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Research article

Remarks on a variant of Lyapunov-LaSalle theorem

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Abstract: The aim of this paper is to give some global stability criteria on a variant of Lyapunov-LaSalle theorem for a class of delay differential system.

Keywords: delay differential equations; Lyapunov-LaSalle theorem; global stability

1. Introduction and motivation

As we know, one of the most common ways to study the asymptotic stability for a system of delay differential equations (DDEs) is the Lyapunov functional method. For DDEs, the Lyapunov-LaSalle theorem (see [6, Theorem 5.3.1] or [11, Theorem 2.5.3]) is often used as a criterion for the asymptotic stability of an autonomous (possibly nonlinear) delay differential system. It can be applied to analyse the dynamics properties for lots of biomathematical models described by DDEs, for example, virus infection models (see, e.g., [2, 3, 10, 14]), microorganism flocculation models (see, e.g., [4, 5, 18]), wastewater treatment models (see, e.g., [16]), etc.

In the Lyapunov-LaSalle theorem, a Lyapunov functional plays an important role. But how to construct an appropriate Lyapunov functional to investigate the asymptotic stability of DDEs, is still a very profound and challenging topic.

To state our purpose, we take the following microorganism flocculation model with time delay in [4] as example:

$$\begin{cases} \dot{x}(t) = 1 - x(t) - h_1 x(t) y(t), \\ \dot{y}(t) = r x(t-\tau) y(t-\tau) - y(t) - h_2 y(t) z(t), \\ \dot{z}(t) = 1 - z(t) - h_3 y(t) z(t), \end{cases} \tag{1.1}$$

where $x(t)$, $y(t)$, $z(t)$ are the concentrations of nutrient, microorganisms and flocculant at time t, respectively. The positive constants h_1, r, h_2 and h_3 represent the consumption rate of nutrient, the growth rate of microorganisms, the flocculating rate of microorganisms and the consumption rate of flocculant, respectively. The phase space of model (1.1) is given by

$$G = \left\{\phi = (\phi_1, \phi_2, \phi_3)^T \in C^+ := C([-\tau, 0], \mathbb{R}^3_+) \ : \ \phi_1 \le 1,\ \phi_3 \le 1\right\}.$$

In model (1.1), there exists a forward bifurcation or backward bifurcation under some conditions [4]. Thus, it is difficult to use the research methods that some virus models used to study the dynamics of such model.

Clearly, (1.1) always has a microorganism-free equilibrium $E_0 = (1, 0, 1)^T$. For considering the global stability of E_0 in G, we define the functional L as follows,

$$L(\phi) = \phi_2(0) + r\int_{-\tau}^{0} \phi_1(\theta)\phi_2(\theta)d\theta,\ \phi \in G. \tag{1.2}$$

The derivative of L along a solution u_t (defined by $u_t(\theta) = u(t+\theta)$ for $\theta \in [-\tau, 0]$) of (1.1) with any $\phi \in G$ is taken as

$$\dot{L}(u_t) = (rx(t) - 1 - h_2 z(t))y(t) \le (r - 1 - h_2 z(t))y(t). \tag{1.3}$$

Obviously, if $r \le 1$, L is a Lyapunov functional on G since $\dot{L} \le 0$ on G.

However, we can not get $r - 1 - h_2 z(t) < 0$ for all $t \ge 0$ and $r > 1$. From this and some conditions on bifurcations of equilibria, L is not a Lyapunov functional on G. But by (3.5), we know that

$$\liminf_{t\to\infty} z(t) \ge \frac{h_1}{h_1 + rh_3}. \tag{1.4}$$

If $r < 1 + h_1h_2/(h_1 + rh_3)$, then there can be found an $\varepsilon > 1$ such that $r - 1 - h_1h_2/\varepsilon(h_1 + rh_3) < 0$. Hence, there exists a $T = T(\phi) > 0$ such that $z(t) \ge h_1/\varepsilon(h_1 + rh_3)$ for all $t \ge T$. By (1.3), we have

$$\dot{V}(u_t) \le \left[r - 1 - \frac{h_1h_2}{\varepsilon(h_1 + rh_3)}\right] y(t) \le 0,\ t \ge T.$$

Obviously, for all $t \ge 0$, the inequality $\dot{V}(u_t) \le 0$ may not hold since we only can obtain $\dot{V}(u_t) \le 0$, $t \ge T$ for some $T > 0$. Therefore, it does not meet the conditions of Lyapunov-LaSalle theorem. But we can give the global stability of E_0 if $r < 1 + h_1h_2/(h_1 + rh_3)$ by using the developed theory in Section 3.

In this paper, we will expand the view of constructing Lyapunov functionals, namely, we first give a new understanding of Lyapunov-LaSalle theorem (including its modified version [9,15,19]), and based on it establish some global stability criteria for an autonomous delay differential system.

