Research article

Traveling waves for SVIR epidemic model with nonlocal dispersal

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Abstract: In this paper, we studied an SVIR epidemic model with nonlocal dispersal and delay, and we find that the existence of traveling wave is determined by the basic reproduction number $\mathfrak{R}_0$ and minimal wave speed c^*. By applying Schauder's fixed point theorem and Lyapunov functional, the existence and boundary asymptotic behaviour of traveling wave solutions is investigated for $\mathfrak{R}_0 > 1$ and $c > c^*$. The existence of traveling waves is obtained for $\mathfrak{R}_0 > 1$ and $c = c^*$ by employing a limiting argument. We also show that the nonexistence of traveling wave solutions by Laplace transform. Our results imply that (i) the diffusion and infection ability of infected individuals can accelerate the wave speed; (ii) the latent period and successful rate of vaccination can slow down the wave speed.

Keywords: traveling wave solutions; nonlocal dispersal; Schauder's fixed point theorem; Lyapunov functional; epidemic model

1. Introduction

As one of the most basic models in modeling infectious diseases, the SIR epidemiological model was introduced by Kermack and McKendrick [1] in 1927. Since then, a lot of differential equations have been studied as models for the spread of infectious diseases. Considering a continuous vaccination strategy, let *V* be a new group of vaccinated individuals, Liu et al. [2] formulated the following system of ordinary differential equations:

$$\begin{cases} \dfrac{dS(t)}{dt} = \Lambda - \beta_1 S(t)I(t) - \alpha S(t) - \mu_1 S(t), \\[2mm] \dfrac{dV(t)}{dt} = \alpha S(t) - \beta_2 V(t)I(t) - (\gamma_1 + \mu_1)V(x,t), \\[2mm] \dfrac{dI(t)}{dt} = \beta_1 S(t)I(t) + \beta_2 V(t)I(t) - \gamma I(x,t) - \mu_3 I(x,t), \\[2mm] \dfrac{dR(t)}{dt} = \gamma_1 V(t) + \gamma I(t) - \mu_1 R(t), \end{cases} \tag{1.1}$$

where $S(t)$, $V(t)$, $I(t)$ and $R(t)$ denote the densities of susceptible, vaccinated, infective and removed individuals at time t, respectively. Λ denote the recruitment rate of susceptible individuals, μ_1 denote the natural death rate. β_1 is the rate of disease transmission between susceptible and infectious individuals, and β_2 is the rate of disease transmission between vaccinated and infected individuals. γ denote the recovery rate, α is the vaccination rate and γ_1 is the rate at which a vaccinated individual obtains immunity. In [2], the authors shown that the global dynamics of model (1.1) is completely determined by the basic reproduction number: that is, if the number is less than unity, then the disease-free equilibrium is globally asymptotically stable, while if the number is greater than unity, then a positive endemic equilibrium exists and it is globally asymptotically stable. Moreover, it was observed in Liu et al. [2] that vaccination has an effect of decreasing the basic reproduction number. By using the classical method of Lyapunov and graph-theoretic approach, Kuniya [3] studied the global stability of a multi-group SVIR epidemic model. Xu et.al [4] formulated a multi-group epidemic model with distributed delay and vaccination age, the authors established the global stability of the model, furthermore, the stochastic perturbation of the model is studied and it is proved that the endemic equilibrium of the stochastic model is stochastically asymptotically stable in the large under certain conditions. In [5–7], the global stability of different SVIR models with age structure are investigated.

On the other hand, in order to understand the geographic spread of infectious disease, the spatial effect would give insights into disease spread and control. Due to this fact, many literatures have studied the spatial effects on epidemics by using reaction-diffusion equations (see, for instance, [8–17] and the references therein). In the study of reaction-diffusion models, the Laplacian operator describes the random diffusion of each individual, but it can not describe the long range diffusion. Therefore, a nonlocal dispersal term has been established, which is by a convolution operator:

$$J * \phi(x) - \phi(x) = \int_{\mathbb{R}} J(x - y)\phi(y)\mathrm{dy} - \phi(\mathrm{x}), \tag{1.2}$$

where $\phi(x)$ denote the densities of individuals at position x, $J(x - y)$ is interpreted as the probability of jumping from position y to position x, the convolution $\int_{\mathbb{R}} J(x-y)\phi(y)\mathrm{dy}$ is the rate at which individuals arrive at position x from all other positions, while $-\int_{\mathbb{R}} J(x-y)\phi(x)\mathrm{dy} = -\phi(\mathrm{x})$ is the rate at which they leave position x to reach any other position. Problems involving such operators are called nonlocal diffusion problems and have appeared in various references [18–31].

Recently, Li et al. [23] proposed a nonlocal dispersal SIR model with delay:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1(J * S(x,t) - S(x,t)) + \Lambda - \dfrac{\beta_1 S(x,t)I(x,t-\tau)}{1 + \theta I(x,t-\tau)} - \mu_1 S(x,t), \\[3mm] \dfrac{\partial I(x,t)}{\partial t} = d_2(J * I(x,t) - I(x,t)) + \dfrac{\beta_1 S(x,t)I(x,t-\tau)}{1 + \theta I(x,t-\tau)} - \gamma I(x,t) - \mu_3 I(x,t), \\[3mm] \dfrac{\partial R(x,t)}{\partial t} = d_3(J * R(x,t) - R(x,t)) + \gamma I(x,t) - \mu_1 R(x,t), \end{cases} \tag{1.3}$$

where $S(x,t)$, $I(x,t)$ and $R(x,t)$ denote the densities of susceptible, infective and removed individuals at position x and time t, respectively. θ measures the saturation level. $d_i(i = 1, 2, 3)$ describes the spatial motility of each compartments. The biological meaning of other parameters are the same as in model (1.1). The authors find that there exists traveling wave solution if the basic reproduction number $\mathfrak{R}_0 > 1$ and the wave speed $c \geq c^*$, where c^* is the minimal wave speed. They also obtain the nonexistence of traveling wave solution for $\mathfrak{R}_0 > 1$ and any $0 < c < c^*$ or $\mathfrak{R}_0 < 1$.

Motivated by [2] and [23], in this paper, we consider a nonlocal dispersal epidemic model with vaccination and delay. Precisely, we study the following model.

$$
\begin{cases}
\dfrac{\partial S(x,t)}{\partial t} = d_1(J * S(x,t) - S(x,t)) + \Lambda - \beta_1 S(x,t)I(x,t-\tau) - \alpha S(x,t) - \mu_1 S(x,t), \\[2mm]
\dfrac{\partial V(x,t)}{\partial t} = d_2(J * V(x,t) - V(x,t)) - \beta_2 V(x,t)I(x,t-\tau) + \alpha S(x,t) - (\gamma_1 + \mu_1)V(x,t), \\[2mm]
\dfrac{\partial I(x,t)}{\partial t} = d_3(J * I(x,t) - I(x,t)) + \beta_1 S(x,t)I(x,t-\tau) + \beta_2 V(x,t)I(x,t-\tau) - \gamma I(x,t) - \mu_3 I(x,t), \\[2mm]
\dfrac{\partial R(x,t)}{\partial t} = d_4(J * R(x,t) - R(x,t)) + \beta_1 S(x,t)I(x,t-\tau) + \gamma_1 V(x,t) + \gamma I(x,t) - \mu_4 R(x,t),
\end{cases}
\tag{1.4}
$$

where $S(x,t)$, $V(x,t)$, $I(x,t)$ and $R(x,t)$ denote the densities of susceptible, vaccinated, infective and removed individuals at position x and time t, respectively. $d_i(i = 1, 2, 3, 4)$ describes the spatial motility of each compartments. The biological meaning of other parameters are the same as in model (1.1). J is the standard convolution operator satisfying the following assumptions.

Assumption 1.1. *[23, 24] The kernel function J satisfies*

(J1) $J \in C^1(\mathbb{R})$, $J(x) \geq 0$, $J(x) = J(-x)$, $\int_{\mathbb{R}} J(x)\mathrm{dx} = 1$ *and J is compactly supported.*

(J2) *There exists a constant $\lambda_M \in (0, +\infty)$ such that*

$$
\int_{\mathbb{R}} J(x)e^{-\lambda x}\mathrm{dx} < +\infty, \ \ for \ any \ \ \lambda \in [0, \lambda_M)
$$

and

$$
\lim_{\lambda \to \lambda_M - 0} \int_{\mathbb{R}} J(x)e^{-\lambda x}\mathrm{dx} \to +\infty.
$$

The organization of this paper is as follows. In section 2, we proved the existence of traveling wave solutions of (1.4) for $c > c^*$ by applying Schauder's fixed point theorem and Lyapunov method. In section 3, we show that the existence of traveling wave solutions of (1.4) for $c = c^*$. Furthermore, we investigate the nonexistence of traveling wave solutions under some conditions in section 4. At last, there is a brief discussion.

2. Existence of traveling wave solutions for $c > c^*$

In this section, we study the existence of traveling wave solutions of system (1.4). Since we have assumed that the recovered have gained permanent immunity and $R(x,t)$ is decoupled from other equations, we indeed need to study the following subsystem of (1.4)

$$
\begin{cases}
\dfrac{\partial S(x,t)}{\partial t} = d_1(J * S(x,t) - S(x,t)) + \Lambda - \beta_1 S(x,t)I(x,t-\tau) - \alpha S(x,t) - \mu_1 S(x,t), \\[2mm]
\dfrac{\partial V(x,t)}{\partial t} = d_2(J * V(x,t) - V(x,t)) - \beta_2 V(x,t)I(x,t-\tau) + \alpha S(x,t) - \mu_2 V(x,t), \\[2mm]
\dfrac{\partial I(x,t)}{\partial t} = d_3(J * I(x,t) - I(x,t)) + (\beta_1 S(x,t) + \beta_2 V(x,t))I(x,t-\tau) - \gamma I(x,t) - \mu_3 I(x,t).
\end{cases}
\tag{2.1}
$$

where $\mu_2 = \gamma_1 + \mu_1$. Obviously, system (2.1) always has a disease-free equilibrium $E_0 = (S_0, V_0, 0) = \left(\frac{\Lambda}{\mu_1+\alpha}, \frac{\Lambda\alpha}{\mu_2(\mu_1+\alpha)}, 0\right)$. Denote the basic reproduction number as following:

$$\mathfrak{R}_0 = \frac{\beta_1 S_0 + \beta_2 V_0}{\mu_3 + \gamma}. \tag{2.2}$$

Furthermore, there exists another equilibrium $E^* = (S^*, V^*, I^*)$ satisfying

$$\begin{cases} \Lambda - \beta_1 S^* I^* - \alpha S^* - \mu_1 S^* = 0, \\ \beta_2 V^* I^* + \alpha S^* - \mu_2 V^* = 0, \\ (\beta_1 S^* + \beta_2 V^*)I^* - \gamma I^* - \mu_3 I^* = 0. \end{cases} \tag{2.3}$$

From [2, Theorem 2.1], system (2.1) has a unique positive equilibrium E^* if $\mathfrak{R}_0 > 1$.

Let $\xi = x + ct$ and substituting ξ into system (2.1), then we obtain the wave form equations as

$$\begin{cases} cS'(\xi) = d_1(J * S(\xi) - S(\xi)) + \Lambda - \beta_1 S(\xi)I(\xi - c\tau) - \alpha S(\xi) - \mu_1 S(\xi), \\ cV'(\xi) = d_2(J * V(\xi) - V(\xi)) + \alpha S(\xi) - \beta_2 V(\xi)I(\xi - c\tau) - \mu_2 V(\xi), \\ cI'(\xi) = d_3(J * I(\xi) - I(\xi)) + \beta_1 S(\xi)I(\xi - c\tau) + \beta_2 V(\xi)I(\xi - c\tau) - \gamma I(\xi) - \mu_3 I(\xi). \end{cases} \tag{2.4}$$

We want to find traveling wave solutions with the following asymptotic boundary conditions:

$$\lim_{\xi \to -\infty} (S(\xi), V(\xi), I(\xi)) = (S_0, V_0, 0) \tag{2.5}$$

and

$$\lim_{\xi \to +\infty} (S(\xi), V(\xi), I(\xi)) = (S^*, V^*, I^*). \tag{2.6}$$

Consider the following linear system of system (2.4) at infection-free equilibrium $(S_0, V_0, 0)$,

$$cI'(\xi) = d_3(J * I(\xi) - I(\xi)) + \beta_1 S_0 I(\xi - c\tau) + \beta_2 V_0 I(\xi - c\tau) - (\gamma + \mu_3)I(\xi). \tag{2.7}$$

Let $I(\xi) = e^{\lambda\xi}$, we have

$$\Delta(\lambda, c) \triangleq d_3 \int_{\mathbb{R}} J(x)e^{-\lambda x}dx - (d_3 + \gamma + \mu_3) - c\lambda + \beta_1 S_0 e^{-c\tau\lambda} + \beta_2 V_0 e^{-c\tau\lambda} = 0. \tag{2.8}$$

By some calculations, we obtain

$$\Delta(0, c) = \beta_1 S_0 + \beta_2 V_0 - \gamma - \mu_3, \quad \lim_{c \to +\infty} \Delta(\lambda, c) = -\infty \text{ for } \lambda > 0,$$

$$\left.\frac{\partial\Delta(\lambda, c)}{\partial\lambda}\right|_{(0,c)} = -c - c\tau(\beta_1 S_0 + \beta_2 V_0) < 0 \text{ for } c > 0,$$

$$\frac{\partial\Delta(\lambda, c)}{\partial c} = -\lambda - \tau\lambda e^{-c\tau\lambda}(\beta_1 S_0 + \beta_2 V_0) < 0 \text{ for } \lambda > 0,$$

$$\frac{\partial^2\Delta(\lambda, c)}{\partial\lambda^2} = d_3 \int_{\mathbb{R}} J(x)x^2 e^{-\lambda x}dx + (c\tau)^2 e^{-c\tau\lambda}(\beta_1 S_0 + \beta_2 V_0) > 0.$$

For any $c \in \mathbb{R}$, $\Delta(0, c) = \mathfrak{R}_0 - 1$, gives us $\Delta(0, c) > 0$ if $\mathfrak{R}_0 > 1$. Then there exist $c^* > 0$ and $\lambda^* > 0$ such that $\left.\frac{\partial\Delta(\lambda,c)}{\partial\lambda}\right|_{(\lambda^*,c^*)} = 0$, we have the following lemma.

