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Research article

A parameterized shift-splitting preconditioner for saddle point problems

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Abstract: Recently, Chen and Ma [A generalized shift-splitting preconditioner for saddle point problems, *Applied Mathematics Letters*,43 (2015) 49-55] introduced a generalized shift-splitting preconditioner for saddle point problems with symmetric positive definite (1,1)-block. In this paper, I establish a parameterized shift-splitting preconditioner for solving the large sparse augmented systems of linear equations. Furthermore, the preconditioner is based on the parameterized shift-splitting of the saddle point matrix, resulting in an unconditional convergent fixed-point iteration, which has the intersection with the generalized shift-splitting preconditioner. In final, one example is provided to confirm the effectiveness.

Keywords: saddle point problem; parameterized shift-splitting; convergence; preconditioner; Eigenvalue

1. Introduction

Consider the following 2×2 block saddle point problems

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} A & B^T \\ -B & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1.1}$$

where $A \in R^{n,n}$ is symmetric positive definite,$B \in R^{m,n}$, $m \le n$. It appears in many different applications of scientific computing, such as constrained optimization [49], the finite element method for solving the Navier-Stokes equation [29, 30, 31], and constrained least squares problems and generalized least squares problems [1, 38, 44, 45] and so on; see [9-17, 19,20,35,39,40] and references therein.

In recent years, there has been a surge of interest in the saddle point problem of the form (1), and a large number of stationary iterative methods have been proposed. For example, Santos et al. [38] studied preconditioned iterative methods for solving the singular augmented system with $A = I$. Yuan et al. [44, 45] proposed several variants of SOR method and preconditioned conjugate gradient methods for solving general augmented system (1) arising from generalized least squares problems where A can be symmetric and positive semidefinite and B can be rank deficient. The SOR-like method requires less arithmetic work per iteration step than other methods but it requires choosing an optimal iteration parameter in order to achieve a comparable rate of convergence. Golub et al. [32] presented SOR-like algorithms for solving linear systems (1). Darvishi et al. [28] studied SSOR method for solving the augmented systems. Bai et al. [2, 3, 23, 49] presented GSOR method, parameterized Uzawa (PU) and the inexact parameterized Uzawa (PIU) methods for solving linear systems (1). Zhang and Lu [46] showed the generalized symmetric SOR method for augmented systems. Peng and Li [37] studied the unsymmetric block overrelaxation-type methods for saddle point. Bai and Golub [4, 5, 6, 7, 8, 33, 40] presented splitting iteration methods such as Hermitian and skew-Hermitian splitting (HSS) iteration scheme and its preconditioned variants, Krylov subspace methods such as preconditioned conjugate gradient (PCG), preconditioned MINRES (PMINRES) and restrictively preconditioned conjugate gradient (RPCG) iteration schemes, and preconditioning techniques related to Krylov subspace methods such as HSS, block-diagonal, block-triangular and constraint preconditioners and so on. Bai and Wang's 2009 LAA paper [40] and Chen and Jiang's 2008 AMC paper [23] studied some general approaches about the relaxed splitting iteration methods. Wu, Huang and Zhao [42] presented modified SSOR (MSSOR) method for augmented systems. Cao, Du and Niu [19] introduced a shift-splitting preconditioner and a local shift-splitting preconditioner for saddle point problems (1). Moreover, the authors studied some properties of the local shift-splitting preconditioned matrix and numerical experiments of a model stokes problem are presented to show the effectiveness of the proposed preconditioners. Recently, Chen and Ma [22] presented a generalized shift-splitting preconditioner for saddle point problems with symmetric positive definite (1, 1)-block and gave theoretical analysis and numerical experiments.

For large, sparse or structure matrices, iterative methods are an attractive option. In particular, Krylov subspace methods apply techniques that involve orthogonal projections onto subspaces of the form

$$\mathcal{K}(\mathcal{A}, b) \equiv \text{span}\left\{b, \mathcal{A}b, \mathcal{A}^2 b, ..., \mathcal{A}^{n-1} b, ...\right\}.$$

The conjugate gradient method (CG), minimum residual method (MINRES) and generalized minimal residual method (GMRES) are common Krylov subspace methods. The CG method is used for symmetric, positive definite matrices, MINRES for symmetric and possibly indefinite matrices and GMRES for unsymmetric matrices [39].