2. Understanding of the Lyapunov-LaSalle theorem

Let $C := C([-\tau, 0], \mathbb{R}^n)$ be the Banach space with the norm defined as $||\phi|| = \max_{\theta\in[-\tau,0]} |\phi(\theta)|$, where $\phi = (\phi_1, \phi_2, \cdots, \phi_n)^T \in C$. We will consider the dynamics of the following system of autonomous DDEs

$$\dot{u}(t) = g(u_t),\ t \ge 0, \tag{2.1}$$

where $\dot{u}(t)$ indicates the right-hand derivative of $u(t)$, $u_t \in C$, $g : C \to \mathbb{R}^n$ is completely continuous, and solutions of system (2.1) which continuously depend on the initial data are existent and unique. For a continuous functional $L : C \to \mathbb{R}$, we define the derivative of L along a solution of system (2.1) in the same way as [6, 11] by

$$\dot{L}(\phi) = \dot{L}(\phi)|_{(2.1)} = \limsup_{s\to 0^+} \frac{L(u_s(\phi)) - L(\phi)}{s}.$$

Let X be a nonempty subset of C and $\overline{X}$ be the closure of X. Let

$$u(t) = u(t, \phi) := (u_1(t, \phi), u_2(t, \phi), \cdots, u_n(t, \phi))^T$$

denote a solution of system (2.1) satisfying $u_0 = \phi \in X$. If the solution $u(t)$ with any $\phi \in X$ is existent on $[0, \infty)$ and X is a positive invariant set of system (2.1), and $u_t(\phi) := (u_{1t}(\phi), u_{2t}(\phi), \cdots, u_{nt}(\phi))^T$, we define the solution semiflow of system (2.1):

$$U(t) := u_t(\cdot) : X \to X \text{ (which also satisfies } U(t) : \overline{X} \to \overline{X}),$$

and for $\phi \in X,\ T \geq 0$, we also define

$$O_T(\phi) := \{u_t(\phi) : t \geq T\}.$$

Let $\omega(\phi)$ be the ω-limit set of ϕ for $U(t)$ and I_m the set $\{1, 2, \cdots, m\}$ for $m \in \mathbb{N}^+$.

The following Definition 2.1 and Theorem 2.1 (see, e.g., [6, Theorem 5.3.1], [11, Theorem 2.5.3]) can be utilized in dynamics analysis of lots of biomathematical models in the form of system (2.1).

Definition 2.1. *We call $L : C \to \mathbb{R}$ is a Lyapunov functional on X for system* (2.1) *if*

(i) *L is continuous on $\overline{X}$,*
(ii) *$\dot{L} \leq 0$ on X.*

Theorem 2.1 (Lyapunov-LaSalle theorem [11])**.** *Let L be a Lyapunov functional on X and $u_t(\phi)$ be a bounded solution of system* (2.1) *that stays in X, then $\omega(\phi) \subset \mathscr{M}$, where $\mathscr{M}$ be the largest invariant set for system* (2.1) *in $\mathscr{E} = \{\phi \in \overline{X} : \dot{L}(\phi) = 0\}$.*

In Theorem 2.1, a Lyapunov functional L on X occupies a decisive position, usually, X is positively invariant with respect to system (2.1). If the condition (i) in Definition 2.1 is changed into the conditions that the functional L is continuous on X and L is not continuous at $\varphi \in \overline{X}$ implies $\lim_{\psi\to\varphi,\psi\in X} L(\psi) = \infty$, then the conclusion of Theorem 2.1 still holds, refer to the slightly modified version of Lyapunov-LaSalle theorem (see [9, 15, 19]). In fact, for analysing the dynamics of many mathematical models with biological background, such functional L in the modified Lyapunov-LaSalle theorem is often used and has become more and more popular. For instance, the functional L constructed with a logarithm (such as $\phi_i(0) - 1 - \ln \phi_i(0)$ for some $i \in I_n$), is defined on

$$X = \{\phi = (\phi_1, \phi_2, \cdots, \phi_n)^T \in C : \phi_i(0) > 0\}, \tag{2.2}$$

which can ensure $u_i(t, \phi)$ is persistent, that is, $\liminf_{t\to\infty} u_i(t, \phi) > \sigma_\phi$, where σ_ϕ is some positive constant (see, e.g., [8, 12]). Clearly, this functional L is not continuous on $\overline{X}$ since $\lim_{\varphi_i(0)\to 0^+,\varphi\in X} L(\varphi) = \infty$. Thus, this does not meet the conditions of Theorem 2.1, but it satisfies the conditions of the modified Lyapunov-LaSalle theorem.