Lemma 2.1. *Let $\mathfrak{R}_0 > 1$, we have*

(i) *If $c = c^*$, then $\Delta(\lambda, c) = 0$ has two same positive real roots λ^*;*

(ii) *If $0 < c < c^*$, then $\Delta(\lambda, c) > 0$ for all $\lambda \in (0, \lambda_{c,\tau})$, where $\lambda_{c,\tau} \in (0, +\infty]$;*

(iii) *If $c > c^*$, then $\Delta(\lambda, c) = 0$ has two positive real roots $\lambda_1(c)$, $\lambda_2(c)$.*

Denote $\lambda_c = \lambda_1(c)$, from Lemma 2.1, we have

$$0 < \lambda_c < \lambda^* < \lambda_2(c) < \lambda_{c,\tau}.$$

For the followings in this section, we always fix $c > c^*$ and $\mathfrak{R}_0 > 1$. Define the following functions:

$$\begin{cases} \overline{S}(\xi) = S_0, \\ \overline{V}(\xi) = V_0, \\ \overline{I}(\xi) = e^{\lambda_c \xi}, \end{cases} \quad \begin{cases} \underline{S}(\xi) = \max\{S_0 - M_1 e^{\varepsilon_1 \xi}, 0\}, \\ \underline{V}(\xi) = \max\{V_0 - M_2 e^{\varepsilon_2 \xi}, 0\}, \\ \underline{I}(\xi) = \max\{e^{\lambda_c \xi}(1 - M_3 e^{\varepsilon_3 \xi}), 0\}, \end{cases}$$

where M_i and $\varepsilon_i (i = 1, 2, 3)$ are some positive constants to be determined in the following lemmas.

Lemma 2.2. *The function $\overline{I}(\xi) = e^{\lambda_c \xi}$ satisfies*

$$cI'(\xi) \geq d_3(J * I(\xi) - I(\xi)) + \beta_1 S_0 I(\xi - c\tau) + \beta_2 V_0 I(\xi - c\tau) - \gamma I(\xi) - \mu_3 I(\xi). \tag{2.9}$$

Lemma 2.3. *The functions $\overline{S}(\xi) = S_0$ and $\overline{V}(\xi) = V_0$ satisfy*

$$\begin{cases} cS'(\xi) \geq d_1(J * S(\xi) - S(\xi)) + \Lambda - \beta_1 S(\xi)\underline{I}(\xi - c\tau) - \alpha S(\xi) - \mu_1 S(\xi), \\ cV'(\xi) \geq d_2(J * V(\xi) - V(\xi)) + \alpha S(\xi) - \beta_2 V(\xi)\underline{I}(\xi - c\tau) - \mu_2 V(\xi). \end{cases} \tag{2.10}$$

The proof is trivial, so we omitted the above two lemmas.

Lemma 2.4. *For each $0 < \varepsilon_1 < \lambda_c$ sufficiently small and M_1 large enough, the function $\underline{S}(\xi) = \max\{S_0 - M_1 e^{\varepsilon_1 \xi}, 0\}$ satisfies*

$$cS'(\xi) \leq d_1(J * S(\xi) - S(\xi)) + \Lambda - \beta_1 S(\xi)\overline{I}(\xi - c\tau) - (\mu_1 + \alpha)S(\xi), \tag{2.11}$$

with $\xi \neq \mathfrak{X}_1 \triangleq \frac{1}{\varepsilon_1} \ln \frac{S_0}{M_1}$.

Proof. See Appendix A. □

Lemma 2.5. *For each $0 < \varepsilon_2 < \lambda_c$ sufficiently small and M_2 large enough, the function $\underline{V}(\xi) = \max\{V_0 - M_2 e^{\varepsilon_2 \xi}, 0\}$ satisfies*

$$cV'(\xi) \leq d_2(J * V(\xi) - V(\xi)) + \alpha\underline{S}(\xi) - \beta_2 V(\xi)\overline{I}(\xi - c\tau) - \mu_2 V(\xi), \tag{2.12}$$

with $\xi \neq \mathfrak{X}_2 \triangleq \frac{1}{\varepsilon_2} \ln \frac{V_0}{M_2}$.

The proof is similar with Lemma 2.4.

Lemma 2.6. *Let* $0 < \varepsilon_3 < \min\{\varepsilon_1/2, \varepsilon_2/2\}$ *and* $M_3 > \max\{S_0, V_0\}$ *is large enough, then the function* $\underline{I}(\xi) = \max\{e^{\lambda_c \xi}(1 - M_3 e^{\varepsilon_3 \xi}), 0\}$ *satisfies*

$$c\underline{I}'(\xi) \leq d_3(J * \underline{I}(\xi) - \underline{I}(\xi)) + \beta_1 \underline{S}(\xi)\underline{I}(\xi - c\tau) + \beta_2 \underline{V}(\xi)\underline{I}(\xi - c\tau) - \gamma \underline{I}(\xi) - \mu_3 \underline{I}(\xi), \qquad (2.13)$$

with $\xi \neq \mathfrak{X}_3 \triangleq \frac{1}{\varepsilon_3} \ln \frac{1}{M_3}$.

Proof. See Appendix B. □

Let $X > \max\{\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3\}$, define

$$\Gamma_X = \left\{ \begin{pmatrix} \phi \\ \varphi \\ \psi \end{pmatrix} \in C([-X, X], \mathbb{R}^3) \; \middle| \; \begin{array}{l} \underline{S}(\xi) \leq \phi(\xi) \leq S_0, \quad \phi(-X) = \underline{S}(-X), \quad \text{for } \xi \in [-X, X]; \\ \underline{V}(\xi) \leq \varphi(\xi) \leq V_0, \quad \varphi(-X) = \underline{V}(X), \quad \text{for } \xi \in [-X, X]; \\ \underline{I}(\xi) \leq \psi(\xi) \leq \bar{I}(\xi), \quad \psi(-X) = \underline{I}(X), \quad \text{for } \xi \in [-X, X]. \end{array} \right\}.$$

For given $(\phi(\xi), \varphi(\xi), \psi(\xi)) \in \Gamma_X$, define

$$\hat{\phi}(\xi) = \begin{cases} \phi(X), & \text{for } \xi > X, \\ \phi(\xi), & \text{for } \xi \in [-X - c\tau, X], \\ \underline{S}(\xi), & \text{for } \xi \leq -X - c\tau, \end{cases} \qquad \hat{\varphi}(\xi) = \begin{cases} \varphi(X), & \text{for } \xi > X, \\ \varphi(\xi), & \text{for } \xi \in [-X - c\tau, X], \\ \underline{V}(\xi), & \text{for } \xi \leq -X - c\tau, \end{cases}$$

and

$$\hat{\psi}(\xi) = \begin{cases} \psi(X), & \text{for } \xi > X, \\ \psi(\xi), & \text{for } \xi \in [-X - c\tau, X], \\ \underline{I}(\xi), & \text{for } \xi \leq -X - c\tau. \end{cases}$$

We have

$$\begin{cases} \underline{S}(\xi) \leq \hat{\phi}(\xi) \leq S_0, \\ \underline{V}(\xi) \leq \hat{\varphi}(\xi) \leq V_0, \\ \underline{I}(\xi) \leq \hat{\psi}(\xi) \leq \bar{I}(\xi). \end{cases}$$

For any $\xi \in [-X, X]$, consider the following initial value problem

$$\begin{cases} cS'(\xi) = d_1 \int_{\mathbb{R}} J(y)\hat{\phi}(\xi - y)\mathrm{d}y + \Lambda - \beta_1 S(\xi)\psi(\xi - c\tau) - (d_1 + \mu_1 + \alpha)S(\xi), \\ cV'(\xi) = d_2 \int_{\mathbb{R}} J(y)\hat{\varphi}(\xi - y)\mathrm{d}y + \alpha\phi(\xi) - \beta_2 V(\xi)\psi(\xi - c\tau) - (d_2 + \mu_2)V(\xi), \\ cI'(\xi) = d_3 \int_{\mathbb{R}} J(y)\hat{\psi}(\xi - y)\mathrm{d}y + \beta_1 \phi(\xi)\psi(\xi - c\tau) + \beta_2 \varphi(\xi)\psi(\xi - c\tau) - (d_3 + \gamma + \mu_3)I(\xi), \\ S(-X) = \underline{S}(-X), \quad V(-X) = \underline{V}(-X), \quad I(-X) = \underline{I}(-X). \end{cases} \qquad (2.14)$$

From the standard theory of functional differential equations (see [32]), the initial value problem (2.14) admits a unique solution $(S_X(\xi), V_X(\xi), I_X(\xi))$ satisfying

$$(S_X, V_X, I_X) \in C^1([-X, X]),$$

this defines an operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) : \Gamma_X \to C^1([-X, X])$ as

$$S_X = \mathcal{A}_1(\phi, \varphi, \psi), \ V_X = \mathcal{A}_2(\phi, \varphi, \psi), \ I_X = \mathcal{A}_3(\phi, \varphi, \psi).$$

Next we show the operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ has a fixed point in Γ_X.

Lemma 2.7. *The operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ maps Γ_X into itself.*

Proof. Firstly, we show that $\underline{S}(\xi) \leq S_X(\xi)$ for any $\xi \in [-X, X]$. If $\xi \in (\mathfrak{X}_1, X)$, $\underline{S}(\xi) = 0$ is a lower solution of the first equation of (2.14). If $\xi \in (-X, \mathfrak{X}_1)$, $\underline{S}(\xi) = S_0 - M_1 e^{\varepsilon_1 \xi}$, by Lemma 2.4, we have

$$c\underline{S}'(\xi) - d_1 \int_{\mathbb{R}} J(y)\hat{\phi}(\xi - y)\mathrm{d}y - \Lambda + \beta_1 \underline{S}(\xi)\psi(\xi - c\tau) - (d_1 + \mu_1 + \alpha)\underline{S}(\xi)$$

$$\leq c\underline{S}'(\xi) - d_1 \int_{\mathbb{R}} J(y)\underline{S}(\xi - y)\mathrm{d}y - \Lambda + \beta_1 \underline{S}(\xi)\bar{I}(\xi - c\tau) - (d_1 + \mu_1 + \alpha)\underline{S}(\xi))$$

$$\leq 0,$$

which implies that $\underline{S}(\xi) = S_0 - M_1 e^{\varepsilon_1 \xi}$ is a lower solution of the first equation of (2.14). Thus $\underline{S}(\xi) \leq S_X(\xi)$ for any $\xi \in [-X, X]$.

Secondly, we show that $S_X(\xi) \leq \overline{S}(\xi) = S_0$ for any $\xi \in [-X, X]$. In fact,

$$d_1 \int_{\mathbb{R}} J(y)\hat{\phi}(\xi - y)\mathrm{d}y + \Lambda - \beta_1 S_0 \psi(\xi - c\tau) - (d_1 + \mu_1 + \alpha)S_0$$

$$\leq d_1 \int_{\mathbb{R}} J(y)S_0 \mathrm{d}y + \Lambda - \beta_1 S_0 \underline{I}(\xi - c\tau) - (d_1 + \mu_1 + \alpha)S_0$$

$$\leq 0,$$

thus $\overline{S}(\xi) = S_0$ is an upper solution to the first equation of (2.14), which gives us $S_X(\xi) \leq S_0$ for any $\xi \in [-X, X]$.

