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Research article

Modeling the impact of sanitation and awareness on the spread of infectious diseases

Rajanish Kumar Rai[1], **Arvind Kumar Misra**[1,*] **and Yasuhiro Takeuchi**[2]

[1] Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi - 221005, India

[2] College of Science and Engineering, Department of Physics and Mathematics, Aoyama Gakuin University, Kanagawa 252-5258, Japan

* **Correspondence:** Email: rajanishrai94@gmail.com.

Abstract: Sanitation and awareness programs play a fundamental role and are much effective public health interventions to control the spread of infectious diseases. In this paper, a nonlinear mathematical model for the control of infectious diseases, such as typhoid fever is proposed and analyzed by considering budget required for sanitation and awareness programs as a dynamic variable. It is assumed that the budget allocation regarding the protection against the disease to warn people and for sanitation increases logistically and its per-capita growth rate increases with the increase in number of infected individuals. In the model formulation, it is assumed that the susceptible individuals contract infection through the direct contact with infected individuals as well as indirectly through bacteria shed in the environment. It is further assumed that a fraction of budget is used to warn people via propagating awareness whereas the remaining part is used for sanitation to reduce the density of bacteria. The condition when budget should spend on sanitation/awareness to reduce the number of infected individuals is obtained. Model analysis reveals that the sanitation and awareness programs have capability to reduce the epidemic threshold and thus control the spread of infection. However, delay in providing funds destabilizes the system and may cause stability switches through Hopf-bifurcation. Numerical simulations are also carried out to support analytical findings.

Keywords: budget allocation; delay; sanitation; awareness; hopf-bifurcation; stability switch

1. Introduction

The poor sanitation is alone responsible for 10 % of global burden of diseases and most of developing countries including India are facing the problem of disease burden due to lack of sanitation, hygiene and adequate drinking water [22]. Across the globe, 2.3 billion people still do not have basic sanitation facilities, of these, 892 million still defecate in the open and annually 842 000 people die in low and middle income countries due to inadequate sanitation, water, and hygiene [40]. In last few decades, various government and non-government organizations have made efforts to improve sanitation coverage and clean drinking water to slow down the spread of diseases [39]. There are several infectious diseases which spread in the population due to poor sanitation and lack of awareness among the individuals regarding the healthy sanitation practices. In particular, typhoid fever is a life-threatening infection caused by *Salmonella* Typhi bacteria, which spreads in the population through contaminated water and food or through close contact with infected individuals. It is estimated that the global disease burden of typhoid fever at 11 to 20 million cases of severe illness and about 128 000 – 161 000 deaths annually [41].

Sanitation coverage plays a crucial role for better hygiene, safe drinking water and is responsible for social and economic development. In 1923, Mahatma Gandhi told that "sanitation is more important than independence". He believed that "cleanliness is next to Godliness" and always encouraged the people to adopt cleanliness as an integral part of life [13]. Awareness programs using different modes like TV, radio, social media also play a very fundamental role to convey information regarding the tools and techniques required for protection against the disease, and address the issues of healthy sanitation practices. To disseminate awareness and improving sanitation require funds, which is provided by the government through budget allocation. Nowadays, government of India is running "Swachh Bharat Abhiyan" (clean India campaign) throughout the India and for this purpose funds are allotted by the government. The main objective of this mission is to achieve the goal of cleanness, and propagate awareness among the individuals to change their behavior regarding healthy sanitation practices [33].

In last few decades, some mathematical models have been proposed to assess the impacts of awareness on the control of infectious diseases by considering transmission rate as decreasing function of infected individuals due to media alerts [2, 3, 18, 19, 20, 30, 34, 37] or awareness programs (via TV, radio, social media etc.) are implemented proportional to the number of infected individuals by considering media as a dynamic variable [4, 6, 12, 16, 23, 26, 28, 29, 32, 35, 38]. In particular, Misra et al. [23] have considered that aware susceptible individuals are fully protected from infection as they use precautionary measures during the infection period whereas Samanta et al. [35] have considered that aware susceptibles are also vulnerable to infection with lower rate than unaware susceptible individuals. Their findings reveal that dissemination of awareness programs is beneficial to slow down the disease prevalence, but it does not affect the epidemic threshold of disease due to media alerts. Pawelek et al. [31] have studied how the information through Twitter impacts the dynamics of influenza epidemic by considering that awareness among the individuals through Twitter messages, change the individual's behavior and reduce the transmission rate of disease. Their findings suggest that social media programs, like Twitter may serve as a better mode of transmission of information and is helpful in reducing the spread of influenza epidemic. Lu et al. [21] have proposed and analyzed an SEI-type infection model to assess the impact of awareness programs on the control of infectious diseases by considering that the media coverage directly influences the individual's behavior and thus reduces their contact rate. It is shown that media coverage reduces the epidemic peak. However, on increasing epidemic threshold above a critical value destabilizes the system through Hopf-bifurcation, which poses challenges to predict and control the spread of infection. Yang et al. [42] have developed two mathematical models to investigate the impact of awareness programs on the prevention and control of cholera epidemic by considering awareness programs as a dynamic variable. In the first model, authors have considered that the transmission rate of disease as well as bacterial shedding rate decreases with implementation of awareness programs whereas in second model, authors have considered that susceptible individuals are divided into unaware and aware classes due to media alerts. The study reveals that awareness programs have positive effect in reducing the spread of disease and the risk of infection. For some diseases, mathematical models are also proposed to assess the effect of time delay in implementation of awareness programs [1, 11, 24, 27, 36]. It is observed that incorporation of time delay in the modeling process destabilizes the system. Recently, Misra et al. [27] have studied how delay in providing funds for implementation of awareness programs changes the dynamics of the system. Authors have considered that the growth rate of budget allocation to warn people increases logistically and its per-capita growth rate increases with the increase in number of infected individuals. The study reveals that the dissemination of awareness among the individuals reduces the spread of infection. However, delay in providing funds destabilizes the system and may cause stability switches through Hopf-bifurcation.