However, we will assume that L is defined on a much smaller subset of X. On the phase space X, we can not ensure that $\dot{L}(u_t) \leq 0$ for all $t \geq 0$. Thus, for each $\phi \in X$, we construct such a bounded subset $O_T(\phi) := \{u_t(\phi) : t \geq T\}$ of X for sufficiently large $T = T(\phi)$. Now, we are in position to give the following Corollary 2.1 equivalent to Theorem 2.1.

Corollary 2.1. *Let the solution $u_t(\phi)$ of system* (2.1) *with $\phi \in X$ be bounded (if and only if the set $O_0(\phi)$ is precompact). If there exists $T = T(\phi) \geq 0$ such that L is a Lyapunov functional on $O_T(\phi)$, then $\dot{L} = 0$ on $\omega(\phi)$.*

Proof. It is clear that if $O_0(\phi)$ is precompact, $u_t(\phi)$ is bounded. Suppose that $\|u_t(\phi)\| < U_\phi,\ t \geq 0$. Clearly, $O_0(\phi)$ is uniformly bounded. Since g is completely continuous, $\dot{u}(t)$ is bounded. Hence, $u(t)$ is uniformly continuous on $[-\tau, \infty)$, from which, it follows that $O_0(\phi)$ is equi-continuous. By Arzelà-Ascoli theorem, we thus have that $O_0(\phi)$ is precompact. Due to L is a Lyapunov functional on $O_T(\phi)$ and $\overline{O_T(\phi)}$ is compact, $L(u_t(\phi))$ is decreasing and bounded on $[T, \infty)$. Thus, there exists some $k < \infty$ such that $\lim_{t\to\infty} L(u_t(\phi)) = L(\varphi) = k$ for all $\varphi \in \omega(\phi)$. Thus, $\dot{L} = 0$ on $\omega(\phi)$. □

Remark 2.1. *It is not difficult to find that in the modified Lyapunov-LaSalle theorem (see, e.g., [9, 15, 19]), if L is not continuous on $\overline{X}$, then $\overline{O_T(\phi)} \subset X$, thus, Corollary 2.1 is an extension of Theorem 2.1 and its modified version.*

Remark 2.2. *In fact, we can see that a bounded $O_T(\phi)$ is positively invariant for system* (2.1)*, L is a Lyapunov functional on it, and then $\dot{L} = 0$ on $\omega(\phi)$ follows from Theorem 2.1. Hence, Corollary 2.1 is also an equivalent variant of Theorem 2.1.*

From Corollary 2.1, we may consider the global properties of system (2.1) on the larger space than X. For example, for such functional L constructed with a logarithm, we can always think about the global properties of the corresponding model in more larger pace $X = \{\phi \in C : \phi_i(0) \geq 0\}$ (povided that $u_i(t, \phi) > 0$ for $t > 0$ and $\phi \in X$) than (2.2).

3. Applications of Lyapunov-LaSalle theorem–global stability criteria

Let X be positively invariant for system (2.1) and E denote the point $(E_1, E_2 \cdots, E_n)^T \in \mathbb{R}^n$. Then we have the following results for the global stability for system (2.1), which are the implementations of Corollary 2.1.

Theorem 3.1. *Suppose that the following conditions hold:*

(i) *Let $u_t(\phi)$ be a bounded solution of system* (2.1) *with any $\phi \in X$. Then there exists $T = T(\phi) \geq 0$ such that L is continuous on $\overline{O_T(\phi)} \subset X$ and for any $\varphi \in \overline{O_T(\phi)}$,*

$$\dot{L}(\varphi) \leq -w(\varphi)b(\varphi), \tag{3.1}$$

where $w^T = (w_1, w_2, \cdots, w_k)^T \in C(\overline{O_T(\phi)}, \mathbb{R}^k),\ b = (b_1, b_2, \cdots, b_k)^T \in C(\overline{O_T(\phi)}, \mathbb{R}^k_+),\ k \geq 1$.

(ii) *There exist $k_i = k_i(\phi) = (k_{i1}, k_{i2}, \cdots, k_{in})^T \in \mathbb{R}^n,\ i = 1, 2$ such that for any $\varphi \in \omega(\phi)$, there hold*

$$k_1 \leq \varphi \leq k_2,\ w(\varphi) \geq (w_{01}, w_{02}, \cdots, w_{0k}) \equiv w_0 = w_0(k_1, k_2) \gg 0,$$

and $w_0 b(\varphi) = 0$ implies that for each $j \in I_n$, $\varphi_j(\theta) = E_j \in \mathbb{R}$ for some $\theta = \theta(j) \in [-\tau, 0]$.

Then E is globally attractive for $U(t)$.