Similarly, $\underline{V}(\xi) \leq V_X(\xi) \leq \overline{V}(\xi)$ and $\underline{I}(\xi) \leq I_X(\xi) \leq \overline{I}(\xi)$ for any $\xi \in [-X, X]$. $\square$

Lemma 2.8. *The operator $\mathcal{A}$ is completely continuous.*

Proof. Suppose $(\phi_i(\xi), \varphi_i(\xi), \psi_i(\xi)) \in \Gamma_X$, $i = 1, 2$.

$$S_{X,i}(\xi) = \mathcal{A}_1(\phi_i(\xi), \varphi_i(\xi), \psi_i(\xi)),$$
$$V_{X,i}(\xi) = \mathcal{A}_2(\phi_i(\xi), \varphi_i(\xi), \psi_i(\xi)),$$
$$I_{X,i}(\xi) = \mathcal{A}_3(\phi_i(\xi), \varphi_i(\xi), \psi_i(\xi)),$$

We show the operator $\mathcal{A}$ is continuous. By direct calculation, we have

$$S_X(\xi) = \underline{S}(-X) \exp\left\{-\frac{1}{c} \int_{-X}^{\xi} (d_1 + \mu_1 + \alpha + \beta_1 \psi(s - c\tau))\mathrm{d}s\right\}$$
$$+ \frac{1}{c} \int_{-X}^{\xi} \exp\left\{-\frac{1}{c} \int_{\eta}^{\xi} (d_1 + \mu_1 + \alpha + \beta_1 \psi(s - c\tau))\mathrm{d}s\right\} f_\phi(\eta)\mathrm{d}\eta,$$

$$V_X(\xi) = \underline{V}(-X) \exp\left\{-\frac{1}{c}\int_{-X}^{\xi}(d_2 + \mu_2 + \beta_2\psi(s - c\tau))\mathrm{d}s\right\}$$
$$+ \frac{1}{c}\int_{-X}^{\xi}\exp\left\{-\frac{1}{c}\int_{\eta}^{\xi}(d_2 + \mu_2 + \beta_2\psi(s - c\tau))\mathrm{d}s\right\}f_\varphi(\eta)\mathrm{d}\eta,$$

and

$$I_X(\xi) = \underline{I}(-X) \exp\left\{-\frac{(d_3 + \gamma + \mu_3)(\xi + X)}{c}\right\}$$
$$+ \frac{1}{c}\int_{-X}^{\xi}\exp\left\{-\frac{(d_3 + \gamma + \mu_3)(\xi - \eta)}{c}\right\}f_\psi(\eta)\mathrm{d}\eta.$$

where

$$f_\phi(\eta) = d_1 \int_{\mathbb{R}} J(\eta - y)\hat{\phi}(y)\mathrm{d}y + \Lambda,$$

$$f_\varphi(\eta) = d_2 \int_{\mathbb{R}} J(\eta - y)\hat{\varphi}(y)\mathrm{d}y + \alpha\phi(\eta),$$

$$f_\psi(\eta) = d_3 \int_{\mathbb{R}} J(\eta - y)\hat{\psi}(y)\mathrm{d}y + (\beta_1\phi(\eta) + \beta_2\varphi(\eta))\psi(\eta - c\tau).$$

For any $(\phi_i, \varphi_i, \psi_i) \in \Gamma_X$, $i = 1, 2$, we have

$$|f_{\phi_1}(\eta) - f_{\phi_2}(\eta)| = d_1 \left|\int_{\mathbb{R}} J(\eta - y)[\hat{\phi}_1(y) - \hat{\phi}_2(y)]\mathrm{d}y\right|$$
$$\leq d_1 \left|\int_{-X}^{X} J(\xi - y)(\phi_1(y) - \phi_2(y))\mathrm{d}y\right| + d_1 \left|\int_{X}^{\infty} J(\xi - y)(\phi_1(X) - \phi_2(X))\mathrm{d}y\right|$$
$$\leq 2d_1 \max_{y\in[-X,X]} |\phi_1(y) - \phi_2(y)|,$$

$$|f_{\varphi_1}(\eta) - f_{\varphi_2}(\eta)| = d_2 \left|\int_{\mathbb{R}} J(\eta - y)[\hat{\varphi}_1(y) - \hat{\varphi}_2(y)]\mathrm{d}y + \alpha(\phi_1(\eta) - \phi_2(\eta))\right|$$
$$\leq 2d_2 \max_{y\in[-X,X]} |\varphi_1(y) - \varphi_2(y)| + \alpha \max_{y\in[-X,X]} |\phi_1(y) - \phi_2(y)|,$$

$$|f_{\psi_1}(\eta) - f_{\psi_2}(\eta)| \leq (2d_2 + \beta_1 S_0 + \beta_2 V_0) \max_{y\in[-X,X]} |\psi_1(y) - \psi_2(y)|$$
$$+ \beta_1 e^{\lambda_c\xi} \max_{y\in[-X,X]} |\phi_1(y) - \phi_2(y)| + \beta_2 e^{\lambda_c\xi} \max_{y\in[-X,X]} |\varphi_1(y) - \varphi_2(y)|.$$

Here we use

$$|\beta_1\phi_2(\xi)\psi_2(\xi - c\tau) - \beta_1\phi_1(\xi)\psi_1(\xi - c\tau)|$$
$$\leq |\beta_1\phi_2(\xi)\psi_2(\xi - c\tau) - \beta_1\phi_2(\xi)\psi_1(\xi - c\tau)| + |\beta_1\phi_2(\xi)\psi_1(\xi - c\tau) - \beta_1\phi_1(\xi)\psi_1(\xi - c\tau)|$$
$$\leq \beta_1 S_0 \max_{y\in[-X,X]} |\psi_1(y) - \psi_2(y)| + \beta_1 e^{\lambda_c\xi} \max_{y\in[-X,X]} |\phi_1(y) - \phi_2(y)|.$$

and

$$|\beta_2\varphi_2(\xi)\psi_2(\xi - c\tau) - \beta_2\varphi_1(\xi)\psi_1(\xi - c\tau)|$$
$$\leq \beta_2 V_0 \max_{y\in[-X,X]} |\psi_1(y) - \psi_2(y)| + \beta_2 e^{\lambda_c \xi} \max_{y\in[-X,X]} |\varphi_1(y) - \varphi_2(y)|.$$

Thus, we obtain that the operator $\mathcal{A}$ is continuous. Next, we show $\mathcal{A}$ is compact. Indeed, since S_X, V_X and I_X are class of $C^1([-X, X])$, note that

$$c(S'_{X,1}(\xi) - S'_{X,2}(\xi)) + (d_1 + \mu_1 + \alpha)(S_{X,1}(\xi) - S_{X,2}(\xi))$$
$$= d_1 \int_{\mathbb{R}} J(\xi - y)(\hat{\phi}_1(y) - \hat{\phi}_2(y))dy + \beta_1\phi_2(\xi)\psi_2(\xi - c\tau) - \beta_1\phi_1(\xi)\psi_1(\xi - c\tau)$$
$$\leq \left(2d_1 + \beta_1 e^{\lambda_c \xi}\right) \max_{y\in[-X,X]} |\phi_1(y) - \phi_2(y)| + \beta_1 S_0 \max_{y\in[-X,X]} |\psi_1(y) - \psi_2(y)|.$$

Same arguments with V'_X and I'_X, give us S'_X, V'_X and I'_X are bounded. Then $\mathcal{A}$ is compact and the operator $\mathcal{A}$ is completely continuous. This ends the proof. $\square$

Obviously, Γ_X is a bounded closed convex set, applying the Schauder's fixed point theorem ([33] Corollary 2.3.10), we have the following theorem.

Theorem 2.1. *There exists $(S_X, V_X, I_X) \in \Gamma_X$ such that*

$$(S_X(\xi), V_X(\xi), I_X(\xi)) = \mathcal{A}(S_X, V_X, I_X)(\xi)$$

for $\xi \in [-X, X]$.

Now we are in position to show the existence of traveling wave solutions, before that we do some estimates for $S_X(\cdot)$, $V_X(\cdot)$ and $I_X(\cdot)$.

Define

$$C^{1,1}([-X, X]) = \{u \in C^1([-X, X]) | u, u' \text{ are Lipschitz continuous}\}$$

with norm

$$\|u\|_{C^{1,1}([-X,X])} = \max_{x\in[-X,X]} |u| + \max_{x\in[-X,X]} |u'| + \sup_{\substack{x,y\in[-X,X]\\x\neq y}} \frac{|u'(x) - u'(y)|}{|x - y|}.$$

Lemma 2.9. *There exists a constant $C(Y) > 0$ such that*

$$\|S_X\|_{C^{1,1}([-Y,Y])} \leq C(Y), \quad \|V_X\|_{C^{1,1}([-Y,Y])} \leq C(Y), \quad \|I_X\|_{C^{1,1}([-Y,Y])} \leq C(Y)$$

for $Y < X$ and $X > \max\{\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3\}$.

Proof. Recall that (S_X, E_X, I_X) is the fixed point of the operator $\mathcal{A}$, then

$$cS'_X(\xi) = d_1 \int_{-\infty}^{+\infty} J(y)\hat{S}_X(\xi - y)dy + \Lambda - \beta_1 S_X(\xi)I_X(\xi - c\tau) - (d_1 + \mu_1 + \alpha)S_X(\xi), \tag{2.15}$$

$$cV'_X(\xi) = d_2 \int_{-\infty}^{+\infty} J(y)\hat{V}_X(\xi - y)dy + \alpha S_X(\xi) - \beta_2 V_X(\xi)I_X(\xi - c\tau) - (d_2 + \mu_2)V_X(\xi), \tag{2.16}$$

$$cI'_X(\xi) = d_3 \int_{-\infty}^{+\infty} J(y)\hat{I}_X(\xi - y)\mathrm{d}y + \beta_1 S_X(\xi)I_X(\xi - c\tau) + \beta_2 V_X(\xi)I_X(\xi - c\tau) - (d_3 + \mu_3)I_X(\xi), \quad (2.17)$$

where

$$(\hat{S}_X(\xi), \hat{V}_X(\xi), \hat{I}_X(\xi)) = \begin{cases} (S_X(X), V_X(X), I_X(X)), & \text{for } \xi > X, \\ (S_X(\xi), V_X(\xi), I_X(\xi)), & \text{for } \xi \in [-X - c\tau, X], \\ (\underline{S}(\xi), \underline{V}(\xi), \underline{I}(\xi)), & \text{for } \xi \leq -X - c\tau, \end{cases}$$

following that $S_X(\xi) \leq S_0$, $V_X(\xi) \leq V_0$, $I_X(\xi) \leq e^{\lambda_c Y}$ for any $\xi \in [-Y, Y]$. Then

$$|S'_X(\xi)| \leq \frac{d_1}{c} \left| \int_{-\infty}^{+\infty} J(y)\hat{S}_X(\xi - y)\mathrm{d}y \right| + \frac{\Lambda}{c} + \frac{d_1 + \mu_1 + \alpha}{c}|S_X(\xi)| + \frac{\beta_1}{c}|S_X(\xi)||I_X(\xi - c\tau)|$$

$$\leq \frac{2d_1 + \mu_1 + \alpha}{c}S_0 + \frac{\Lambda}{c} + \frac{\beta_1 S_0}{c}e^{\lambda_c Y},$$

$$|V'_X(\xi)| \leq \frac{2d_2 + \mu_2}{c}V_0 + \frac{\alpha S_0}{c} + \frac{\beta_2 V_0}{c}e^{\lambda_c Y},$$

$$|I'_X(\xi)| \leq \left(\frac{d_3 + \mu_3}{c} + \frac{\beta_1 S_0}{c} + \frac{\beta_2 V_0}{c} \right)e^{\lambda_c Y}.$$

Thus, there exists some constant $C_1(Y) > 0$ such that

$$\|S_X\|_{C^1([-Y,Y])} \leq C_1(Y), \quad \|V_X\|_{C^1([-Y,Y])} \leq C_1(Y), \quad \|I_X\|_{C^1([-Y,Y])} \leq C_1(Y).$$

Then for any $\xi_1, \xi_2 \in [-Y, Y]$ such that

$$|S_X(\xi_1) - S_X(\xi_2)| \leq C_1(Y)|\xi_1 - \xi_2|, \quad |V_X(\xi_1) - V_X(\xi_2)| \leq C_1(Y)|\xi_1 - \xi_2|, \quad |I_X(x_1) - I_X(x_2)| \leq C_1(Y)|\xi_1 - \xi_2|.$$

From (2.15), we have

$$c|S'_X(\xi_1) - S'_X(\xi_2)| \leq d_1 \left| \int_{-\infty}^{+\infty} J(y)(\hat{S}_X(\xi_1 - y) - \hat{S}_X(\xi_2 - y))\mathrm{d}y \right|$$

$$+ (d_1 + \mu_1 + \alpha)|S_X(\xi_1) - S_X(\xi_2)| + S_0|I_X(\xi_1) - I_X(\xi_2)|.$$

Recall (J1) of Assumption 1.1, we know J is Lipschitz continuous and compactly supported on $\mathbb{R}$, let L be the Lipschitz constant for J and R be the radius of supp$J$. Then

$$d_1 \left| \int_{-\infty}^{+\infty} J(y)(\hat{S}_X(\xi_1 - y) - \hat{S}_X(\xi_2 - y))\mathrm{d}y \right|$$

$$= d_1 \left| \int_{-R}^{R} J(y)\hat{S}_X(\xi_1 - y)\mathrm{d}y - \int_{-R}^{R} J(y)\hat{S}_X(\xi_2 - y)\mathrm{d}y \right|$$

$$= d_1 \left| \int_{\xi_1-R}^{\xi_1+R} J(\xi_1 - y)\hat{S}_X(y)\mathrm{d}y - \int_{\xi_2-R}^{\xi_2+R} J(y)\hat{S}_X(y)\mathrm{d}y \right|$$

$$= d_1 \left| \left(\int_{\xi_1-R}^{\xi_2-R} + \int_{\xi_2-R}^{\xi_2+R} + \int_{\xi_2+R}^{\xi_1+R} \right) J(\xi_1 - y)\hat{S}_X(y)\mathrm{d}y - \int_{\xi_2-R}^{\xi_2+R} J(y)\hat{S}_X(y)\mathrm{d}y \right|$$

$$\leq d_1 \left| \int_{\xi_2+R}^{\xi_1+R} J(\xi_1 - y)\hat{S}_X(y)\mathrm{d}y \right| + d_1 \left| \int_{\xi_1-R}^{\xi_2-R} J(\xi_1 - y)\hat{S}_X(y)\mathrm{d}y \right|$$

$$+ d_1 \left| \int_{\xi_2-R}^{\xi_2+R} (J(\xi_1 - y) - J(\xi_2 - y)) \hat{S}_X(y) dy \right|$$

$$\leq d_1 (2S_0 \|J\|_{L^\infty} + 2RLS_0)|\xi_1 - \xi_2|.$$

Thus there exists some constant $C_2(Y) > 0$ such that

$$|S'_X(\xi_1) - S'_X(\xi_2)| \leq C_2(Y)|\xi_1 - \xi_2|.$$

Similarly

$$|V'_X(\xi_1) - V'_X(\xi_2)| \leq C_2(Y)|\xi_1 - \xi_2|, \quad |I'_X(\xi_1) - I'_X(\xi_2)| \leq C_2(Y)|\xi_1 - \xi_2|.$$

