Σ_0 are directed toward each other. Through simple calculating we can certificate that the sliding mode exists for model (1.1) with (1.2) and we denote the sliding domain as

$$\left\{(S,I)\in R^2\left|\frac{I_c}{\frac{\beta}{d+\alpha+\gamma}-1}\le S\le\frac{I_c}{\frac{\beta}{d+\alpha+\gamma+f}-1},I=I_c\right.\right\}. \tag{3.1}$$

We also denote the end points of the sliding domain by A and B, and let M and N be the intersection points of sliding domain with line $S(t)=0$ and line $S+I=\frac{\Lambda}{d}$. Still with Utkins method we eliminate the control item ε in the alternate surface Σ_0. In Σ_0, $I'=0$ holds, then we have

$$(\gamma+\varepsilon f)I=\beta\frac{SI}{S+I}-dI-\alpha I.$$

Substituting the above expression and $I=I_c$ into the first equation of (1.1) provides a differential equation

$$\begin{cases}\frac{dS(t)}{dt}=\Lambda-dS-(d+\alpha)I\\ I=I_c\end{cases} \tag{3.2}$$

which describes the dynamics of model (1.1) with (1.2) in Σ_0. The system (3.2) has a unique equilibrium (S^*,I_c) where $S^*=\frac{\Lambda-(d+\alpha)I_c}{d}$, and it is locally asymptotically stable. We can prove that (S^*,I_c) located in the sliding domain when

$$\frac{I_c}{\frac{\beta}{d+\alpha+\gamma}-1}<S^*<\frac{I_c}{\frac{\beta}{d+\alpha+\gamma+f}-1}$$

that is

$$I_2<I_c<I_1. \tag{3.3}$$

With that condition model (1.1) with (1.2) has a sliding equilibrium denoted by $E_S(S^*,I_c)$.

To explore the global stability of system (1.1) with (1.2), we consider the relationship of sliding domain Ω and the attraction region D. When the right end of Ω belongs to D, i.e.

$$\frac{I_c}{\frac{\beta}{d+\alpha+\gamma+f}-1}\le\frac{\Lambda}{d}-I_c\Longleftrightarrow I_c\le\frac{\Lambda}{d}\cdot\left(1-\frac{d+\alpha+\gamma+f}{\beta}\right)\triangleq I_3, \tag{3.4}$$

sliding domain Ω is included in the attraction region D. While the left end of Ω does not belong to D, i.e.

$$\frac{I_c}{\frac{\beta}{d+\alpha+\gamma}-1}\ge\frac{\Lambda}{d}-I_c\Longleftrightarrow I_c\ge\frac{\Lambda}{d}\cdot\left(1-\frac{d+\alpha+\gamma}{\beta}\right)\triangleq I_4, \tag{3.5}$$

sliding domain Ω is excluded from the attraction region D. It is obvious that the sliding domain Ω is partly in the attraction region D when $I_3<I_c<I_4$. We can also get that $I_1<I_4$ for $R_{01}>1$ and $I_2<I_3$ for $R_{02}>1$. We only investigate the case of $R_{01}>1$ and $R_{02}>1$ because the disease will die out without any control when the condition does not hold. Here condition (3.3) control the existence of sliding equilibrium, and conditions (3.4) and (3.5) describe the relationship of the sliding domain Ω and the attraction region D (see in Figure 1).

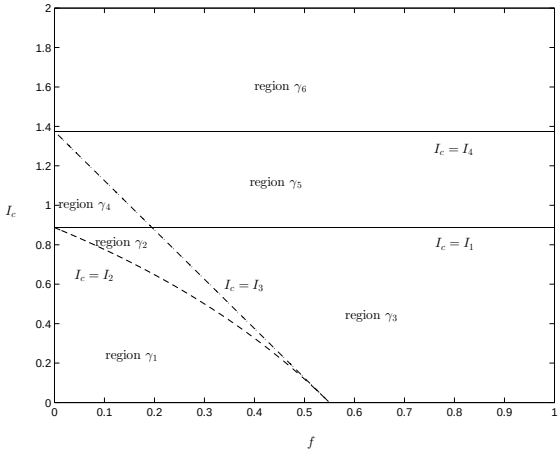


Figure 1. Bifurcation set for the model (1.1) with (1.2) with respect to the control intensity f and threshold level I_c. Let γ_1 be the domain bounded by the curve $I_c = I_2$, $I_c = 0$ and $f = 0$; γ_2 be the domain bounded by the curve $I_c = I_1$, $I_c = I_2$ and $I_c = I_3$; γ_3 be the domain bounded by the curve $I_c = I_1$, $I_c = I_3$, $I_c = 0$ and $f = 1$; γ_4 be the domain bounded by the curve $I_c = I_1$, $I_c = I_3$ and $f = 0$; γ_5 be the domain bounded by the curve $I_c = I_1$, $I_c = I_3$, $I_c = I_4$ and $f = 1$; γ_6 be the domain bounded by the curve $I_c = I - 4$, $f = 0$, $f = 1$ and $I_c = \frac{\Lambda}{d}$. Parameter values are $\Lambda = 0.5$, $d = 0.2$, $\gamma = 0.05$, $\alpha = 0.2$ and $\beta = 1$.