Proof. To obtain E is globally attractive for $U(t)$ in X, we only need to prove $\omega(\phi) = \{E\}$ for any $\phi \in X$. Since $u_t(\phi)$ is bounded, $w(u_t(\phi))$ is also bounded. Let $w_i(u_t(\phi)) = f_i(t)$ for each $i \in I_k$; then there exist sequences $\{t^i_m\} \subset \mathbb{R}_+$ such that

$$\begin{aligned}\liminf_{t\to\infty} w(u_t(\phi)) &:= \left(\liminf_{t\to\infty} w_1(u_t(\phi)), \liminf_{t\to\infty} w_2(u_t(\phi)), \cdots, \liminf_{t\to\infty} w_k(u_t(\phi))\right)\\ &= \left(\lim_{m\to\infty} f_1(t^1_m), \lim_{m\to\infty} f_2(t^2_m), \cdots, \lim_{m\to\infty} f_k(t^k_m)\right).\end{aligned}$$

For each sequence $\{t^i_m\}$, $\{u_{t^i_m}(\phi)\}$ contains a convergent subsequence; for simplicity of notation let us assume that $\{u_{t^i_m}(\phi)\}$ is this subsequence and let $\lim_{m\to\infty} u_{t^i_m}(\phi) = \phi^i \in \omega(\phi)$. Since w_i $(i \in I_k)$ is continuous on $\overline{O_T(\phi)}$,

$$\liminf_{t\to\infty} w_i(u_t(\phi)) = \lim_{m\to\infty} w_i\left(u_{t^i_m}(\phi)\right) = w_i\left(\phi^i\right).$$

By the condition (ii), $w_i\left(\phi^i\right) \geq w_{0i} > 0$, $i \in I_k$. Thus, $\liminf_{t\to\infty} w(u_t(\phi)) \geq w_0 \gg 0$. Hence, there exists $T_1 = T_1(\phi) \geq T$ such that $w(u_t(\phi)) \geq w_0/2$ for all $t \geq T_1$. Accordingly, for any $\varphi \in O_{T_1}(\phi)$,

$$\dot{L}(\varphi) \leq -w(\varphi)b(\varphi) \leq -\frac{w_0 b(\varphi)}{2} \leq 0.$$

Hence, L is a Lyapunov functional on $O_{T_1}(\phi)$. By Corollary 2.1, we have that $\dot{L} = 0$ on $\omega(\phi)$.

Next, we show that $\omega(\phi) = \{E\}$. Let $u_t(\psi)$ be a solution of system (2.1) with any $\psi \in \omega(\phi) \subset \overline{O_T(\phi)}$. Then, from (i) and the invariance of $\omega(\phi)$, it follows

$$\dot{L}(u_t(\psi)) \leq -w(u_t(\psi))b(u_t(\psi)), \ \forall t \geq 0.$$

By (ii), $\dot{L}(u_t(\psi)) \leq -w_0 b(u_t(\psi)) \leq 0$, $\forall t \geq 0$. Furthermore, $u_\tau(\psi) = E$ for any $\psi \in \omega(\phi)$, and then $\omega(\phi) = \{E\}$. Thus E is globally attractive for $U(t)$. □

Remark 3.1. *By $\omega(\phi) = \{E\}$, it is clear that E is an equilibrium of system* (2.1). *In theorem 3.1, a Lyapunov functional L on $w(\phi)$ implies a Lyapunov functional L on $O_T(\phi)$ for some $T = T(\phi)$.*

Next, we will give an illustration for Theorem 3.1. Now, we reconsider the global stability for the infection-free equilibrium $E_0 = (x_0, 0, 0)^T$ $(x_0 = s/d)$ of the following virus infection model with inhibitory effect proposed in [1],

$$\begin{cases} \dot{x}(t) = s - dx(t) - cx(t)y(t) - \beta x(t)v(t), \\ \dot{y}(t) = e^{-\mu\tau}\beta x(t-\tau)v(t-\tau) - py(t), \\ \dot{v}(t) = ky(t) - uv(t), \end{cases} \tag{3.2}$$

where $x(t)$, $y(t)$, and $v(t)$ denote the population of uninfected cells, infected cells and viruses at time t, respectively. The positive constant c is the apoptosis rate at which infected cells induce uninfected cells. All other parameters in model (3.2) have the same biological meanings as that in the model of [7].