From the above discussion, there exists some constant $C(Y) > 0$ for any $Y < X$ that is independent of X such that

$$\|S_X\|_{C^{1,1}([-Y,Y])} \leq C(Y), \quad \|V_X\|_{C^{1,1}([-Y,Y])} \leq C(Y), \quad \|I_X\|_{C^{1,1}([-Y,Y])} \leq C(Y).$$

$\square$

Now let $\{X_n\}_{n=1}^{+\infty}$ be an increasing sequence such that $X_n \geq \{\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3\}$, $X_n > Y + R$ for each n and $\lim X_n = +\infty$, where R is the radius of supp$J$. For every $c > c^*$, we have $(S_{X_n}, V_{X_n}, I_{X_n}) \in \Gamma_{X_n}$ satisfying Lemma 2.9 and Equations (2.15)-(2.17). For the sequence $(S_{X_n}, V_{X_n}, I_{X_n})$, we can extract a subsequence denoted by $\{S_{X_{n_k}}\}_{k\in\mathbb{N}}$, $\{V_{X_{n_k}}\}_{k\in\mathbb{N}}$ and $\{I_{X_{n_k}}\}_{k\in\mathbb{N}}$ tending to functions $(S, V, I) \in C^1(\mathbb{R})$ in the following topologies

$$S_{X_{n_k}} \to S, \quad V_{X_{n_k}} \to V \text{ and } I_{X_{n_k}} \to I \text{ in } C^1_{\text{loc}}(\mathbb{R}) \text{ as } k \to +\infty.$$

Since J is compactly supported, applying the dominated convergence theorem, thus

$$\lim_{k\to+\infty} \int_{\mathbb{R}} J(y) \hat{S}_{X_{n_k}}(\xi - y) dy = \int_{\mathbb{R}} J(y) S(\xi - y) dy = J * S(\xi),$$

$$\lim_{k\to+\infty} \int_{\mathbb{R}} J(y) \hat{V}_{X_{n_k}}(\xi - y) dy = \int_{\mathbb{R}} J(y) V(\xi - y) dy = J * V(\xi)$$

and

$$\lim_{k\to+\infty} \int_{\mathbb{R}} J(y) \hat{I}_{X_{n_k}}(\xi - y) dy = \int_{\mathbb{R}} J(y) I(\xi - y) dy = J * I(\xi).$$

Moreover, (S, V, I) satisfies system (2.4) and

$$\underline{S}(\xi) \leq S(\xi) \leq S_0, \quad \underline{V}(\xi) \leq V(\xi) \leq V_0, \quad \underline{I}(\xi) \leq I(\xi) \leq e^{\lambda_c \xi}.$$

Next, we show that $I(\xi)$ is bounded in $\mathbb{R}$ by the method in [34] (see also [35, 36]).

Lemma 2.10. *There exists some positive constant C such that*

$$\int_{\mathbb{R}} J(y) \frac{I(\xi - y)}{I(\xi)} dy < C, \quad \frac{I(\xi - c\tau)}{I(\xi)} < C \quad and \quad \left| \frac{I'(\xi)}{I(\xi)} \right| < C.$$

Proof. Let $\theta(\xi) = \frac{I'(\xi)}{I(\xi)}$, from the third equation of (2.4), we have

$$\theta(\xi) \geq \frac{d_3}{c} \left(\int_{\mathbb{R}} J(y) \frac{I(\xi - y)}{I(\xi)} dy - 1 \right) - \frac{\gamma + \mu_3}{c}$$

$$= \frac{d_3}{c} \int_{\mathbb{R}} J(y) e^{\int_\xi^{\xi-y} \theta(s) ds} dy - \left(\frac{d_3 + \gamma + \mu_3}{c} \right).$$

Set $\varpi = \left(\frac{d_3 + \gamma + \mu_3}{c} \right)$ and $W(\xi) = \exp \left\{ \varpi \xi + \int_0^\xi \theta(s) ds \right\}$, thus

$$W'(\xi) = (\varpi + \theta(\xi)) W(\xi) \geq \frac{d_3}{c} \int_{\mathbb{R}} J(y) e^{\int_\xi^{\xi-y} \theta(s) ds} dy W(\xi),$$

that is $W(\xi)$ is non-decreasing. We can take some $R_0 > 0$ with $2R_0 < R$, where R is the radius of supp$J$. Then by the same argument in [34, Lemma 2.2], we have

$$W(\xi) \geq \frac{d_3}{c} R_0 \int_{\mathbb{R}} J(y) e^{\varpi y} W(\xi - R_0 - y) dy$$

and

$$W(\xi + R_0) \leq \sigma_0 W(\xi) \quad \text{for all} \quad \xi \in \mathbb{R},$$

where

$$\sigma_0 \triangleq \frac{d_3}{c R_0 \int_{-\infty}^{-2R_0} J(y) e^{\varpi y} dy}.$$

Thus

$$\int_{\mathbb{R}} J(y) \frac{I(\xi - y)}{I(\xi)} dy = \int_{-\infty}^0 J(y) \frac{I(\xi - y)}{I(\xi)} dy + \int_0^{+\infty} J(y) \frac{I(\xi - y)}{I(\xi)} dy$$

$$= \int_{-\infty}^0 J(y) e^{\varpi y} \frac{W(\xi - y)}{W(\xi)} dy + \int_0^{+\infty} J(y) e^{\varpi y} \frac{W(\xi - y)}{W(\xi)} dy$$

$$\leq \sigma_0 \int_{-\infty}^0 J(y) e^{\varpi y} \frac{W(\xi - y - R_0)}{W(\xi)} dy + \int_0^{+\infty} J(y) e^{\varpi y} dy$$

$$\leq \frac{c \sigma_0}{d_3 R_0} + \int_0^{+\infty} J(y) e^{\varpi y} dy.$$

Again with the third equation of (2.4), we have

$$I'(\xi) + \varpi I(\xi) = d_3 J * I(\xi) + \beta_1 S(\xi) I(\xi - c\tau) + \beta_2 V(\xi) I(\xi - c\tau) > 0 \quad \text{for all} \quad \xi \in \mathbb{R}.$$

Let $U(\xi) = e^{\varpi \xi} I(\xi)$, then $U'(\xi) \geq 0$, it follows that

$$\frac{I(\xi - c\tau)}{I(\xi)} \leq e^{\varpi c\tau}.$$

Furthermore,

$$\left| \frac{I'(\xi)}{I(\xi)} \right| \leq \frac{d_3}{c} \int_{\mathbb{R}} J(y) \frac{I(\xi - y)}{I(\xi)} dy + (\beta_1 S_0 + \beta_2 V_0) \frac{I(\xi - c\tau)}{I(\xi)} + \varpi.$$

This completes the proof. □

Lemma 2.11. *Choose $c_k \in (c^*, c^* + 1)$ and let $\{c_k, S_k, V_k, I_k\}$ be a sequence of traveling waves of (2.1) with speeds $\{c_k\}$. If there is a sequence $\{\xi_k\}$ such that $I_k(\xi_k) \to +\infty$ as $k \to +\infty$, then $S_k(\xi_k) \to 0$ and $V_k(\xi_k) \to 0$ as $k \to +\infty$.*

Proof. Assume that there is a subsequence of $\{\xi_k\}_{k\in\mathbb{N}}$ again denoted by ξ_k, such that $I_k(\xi_k) \to +\infty$ as $k \to +\infty$ and $S_k(\xi_k) \geq \varepsilon$ in $\mathbb{R}$ for all $k \in \mathbb{N}$ with some positive constant ε. From the first equation of (2.4), we have

$$S'_k(\xi) \leq \frac{2d_1 S_0 + \Lambda}{c^*} \triangleq \text{delta}_0 \ \text{ in } \ \mathbb{R}.$$

It follows that

$$S_k(\xi) \geq \frac{\varepsilon}{2}, \quad \forall \xi \in [\xi_k - \text{delta}, \xi_k],$$

for all $k \in \mathbb{N}$, where delta $= \frac{\varepsilon}{\text{delta}_0}$. By Lemma 2.10, we have $\left|\frac{I'_k}{I_k}\right| < C_0$ for some $C_0 > 0$. Then

$$\frac{I_k(\xi_k)}{I_k(\xi - c\tau)} = \exp\left\{\int_{\xi-c\tau}^{\xi_k} \frac{I'_k(s)}{I_k(s)}\text{ds}\right\} \leq e^{C_0(c\tau+\text{delta})}, \quad \forall \xi \in [\xi_k - \text{delta}, \xi_k]$$

for all $k \in \mathbb{N}$. Thus

$$\min_{\xi\in[\xi_k-\text{delta},\xi_k]} I_k(\xi - c\tau) \geq e^{-C_0(c\tau+\text{delta})} I_k(\xi_k),$$

which give us

$$\min_{\xi\in[\xi_k-\text{delta},\xi_k]} I_k(\xi - c\tau) \to +\infty \ \text{ as } \ k \to +\infty$$

since $I_k(\xi_k) \to +\infty$ as $k \to +\infty$. Recalling the first equation of (2.4), one can have

$$\max_{\xi\in[\xi_k-\text{delta},\xi_k]} S'_k(\xi) \leq \text{delta}_0 - \frac{\beta_1\varepsilon}{2} \min_{\xi\in[\xi_k-\text{delta},\xi_k]} I_k(\xi - c\tau) \to -\infty \ \text{ as } \ k \to +\infty.$$

Moreover, there exists some $K > 0$ such that

$$S'_k(\xi) \leq -\frac{2S_0}{\text{delta}}, \quad \forall k \geq K \ \text{ and } \ \xi \in [\xi_k - \text{delta}, \xi_k].$$

Note that $S_k < S_0$ in $\mathbb{R}$ for each $k \in \mathbb{N}$. Hence $S_k(\xi_k) \leq -S_0$ for all $k \geq K$, which reduces to a contradiction since $S_k(\xi_k) \geq \varepsilon$ in $\mathbb{R}$ for all $k \in \mathbb{N}$ with some positive constant ε. Similarly, we can show that $V_k(\xi_k) \to 0$ as $k \to +\infty$. This completes the proof. $\square$

Lemma 2.12. *If* $\limsup_{\xi\to\infty} I(\xi) = \infty$, *then* $\lim_{\xi\to\infty} I(\xi) = \infty$.

The proof is similar to that of [34, Lemma 2.4], so we omit the details. With the previous lemmas, we can show that $I(\xi)$ is bounded in $\mathbb{R}$.

Theorem 2.2. *$I(\xi)$ is bounded in $\mathbb{R}$.*

Proof. Assume that $\limsup_{\xi\to\infty} I(\xi) = \infty$, then we have $\lim_{\xi\to\infty} S(\xi) = 0$ and $\lim_{\xi\to\infty} V(\xi) = 0$ from Lemma 2.11 and Lemma 2.12. Set $\theta(\xi) = \frac{I'(\xi)}{I(\xi)}$, from the third equation of (2.4), we have

$$c\theta(\xi) = d_3 \int_{\mathbb{R}} J(y)e^{\int_\xi^{\xi-y} \theta(s)\text{ds}} - (d_3 + \gamma + \mu_3) + B(\xi),$$

where

$$B(\xi) = [\beta_1 S(\xi) + \beta_2 V(\xi)]\frac{I(\xi - c\tau)}{I(\xi)}.$$

Since $\frac{I(\xi-c\tau)}{I(\xi)} < C$ for some positive constant C from Lemma 2.10. By using [34, Lemma 2.5], we can get that $\lim\limits_{\xi\to+\infty} \theta(\xi)$ exists and satisfies the following equation

$$f(\lambda, c) \triangleq d_3\left(\int_{\mathbb{R}} J(y)e^{-\lambda y} - 1\right) - c\lambda - (\gamma + \mu_3).$$

By some calculations, we obtain

$$f(0, c) < 0, \quad \left.\frac{\partial f(\lambda, c)}{\partial \lambda}\right|_{\lambda=0} < 0, \quad \frac{\partial^2 f(\lambda, c)}{\partial \lambda^2} > 0 \text{ and } \lim_{\lambda\to+\infty} f(\lambda, c) = -\infty.$$

Thus, $I(\xi)$ is bounded by using the same arguments in [34, Theorem 2.6]. This ends the proof. $\qquad\square$

Since $I(\xi)$ is bounded in $\mathbb{R}$, we assume that there exists a positive constant $\rho < \infty$ such that $I(\xi) < \rho$. Furthermore, it can be verified $\frac{\Lambda}{\mu_1+\alpha+\beta_1\rho}$ is a lower solution of S and $\frac{\alpha\Lambda}{(\mu_1+\alpha+\beta_1\rho)(\mu_2+\beta_2\rho)}$ is a lower solution of V. Then we have the following proposition.

Proposition 2.1. *$S(\xi), V(\xi)$ and $I(\xi)$ satisfy*

$$\frac{\Lambda}{\mu_1 + \alpha + \beta_1\rho} \leq S(\xi) \leq S_0, \quad \frac{\alpha\Lambda}{(\mu_1 + \alpha + \beta_1\rho)(\mu_2 + \beta_2\rho)} \leq V(\xi) \leq V_0, \quad \underline{I}(\xi) \leq I(\xi) \leq \rho$$

for $\xi \in \mathbb{R}$.

The following lemma is to show that $I(\xi)$ cannot approach 0.