In this paper, based on generalized shift-splitting preconditioners presented by Chen and Ma [22], I establish a parameterized shift-splitting preconditioner for saddle point problems with symmetric positive definite (1,1)-block. Furthermore, the preconditioner is based on a parameterized shift-splitting of the saddle point matrix, resulting in an unconditional convergent fixed-point iteration, which has the intersection with the generalized shift-splitting preconditioner. However, the relaxed parameters of the parameterized shift-splitting methods are not optimal and only lie in the convergence region of the method.

2. Parameterized shift-splitting (PSS) preconditioner

Recently, for the coefficient matrix of the augmented system (1.1), Chen and Ma [22] made the following splitting

$$\mathcal{A} = \frac{1}{2}\begin{pmatrix} \alpha I + A & B^T \\ -B & \beta I \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \alpha I - A & -B^T \\ B & \beta I \end{pmatrix}, \tag{2.1}$$

where $\alpha > 0, \beta > 0$ are two constant numbers and I is the identity matrix (with appropriate dimension). Based on the iteration methods studied in [19, 22], a parameterized shift-splitting of the saddle point matrix $\mathcal{A}$ can be constructed as follows:

$$\mathcal{A} = \frac{1}{2}\begin{pmatrix} \alpha I + A & (1-\alpha\beta)B^T \\ -B & \beta I \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \alpha I - A & -(1+\alpha\beta)B^T \\ B & \beta I \end{pmatrix}, \tag{2.2}$$

where $\alpha > 0, \beta > 0$ are two constant numbers and I is the identity matrix (with appropriate dimension). By this special splitting, the following parameterized shift-splitting method can be defined for solving the saddle point problem (1.1):

Parameterized shift-splitting (PSS) method: Given initial vectors $u^0 \in R^{m+n}$, and two relaxed parameters $\alpha > 0$ and $\beta > 0$. For $k = 0, 1, 2, ...$ until the iteration sequence $\{u^k\}$ converges, compute

$$\frac{1}{2}\begin{pmatrix} \alpha I + A & (1-\alpha\beta)B^T \\ -B & \beta I \end{pmatrix} u^{k+1} = \frac{1}{2}\begin{pmatrix} \alpha I - A & -(1+\alpha\beta)B^T \\ B & \beta I \end{pmatrix} u^k + \begin{pmatrix} f \\ g \end{pmatrix}, \tag{2.3}$$

where $\alpha > 0, \beta > 0$ are two constant numbers. It is easy to see that the iteration matrix of the Parameterized shift-splitting iteration is

$$\Gamma = \begin{pmatrix} \alpha I + A & (1-\alpha\beta)B^T \\ -B & \beta I \end{pmatrix}^{-1} \begin{pmatrix} \alpha I - A & -(1+\alpha\beta)B^T \\ B & \beta I \end{pmatrix}. \tag{2.4}$$

If we use a Krylov subspace method such as GMRES (Generalized Minimal Residual) method or its restarted variant to approximate the solution of this system of linear equations, then

$$\mathcal{P}_{PSS} = \frac{1}{2}\begin{pmatrix} \alpha I + A & (1-\alpha\beta)B^T \\ -B & \beta I \end{pmatrix}, \tag{2.5}$$

can be served as a preconditioner. We call the preconditioner $\mathcal{P}_{PSS}$ the parameterized shift-splitting preconditioner for the nonsymmetric saddle point matrix $\mathcal{A}$.

In every iteration of the parameterized shift-splitting iteration (2.5) or the preconditioned Krylov subspace method, we need solve a residual equation

$$\begin{pmatrix} \alpha I + A & (1+\alpha\beta)B^T \\ -B & \beta I \end{pmatrix} z = r \tag{2.6}$$

needs to be solved for a given vector r at each step. Since the matrix $\mathcal{P}_{PSS}$ has the following matrix factorization

$$\mathcal{P}_{PSS} = \frac{1}{2}\begin{pmatrix} I & \frac{1-\alpha\beta}{\beta}B^T \\ 0 & I \end{pmatrix}\begin{pmatrix} A + \alpha I + \frac{1-\alpha\beta}{\beta}B^T B & 0 \\ 0 & \beta I \end{pmatrix}\begin{pmatrix} I & 0 \\ -\frac{1}{\beta}B & I \end{pmatrix}. \tag{2.7}$$

Let $r = [r_1^T, r_2^T]^T$ and $z = [z_1^T, z_2^T]^T$, where $r_1, z_1 \in R^n$ and $r_2, z_2 \in R^m$. Then by (8), it can resuly in

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ \frac{1-\alpha\beta}{\beta} & I \end{pmatrix} \begin{pmatrix} A + \alpha I + \frac{1-\alpha\beta}{\beta} B^T B & 0 \\ 0 & \beta I \end{pmatrix} \begin{pmatrix} I & -\frac{1}{\beta} B^T \\ 0 & I \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}. \tag{2.8}$$

Hence, analogous to Algorithm 2.1 in [19], we can derive the following algorithmic version of the generalized shift-splitting iteration method.