In [1], we know E_0 is globally asymptotically stable if the basic reproductive number of (3.2) $R_0 = e^{-\mu\tau}k\beta x_0/pu < 1$ in the positive invariant set

$$G = \left\{\phi \in C([-\tau, 0], \mathbb{R}^3_+) : \phi_1 \leq x_0\right\} \subset C^+ := C([-\tau, 0], \mathbb{R}^3_+).$$

Indeed, by Theorem 3.1, we can extend the result of [1] to the larger set C^+. Thus, we have

Corollary 3.1. *If $R_0 < 1$, then E_0 is globally asymptotically stable in C^+.*

Proof. It is not difficult to obtain E_0 is locally asymptotically stable. Thus, we only need to prove that E_0 is globally attractive in C^+. With the aid of the technique of constructing Lyapunov functional in [3], we define the following functional:

$$L(\phi) = \phi_1(0) - x_0 - x_0 \ln \frac{\phi_1(0)}{x_0} + a_1\phi_2(0) + a_1 e^{-\mu\tau} \int_{-\tau}^{0} \beta\phi_1(\theta)\phi_3(\theta)d\theta + a_2\phi_3(0), \tag{3.3}$$

where

$$a_1 = \frac{2(k\beta x_0 + ucx_0)}{pu - e^{-\mu\tau}k\beta x_0},$$
$$a_2 = \frac{2(p\beta x_0 + e^{-\mu\tau}c\beta x_0^2)}{pu - e^{-\mu\tau}k\beta x_0}.$$

Let $u_t = (x_t, y_t, v_t)^T$ be the solution of (3.2) with any $\phi \in C^+$. From [1, Lemma 2.1] and its proof, it follows $\omega(\phi) \subset G$. Thus, for $\varphi \in \omega(\phi)$, we have

$$\begin{aligned} w(\varphi) &\equiv \left(\frac{d}{\varphi_1(0)}, a_1 p - a_2 k - cx_0, a_2 u - a_1 e^{-\mu\tau}\beta\varphi_1(0) - \beta x_0\right) \\ &\geq \left(\frac{d}{x_0}, a_1 p - a_2 k - cx_0, a_2 u - a_1 e^{-\mu\tau}\beta x_0 - \beta x_0\right) \\ &= \left(\frac{d}{x_0}, cx_0, \beta x_0\right) \equiv w_0 \gg 0, \end{aligned}$$

where $c, \beta > 0$ and $\varphi_1(0) \leq x_0$ are used. Let $b(\varphi) \equiv ((x_0 - \varphi_1(0))^2, \varphi_2(0), \varphi_3(0))^T$. Then $w_0 b(\varphi) = 0$ implies $\varphi(0) = E_0$.

The derivative of L_1 along this solution u_t of (3.2) for $t \geq \tau$ is given as

$$\begin{aligned} \dot{L}_1(u_t) &= d\,(x_0 - x(t))\left(1 - \frac{x_0}{x(t)}\right) \\ &\quad + x_0\,(cy(t) + \beta v(t)) - x(t)\,(cy(t) + \beta v(t)) \\ &\quad + a_1 e^{-\mu\tau}\beta x(t)v(t) - a_1 p y(t) + a_2 k y(t) - a_2 u v(t) \\ &\leq -\frac{d}{x(t)}(x_0 - x(t))^2 - (a_1 p - a_2 k - cx_0)\,y(t) \\ &\quad - (a_2 u - a_1 e^{-\mu\tau}\beta x(t) - \beta x_0)v(t) \\ &= -w(u_t)b(u_t). \end{aligned}$$

Therefore, it follows from Theorem 3.1 that E_0 is globally attractive in C^+. □

In [3, Theorem 3.1], the infection-free equilibrium $E_0 = (x_0, 0, 0)^T$ ($x_0 = s/g(0)$) is globally asymptotically stable in G under some conditions. By Theorem 3.1, we can show that E_0 is globally asymptotically stable in C^+. The global dynamical properties of the model in [10] have been discussed in the positive invariant set $\Omega \subset C^+$, thus we can discuss them in the larger set C^+.

Theorem 3.2. *In the condition (ii) of Theorem 3.1, if the condition that* $w_0 b(\varphi) = 0$ *implies that for each* $j \in I_n$*,* $\varphi_j(\theta) = E_j \in \mathbb{R}$ *for some* $\theta = \theta(j) \in [-\tau, 0]$ *is replaced by the condition that* $w_0 b(\varphi) = 0$ *implies that* $\varphi_j(\theta) = E_j \in \mathbb{R}$ *for some* $j \in I_n$ *and some* $\theta = \theta(j) \in [-\tau, 0]$*, then* $\lim_{t\to\infty} u_{jt}(\phi) = E_j$ *for any* $\phi \in X$.