Although awareness plays a very important role to reduce the contact rate between susceptible and infected individuals, at the same time awareness regarding healthy sanitation practices is also important to reduce the number of infected individuals. In this context, budget allocation by the government for sanitation coverage is a rational step for the control of bacteria shed in the environment. For this purpose, Mara et al. [22] have studied the impacts of sanitation for the control of infectious diseases. Their findings suggest that improved sanitation is responsible for good human health and also for social and economic development, particularly in developing countries. For sanitation coverage and implementation of information campaigns, funds are needed, which are provided by the government through budget allocation. Nationwide policies are required for sanitation coverage and budget should be allotted to the responsible organizations for effective implementation [15].

Thus, there is a need of mathematical model which explicitly incorporate the combined effects of sanitation and awareness programs through budget allocation for the control of infectious diseases, such as typhoid fever which spreads in the population through the direct contact of susceptible with infected individuals as well as indirectly through bacteria present in the environment. In this paper, it is assumed that the growth rate of budget required for sanitation and awareness programs follows logistic model and increment in per-capita growth rate of budget allocation is proportional to the number of infected individuals. It is also assumed that a fraction of budget is used to warn people regarding the protection against the disease and the remaining part of funds is used for sanitation coverage to reduce the density of bacteria shed in the environment.

2. Mathematical model and description

In the region under consideration, it is assumed that the population is homogeneously mixed and disease spreads via the direct contact between susceptible and infected individuals as well as indirectly through bacteria shed in the environment. Some epidemiologists are in opinion that the bacterial infectious diseases, such as typhoid fever do not confer proper immunity after the disease and have higher infective dose (number of bacteria needed to produce an infection). Here, we assume that mean value of infective dose in the susceptible population does not change significantly. Thus, we consider an SIS model with immigration where infected individuals recover from infection after certain period of time and join the susceptible class [9, 25]. The information regarding the protection against the disease is propagated through TV, radio, social media etc. As a result, aware people use precautionary measures for their protection during the infection period and thus reduce their contacts with infected individuals. Nowadays, government of India is propagating Swachh Bharat campaigns throughout the India with the slogan "*Ek Kadam Swachhata Ki Ore*" (A step towards sanitation) [33]. The sanitation coverage and public health information campaigns can help to prevent and control the spread of communicable diseases. For this purpose, funds are allotted by the government through budget allocation, a fraction of budget is used to warn people regarding the tools and techniques required for protection and remaining part of funds is spent for sanitation coverage. The sanitation coverage plays a fundamental role to reduce bacteria in the environment. Let $N(t)$ be the total population at any time t in the region under consideration, which is divided into two sub-populations (i) susceptible population $S(t)$ (ii) infected population $I(t)$. It is assumed that all the susceptible individuals living in habitat are affected by bacteria population. Let $B(t)$ be the density of bacteria shed in the environment at time t. The variable $M(t)$ addresses the budget allocation by the government to warn people and for sanitation coverage. It is assumed that the growth rate of budget allocation follows logistic model with intrinsic growth rate 'r' and carrying capacity 'K'. Further, it is assumed that the increment in per-capita growth rate of budget allocation is proportional to the number of infected individuals in the region. It is also assumed that the density of bacteria increases with the increase in number of infected individuals apart from its self-growth rate and natural death rate. For the feasibility of model system, it is assumed that natural death rate of bacteria is higher than its self-growth rate, i.e., $(\phi_0 - \phi) > 0$. The interaction between susceptible and infected individuals is assumed to follow the simple law of mass action and the interaction of susceptible with bacteria shed in the environment as saturated function of bacterial density because bacteria shed in the environment can infect the susceptible individuals to a limited extent and this interaction is represented by the term $\eta \frac{B}{L+B} S$.