Algorithm 2.1. For a given vector $r = [r_1^T, r_2^T]^T$, the vector $z = [z_1^T, z_2^T]^T$ can be computed by (9) from the following steps:

Step 1: $t_1 = r_1 - \frac{1-\alpha\beta}{\beta} B^T r_2$;

Step 2: Solve $(A + \alpha I + \frac{1-\alpha\beta}{\beta} B^T B) z_1 = t_1$;

Step 3: $z_2 = \frac{1}{\beta}(B z_1 + r_2)$.

Now, we turn to study the convergence of the parameterized shift-splitting iteration for solving symmetric saddle point problems. It is well known that the iteration method (2.5) is convergent for every initial guess if and only if $\rho(\Gamma) < 1$, where $\rho(\Gamma)$ denotes the spectral radius of Γ). Let λ be an eigenvalue of Γ) and $[x^*, y^*]^*$ be the corresponding eigenvector. Then we have

$$\begin{pmatrix} \alpha I - A & -(1+\alpha\beta) B^T \\ B & \beta I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} \alpha I + A & (1-\alpha\beta) B^T \\ -B & \beta I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2.9}$$

or equivalently,

$$\begin{aligned} &(\lambda - 1)\alpha x + (\lambda + 1) A x + (1 + \alpha\beta + \lambda - \lambda\alpha\beta) B^T y = 0, \\ &(\lambda + 1) B x + (1 - \lambda)\beta y = 0. \end{aligned} \tag{2.10}$$

To get the convergence of the parameterized shift-splitting iteration, we first give some lemmas.

Lemma 2.1. *Let A be a symmetric positive definite matrix, and B has full row rank. Let* Γ *be defined as in (5) with* $\alpha > 0$ *and* $\beta > 0$. *If* λ *be an eigenvalue of* Γ, *then* $\lambda \neq \pm 1$.

Proof. Let $[x^*, y^*]^*$ be the corresponding eigenvector of λ. if $\lambda = 1$, then from (11) we have

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0. \tag{2.11}$$

It is easy to get that the coefficient matrix of (2.11) is nonsingular. Hence $x = 0$ and $y = 0$. which contradicts the assumption that $[x^*, y^*]^*$ is an eigenvector of the iteration matrix Γ. So $\lambda \neq 1$.

Now, we prove that $\lambda \neq -1$. If $\lambda = -1$, then from (2.10) we can obtain

$$-2\alpha x + 2\alpha\beta B^T y = 0, \quad \text{and} \quad -2\beta y = 0. \tag{2.12}$$

Since $\alpha > 0, \beta > 0$, from (2.12) we get $x = 0$ and $y = 0$, which also contradict the assumption that $[x^*, y^*]^*$ is an eigenvector of the iteration matrix Γ. So $\lambda \neq -1$. This completes the proof. □

Lemma 2.2. *Assume A be a symmetric positive definite matrix, and B has full row rank. Let* λ *be an eigenvalue of* Γ*(with* $\alpha > 0$ *and* $\beta > 0$*) and* $[x^*, y^*]^*$ *be the corresponding eigenvector with* $x \in C^n$ *and* $y \in C^m$. *Then* $x \neq 0$. *Moreover, if* $y = 0$, *then* $|\lambda| < 1$.

Proof. If $x = 0$, then from (11) we have $(1 + \alpha\beta + \lambda - \lambda\alpha\beta) B^T y = 0$. By Lemma 2.1 we know that $\lambda \neq -1$ and $\alpha > 0, \beta > 0$. Thus we have $B^T y = 0$. Because B^T has full column rank, we get $y = 0$, which contradicts with the assumption that $[x^*, y^*]^*$ is an eigenvector. Thus $x \neq 0$. From Lemma 2.2 [21], we easy know $|\lambda| \leq \| (\alpha I + A)^{-1}(\alpha I - A) \|_2$.