Lemma 2.13. *Assume that $\mathfrak{R}_0 > 1$, then for each $c > c^*$, we have*

$$\liminf_{\xi\to\infty} I(\xi) > 0.$$

Proof. We only need to show that if $I(\xi) \leq \varepsilon_0$ for some small enough constant $\varepsilon_0 > 0$, then $I'(\xi) > 0$ for all $\xi \in \mathbb{R}$. Assume by way of contradiction that there is no such ε_0, that is there exist some sequence $\{\xi_k\}_{k\in\mathbb{N}}$ such that $I(\xi_k) \to 0$ as $k \to +\infty$ and $I'(\xi_k) \leq 0$. Denote

$$S_k(\xi) \triangleq S(\xi_k + \xi), \quad V_k(\xi) \triangleq V(\xi_k + \xi) \text{ and } I_k(\xi) \triangleq I(\xi_k + \xi).$$

Thus we have $I_k(0) \to 0$ as $k \to +\infty$ and $I_k(\xi) \to 0$ locally uniformly in $\mathbb{R}$ as $k \to +\infty$. As a consequence, there also holds that $I'_k(\xi) \to 0$ locally uniformly in $\mathbb{R}$ as $k \to +\infty$ by the third equation of (2.4). From the argument in [25, Theorem 2.9], we can obtain that $S_\infty = S_0$ and $V_\infty = V_0$.

Let $\psi_k(\xi) \triangleq \frac{I_k(\xi)}{I_k(0)}$. By Lemma 2.10, and in the view of

$$\psi'_k(\xi) = \frac{I'_k(\xi)}{I_k(0)} = \frac{I'_k(\xi)}{I_k(\xi)}\psi_k(\xi),$$

we have $\psi_k(\xi)$ and $\psi'_k(\xi)$ are also locally uniformly in $\mathbb{R}$ as $k \to +\infty$. Letting $k \to +\infty$, thus

$$c\psi'_\infty(\xi) = d_3 \int_{\mathbb{R}} J(y)\psi_\infty(\xi - y)\mathrm{d}y + (\beta_1 S_0 + \beta_2 V_0)\psi_\infty(\xi - c\tau) - (d_3 + \gamma + \mu_3)\psi_\infty(\xi).$$

One can have $\psi_\infty(\xi) > 0$ in $\mathbb{R}$. In fact, if there exist some ξ_0 such that $\psi_\infty(\xi_0) = 0$ and $\psi_\infty(\xi) > 0$ for all $\xi < \xi_0$, then

$$0 = d_3 \int_\mathbb{R} J(y)\psi_\infty(\xi_0 - y)\mathrm{d}y + (\beta_1 S_0 + \beta_2 V_0)\psi_\infty(\xi_0 - c\tau) > 0,$$

which is a contradiction.

Denote $Z(\xi) \triangleq \frac{\psi'_\infty(\xi)}{\psi_\infty(\xi)}$, it is easy to verify $Z(\xi)$ satisfies

$$cZ(\xi) = d_3 \int_\mathbb{R} J(y)e^{\int_\xi^{\xi-y} Z(s)\mathrm{d}s}\mathrm{d}y + (\beta_1 S_0 + \beta_2 V_0)e^{\int_\xi^{\xi-c\tau} Z(s)\mathrm{d}s} - (d_3 + \gamma + \mu_3). \qquad (2.18)$$

Then by similar discussion in [25, Theorem 2.9], for $\mathfrak{R}_0 > 1$ and $c > c^*$, we have

$$0 < \psi'_\infty(0) = \lim_{k\to+\infty} \psi'_n(0) = \lim_{k\to+\infty} \frac{I'_n(0)}{I_n(0)}.$$

Thus, $I'(\xi_k) = I'_n(0) > 0$, which is a contradiction. This completes the proof. □

Remark 2.1. *In the proof of Lemma 2.13, we need to show that $Z(\pm\infty)$ exist in Equation (2.18). In [25], the authors applying [37, Lemma 3.4] to show that $Z(\pm\infty)$ exist. There is a time delay term in Equation (2.18) which is different from [37, Lemma 3.4], but we can still using the method in [37, Lemma 3.4] to proof $Z(\pm\infty)$ exist. The proof is trivial, so we omitted it.*

Now, we can give the main result in this section.

Theorem 2.3. *Suppose $\mathfrak{R}_0 > 1$, then for every $c > c^*$, system (2.1) admits a nontrivial traveling wave solution $(S(x + ct), V(x + ct), I(x + ct))$ satisfying the asymptotic boundary condition (2.5) and (2.6).*

Proof. First, it is easy to verify that $S(-\infty) = S_0, V(-\infty) = V_0, I(-\infty) = 0$ by Lemmas 2.4, 2.5 and 2.6.

Next, we will show $(S(\xi), V(\xi), I(\xi)) = (S^*, V^*, I^*)$ as $\xi \to +\infty$ by using Lyapunov function. From Proposition 2.1 and Lemma 2.13, we have $S(\xi) > 0$, $V(\xi) > 0$ and $I(\xi) > 0$.

Let $g(x) = x - 1 - \ln x$, $\alpha^+(y) = \int_y^{+\infty} J(x)\mathrm{d}x$, $\alpha^-(y) = \int_{-\infty}^y J(x)\mathrm{d}x$. Since J is compactly supported, and recall that R is the radius of supp$J$, hence

$$\alpha^+(y) \equiv 0 \text{ and } \alpha^-(y) \equiv 0 \text{ for } |y| \geq R. \qquad (2.19)$$

Define the following Lyapunov functional

$$L(S, V, I)(\xi) = cS^*L_1(\xi) + cV^*L_2(\xi) + cI^*L_3(\xi) + d_1S^*U_1(\xi) + d_2V^*U_2(\xi) + d_3I^*U_3(\xi)$$

where

$$L_1(\xi) = g\left(\frac{S(\xi)}{S^*}\right); \quad L_2(\xi) = g\left(\frac{V(\xi)}{V^*}\right);$$

$$L_3(\xi) = g\left(\frac{I(\xi)}{I^*}\right) + (\mu_3 + \gamma)I^* \int_0^{c\tau} g\left(\frac{I(\xi - \theta)}{I^*}\right)\mathrm{d}\theta;$$

$$U_1(\xi) = \int_0^{+\infty} \alpha^+(y)g\left(\frac{S(\xi - y)}{S^*}\right)\mathrm{d}y - \int_{-\infty}^0 \alpha^-(y)g\left(\frac{S(\xi - y)}{S^*}\right)\mathrm{d}y;$$

$$U_2(\xi) = \int_0^{+\infty} \alpha^+(y)g\left(\frac{V(\xi-y)}{V^*}\right)dy - \int_{-\infty}^0 \alpha^-(y)g\left(\frac{V(\xi-y)}{V^*}\right)dy;$$

$$U_3(\xi) = \int_0^{+\infty} \alpha^+(y)g\left(\frac{I(\xi-y)}{I^*}\right)dy - \int_{-\infty}^0 \alpha^-(y)g\left(\frac{I(\xi-y)}{I^*}\right)dy.$$

Thanks to [38, Theorem 1] and $S(\xi) > 0$, $V(\xi) > 0$, $I(\xi) > 0$, we can get that $L_1(\xi)$, $L_2(\xi)$ and $L_3(\xi)$ are bounded from below. Furthermore, by using (2.19), Proposition 2.1 and Lemma 2.13, we can claim that $U_1(\xi)$, $U_2(\xi)$ and $U_3(\xi)$ is bounded from below. Thus $L(S, V, I)(\xi)$ is well defined and bounded from below. Note that $\alpha^\pm = \frac{1}{2}$, $\frac{d\alpha^+(y)}{dy} = J(y)$ and $\frac{d\alpha^-(y)}{dy} = -J(y)$, we have

$$
\begin{aligned}
\frac{dU_1(\xi)}{d\xi} &= \frac{d}{d\xi}\int_0^{+\infty} \alpha^+(y)g\left(\frac{S(\xi-y)}{S^*}\right)dy - \frac{d}{d\xi}\int_{-\infty}^0 \alpha^-(y)g\left(\frac{S(\xi-y)}{S^*}\right)dy \\
&= \int_0^{+\infty} \alpha^+(y)\frac{d}{d\xi}g\left(\frac{S(\xi-y)}{S^*}\right)dy - \int_{-\infty}^0 \alpha^-(y)\frac{d}{d\xi}g\left(\frac{S(\xi-y)}{S^*}\right)dy \\
&= -\int_0^{+\infty} \alpha^+(y)\frac{d}{dy}g\left(\frac{S(\xi-y)}{S^*}\right)dy + \int_{-\infty}^0 \alpha^-(y)\frac{d}{dy}g\left(\frac{S(\xi-y)}{S^*}\right)dy \\
&= g\left(\frac{S(\xi)}{S^*}\right) - \int_{-\infty}^{+\infty} J(y)g\left(\frac{S(\xi-y)}{S^*}\right)dy.
\end{aligned}
$$

Similarly,

$$\frac{dU_2(\xi)}{d\xi} = g\left(\frac{V(\xi)}{V^*}\right) - \int_{-\infty}^{+\infty} J(y)g\left(\frac{V(\xi-y)}{V^*}\right)dy;$$

$$\frac{dU_3(\xi)}{d\xi} = g\left(\frac{I(\xi)}{I^*}\right) - \int_{-\infty}^{+\infty} J(y)g\left(\frac{I(\xi-y)}{I^*}\right)dy.$$

By some calculations, it can be shown that

$$
\frac{d}{d\xi}\int_0^{c\tau} g\left(\frac{I(\xi-\theta)}{I^*}\right)d\theta = \int_0^{c\tau} \frac{d}{d\xi}g\left(\frac{I(\xi-\theta)}{I^*}\right)d\theta = -\int_0^{c\tau} \frac{d}{d\theta}g\left(\frac{I(\xi-\theta)}{I^*}\right)d\theta
$$

$$= \frac{I(\xi)}{I^*} - \frac{I(\xi-c\tau)}{I^*} + \ln\frac{I(\xi-c\tau)}{I(\xi)}.$$

Thus

$$
\begin{aligned}
\frac{dL(\xi)}{d\xi} =& \left(1 - \frac{S^*}{S(\xi)}\right)(d_1(J * S(\xi) - S(\xi)) + \Lambda - \beta_1 S(\xi)I(\xi-c\tau) - (\alpha+\mu_1)S(\xi)) \\
&+ \left(1 - \frac{V^*}{V(\xi)}\right)(d_2(J * V(\xi) - V(\xi)) + \alpha S(\xi) - \beta_2 V(\xi)I(\xi-c\tau) - \mu_2 E(\xi)) \\
&+ \left(1 - \frac{I^*}{I(\xi)}\right)(d_3(J * I(\xi) - I(\xi)) + \beta_1 S(\xi)I(\xi-c\tau) + \beta_2 V(\xi)I(\xi-c\tau) - (\gamma+\mu_3)I(\xi)) \\
&+ (\mu_3+\gamma)I^*\left(\frac{I(\xi)}{I^*} - \frac{I(\xi-c\tau)}{I^*} + \ln\frac{I(\xi-c\tau)}{I(\xi)}\right) \\
&+ d_1 S^* g\left(\frac{S(\xi)}{S^*}\right) - d_1 S^*\int_{-\infty}^{+\infty} J(y)g\left(\frac{S(\xi-y)}{S^*}\right)dy
\end{aligned}
$$

$$+ d_2 V^* g\left(\frac{V(\xi)}{V^*}\right) - d_2 V^* \int_{-\infty}^{+\infty} J(y) g\left(\frac{V(\xi - y)}{V^*}\right) dy$$

$$+ d_3 I^* g\left(\frac{I(\xi)}{I^*}\right) - d_3 I^* \int_{-\infty}^{+\infty} J(y) g\left(\frac{I(\xi - y)}{I^*}\right) dy$$

$$\triangleq B_1 + B_2,$$

where

$$B_1 = \left(1 - \frac{S^*}{S(\xi)}\right) d_1 (J * S(\xi) - S(\xi)) + d_1 S^* g\left(\frac{S(\xi)}{S^*}\right) - d_1 S^* \int_{-\infty}^{+\infty} J(y) g\left(\frac{S(\xi - y)}{S^*}\right) dy$$

$$+ \left(1 - \frac{V^*}{V(\xi)}\right) d_2 (J * V(\xi) - V(\xi)) + d_2 V^* g\left(\frac{V(\xi)}{V^*}\right) - d_2 V^* \int_{-\infty}^{+\infty} J(y) g\left(\frac{V(\xi - y)}{V^*}\right) dy$$

$$+ \left(1 - \frac{I^*}{I(\xi)}\right) d_3 (J * I(\xi) - I(\xi)) + d_3 I^* g\left(\frac{I(\xi)}{I^*}\right) - d_3 I^* \int_{-\infty}^{+\infty} J(y) g\left(\frac{I(\xi - y)}{I^*}\right) dy,$$

and

$$B_2 = \left(1 - \frac{S^*}{S(\xi)}\right)(\Lambda - \beta_1 S(\xi) I(\xi - c\tau) - (\alpha + \mu_1) S(\xi))$$

$$+ \left(1 - \frac{V^*}{V(\xi)}\right)(\alpha S(\xi) - \beta_2 V(\xi) I(\xi - c\tau) - \mu_2 E(\xi))$$

$$+ \left(1 - \frac{I^*}{I(\xi)}\right)(\beta_1 S(\xi) I(\xi - c\tau) + \beta_2 V(\xi) I(\xi - c\tau) - (\gamma + \mu_3) I(\xi))$$

$$+ (\mu_3 + \gamma) I^* \left(\frac{I(\xi)}{I^*} - \frac{I(\xi - c\tau)}{I^*} + \ln \frac{I(\xi - c\tau)}{I(\xi)}\right).$$