In view of the above considerations, the dynamics of the model is governed by the following system of nonlinear differential equations:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \left(\beta - \beta_1 \frac{k_1 M(t)}{p + k_1 M(t)}\right) S(t) I(t) - \eta \frac{B(t)}{L + B(t)} S(t) + \nu I(t) - d S(t), \\
\frac{dI(t)}{dt} &= \left(\beta - \beta_1 \frac{k_1 M(t)}{p + k_1 M(t)}\right) S(t) I(t) + \eta \frac{B(t)}{L + B(t)} S(t) - (\nu + \alpha + d) I(t), \\
\frac{dB(t)}{dt} &= \phi_1 I(t) + \phi B(t) - \phi_0 B(t) - \Phi \frac{(1 - k_1) M(t)}{q + (1 - k_1) M(t)} B(t), \\
\frac{dM(t)}{dt} &= r\left(1 - \frac{M(t)}{K}\right) M(t) + \theta I(t) M(t). \qquad (2.1)
\end{aligned}$$

In the above model system, it is assumed that a fraction of the total allotted budget M, (i.e., $k_1 M$, $0 < k_1 < 1$) is used for implementation of awareness programs regarding the tools and techniques required for protection due to which they keep themselves away from infected individuals and thus the direct contact between susceptible and infected individuals decreases at a factor $f_1(M) = \beta_1 \frac{k_1 M}{p + k_1 M}$. The reason behind considering this saturated type of functional response is that the amount of budget used to warn people via propagating awareness has limited impact to reduce the contact rate between susceptible and infected individuals [19] and in this case the effective contact rate between susceptible and infected individuals becomes $\beta(M) = \left(\beta - \beta_1 \frac{k_1 M}{p + k_1 M}\right)$. For the feasibility of model system, $\beta_1 < \beta$. Further, it is assumed that the remaining part of budget, i.e., $(1 - k_1)M$ is used for sanitation coverage and due to this, the density of bacteria shed in the environment decreases at a factor $f_2(M) = \Phi\frac{(1-k_1)M}{q+(1-k_1)M}$. Here, we have considered a saturated type of function because the amount of funds which is used for sanitation coverage can reduce the bacteria shed in the environment to a limited extent. All the parameters of model system (2.1) are assumed to be positive and description of model parameters is given in Table 1.

Table 1. Parameter description for the model system.

Parameters	Description	Units
Λ	Immigration in the class of susceptible population	$person\ day^{-1}$
β	Contact rate of susceptible with infected individuals in absence of funds	$person^{-1}day^{-1}$
β_1	Efficacy of budget allocation to reduce the contact rate via propagating awareness	$person^{-1}day^{-1}$
k_1	Fractional constant determining the budget allocation to warn susceptibles via propagating awareness	
p	Half saturation point for $f_1(M)$ as it attains half of its maximum possible value β_1 when budget allocation arrives at $\frac{p}{k_1}$	
η	Transmission rate of susceptible to infected class due to interaction of susceptible with bacteria present in the environment	day^{-1}
L	Half saturation constant	$cells/mm^3$
ν	Recovery rate of human population	day^{-1}
α	Disease induced death rate of human population	day^{-1}
d	Natural death rate of human population	day^{-1}
ϕ_1	Growth rate of bacteria due to increase in infected individuals	$cells/(mm^3\ person\ day)$
ϕ	Self-growth rate of bacteria	day^{-1}
ϕ_0	Natural death rate of bacteria	day^{-1}
Φ	Efficacy of sanitation coverage to reduce bacteria in the environment due to budget allocation	day^{-1}
q	Half saturation point for $f_2(M)$ as it attains half of its maximum possible value Φ when budget allocation arrives at $\frac{q}{1-k_1}$	
r	Intrinsic growth rate of budget allocation	day^{-1}
K	Carrying capacity of budget allocation	
θ	Per-capita growth rate of budget allocation due to increase in infected individuals	$(person\ day)^{-1}$

In model system (2.1) at time t, it is assumed that the per-capita growth rate of budget allocation to warn people and for sanitation increases proportional to the number of infected individuals. However, it may be noted that the number of reported cases of infected individuals known to the government may be some days old and increment in per-capita growth rate of budget allocation depends upon this data, which leads to the incorporation of time delay in per-capita growth rate of budget allocation due to increase in infected individuals. In order to this, we have considered that at time t, the per-capita growth rate of budget allotted to warn people and for sanitation is in accordance with the number of infected individuals reported at time $t-\tau$ (for some $\tau > 0$).