4. Global behavior of the sliding mode

In this section we consider the asymptotical behavior of the system (1.1) with (1.2).

Theorem 1. *The equilibrium E_2 is globally asymptotically stable if $I_c < I_2$.*

Proof. This case corresponds to area γ_1 in Figure 1. Here the two endemic equilibria for models (2.2) and (2.3) belong to the same sides of the switching surface Σ_0. The sliding domain Ω is totally in the attraction region D as shown in Figure 2(a). Here E_2 is a real equilibrium while E_1 is a virtual one and there is no sliding equilibrium in Ω. Since $R_{02} > 1$, the equilibrium E_2 is asymptotically stable in Σ_2. The trajectories initiating from the region Σ_1 tend to the virtual equilibrium E_1 before hitting the switching surface Σ_0. In the sliding domain Ω (segment$\overline{AB}$ in Figure 2(a)), we have

$$\begin{aligned}
S' &= \Lambda - dS - (d+\alpha)I_c \\
&> \Lambda - d\frac{I_c}{\frac{\beta}{d+\alpha+\gamma+f} - 1} - (d+\alpha)I_c \\
&> \Lambda - \left(\frac{d}{\frac{\beta}{d+\alpha+\gamma+f} - 1} + d + \alpha\right)\Lambda\frac{\beta - (d+\alpha+\gamma+f)}{\beta(d+\alpha) - \alpha(d+\alpha+\gamma+f)} \\
&= \Lambda - \left(\frac{d}{R_{02}-1} + d + \alpha\right)\Lambda\frac{R_{02}-1}{R_{02}(d+\alpha) - \alpha} \\
&= 0.
\end{aligned}$$