Theorem 2.3. *Assume $A \in R^{n\times n}$ be a symmetric positive definite matrix, and $B \in R^{m\times n}$ has full row rank, and let α, β be two positive constants. Let $\rho(\Gamma)$ denote the spectral radius of the parameterized shift-splitting iteration matrix. Then it holds that*

$$\rho(\Gamma) < 1, \quad \forall \alpha > 0, \beta > 0, \tag{2.13}$$

i.e., the parameterized shift-splitting iteration converges to the unique solution of the saddle point problem (1.1).

Proof. Let λ be an eigenvalue of Γ and $[x^*, y^*]^*$ be the corresponding eigenvector with $x \in C^n$ and $y \in C^m$. By Lemma 2.1 we obtain $\lambda \neq 1$. Then we can obtain from (2.10) that

$$y = \frac{\lambda + 1}{\beta(\lambda - 1)} Bx. \tag{2.14}$$

Substituting (2.14) into the first equation of (2.10), we get

$$(\lambda - 1)\alpha x + (\lambda + 1)Ax + \frac{(1 + \alpha\beta + \lambda - \lambda\alpha\beta)(\lambda + 1)}{\beta(\lambda - 1)} B^T Bx = 0. \tag{2.15}$$

By Lemma 2.2, we obtain that $x \neq 0$. Multiplying $\beta(\lambda - 1)$ as well as $\frac{x^*}{x^*x}$ on both sides of Eq. (2.15), we have

$$\alpha\beta(\lambda - 1)^2 + \beta(\lambda^2 - 1)\frac{x^*Ax}{x^*x} + (1 + \alpha\beta + \lambda - \lambda\alpha\beta)(\lambda + 1)\frac{x^*B^TBx}{x^*x} = 0. \tag{2.16}$$

Let

$$a = \frac{x^*Ax}{x^*x}, b = \frac{x^*B^TBx}{x^*x}. \tag{2.17}$$

Because A is a symmetric positive definite matrix and B has full row rank, we get $a > 0$ and $b \geq 0$. Substituting (2.17) into (2.16), we know that λ satisfies the following real quadratic equation

$$\lambda^2 + \frac{2b - 2\alpha\beta}{\alpha\beta + \beta a + b + \alpha\beta b}\lambda + \frac{\alpha\beta - \beta a + b + \alpha\beta b}{\alpha\beta + \beta a + b + \alpha\beta b} = 0. \tag{2.18}$$

Then from Lemma 2.2, we know that a sufficient and necessary condition for the roots of the real quadratic equation (2.18) to satisfy $|\lambda| < 1$ is

$$\left|\frac{\alpha\beta - \beta a + b + \alpha\beta b}{\alpha\beta + \beta a + b + \alpha\beta b}\right| < 1, \tag{2.19}$$

and

$$\left|\frac{2b - 2\alpha\beta}{\alpha\beta + \beta a + b + \alpha\beta b}\right| < 1 + \frac{\alpha\beta - \beta a + b + \alpha\beta b}{\alpha\beta + \beta a + b + \alpha\beta b}. \tag{2.20}$$

It is easy to find that (2.19) and (2.20) hold for all $\alpha > 0$ and $\beta > 0$ when $a > 0$ and $b > 0$. Also, if $b = 0$, there is a $x \neq 0$ such that $Bx = 0$. Then by Lemma 2.1, from (2.10) we have $y = 0$. Hence, by Lemma 2.2 we have $|\lambda| < 1$. Thus $\rho(\Gamma) < 1$. Then, we get (2.13), e.e., the parameterized shift-splitting iteration converges to the unique solution of the saddle point problem (1.1). □

Remark 2.1. When $\alpha = 0$, The parameterized shift-splitting preconditioner reduces to the local shift-splitting preconditioner. Moreover, the parameterized shift-splitting preconditioner in this paper and the generalized shift-splitting preconditioner in [21] are two different preconditioning modes. Moreover, they have an intersection.

Remark 2.2 From Theorem 2.3, we know that the parameterized shift-splitting iteration method is uncondit0inally convergent.