For B_1, using $\ln \frac{S(\xi)}{S^*} = \ln \frac{S(\xi - y)}{S^*} - \ln \frac{S(\xi - y)}{S(\xi)}$, thus

$$\left(1 - \frac{S^*}{S(\xi)}\right) d_1 (J * S(\xi) - S(\xi)) + d_1 S^* g\left(\frac{S(\xi)}{S^*}\right) - d_1 S^* \int_{-\infty}^{+\infty} J(y) g\left(\frac{S(\xi - y)}{S^*}\right) dy$$

$$= d_1 S^* \int_{-\infty}^{+\infty} J(y) \left[\frac{S(\xi - y)}{S^*} - \frac{S(\xi - y)}{S(\xi)} - \ln \frac{S(\xi)}{S^*}\right] - d_1 S^* \int_{-\infty}^{+\infty} J(y) g\left(\frac{S(\xi - y)}{S^*}\right) dy$$

$$= d_1 S^* \int_{-\infty}^{+\infty} J(y) \left[g\left(\frac{S(\xi - y)}{S^*}\right) - g\left(\frac{S(\xi - y)}{S(\xi)}\right)\right] - d_1 S^* \int_{-\infty}^{+\infty} J(y) g\left(\frac{S(\xi - y)}{S^*}\right) dy$$

$$= - d_1 S^* \int_{-\infty}^{+\infty} J(y) g\left(\frac{S(\xi - y)}{S(\xi)}\right) dy.$$

Then

$$B_1 = - d_1 S^* \int_{-\infty}^{+\infty} J(y) g\left(\frac{S(\xi - y)}{S(\xi)}\right) dy - d_2 V^* \int_{-\infty}^{+\infty} J(y) g\left(\frac{V(\xi - y)}{V(\xi)}\right) dy$$

$$- d_3 I^* \int_{-\infty}^{+\infty} J(y) g\left(\frac{I(\xi - y)}{I(\xi)}\right) dy. \tag{2.20}$$

For B_2, by some calculation yields

$$
\begin{aligned}
B_2 =& \mu_1 S^* \left(2 - \frac{S(\xi)}{S^*} - \frac{S^*}{S(\xi)} \right) - \beta_1 S^* I^* g \left(\frac{S(\xi)I(\xi - c\tau)}{S^* I(\xi)} \right) \\
& - \beta_2 V^* I^* \left[g \left(\frac{V(\xi)I(\xi - c\tau)}{V^* I(\xi)} \right) + g \left(\frac{S(\xi)V^*}{S^* V(\xi)} \right) \right] \\
& - \mu_2 V^* \left[g \left(\frac{V(\xi)}{V^*} \right) + g \left(\frac{S(\xi)V^*}{S^* V(\xi)} \right) \right] \\
& - (\alpha S^* + \beta_1 S^* I^*) g \left(\frac{S^*}{S(\xi)} \right),
\end{aligned}
$$

here we use $(\mu_3 + \gamma)I^* = \beta_1 S^* I^* + \beta_2 V^* I^*$ and $\alpha S(\xi)\frac{V^*}{V(\xi)} = (\beta_2 V^* I^* + \mu_2 V^*)\frac{S(\xi)V^*}{S^* V(\xi)}$. Combining B_1 and B_2, we obtain $L(\xi)$ is decreasing in ξ.

Consider an increasing sequence $\{\xi_n\}_{n \geq 0}$ with $\xi_n > 0$ such that $\xi_n \to +\infty$ when $n \to +\infty$ and denote

$$
\{S_n(\xi) = S(\xi + \xi_n)\}_{n \geq 0}, \quad \{V_n(\xi) = V(\xi + \xi_n)\}_{n \geq 0,} \quad \text{and} \quad \{I_n(\xi) = I(\xi + \xi_n)\}_{n \geq 0}.
$$

We can assume that S_n, V_n and I_n converge to some nonnegative functions S_∞, V_∞ and I_∞. Furthermore, since $L(S, V, I)(\xi)$ is non-increasing on ξ, then there exists a constant $\hat{C}$ and large n such that

$$
\hat{C} \leq L(S_n, V_n, I_n)(\xi) = L(S, V, I)(\xi + \xi_n) \leq L(S, V, I)(\xi).
$$

Therefore there exists some de͠lta $\in \mathbb{R}$ such that $\lim_{n \to \infty} L(S_n, V_n, I_n)(\xi) = $ de͠lta for any $\xi \in \mathbb{R}$. By Lebegue dominated convergence theorem, gives us

$$
\lim_{n \to +\infty} L(S_n, V_n, I_n)(\xi) = L(S_\infty, V_\infty, I_\infty)(\xi), \ \xi \in \mathbb{R}.
$$

Thus

$$
L(S_\infty, V_\infty, I_\infty)(\xi) = \text{de͠lta}.
$$

Note that $\frac{dL}{d\xi} = 0$ if and only if $S(\xi) \equiv S^*$, $V(\xi) \equiv V^*$ and $I(\xi) \equiv I^*$, it follows that

$$
(S_\infty, V_\infty, I_\infty) \equiv (S^*, V^*, I^*).
$$

This completes the proof. $\qquad \square$

3. Existence of traveling wave solutions for $c = c^*$

In this section, we investigate the existence of traveling wave solutions for the case $c = c^*$ by a limiting argument(see [23, 39]).

Theorem 3.1. *Suppose $\mathfrak{R}_0 > 1$, then for every $c = c^*$, system (2.1) admits a nontrivial traveling wave solution $(S(x + c^*t), V(x + c^*t), I(x + c^*t))$ satisfying*

$$
\lim_{\xi \to +\infty} (S(\xi), V(\xi), I(\xi)) = (S^*, V^*, I^*).
$$

Furthermore, if we assume that $S(-\infty)$ and $V(-\infty)$ exist, then $(S(x + c^*t), V(x + c^*t), I(x + c^*t))$ also satisfying

$$
\lim_{\xi \to -\infty} (S(\xi), V(\xi), I(\xi)) = (S_0, V_0, 0).
$$

Proof. Let $\{c_n\} \subset (c^*, c^* + 1)$ be a decreasing sequence such that $\lim_{n\to\infty} c_n = c^*$. Then for each c_n, there exists a traveling wave solution $(S_n(\cdot), V_n(\cdot), I_n(\cdot))$ of system (2.4) with asymptotic boundary condition (2.5) and (2.6). Since $(S_n(\cdot + a), V_n(\cdot + a), I_n(\cdot + a))$ are also solutions of (2.4) for any $a \in \mathbb{R}$, we can assume that

$$I_n(0) = \text{delta}^*, \quad I_n(\xi) \leq \text{delta}^*, \quad \xi < 0$$

with $0 < \text{delta} < I^*$ is small enough.

Similar to [23, 39], we can find a subsequence of (S_n, V_n, I_n), again denoted by (S_n, V_n, I_n), such that (S_n, V_n, I_n) and (S'_n, V'_n, I'_n) converge uniformly on every bounded interval to function (S, V, I) and (S', V', I'), respectively. Applying the Lebesgue dominated convergence theorem, it then follows that

$$\lim_{n\to\infty} J * S_n = J * S, \quad \lim_{n\to\infty} J * V_n = J * V, \quad \text{and} \quad \lim_{n\to\infty} J * I_n = J * I$$

on every bounded interval. Then we get that (S, V, I) satisfies system (2.4). From the proof of Theorem 2.3, the Lyapunov functional is independent of c. By the same argument in the proof of Theorem 2.3, we claim that $I(\xi) > 0$ for any $\xi \in \mathbb{R}$. Hence, we can still get that

$$\lim_{\xi\to+\infty} S(\xi) = S^*, \quad \lim_{\xi\to+\infty} V(\xi) = V^*, \quad \lim_{\xi\to+\infty} I(\xi) = I^*.$$

Moreover, we have

$$I(0) = \text{delta}^*, \quad I(\xi) \leq \text{delta}^*, \quad \xi < 0.$$

Let

$$S_{sup} = \limsup_{\xi\to-\infty} S(\xi), \quad V_{sup} = \limsup_{\xi\to-\infty} V(\xi), \quad I_{sup} = \limsup_{\xi\to-\infty} I(\xi)$$

and

$$S_{inf} = \liminf_{\xi\to-\infty} S(\xi), \quad V_{inf} = \liminf_{\xi\to-\infty} V(\xi), \quad I_{inf} = \liminf_{\xi\to-\infty} I(\xi).$$

Next, we show that $I(-\infty)$ exists. By way of contradiction, assume that $I_{inf} < I_{sup}$. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ satisfying $x_n, y_n \to -\infty$ as $n \to +\infty$ such that

$$\lim_{n\to+\infty} I(x_n) = I_{inf} \quad \lim_{n\to+\infty} I(y_n) = I_{sup}.$$

Since we assumed that $S(-\infty)$ and $V(-\infty)$ exist, then $S_{sup} = S_{inf} = S(-\infty)$ and $V_{sup} = V_{inf} = V(-\infty)$. From [40, Lemma 2.3], we can obtain that $S'(-\infty) = 0$ and $V'(-\infty) = 0$. For any sequence $\{\xi_n\}$, $\xi_n \to -\infty$ as $n \to +\infty$, using Fatou Lemma, one have that

$$S(-\infty) \leq \liminf_{n\to\infty} J * S(\xi_n) \leq \limsup_{n\to\infty} J * S(\xi_n) \leq S(-\infty).$$

and

$$V(-\infty) \leq \liminf_{n\to\infty} J * V(\xi_n) \leq \limsup_{n\to\infty} J * V(\xi_n) \leq V(-\infty).$$

Thus, we have

$$\lim_{n\to\infty} [J * S(\xi_n) - S(\xi_n)] = 0 \quad \text{and} \quad \lim_{n\to\infty} [J * V(\xi_n) - V(\xi_n)] = 0$$

Taking $\xi = x_n$ and $\xi = y_n$ in the first equation of system 2.4, and letting $n \to \infty$, we obtain that $I_{inf} = I_{sup}$, which is a contradiction. Hence, $I(-\infty)$ exists and $I(-\infty) <$ delta*. From system (2.4) and [40, Lemma 2.3], we obtain

$$\begin{cases} \Lambda - \beta_1 S(-\infty)I(-\infty) - \alpha S(-\infty) - \mu_1 S(-\infty = 0), \\ \alpha S(-\infty) - \beta_2 V(-\infty)I(-\infty) - \mu_2 V(-\infty) = 0, \\ \beta_1 S(-\infty)I(-\infty) + \beta_2 V(-\infty)I(-\infty)) - \gamma I(-\infty) - \mu_3 I(-\infty) = 0. \end{cases} \quad (3.1)$$

In the view of delta$^* <$ I*, it follows that

$$\lim_{\xi \to -\infty} S(\xi) = S_0, \quad \lim_{\xi \to -\infty} V(\xi) = V_0, \quad \lim_{\xi \to -\infty} I(\xi) = 0.$$

This completes the proof. □

Remark 3.1. *For the case $c = c^*$, there is a priori condition assuming $S(-\infty)$ and $V(-\infty)$ exist. This condition is only necessary for the difficulty in mathematics. In [34], the authors have given some results for the case $c = c^*$ in a nonlocal diffusive SIR model without constant recruitment, but some estimates is much more difficult for our model with constant recruitment and time delay as in [34, Section 3]. Thus, how to extend the methods in [34] to our model, it will be an interesting problem for further investigation.*

4. Nonexistence of traveling wave solutions

In this section, we show the nonexistence of traveling waves when $\mathfrak{R}_0 > 1$ with $0 < c < c^*$.