Proof. In the foundation of the similar argument as in the proof of Theorem 3.1, we have that $\dot{L} = 0$ on $\omega(\phi)$. Let $u_t(\psi)$ be a solution of system (2.1) with any $\psi \in \omega(\phi)$. Then it follows from the invariance of $\omega(\phi)$ that $u_t(\psi) \in \omega(\phi)$ for any $t \in \mathbb{R}$, and

$$\dot{L}(u_t(\psi)) \le -w_0 b(u_t(\psi)) \le 0.$$

Hence, $u_{jt}(\psi) = E_j$ for $t \in \mathbb{R}$, and then $\psi_j = u_{j0}(\psi) = E_j$. Therefore, $\lim_{t\to\infty} u_{jt}(\phi) = E_j$ for any $\phi \in X$. □

Next, by using Theorem 3.2, we will give the global stability of the equilibrium E_0 of (1.1) under certain conditions. By (1.3),

$$\dot{L}(u_t) \le -w(u_t)b(u_t), \tag{3.4}$$

where

$$\begin{aligned} w(u_t) &= 1 + h_2 z_t(0) - r = 1 + h_2 z(t) - r, \\ b(u_t) &= y_t(0) = y(t). \end{aligned}$$

Let

$$p(t) = \frac{r}{h_1} x_t(-\tau) + y_t(0) = \frac{r}{h_1} x(t-\tau) + y(t), \ t \ge \tau.$$

Then we have $\dot{p}(t) \le r/h_1 - p(t)$, and it holds $\limsup_{t\to\infty} y(t) \le r/h_1$. Hence, it follows from the first and the third equations of (1.1) that

$$\liminf_{t\to\infty} x(t) \ge \frac{1}{r+1}, \ \liminf_{t\to\infty} z(t) \ge \frac{h_1}{h_1 + rh_3}. \tag{3.5}$$

Thus, for any $\varphi \in \omega(\phi)$, there hold

$$\begin{aligned} &(1/(r+1), 0, h_1/(h_1 + rh_3))^T \le \varphi \le (1, r/h_1, 1)^T, \\ &w(\varphi) = 1 + h_2\varphi_3(0) - r \ge 1 + h_1 h_2/(h_1 + rh_3) - r \equiv w_0 > 0, \end{aligned}$$

and $w_0 b(\varphi) = w_0 \varphi_2(0) = 0$ implies $\varphi_2(0) = 0$. Therefore, it follows from Theorem 3.2 $\lim_{t\to\infty} u_{2t}(\phi) = \lim_{t\to\infty} y_t = 0$. The first and the third equations of (1.1) together with the invariance of $w(\phi)$ yield that $w(\phi) = \{E_0\}$ for any $\phi \in G$. Thus, E_0 is globally stable with the local stability of E_0 if $1 + h_1 h_2/(h_1 + rh_3) > r$.

Thus, we only need to obtain the solutions of a system are bounded and then may establish the upper- and lower-bound estimates of ω-limit set of this system. Thereupon, for $\phi \in X$, we can identify its ω-limit set $\omega(\phi)$ by considering a Lyapunov functional L on the orbit through ϕ after some large time $T = T(\phi)$. In consequence, by Theorem 3.2, the global stability result of the equilibrium E_0 of (1.1) in G can also be extended to the larger set C^+.

Corollary 3.2. *Let $a : \mathbb{R}^n \to \mathbb{R}_+$ be continuous and $a(s) \to \infty$ ($|s| \to \infty$), and let $O'_T(\phi) := \{u_t(\phi) : t \in [T, \varepsilon_\phi)\}$, $T \in [0, \varepsilon_\phi)$, where $[0, \varepsilon_\phi)$ ($\varepsilon_\phi > \tau$) is the maximal interval of existence of $u(t, \phi)$. If there exists $T = T(\phi) \in (0, \varepsilon_\phi)$ such that L is continuous on $O'_T(\phi)$, and for any $\varphi \in O'_T(\phi)$,*

$$a(\varphi(0)) \leq L(\varphi),\ \dot{L}(\varphi) \leq -w_0 b(\varphi),\ 0 \ll w_0^T \in \mathbb{R}^k, \tag{3.6}$$

where $b \in C(O'_T(\phi), \mathbb{R}^k_+)$, then $u_t(\phi)$ is a bounded solution of system (2.1) with any $\phi \in X$. In addition, if L is continuous on $\overline{O_T(\phi)} \subset X$, (3.6) holds for any $\varphi \in \overline{O_T(\phi)}$, and $w_0 b(\varphi) = 0$ implies that for each $j \in I_n$, $\varphi_j(\theta) = E_j \in \mathbb{R}$ for some $\theta = \theta(j) \in [-\tau, 0]$, then E is globally attractive for $U(t)$.