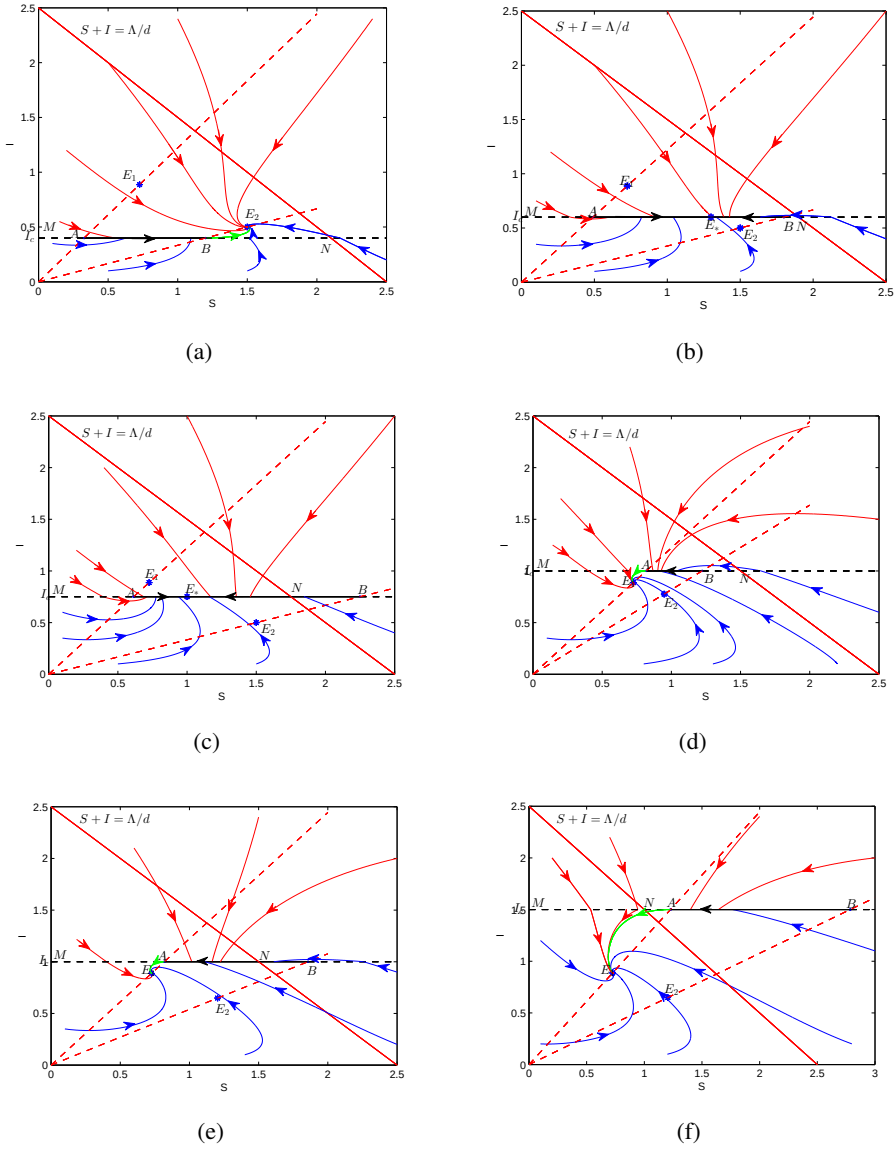


Figure 2. Phase plane S-I for model (1.1) with (1.2), showing the sliding domain $\overline{AB}$ and asymptotical equilibrium (regular equilibrium, virtual equilibrium and sliding equilibrium). Let $\Lambda = 0.5$, $d = 0.2$, $\gamma = 0.05$, $\alpha = 0.2$, $\beta = 1$, and other parameters are chosen following the conditions of area $\gamma_i, i = 1, 2, \ldots, 6$ as shown in Figure 1. (**a**) $I_c = 0.4$, $f = 0.3$ for γ_1; (**b**) $I_c = 0.6$, $f = 0.3$ for γ_2; (**c**) $I_c = 0.75$, $f = 0.3$ for γ_3; (**d**) $I_c = 1$, $f = 0.1$ for γ_4; (**e**) $I_c = 1$, $f = 0.2$ for γ_5; (**f**) $I_c = 1.5$, $f = 0.2$ for γ_6.

It indicates that the trajectories move to the right end point B along the sliding domain once they hit Ω.

We claim that the trajectory initiating from the right end point $B\left(\frac{I_c}{\frac{\beta}{d+\alpha+\gamma+f}-1}, I_c\right)$ will not hit the switching surface again. From the analyzing of control system (2.3) we have that the segment $\overline{BE_2}$ is a part of the isoclinic line along with $I' = 0$. If E_2 is a node the trajectory initiating from B tends to the stable equilibrium E_2 directly in Σ_2. Else it runs spirally to E_2 if E_2 is a focus.

Assume the right sides of model (2.3) as P and Q, define Dulac function as $D(S, I) = \frac{1}{SI}$,then

$$\frac{\partial(DP)}{\partial S} + \frac{\partial(DQ)}{\partial I} = -\frac{\Lambda}{S^2 I} - \frac{\gamma + f}{S^2} < 0.$$

It can be proved that model (2.2) has the similar result with the same Dulac function. Following the non-existence theorem of limit cycle to non-smooth system [24], we can also claim that no limit cycle surrounds the real equilibrium E_2 and the sliding domain$\overline{AB}$. So we can claim that E_2 is globally asymptotically stable. □

Theorem 2. *The sliding equilibrium E_S is globally stable if $I_2 < I_c < I_1$.*

Proof. There are two situations that the sliding domain is totally included in the attraction area D(see Figure 2(b)), i.e. area γ_2 in Figure 1 or partly included in the attraction area$D$(see Figure 2(c)), i.e.γ_3 in Figure 1 under this condition. From the discussion in section 3 we know that in this case the sliding equilibrium E_S exists and is stable in the sliding domain. The equilibrium points E_1 and E_2 are all virtual ones. If the sliding domain is totally within the attraction region D (see Figure 2(b)), we can prove that no limit cycle surrounds the sliding domain with similar method in Theorem 1.