Theorem 4.1. *If $\mathfrak{R}_0 > 1$ and $0 < c < c^*$, then there exists no nontrivial positive solutions of (2.4) with (2.5) and (2.6).*

Proof. Since $\mathfrak{R}_0 > 1$ gives us $\beta_1 S_0 + \beta_2 V_0 > \mu_3 + \gamma$. Assume there exists nontrivial positive solution (S, V, I) of (2.4) with (2.5) and (2.6). Then there exists a positive constant $K > 0$ large enough such that, for any $\xi < -K$, we have

$$cI'(\xi) \geq d_3(J * I(\xi) - I(\xi)) + \frac{\beta_1 S_0 + \beta_2 V_0 - (\gamma + \mu_3)}{2} I(\xi - c\tau) + (\gamma + \mu_3)(I(\xi - c\tau) - I(\xi)) \quad (4.1)$$

holds. Let $K(\xi) = \int_{-\infty}^{\xi} I(\eta)d\eta$. By Fubini theorem, thus

$$\begin{aligned} d_3 \int_{-\infty}^{\xi} J * I(s)ds &= d_3 \int_{-\infty}^{\xi} \int_{\mathbb{R}} J(y)I(s - y)dyds \\ &= d_3 \int_{\mathbb{R}} \int_{-\infty}^{\xi} J(y)I(s - y)dsdy \\ &= d_3 \int_{\mathbb{R}} J(y) \int_{-\infty}^{\xi} I(s - y)dsdy \\ &= d_3 J * K(\xi). \end{aligned} \quad (4.2)$$

Integrating the both sides of (4.1) from $-\infty$ to ξ with $\xi \leq -K$, we have

$$cI(\xi) \geq d_3(J * K(\xi) - K(\xi)) + (\gamma + \mu_3)[K(\xi - c\tau) - K(\xi)]$$
$$+ \frac{\beta_1 S_0 + \beta_2 V_0 - (\gamma + \mu_3)}{2} K(\xi - c\tau). \tag{4.3}$$

Furthermore, the following two equations hold.

$$\int_{-\infty}^{\xi} [K(\eta - c\tau) - K(\eta)]\mathrm{d}\eta = \int_{-\infty}^{\xi} (-c\tau) \int_0^1 \frac{\partial K(\eta - c\tau s)}{\partial s} \mathrm{d}s\mathrm{d}\eta$$
$$= -c\tau \int_0^1 K(\xi - c\tau s)\mathrm{d}s \tag{4.4}$$

and

$$d_3 \int_{-\infty}^{\xi} [J * K(\eta) - K(\eta)]\mathrm{d}\eta = d_3 \int_{-\infty}^{\xi} \int_{-\infty}^{+\infty} (-x)J(x) \int_0^1 \frac{\partial K(\eta - xs)}{\partial s} \mathrm{d}s\mathrm{d}x\mathrm{d}\eta$$
$$= d_3 \int_{-\infty}^{+\infty} (-x)J(x) \int_0^1 K(\xi - xs)\mathrm{d}s\mathrm{d}x. \tag{4.5}$$

Integrating both sides of inequality (4.3) from $-\infty$ to ξ, and combining Equations (4.4) and (4.5) yield

$$\frac{\beta_1 S_0 + \beta_2 V_0 - (\gamma + \mu_3)}{2} \int_{-\infty}^{\xi} K(\eta - c\tau)\mathrm{d}\eta$$
$$\leq cK(\xi) + (\gamma + \mu_3)c\tau \int_0^1 K(\xi - c\tau s)\mathrm{d}s$$
$$+ d_3 \int_{-\infty}^{+\infty} xJ(x) \int_0^1 K(\xi - xs)\mathrm{d}s\mathrm{d}x$$
$$\leq \left(c + d_3 \int_{\mathbb{R}} xJ(x)\mathrm{d}x + (\gamma + \mu_3)c\tau \right) K(\xi), \tag{4.6}$$

Here we use $xK(\xi - sx)$ as a non-increasing function with $s \in (0, 1)$. By (J1) of Assumption 1.1, we have $\int_{\mathbb{R}} xJ(x)\mathrm{d}x = 0$. Then for $\xi < -K$, we have

$$\frac{\beta_1 S_0 + \beta_2 V_0 - (\gamma + \mu_3)}{2} \int_0^{+\infty} K(\xi - \eta - c\tau)\mathrm{d}\eta$$
$$\leq (c + (\gamma + \mu_3)c\tau)K(\xi), \tag{4.7}$$

For the non-decreasing function $K(\xi)$, there exists some $\tilde{\eta}$ with $\tilde{\eta} + c\tau > 0$ such that

$$\frac{\beta_1 S_0 + \beta_2 V_0 - (\gamma + \mu_3)}{2}(\tilde{\eta} + c\tau)K(\xi - \tilde{\eta} - c\tau)$$
$$\leq (c + (\gamma + \mu_3)c\tau)K(\xi), \tag{4.8}$$

Thus there exists a sufficiently large constant $\theta > -c\tau$ and some constant $\varepsilon \in (0, 1)$, such that

$$K(\xi - \theta - c\tau) \leq \varepsilon K(\xi), \quad \xi \leq -M.$$

Let

$$p(\xi) = K(\xi)e^{-\nu\xi},$$

where

$$0 < \nu \triangleq \frac{1}{\theta + c\tau} \ln \frac{1}{\varepsilon} < \lambda_c,$$

By some simple calculation, we have

$$p(\xi - \theta - c\tau) \le p(\xi).$$

Using L'Hospital's rule yields

$$\lim_{\xi \to +\infty} p(\xi) = \lim_{\xi \to +\infty} \frac{K(\xi)}{e^{\nu\xi}} = \lim_{\xi \to +\infty} \frac{I(\xi)}{\nu e^{\nu\xi}} = 0,$$

Note that $p(\xi) \ge 0$, thus there exists a constant p_0 such that

$$p(\xi) = K(\xi)e^{-\nu\xi} \le p_0, \quad \xi \in \mathbb{R}. \tag{4.9}$$

On the other hand, since $S(\xi) \le S_0$ and $V(\xi) \le V_0$ for $\xi \in \mathbb{R}$, recall the third equation of (2.4), we have

$$
\begin{aligned}
cI'(\xi) &= d_3(J * I(\xi) - I(\xi)) + \beta_1 S(\xi)I(\xi - c\tau) + \beta_2 V(\xi)I(\xi - c\tau) - \gamma I(\xi) - \mu_3 I(\xi) \\
&\le d_3(J * I(\xi) - I(\xi)) + \beta_1 S_0 I(\xi - c\tau) + \beta_2 V_0 I(\xi - c\tau) - \gamma I(\xi) - \mu_3 I(\xi).
\end{aligned} \tag{4.10}
$$

Integrating the both sides of (4.10) from $-\infty$ to ξ yields

$$cI(\xi) \le d_3 J * K(\xi) - (\gamma + \mu_3 + d_3)K(\xi) + (\beta_1 S_0 + \beta_2 V_0)K(\xi - c\tau). \tag{4.11}$$

From (4.9), using J is compactly supported, for $\xi \in \mathbb{R}$, there exists a positive constant M_1 such that

$$
\begin{aligned}
(d_3 J * K(\xi))e^{-\nu\xi} &= d_3 \int_{\mathbb{R}} J(y)e^{-\nu\xi}K(\xi - y)\mathrm{d}y \\
&= d_3 \int_{\mathbb{R}} J(y)e^{-\nu y}K(\xi - y)e^{-\nu(\xi - y)}\mathrm{d}y \\
&\le d_3 p_0 \int_{\mathbb{R}} J(y)e^{-\nu y}\mathrm{d}y \\
&\le M_1.
\end{aligned} \tag{4.12}
$$

Thus there exists a constant $M_2 > 0$ such that

$$I(\xi)e^{-\nu\xi} \le M_2, \quad \xi \in \mathbb{R}, \tag{4.13}$$

since (4.9), (4.11) and (4.12) hold. Then

$$\sup_{\xi \in \mathbb{R}}\{I(\xi)e^{-\nu\xi}\} < +\infty. \tag{4.14}$$

By the same procedure in (4.12), there exists a positive constant M_2 such that

$$(d_3 J * I(\xi))e^{-\nu\xi} \le M_2. \tag{4.15}$$

Hence

$$\sup_{\xi \in \mathbb{R}} \{I'(\xi)e^{-\nu\xi}\} < +\infty. \tag{4.16}$$

For $\lambda \in \mathbb{C}$ with $0 < \mathrm{Re}\lambda < \nu$, define the following two-side Laplace transform of $I(\xi)$,

$$\mathcal{L}_I(\lambda) := \int_{\mathbb{R}} I(\xi)e^{-\lambda\xi}\mathrm{d}\xi.$$

From (2.4), we have

$$d_3(J * I(\xi) - I(\xi)) - cI'(\xi) + (\beta_1 S_0 + \beta_2 V_0)I(\xi - c\tau) - (\gamma + \mu_3)I(\xi)$$
$$= \beta_1(S_0 - S(\xi))I(\xi - c\tau) + \beta_2(V_0 - V(\xi))I(\xi - c\tau). \tag{4.17}$$

Take the two-side Laplace transform to the above equation, thus

$$\Delta(\lambda, c)\mathcal{L}_I(\lambda) = \int_{\mathbb{R}} e^{-\lambda\xi}[\beta_1(S_0 - S(\xi))I(\xi - c\tau) + \beta_2(V_0 - V(\xi))I(\xi - c\tau)]\mathrm{d}\xi \tag{4.18}$$

for $\lambda \in \mathbb{C}$ with $0 < \mathrm{Re}\lambda < \nu$. Let $L(\xi) = S_0 - S(\xi)$, we have $0 \le L(\xi) \le S_0$ and $\lim_{\xi \to -\infty} L(\xi) = 0$. Then from the first equation of (2.4), we have

$$cL'(\xi) = d_1(J * L(\xi) - L(\xi)) + \beta_1 S(\xi)I(\xi - c\tau) + (\alpha + \mu_1)S(\xi).$$

Let $\eta \in C^\infty(\mathbb{R}, [0, 1])$ be a nonnegative nondecreasing function, $\eta(x) \equiv 0$ in $(-\infty, -2]$ and $\eta(x) \equiv 1$ in $[-1, +\infty)$. For $N \in \mathbb{N}$, set $\eta_N = \eta\left(\frac{x}{N}\right)$. Then, taking $0 \le \nu_0 \le \nu$, we have

$$c\int_{\mathbb{R}} L'(\xi)e^{-\nu_0\xi}\eta_N\mathrm{d}\xi = d_1\int_{\mathbb{R}} (J * L(\xi) - L(\xi))e^{-\nu_0\xi}\eta_N\mathrm{d}\xi + \int_{\mathbb{R}} S(\xi)[\beta_1 I(\xi - c\tau) + \alpha + \mu_1]e^{-\nu_0\xi}\eta_N\mathrm{d}\xi.$$

By the argument in [22, Theorem 3.1], there exists a constant $\Xi > 0$ dependent on ν_0 such that

$$\int_{\mathbb{R}} L(\xi)e^{-\nu_0\xi}\mathrm{d}\xi \le \Xi.$$

Thus,

$$\int_{\mathbb{R}} \beta_1(S_0 - S(\xi))I(\xi - c\tau)e^{-(\nu+\nu_0)\xi}\mathrm{d}\xi \le \beta_1 \sup_{\xi \in \mathbb{R}}\{I(\xi)e^{-\nu\xi}\} \int_{\mathbb{R}} L(\xi)e^{-\nu_0\xi}\mathrm{d}\xi < \infty.$$

Similarly,

$$\int_{\mathbb{R}} \beta_2(V_0 - V(\xi))I(\xi - c\tau)e^{-(\nu+\nu_0)\xi}\mathrm{d}\xi < \infty.$$

From the property of Laplace transform [41], $\mathcal{L}_I(\lambda)$ is well defined with $\mathrm{Re}\lambda > 0$. Note that Equation (4.18) can be rewritten as

$$\int_{\mathbb{R}} e^{-\lambda\xi}\left[\Delta(\lambda, c)I(\xi) + \beta_1(S_0 - S(\xi))I(\xi - c\tau) + \beta_2(V_0 - V(\xi))I(\xi - c\tau)\right]\mathrm{d}\xi = 0. \tag{4.19}$$

Recall (J2) of Assumption 1.1, then $\Delta(\lambda, c) \to +\infty$ as $\xi \to +\infty$ for $c \in (0, c^*)$ which is a contradiction of (4.19). This completes the proof. $\qquad \square$

5. Discussion

As traveling wave solutions describe the transition from disease-free equilibrium to endemic equilibrium when the wave speed is larger than the minimal wave speed. Now, we focus on how the parameters in system (2.1) can affect the wave speed. Suppose $(\hat{\lambda}, \hat{c})$ be a zero root of $\Delta(\lambda, c)$, recall that $V_0 = \frac{\Lambda\alpha}{\mu_2(\mu_1+\alpha)}$ and $\mu_2 = \mu_1 + \gamma_1$, we have

$$\Delta(\hat{\lambda}, \hat{c}) = d_3 \int_{\mathbb{R}} J(x)e^{-\hat{\lambda}x}\mathrm{d}x - (d_3 + \gamma + \mu_3) - \hat{c}\hat{\lambda} + \beta_1 S_0 e^{-\hat{c}\tau\hat{\lambda}} + \frac{\beta_2\Lambda\alpha}{(\mu_1 + \gamma_1)(\mu_1 + \alpha)}e^{-\hat{c}\tau\hat{\lambda}} = 0.$$

By some calculations, we obtain

$$\frac{\mathrm{d}\hat{c}}{\mathrm{d}d_3} = \frac{\int_{\mathbb{R}} J(x)[e^{-\hat{\lambda}x} - 1]\mathrm{d}x}{\hat{\lambda}(1 + [\beta_1 S_0 + \beta_2 V_0]\tau e^{-\hat{c}\tau\hat{\lambda}})} > 0, \quad \frac{\mathrm{d}\hat{c}}{\mathrm{d}\tau} = -\frac{\beta_1 S_0 + \beta_2 V_0}{e^{\hat{c}\tau\hat{\lambda}} + \beta_1 S_0\tau + \beta_2 V_0\tau} < 0,$$

$$\frac{\mathrm{d}\hat{c}}{\mathrm{d}\beta_1} = \frac{S_0 e^{-\hat{c}\tau\hat{\lambda}}}{\hat{\lambda}(1 + [\beta_1 S_0 + \beta_2 V_0]\tau e^{-\hat{c}\tau\hat{\lambda}})} > 0, \quad \frac{\mathrm{d}\hat{c}}{\mathrm{d}\beta_2} = \frac{V_0 e^{-\hat{c}\tau\hat{\lambda}}}{\hat{\lambda}(1 + [\beta_1 S_0 + \beta_2 V_0]\tau e^{-\hat{c}\tau\hat{\lambda}})} > 0,$$

and

$$\frac{\mathrm{d}\hat{c}}{\mathrm{d}\gamma_1} = -\frac{\beta_2 V_0 e^{-\hat{c}\tau\hat{\lambda}}}{(\mu_1 + \gamma_1)\hat{\lambda}(1 + [\beta_1 S_0 + \beta_2 V_0]\tau e^{-\hat{c}\tau\hat{\lambda}})} < 0,$$

that is, $\hat{c}$ is a decreasing function on γ_1 and τ, while $\hat{c}$ is an increasing function on d_3, β_1 and β_2. From the biological point of view, this indicates the following four scenarios:

I. The more successful the vaccination, the slower the disease spreads;

II. The longer the latent period, the slower the disease spreads;

III. The faster infected individuals move, the faster the disease spreads;

IV. The more effective the infections are, the faster the disease spreads.

Now, we are in a position to make the following summary:

Mathematically, we investigated a nonlocal dispersal epidemic model with vaccination and delay; The existence of traveling wave solutions is studied by applying Schauder fixed point theorem with upper-lower solutions, that is there exists traveling wave solutions when $\Re_0 > 1$ with $c > c^*$. Furthermore, the boundary asymptotic behaviour of traveling wave solutions at $+\infty$ was established by the methods of constructing suitable Lyapunov like function. We also showed that there exists traveling wave solutions when $\Re_0 > 1$ with $c = c^*$. Finally, we proved the nonexistence of traveling wave solutions under the assumptions $\Re_0 > 1$ and $0 < c < c^*$.