Proof. Since

$$a(u(t, \phi)) \leq L(u_t(\phi)) \leq L(u_T(\phi)),\ t \in [T, \varepsilon_\phi),$$

and the fact that $a(s) \to \infty$ ($|s| \to \infty$), $u_t(\phi)$ is bounded on $[0, \varepsilon_\phi)$, and then $\varepsilon_\phi = \infty$. Thus, Corollary 3.2 follows from Theorem 3.1. □

Corollary 3.3. *Assume that E is an equilibrium of system (2.1). Let $a : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and increasing, $a(s) > 0$ for $s > 0$, $a(0) = 0$, and $\lim_{s\to\infty} a(s) \to \infty$, and let $O'_T(\phi) = \{u_t(\phi) : t \in [T, \varepsilon_\phi)\}$, $T \in [0, \varepsilon_\phi)$, where $[0, \varepsilon_\phi)$ ($\varepsilon_\phi > \tau$) is the maximal interval of existence of $u(t, \phi)$. If there exists $T \in (0, \varepsilon_\phi)$ which is independent of ϕ such that L is continuous on $O'_T(\phi)$ and $u_T(X)$, respectively, and for any $\varphi \in O'_T(\phi)$,*

$$a(|\varphi(0) - E|) \leq L(\varphi),\ \dot{L}(\varphi) \leq -w_0 b(\varphi),\ 0 \ll w_0^T \in \mathbb{R}^k, \tag{3.7}$$

where $b \in C(O'_T(\phi), \mathbb{R}^k_+)$ and $L(E) = 0$, then $u_t(\phi)$ is a bounded solution of system (2.1) with any $\phi \in X$ and E is uniformly stable. In addition, if L is continuous on $\overline{O_T(\phi)} \subset X$, (3.7) holds for any $\varphi \in \overline{O_T(\phi)}$, and $w_0 b(\varphi) = 0$ implies that for each $j \in I_n$, $\varphi_j(\theta) = E_j \in \mathbb{R}$ for some $\theta = \theta(j) \in [-\tau, 0]$, then E is globally asymptotically stable for $U(t)$.

Proof. It follows from Corollary 3.2 that the boundedness of $u_t(\phi)$ and the global attractivity of E are immediate. Thus, we only need to prove E is uniformly stable. By the continuity of solutions with respect to the initial data for compact intervals, for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$u_t(\phi) \in B(u_t(E), \varepsilon) = B(E, \varepsilon),$$

where $\phi \in B(E, \delta_1) \cap X$ and $t \in [0, T]$. Hence, $|u(t, \phi) - E| < \varepsilon$ for any $t \in [0, T]$. Since L is continuous on $u_T(X)$ and $L(E) = 0$, there exists $\delta_2 > 0$ such that for any $\phi \in B(E, \delta_2) \cap X$, $L(u_T(\phi)) < a(\varepsilon)$. Accordingly, from (3.7), it follows

$$a(|u(t, \phi) - E|) \leq L(u_t(\phi)) \leq L(u_T(\phi)) < a(\varepsilon),$$

which yields $|u(t, \phi) - E| < \varepsilon$ for any $t \geq T$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then for any $\phi \in B(E, \delta) \cap X$, it follows $|u(t, \phi) - E| < \varepsilon$ for any $t \geq 0$. Therefore, E is uniformly stable. □

Lemma 3.1. *([13, Lemma 1.4.2]) For any infinite positive definite function $c \in C(\mathbb{R}^n, \mathbb{R}_+)$ with respect to origin, there must exist a radially unbounded class K (Kamke) function $a \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $a(|x|) \leq c(x)$.*

By Lemma 3.1, we have the following remark.

Remark 3.2. *If there exists an infinite positive definite function $c \in C(\mathbb{R}^n, \mathbb{R}_+)$ with respect to E such that $c(\varphi(0)) \leq L(\varphi)$, then there is such function a in Corollary 3.3, or rather there is a radially unbounded class K function $a \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $a(|\varphi(0) - E|) \leq c(\varphi(0)) \leq L(\varphi)$.*

Corollary 3.4. *In Corollary 3.2, if the condition $w_0 b(\varphi) = 0$ implies that for each $j \in I_n$, $\varphi_j(\theta) = E_j \in \mathbb{R}$ for some $\theta = \theta(j) \in [-\tau, 0]$ is replaced by the condition $w_0 b(\varphi) = 0$ implies that $\varphi_j(\theta) = E_j \in \mathbb{R}$ for some $j \in I_n$ and some $\theta = \theta(j) \in [-\tau, 0]$, then $\lim_{t\to\infty} u_{jt}(\phi) = E_j$ for any $\phi \in X$.*

For a dissipative system (2.1), we will give the upper- and lower-bound estimates of $M := \bigcup_{\phi\in X}\omega(\phi)$. To this end, we need the following Lemma 3.2.