3. Numerical examples

In this section, I present one example to illustrate the effectiveness of the parameter shift-splitting preconditioner for GMRES(m) method and MINRES to solve the linear systems (1) in the sense of iteration step (denoted as It), elapsed CPU time in seconds (denoted as CPU), and relative residual error (denoted as RES). All numerical examples are carried out in Matlab 7.0. In our experiments, all runs with respect to both GSS method and PSS method are started from initial vector $((x^{(0)})^T, (y^{(0)})^T)^T = 0$, and terminated if the current iteration satisfy RES $< 10^{-6}$.

Table 1. Iteration counts, relative residual and CPU time about preconditioned matrices $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ and $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ when choosing different parameters. Here, $p = 16$.

$\mathcal{T}_{PSS}^{-1}\mathcal{A}$	α	β	$It_{BiCGSTAB}$	$Res_{BiCGSTAB}$	CPU(s)
	0.001	0.002	2	6.5007×10^{-7}	0.147
	0.003	0.001	1	8.0657×10^{-7}	0.074
	0.005	0.006	5.5	6.5785×10^{-7}	0.367
	0.007	0.004	4	5.9859×10^{-7}	0.262
	0.009	0.008	2	6.4956×10^{-7}	0.142
$\mathcal{T}_{GSS}^{-1}\mathcal{A}$	α	β	$It_{BiCGSTAB}$	$Res_{BiCGSTAB}$	CPU(s)
	0.001	0.002	2	6.5008×10^{-7}	0.146
	0.003	0.001	1	8.0664×10^{-7}	0.072
	0.005	0.006	5.5	4.6159×10^{-7}	0.329
	0.007	0.004	4	6.1723×10^{-7}	0.268
	0.009	0.008	2	6.4961×10^{-7}	0.151

Example 3.1. [18] Consider the linear system of equations (1) with

$$T = I \otimes V + V \otimes I \text{ and } \mathrm{W} = 10(\mathrm{I} \otimes \mathrm{V_C} + \mathrm{V_C} \otimes \mathrm{I}) + 9(\mathrm{e_1 e_1^T} + \mathrm{e_l e_1^T}) \otimes \mathrm{I},$$

where $V = \text{tridiag}(-1, 2, -1) \in \mathcal{R}^{l \times l}, \mathrm{V_C} = \mathrm{V} - \mathrm{e_1 e_1^T} - \mathrm{e_l e_1^T} \in \mathcal{R}^{l \times l}$ and e_1 and e_l are the first and last unit vectors in $\mathcal{R}^l$, respectively. Here T and K correspond to the five-point centered difference matrices approximating the negative Laplacian operator with homogeneous Dirichlet boundary conditions and periodic boundary conditions, respectively, on a uniform mesh in the unit square $[0, 1] \times [0, 1]$ with the mesh-size $h = \frac{1}{l+1}$.

In Figures 1 ∼ 4, I report the eigenvalue distribution for the generalized shift-splitting preconditioned matrix $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ and the parameter shift-splitting preconditioned matrix for different parameter, respectively. In Tables 1 ∼ 4, we report iteration counts, relative residual and cpu time about preconditioned matrices $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ and $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ with $l = 16$ and $l = 24$ when choosing different parameters. Figures 1 ∼ 4 and Tables 1 ∼ 4 show that the GSS preconditioner and PSS preconditioner have the same eigenvalue distribution and the convergence when choosing different parameters.