Since E_2 is in the opposite side of the switching surface with respect to Σ_2, the trajectories initiating from Σ_2 tend to E_2 before hitting the switching surface Σ_0. Some of the trajectories will hit the sliding domain and then move to the sliding equilibrium E_S along Ω. While other trajectories will enter the region Σ_1 by crossing through the switching surface in $\overline{MA}$ or $\overline{BN}$ and then hit the sliding domain. After that they will tend to the switching surface Σ_0 following the rules of the area and run to the sliding equilibrium E_S along Ω ultimately. Similarly the trajectories initiating from Σ_1 tends to the sliding equilibrium E_S finally. As previous statement, E_S is globally asymptotically stable. □

Theorem 3. *The equilibrium E_1 is globally stable if $I_c > I_1$.*

Proof. There are three subsituations of the sliding domain is totally, partly and absolutely not included in the attraction area D. These three subsituations correspond to the area γ_4, γ_5 and γ_6 in Figure 1. In this case, E_1 is a real equilibrium and E_2 is a virtual one. In the sliding domain, we have

$$\begin{aligned} S' &= \Lambda - dS - (d+\alpha)I_c \\ &< \Lambda - d\frac{I_c}{\frac{\beta}{d+\alpha+\gamma} - 1} - (d+\alpha)I_c \\ &< \Lambda - \left(\frac{d}{\frac{\beta}{d+\alpha+\gamma} - 1} + d + \alpha\right)\Lambda\frac{\beta - (d+\alpha+\gamma)}{\beta(d+\alpha) - \alpha(d+\alpha+\gamma)} \end{aligned}$$

$$= \Lambda - \left(\frac{d}{R_{01}-1} + d + \alpha\right)\Lambda\frac{R_{01}-1}{R_{01}(d+\alpha)-\alpha}$$
$$= 0.$$

This indicates that once the trajectories hit the sliding domain they will move to the left end point A.

When $I_c < I_3$, from section 3 we have that the sliding domain is totally included in the attraction region (see Fig.2(d)). It is similar to the proof of Theorem 1 that E_1 is globally stable.

When $I_3 < I_c < I_4$, the sliding domain is partly in the attraction region D (see Figure 2(e)). All the trajectories initiating from Σ_1 will either tends to E_1 or hit the sliding domain Ω. After hitting Ω, the trajectories move to the left end point A along the sliding domain and then converge to the equilibrium E_1 instead of hitting the sliding domain again.

When $I_c > I_4$, the sliding domain is absolutely excluded from attraction region D (see Figure 2(f)). All the trajectories initiating from Σ_2 will enter Σ_1 no matter hitting the sliding domain or not for its virtual equilibrium E_2 located in Σ_1. Since E_1 is the only equilibrium in Σ_1 and no limit cycle exists, equilibrium E_1 is stable in Σ_1. Therefore E_1 is globally stable. □

From Theorem 4.2 we can see that the solutions of discontinuous model (1.1) with (1.2) converge to either endemic equilibria or the sliding equilibria. So the final size of the infected patient depends a lot on the threshold value I_c the sign of compulsory treatment. In addition, the rate of compulsory treatment also has effect on it. Those results are obvious from the proof of Theorem 4.2.

5. Conclusion and discussion

Theorem in section 4 certificates that the final size of the infectious disease can be controlled by the threshold value I_c and the compulsory treatment rate f. If the compulsory treatment is implied when the number of infectious patient is small, then the final size of the disease will be stable around the equilibria of the control model. In opposite, if the compulsory strategy is carried out a little later (i.e. after the number of infectious patient exceeds the vertical coordinate of equilibria of controlled model and before it reaches the vertical coordinate of equilibria of free model), the final size will be stable at the threshold value. And if the strategy is carried out too late (i.e. the number of infectious patient exceeds the vertical coordinate of the equilibria of the free model), the final size can not be decreased effectively but the maximum of the infected patient can be cut down and it will be postponed for some time.

From the conclusion we can see that the threshold value I_c of implementing the compulsory treatment together with the rate f both act as important roles in the control of the infectious disease. According to the aim of controlling, the proper threshold value I_c and the rate f can be decided and the public agency can make their policy following the criterion they want. However the threshold value I_c and rate f of compulsory treatment influence the final size of infected patient together, it is necessary to analyze the character of the influence.

During the controlling of the infectious disease with compulsory treatment, the final number of infected patient is the critical value that the public healthy agency cares about, and the final number is impacted by the compulsory treatment rate f and the threshold value I_c. From the curves in Figure 3, we