Lemma 3.2. *Let $Q \subset \overline{X}$ be a precompact invariant set. Then $\overline{Q}$ is a compact invariant set.*

Proof. For any $\phi \in \overline{Q}$, there is a sequence $\{\phi_n\} \subset Q$ such that $\lim_{n\to\infty} \phi_n = \phi$. Consider that $u_t(\phi_n) \in Q$, and $u_t(\phi)$ is continuous in ϕ, t for $\phi \in \overline{X}$ and $t \geq 0$. Thus, for any $t \geq 0$, $\lim_{n\to\infty} u_t(\phi_n) = u_t(\phi) \in \overline{Q}$. Consequently, $u_t(\overline{Q}) \subset \overline{Q}$ for all $t \geq 0$. Since for any $t \geq 0$, there can be found $\psi_n \in Q$ such that $u_t(\psi_n) = \phi_n$, $\psi_n \in Q$. By the compactness of $\overline{Q}$ (since Q is precompact), the sequence $\{\psi_n\}$ contains a convergent subsequence; in order not to complicate the notation, we still assume that $\{\psi_n\}$ is this subsequence and let $\lim_{n\to\infty} \psi_n = \psi \in \overline{Q}$. Thus, $\lim_{n\to\infty} u_t(\psi_n) = u_t(\psi) = \phi \in u_t(\overline{Q})$, which shows that $\overline{Q} \subset u_t(\overline{Q})$. Therefore, $\overline{Q} = u_t(\overline{Q})$ for any $t \geq 0$. □

Theorem 3.3. *Suppose that there exist $k_1, k_2 \in \mathbb{R}^n$ such that*

$$k_1 \leq \liminf_{t\to\infty} u_t(\phi)(\theta) \leq \limsup_{t\to\infty} u_t(\phi)(\theta) \leq k_2, \ \forall\phi \in X, \ \forall\theta \in [-\tau, 0], \tag{3.8}$$

where

$$\liminf_{t\to\infty} u_t(\phi)(\theta) := \left(\liminf_{t\to\infty} u_{1t}(\phi)(\theta), \cdots, \liminf_{t\to\infty} u_{nt}(\phi)(\theta)\right)^T,$$
$$\limsup_{t\to\infty} u_t(\phi)(\theta) := \left(\limsup_{t\to\infty} u_{1t}(\phi)(\theta), \cdots, \limsup_{t\to\infty} u_{nt}(\phi)(\theta)\right)^T.$$

Then $\overline{M} \subset \overline{X}$ is compact, and $\overline{M} \subset \left\{\varphi \in \overline{X} : k_1 \leq \varphi \leq k_2\right\}$.

Proof. Clearly, $\overline{M} \subset \left\{\varphi \in \overline{X} : k_1 \leq \varphi \leq k_2\right\}$. The fact $O_0(\phi)$ is precompact for any $\phi \in X$ follows from (3.8) and Corollary 2.1. Hence, for any $\varphi \in M$, $k_1 \leq \varphi \leq k_2$. Obviously, M is uniformly bounded. Since g is completely continuous and $u_t(M) = M$, there exists $M_1 > 0$ such that

$$|\dot{u}(t, \varphi)| \leq M_1, \ \forall t \geq 0, \ \forall\varphi \in M.$$

It follows from the invariance of M that for any $\varphi \in M$ and any $t \geq \tau$, there exists $\psi \in M$ such that $u_t(\psi) = \varphi$, it can be shown that M is a precompact invariant set. Accordingly, $\overline{M}$ is a compact invariant set from Lemma 3.2. Thus $\overline{M}$ is compact in $\overline{X}$. □

4. Conclusions

In this paper, we first give a variant of Theorem 2.1, see Corollary 2.1. In fact, the modified version of Lyapunov-LaSalle theorem (see, e.g., [9, 15, 19]) is to expand the condition (i) of Definition 2.1, while Corollary 2.1 is mainly to expand the condition (ii) of Definition 2.1. More specifically, we assume that L is defined on a much smaller subset of X, indeed, we only need L is a Lyapunov functional on $\bar{O}_T(\phi)$, where $u_t(\phi)$ is bounded and $T = T(\phi)$ is a nonnegative constant. In such a way, we may consider the global properties of system (2.1) on the larger space than X and even consider one of system (2.1) on the much lager C (or the nonnegative cone C^+ of C).

As a result, the criteria for the global attractivity of equilibria of system (2.1) are given in Theorem 3.1 and Theorem 3.2, respectively. As direct consequences, we also give the corresponding particular cases of Theorem 3.1 and Theorem 3.2, see Corollaries 3.2, 3.3 and 3.4, respectively. The developed theory can be utilized in many models (see, e.g., [2, 3, 9, 10, 14]). The compactness and the upper- and lower-bound estimates of M for a dissipative system (2.1) are given in Theorem 3.3. Further, by employing the results of this paper, the recent research results in [3, 10, 16–18, 20, 21] can be extended and some techniques in [4, 5] can be simplified.

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Conflict of interest

The authors declare there is no conflict of interest in this paper.

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