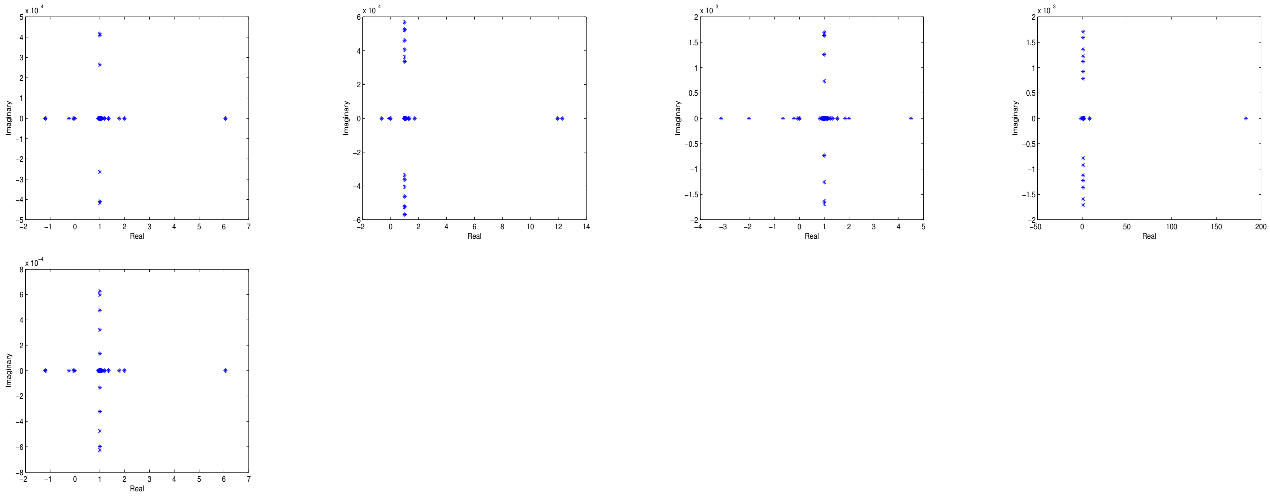

Figure 1. The eigenvalue distribution for the parameter shift-splitting preconditioned matrix $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ when $\alpha = 0.001, \beta = 0.002$(the first), $\alpha = 0.003, \beta = 0.001$ (the second),$\alpha = 0.005, \beta = 0.006$(the third), $\alpha = 0.007, \beta = 0.004$(the fourth) and $\alpha = 0.009, \beta = 0.008$ (the fifth), respectively. Here, $l = 16$.

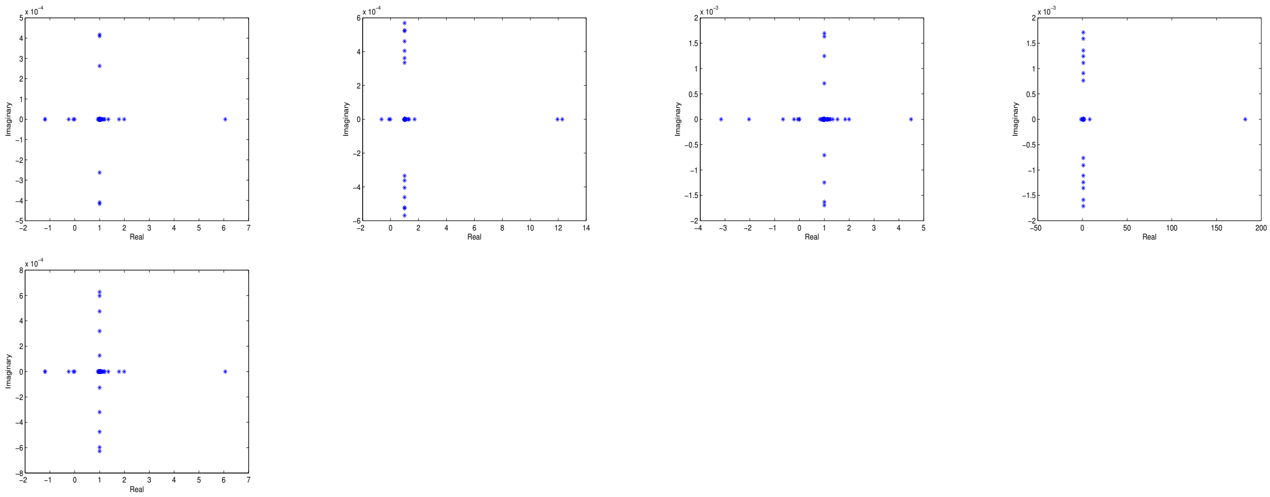

Figure 2. The eigenvalue distribution for the generalized shift-splitting preconditioned matrix $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ when $\alpha = 0.001, \beta = 0.002$(the first), $\alpha = 0.003, \beta = 0.001$ (the second),$\alpha = 0.005, \beta = 0.006$(the third), $\alpha = 0.007, \beta = 0.004$(the fourth) and $\alpha = 0.009, \beta = 0.008$ (the fifth), respectively. Here, $l = 16$.