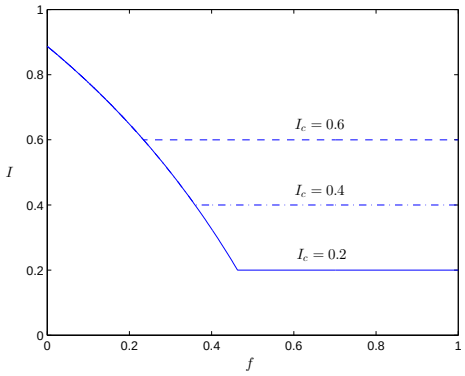
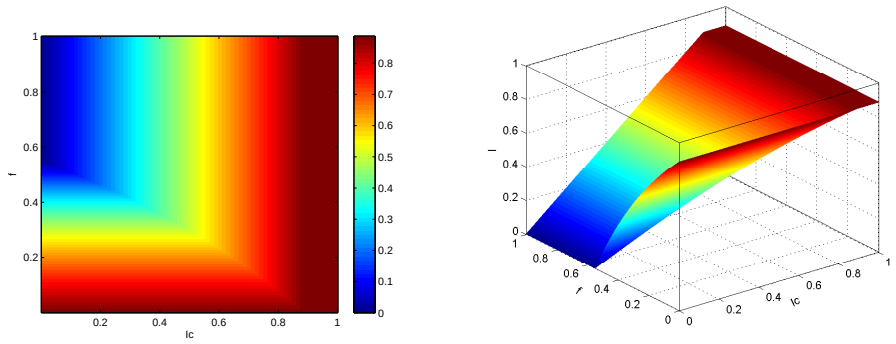


Figure 3. Change of infected patient number I with compulsory treatment ratio f in the epidemic equilibrium. Solid line shows the case of $I_c = 0.2$;dash line shows the case of $I_c = 0.4$; dash-dot line shows the case of $I_c = 0.6$. Let $\Lambda = 0.5$, $d = 0.2$, $\gamma = 0.05$, $\alpha = 0.2$, and $\beta = 1$.

can find that the final infected patient number decreases with the compulsory treatment rate f in the first stage no matter which value does the threshold value I_c take. This means that the compulsory treatment is an effective method to control the disease and it can help to cut down the transmission. However the infected patient number keeps solid when the compulsory treatment rate reaches and transcends a certain value. This phenomenon indicates that the sustained increase of compulsory treatment rate dose not work all through. So when the compulsory treatment rate reaches the critical value, the expense in raising the rate is unproductive. In Figure 3 the finale infected patient number curves with $I_c = 0.2$, $I_c = 0.4$ and $I_c = 0.6$ are represented by different kinds of lines. From these curves we can find that the critical value of f is related to the threshold value of I_c. The following content focuses on the influence of both compulsory treatment rate f and threshold value I_c to the final infected patient number.

Figure 4 shows the influence of compulsory treatment ratio f and the threshold value I_c to the number of final infected patients. We can find that when the threshold value I_c is small, the cases of final infected patients number in Figure 4 is coincident with the result showed in Figure 3, i.e. the infected patients number decreases with compulsory treatment rate f at the initial stage and then remains unchanged. However when the threshold value I_c located around 1, the final infected patients number is nearly unchanged with the compulsory treatment rate f. This circumstance implies that if the threshold value I_c to trigger the compulsory treatment is relatively big, no matter what ratio is the compulsory treatment carried out, the final infected patients number can not be reduced effectively. Analyzing from the angle of compulsory treatment ratio f, when f is relatively small, the final infected patients number stays in a high level no matter which number does the threshold value I_c takes. That case indicates that if the compulsory treatment rate is quite low, no matter when to imply the compulsory treatment, the control policy is destined to be a failure. For only small part of infected patients is forced to take the treatment and most of the infected patients are still in connections with the sensitive people, the transmission can not be cut down and the infectious disease can not be controlled. With the aim of controlling the infectious disease, the compulsory treatment rate f should not be too small while the threshold value I_c should not be too high. Otherwise the policy of compulsory treatment would be a useless measure.



(a) Projected picture of infected patient number I.

(b) Three-dimensional image of infected patient number I.

Figure 4. Influence of compulsory treatment ratio f and the threshold value I_c to the number of infected patient in the epidemic equilibrium. Let $\Lambda = 0.5$, $d = 0.2$, $\gamma = 0.05$, $\alpha = 0.2$, and $\beta = 1$.

From Figure 4, we can also find that when the threshold value I_c is confirmed at a certain level, the final infected patients number can not decrease with the improvement of the compulsory treatment rate f all the time. When the compulsory treatment f reaches a level (the level is a function of threshold value I_c), the improvement of the compulsory treatment rate can not minimize the final infected patients number at all. That appearance coincides with the truth in medical techniques that too much abundant treatment is ineffective to the disease control.

Although a higher compulsory treatment rate f and a lower threshold value I_c to start the compulsory treatment are advantageous to the control of the infectious disease, the quite high rate f and quite low threshold value I_c need much more expense of money and manual power. In practice, the investment for controlling the infectious disease is always limited. From the result of this paper, the optimum policy existed. Finding the best strategy with the limited resource is the top-drawer problem in the controlling of infectious disease.

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Conflict of interest

The authors declare that they have no conflict of interest.

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