Biologically, our results imply that the nonlocal dispersal and infection ability of infected individuals can accelerate the spreading of infectious disease, while the latent period and successful rate of vaccination can slow down the disease spreads.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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Appendix A: Proof of Lemma 2.4

Proof. If $\xi > \mathfrak{X}_1$, then $\underline{S}(\xi) = 0$, equation (2.11) holds. If $\xi < \mathfrak{X}_1$, then $\underline{S}(\xi) = S_0 - M_1 e^{\varepsilon_1 \xi}$, we have

$$c\underline{S}'(\xi) - d_1(J * \underline{S}(\xi) - \underline{S}(\xi)) - \Lambda + \beta_1 \underline{S}(\xi)\overline{I}(\xi - c\tau) + (\mu_1 + \alpha)\underline{S}(\xi)$$

$$= -c\varepsilon_1 M_1 e^{\varepsilon_1 \xi} + d_1 M_1 e^{\varepsilon_1 \xi} \int_{\mathbb{R}} J(x)e^{-\varepsilon_1 x}\mathrm{d}x - d_1 M_1 e^{\varepsilon_1 \xi} - \Lambda$$

$$\quad + \beta_1(S_0 - M_1 e^{\varepsilon_1 \xi})e^{\lambda_c(\xi - c\tau)} + (\mu_1 + \alpha)(S_0 - M_1 e^{\varepsilon_1 \xi})$$

$$\leq e^{\varepsilon_1 \xi}\left[-c\varepsilon_1 M_1 e^{\varepsilon_1 \xi} + d_1 M_1 e^{\varepsilon_1 \xi}\int_{\mathbb{R}} J(x)e^{-\varepsilon_1 x}\mathrm{d}x - d_1 M_1 e^{\varepsilon_1 \xi} + \beta_1 S_0\left(\frac{S_0}{M_1}\right)^{\frac{\lambda - \varepsilon_1}{\varepsilon_1}}\right].$$

Here we use

$$e^{(\lambda_c - \varepsilon_1)\xi} < \left(\frac{S_0}{M_1}\right)^{\frac{\lambda_c - \varepsilon_1}{\varepsilon_1}} \quad \text{for} \quad \xi < \mathfrak{X}_1.$$

Keeping $\varepsilon_1 M_1 = 1$, letting $M_1 \to +\infty$ for some $M_1 > S_0$ large enough and ε_1 small enough, we have

$$c\underline{S}'(\xi) - d_1(J * \underline{S}(\xi) - \underline{S}(\xi)) - \Lambda + \beta_1 \underline{S}(\xi)\overline{I}(\xi - c\tau) + (\mu_1 + \alpha)\underline{S}(\xi) \leq 0.$$

This completes the proof. □

Appendix B: Proof of Lemma 2.6

Proof. If $\xi > \frac{1}{\varepsilon_3} \ln \frac{1}{M_3}$, the Equation (2.13) holds since $\underline{I}(\xi) = 0$. If $\xi < \frac{1}{\varepsilon_3} \ln \frac{1}{M_3}$, then $\underline{I}(\xi) = e^{\lambda_c \xi}(1 - M_3 e^{\varepsilon_3 \xi})$, we have the following four cases.

Case I: $\xi > \max\{\mathfrak{X}_1, \mathfrak{X}_2\}$.

In this case, $\underline{S}(\xi) = \underline{V}(\xi) = 0$. Thus, Equation (2.13) is equivalent to

$$c\underline{I}'(\xi) \le d_3(J * \underline{I}(\xi) - \underline{I}(\xi)) - \gamma \underline{I}(\xi) - \mu_3 \underline{I}(\xi),$$

that is

$$c\lambda_c - d_3 \int_{\mathbb{R}} J(y) e^{-\lambda_c y} dy + d_3 + \gamma + \mu_3$$
$$\le M_3 e^{\varepsilon_3 \xi} \left[c(\lambda + \varepsilon_3) - d_3 \int_{\mathbb{R}} J(y) e^{-(\lambda_c + \varepsilon_3)y} dy + d_3 + \gamma + \mu_3 \right].$$

From $\Delta(\lambda_c, c) = 0$ and $\Delta(\lambda_c + \varepsilon_3, c) < 0$, we have

$$\beta_1 S_0 e^{-c\tau\lambda_c} + \beta_2 V_0 e^{-c\tau\lambda_c} \le M_3 e^{\varepsilon_3 \xi} \left[-\Delta(\lambda_c + \varepsilon_3, c) + \beta_1 S_0 e^{-c\tau(\lambda_c + \varepsilon_3)} + \beta_2 V_0 e^{-c\tau(\lambda_c + \varepsilon_3)} \right],$$

Because $\tau > 0$, $\lambda_c > 0$, it suffices to prove

$$\beta_1 S_0 + \beta_2 V_0 \le M_3 e^{\varepsilon_3 \xi} \left[-\Delta(\lambda_c + \varepsilon_3, c) + \beta_1 S_0 e^{-c\tau(\lambda_c + \varepsilon_3)} + \beta_2 V_0 e^{-c\tau(\lambda_c + \varepsilon_3)} \right].$$

Since $\xi > \max\{\mathfrak{X}_1, \mathfrak{X}_2\}$, $M_3 > \max\{S_0, V_0\}$ and $0 < \varepsilon_3 < \min\{\varepsilon_1/2, \varepsilon_2/2\}$, note that $e^{\varepsilon_3 \xi} \ge \left(\frac{S_0}{M_1}\right)^{\frac{1}{2}} \left(\frac{V_0}{M_2}\right)^{\frac{1}{2}}$, then we only need to ensure

$$\beta_1 S_0 + \beta_2 V_0 \le -\Delta(\lambda_c + \varepsilon_3, c) M_3 \left(\frac{S_0}{M_1}\right)^{\frac{1}{2}} \left(\frac{V_0}{M_2}\right)^{\frac{1}{2}}.$$

Thus, Equation (2.13) holds for sufficiently large $M_3 > 0$ with

$$M_3 \ge \frac{\beta_1 S_0 + \beta_2 V_0}{-\Delta(\lambda_c + \varepsilon_3, c)} \sqrt{\frac{S_0}{M_1}} \sqrt{\frac{V_0}{M_2}} \triangleq \Pi_1.$$

Case II: $\mathfrak{X}_1 > \xi > \mathfrak{X}_2$.

In this case, $\underline{S}(\xi) = S_0 - M_1 e^{\varepsilon_1 \xi}$ and $\underline{V}(\xi) = 0$. Hence, Equation (2.13) is equivalent to

$$c\underline{I}'(\xi) \le d_3(J * \underline{I}(\xi) - \underline{I}(\xi)) - \gamma \underline{I}(\xi) - \mu_3 \underline{I}(\xi) + \beta_1 \underline{S}(\xi) \underline{I}(\xi - c\tau),$$

that is

$$c\lambda_c - d_3 \int_{\mathbb{R}} J(y) e^{-\lambda_c y} dy + d_3 + \gamma + \mu_3 - \beta_1 S_0 e^{-\lambda_c c\tau} + \beta_1 M_1 e^{\varepsilon_1 \xi - \lambda_c c\tau}$$
$$\le M_3 e^{\varepsilon_3 \xi} \left[c(\lambda + \varepsilon_3) - d_3 \int_{\mathbb{R}} J(y) e^{-(\lambda_c + \varepsilon_3)y} dy + d_3 + \gamma + \mu_3 - \beta_1 S_0 e^{-(\varepsilon_3 + \lambda_c)c\tau} + \beta_1 M_1 e^{\varepsilon_1 \xi - (\varepsilon_3 + \lambda_c)c\tau} \right],$$

we need to prove

$$\beta V_0 \le -\Delta(\lambda_c + \varepsilon_3, c) M_3 e^{\varepsilon_3 \xi}.$$

Choose M_3 large enough with

$$M_3 \ge \frac{\beta_2 \sqrt{V_0 M_2}}{-\Delta(\lambda_c + \varepsilon_3, c)} \triangleq \Pi_2.$$

Case III: $\mathfrak{X}_2 > \xi > \mathfrak{X}_1$.

In this case, $\underline{V}(\xi) = V_0 - M_2 e^{\varepsilon_2 \xi}$ and $\underline{S}(\xi) = 0$. Similar to Case II, Equation (2.13) holds if we choose

$$M_3 \ge \frac{\beta_1 \sqrt{S_0 M_1}}{-\Delta(\lambda_c + \varepsilon_3, c)} \triangleq \Pi_3$$

large enough.

Case VI: $\xi < \min\{\mathfrak{X}_1, \mathfrak{X}_2\}$.

In this case, $\underline{S}(\xi) = S_0 - M_1 e^{\varepsilon_1 \xi}$ and $\underline{V}(\xi) = V_0 - M_2 e^{\varepsilon_2 \xi}$, Equation (2.13) is equivalent to

$$c\underline{I}'(\xi) \le d_3(J * \underline{I}(\xi) - \underline{I}(\xi)) - \gamma \underline{I}(\xi) - \mu_3 \underline{I}(\xi) + \beta_1 \underline{S}(\xi)\underline{I}(\xi - c\tau) + \beta_2 \underline{V}(\xi)\underline{I}(\xi - c\tau),$$

that is

$$c\lambda_c - d_3 \int_{\mathbb{R}} J(y)e^{-\lambda_c y}\mathrm{dy} + \mathrm{d}_3 + \gamma + \mu_3 - \beta_1 \mathrm{S}_0 \mathrm{e}^{-\lambda_c c\tau} - \beta_2 \mathrm{V}_0 \mathrm{e}^{-\lambda_c c\tau} + \beta_1 \mathrm{M}_1 \mathrm{e}^{\varepsilon_1 \xi - \lambda_c c\tau} + \beta_2 \mathrm{M}_2 \mathrm{e}^{\varepsilon_2 \xi - \lambda_c c\tau}$$

$$\le M_3 e^{\varepsilon_3 \xi}\Bigg(c(\lambda + \varepsilon_3) - d_3 \int_{\mathbb{R}} J(y)e^{-(\lambda_c + \varepsilon_3)y}\mathrm{dy} + \mathrm{d}_3 + \gamma + \mu_3 - \beta_1 \mathrm{S}_0 \mathrm{e}^{-(\varepsilon_3 + \lambda_c)c\tau}$$

$$+ \beta_1 M_1 e^{\varepsilon_1 \xi - (\varepsilon_3 + \lambda_c)c\tau} - \beta_2 V_0 e^{-(\varepsilon_3 + \lambda_c)c\tau} + \beta_2 M_2 e^{\varepsilon_1 \xi - (\varepsilon_3 + \lambda_c)c\tau}\Bigg)$$

we only need to ensure

$$M_3 \ge \frac{\beta_1 M_1 e^{(\varepsilon_1 - \varepsilon_3)\xi - \lambda_c c\tau} + \beta_2 M_2 e^{(\varepsilon_2 - \varepsilon_3)\xi - \lambda_c c\tau}}{-\Delta(\lambda_c + \varepsilon_3, c) + \beta_1 M_1 e^{\varepsilon_1 \xi - (\varepsilon_3 + \lambda_c)c\tau} + \beta_2 M_2 e^{\varepsilon_2 \xi - (\varepsilon_3 + \lambda_c)c\tau}}$$

Since $\xi < \min\{\mathfrak{X}_1, \mathfrak{X}_2\}$, $0 < S_0 < M_3$, $0 < V_0 < M_3$, $\varepsilon_3 < \min\{\varepsilon_1/2, \varepsilon_2/2\}$ and $\tau > 0$, we have

$$\frac{\beta_1 M_1 e^{(\varepsilon_1 - \varepsilon_3)\xi - \lambda_c c\tau} + \beta_2 M_2 e^{(\varepsilon_2 - \varepsilon_3)\xi - \lambda_c c\tau}}{-\Delta(\lambda_c + \varepsilon_3, c) + \beta_1 M_1 e^{\varepsilon_1 \xi - (\varepsilon_3 + \lambda_c)c\tau} + \beta_2 M_2 e^{\varepsilon_2 \xi - (\varepsilon_3 + \lambda_c)c\tau}} < \frac{\beta_1 \sqrt{S_0 M_1} + \beta_2 \sqrt{V_0 M_2}}{-\Delta(\lambda_c + \varepsilon_3, c)}.$$

Then Equation (2.13) holds if we choose M_3 large enough with

$$M_3 \ge \frac{\beta_1 \sqrt{S_0 M_1} + \beta_2 \sqrt{V_0 M_2}}{-\Delta(\lambda_c + \varepsilon_3, c)} \triangleq \Pi_4.$$

Through the above discussion, Equation (2.13) holds if we choose $M_3 \ge \max\{\Pi_1, \Pi_2, \Pi_3, \Pi_4\}$ large enough for all $\xi \in \mathbb{R}$. Here we completes the proof. $\qquad\square$

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