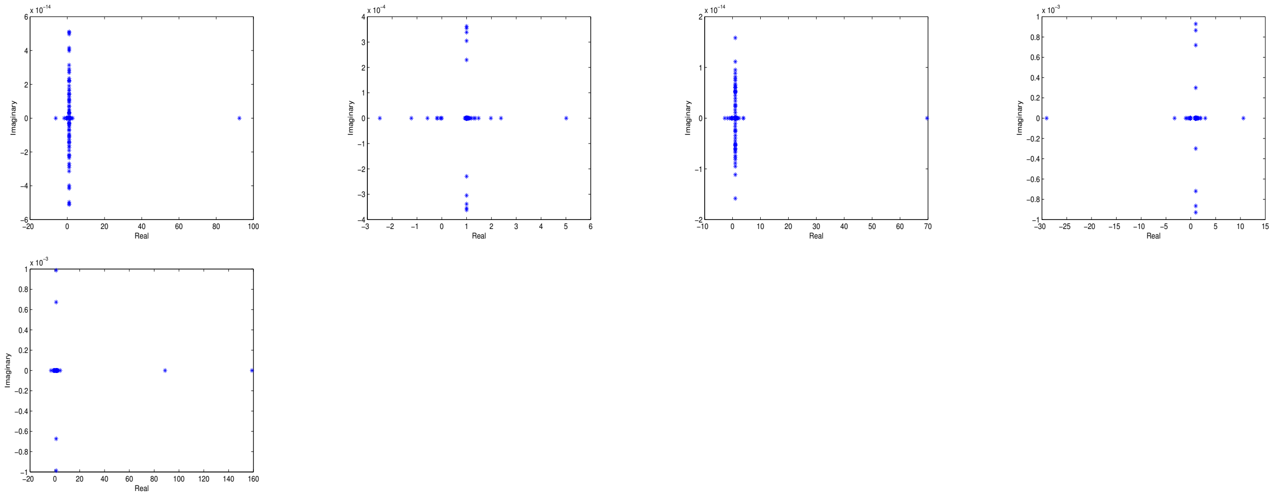

Figure 3. The eigenvalue distribution for the parameter shift-splitting preconditioned matrix $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ when $\alpha = 0.001, \beta = 0.002$(the first), $\alpha = 0.003, \beta = 0.001$ (the second),$\alpha = 0.005, \beta = 0.006$(the third), $\alpha = 0.007, \beta = 0.004$(the fourth) and $\alpha = 0.009, \beta = 0.008$ (the fifth), respectively. Here, $l = 24$.

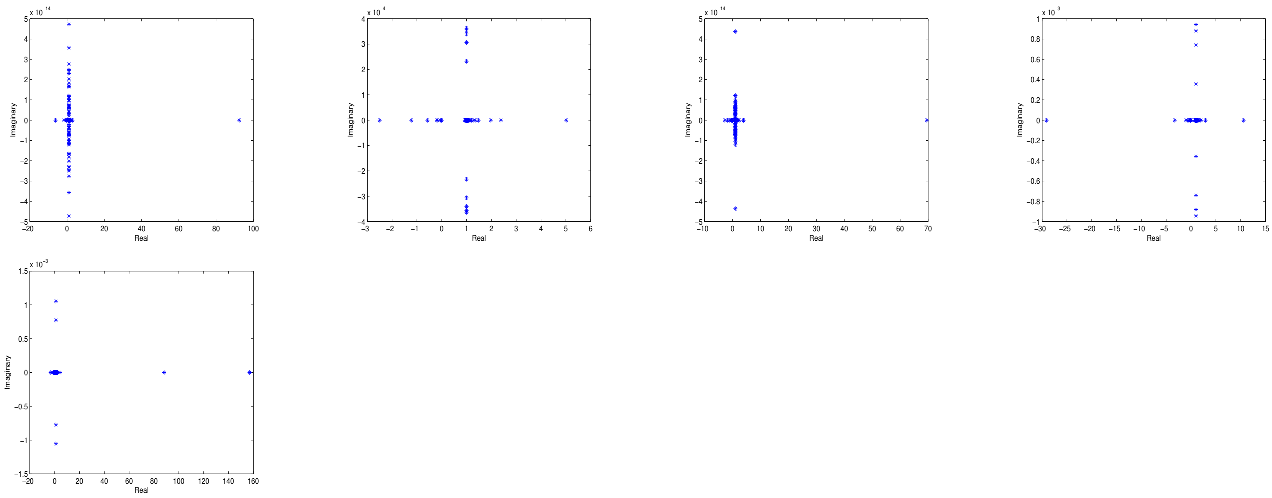

Figure 4. The eigenvalue distribution for the generalized shift-splitting preconditioned matrix $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ when $\alpha = 0.001, \beta = 0.002$(the first), $\alpha = 0.003, \beta = 0.001$ (the second),$\alpha = 0.005, \beta = 0.006$(the third), $\alpha = 0.007, \beta = 0.004$(the fourth) and $\alpha = 0.009, \beta = 0.008$ (the fifth), respectively. Here, $l = 24$.

Table 2. Iteration counts, relative residual and CPU time about preconditioned matrices $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ and $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ when choosing different parameters. Here, $p = 16$.

$\mathcal{T}_{PSS}^{-1}\mathcal{A}$	α	β	It_{GMRES}	Res_{GMRES}	CPU(s)
	0.001	0.002	9(1)	1.8872×10^{-8}	0.394
	0.003	0.001	8(1)	3.4887×10^{-7}	0.359
	0.005	0.006	13(1)	3.2668×10^{-7}	0.499
	0.007	0.004	11(1)	4.7948×10^{-7}	0.432
	0.009	0.008	13(1)	8.5534×10^{-7}	0.530
$\mathcal{T}_{GSS}^{-1}\mathcal{A}$	α	β	It_{GMRES}	Res_{GMRES}	CPU(s)
	0.001	0.002	9(1)	1.8871×10^{-8}	0.402
	0.003	0.001	8(1)	3.4834×10^{-7}	0.358
	0.005	0.006	13(1)	3.2640×10^{-7}	0.510
	0.007	0.004	11(1)	4.7939×10^{-7}	0.432
	0.009	0.008	13(1)	8.5183×10^{-7}	0.493

Table 3. Iteration counts, relative residual and CPU time about preconditioned matrices $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ and $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ when choosing different parameters. Here, $p = 24$.

$\mathcal{T}_{PSS}^{-1}\mathcal{A}$	α	β	$It_{BiCGSTAB}$	$Res_{BiCGSTAB}$	CPU(s)
	0.001	0.002	2.5	9.8445×10^{-7}	1.557
	0.003	0.001	1	8.6139×10^{-7}	0.583
	0.005	0.006	5.5	7.2770×10^{-7}	3.166
	0.007	0.004	4	9.6488×10^{-7}	2.366
	0.009	0.008	17	9.2708×10^{-7}	10.313
$\mathcal{T}_{GSS}^{-1}\mathcal{A}$	α	β	$It_{BiCGSTAB}$	$Res_{BiCGSTAB}$	CPU(s)
	0.001	0.002	2.5	9.8383×10^{-7}	1.551
	0.003	0.001	1	8.6147×10^{-7}	0.580
	0.005	0.006	5.5	6.9932×10^{-7}	3.382
	0.007	0.004	4	9.4912×10^{-7}	2.391
	0.009	0.008	20.5	9.0799×10^{-7}	12.469

4. Discussion and conclusion

In this paper, the author presents a parameterized shift-splitting preconditioner for saddle point problems with symmetric positive definite $(1,1)-$block. Theoretical analysis shows that PSS method is an unconditional convergent fixed-point iteration. Furthermore, numerical example indicates that the GSS preconditioner and PSS preconditioner have the same eigenvalue distribution and the convergence when choosing different parameters.

Table 4. Iteration counts, relative residual and CPU time about preconditioned matrices $\mathcal{T}_{GSS}^{-1}\mathcal{A}$ and $\mathcal{T}_{PSS}^{-1}\mathcal{A}$ when choosing different parameters. Here, $p = 24$.

$\mathcal{T}_{PSS}^{-1}\mathcal{A}$	α	β	It_{GMRES}	Res_{GMRES}	CPU(s)
	0.001	0.002	16(1)	7.7884×10^{-8}	5.527
	0.003	0.001	13(1)	7.5976×10^{-7}	4.262
	0.005	0.006	20(1)	6.4167×10^{-7}	6.343
	0.007	0.004	16(1)	8.7812×10^{-7}	5.338
	0.009	0.008	21(1)	6.5959×10^{-7}	6.970
$\mathcal{T}_{GSS}^{-1}\mathcal{A}$	α	β	It_{GMRES}	Res_{GMRES}	CPU(s)
	0.001	0.002	16(1)	7.7882×10^{-8}	5.500
	0.003	0.001	13(1)	7.5943×10^{-7}	4.400
	0.005	0.006	20(1)	6.4171×10^{-7}	6.298
	0.007	0.004	16(1)	8.7812×10^{-7}	5.288
	0.009	0.008	21(1)	6.6135×10^{-7}	6.897

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Conflict of interest

All authors confirm that there is no conflict of interest in this paper.

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