In view of this, the dynamics of the model is governed by the following system of nonlinear delay differential equations:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \left(\beta - \beta_1\frac{k_1M(t)}{p+k_1M(t)}\right)S(t)I(t) - \eta\frac{B(t)}{L+B(t)}S(t) + \nu I(t) - dS(t),\\
\frac{dI(t)}{dt} &= \left(\beta - \beta_1\frac{k_1M(t)}{p+k_1M(t)}\right)S(t)I(t) + \eta\frac{B(t)}{L+B(t)}S(t) - (\nu+\alpha+d)I(t),\\
\frac{dB(t)}{dt} &= \phi_1 I(t) + \phi B(t) - \phi_0 B(t) - \Phi\frac{(1-k_1)M(t)}{q+(1-k_1)M(t)}B(t),
\end{aligned}$$

$$\frac{dM(t)}{dt} = r\left(1-\frac{M(t)}{K}\right)M(t)+\theta I(t-\tau)M(t). \tag{2.2}$$

where, $S(0) = S_0 > 0$, $I(\vartheta) = I_0 \geq 0$ for $\vartheta \in [-\tau, 0)$, $B(0) = B_0 \geq 0$ and $M(0) = M_0 \geq 0$.
Using the fact that $S(t) + I(t) = N(t)$, the above model system (2.2) reduces to following system of nonlinear delay differential equations:

$$\begin{aligned}
\frac{dI(t)}{dt} &= \left(\beta-\beta_1\frac{k_1M(t)}{p+k_1M(t)}\right)(N(t)-I(t))I(t)+\eta\frac{B(t)}{L+B(t)}(N(t)-I(t)) \\
&\quad -(\nu+\alpha+d)I(t), \\
\frac{dN(t)}{dt} &= \Lambda - dN(t) - \alpha I(t), \\
\frac{dB(t)}{dt} &= \phi_1 I(t) - (\phi_0-\phi)B(t) - \Phi\frac{(1-k_1)M(t)}{q+(1-k_1)M(t)}B(t), \\
\frac{dM(t)}{dt} &= r\left(1-\frac{M(t)}{K}\right)M(t)+\theta I(t-\tau)M(t).
\end{aligned} \tag{2.3}$$

3. Positivity of solutions and boundedness

To show that the model system (2.1) is epidemiologically feasible, we show that all the variables of model system (2.1) are non-negative for all time t. In order to this, we have the following Lemma, which is stated as follows:

Lemma 3.1. *The solution $S(t)$, $I(t)$, $B(t)$ and $M(t)$ of model system (2.1) with initial conditions $S(0) > 0$, $I(0) \geq 0$, $B(0) \geq 0$ and $M(0) \geq 0$ are positive for all $t > 0$.*

Now, it is sufficient to study model system (2.3) in detail rather than system (2.2). For the solution of model system (2.3), the region of attraction [7] is given by the set:

$$\Omega = \{(I,N,B,M) \in \mathbb{R}^4_+ \ : \ 0 \leq I \leq N \leq \frac{\Lambda}{d}, 0 \leq B \leq B_m, 0 \leq M \leq M_m\},$$

where, $B_m = \frac{\phi_1\Lambda}{d(\phi_0-\phi)}$, $M_m = \frac{K}{r}\left(r+\theta\frac{\Lambda}{d}\right)$ and it attracts all solutions initiating in the interior of positive orthant.
For the proof of positivity of solutions and boundedness, see **Appendix A**.

4. Equilibrium analysis

The model system (2.3) is nonlinear and so it is difficult to find exact solution. Thus, we discuss its qualitative behavior around the equilibrium using stability theory of differential equation to get insight regarding the long-term disease dynamics and its control strategies. In this section, we show the feasibility of all equilibria by setting rate of change with respect to time t of all dynamical variables to zero. For the model system (2.3), we obtain four feasible equilibria, which are listed as follows:
(i) The disease and budget-free equilibrium (DBFE) $E_1\left(0, \frac{\Lambda}{d}, 0, 0\right)$.

(ii) The budget-free endemic equilibrium (BFEE) $E_2(I_2, N_2, B_2, 0)$. This equilibrium is feasible if $R_0 > 1$.
(iii) The disease-free equilibrium (DFE) $E_3(0, \frac{\Lambda}{d}, 0, K)$.
(iv) The interior equilibrium (IE) $E^*(I^*, N^*, B^*, M^*)$. This equilibrium is feasible if $R_1 > 1$,
where, $R_0 = \frac{\beta\Lambda}{d(\nu+\alpha+d)} + \frac{\eta\phi_1\Lambda}{dL(\nu+\alpha+d)(\phi_0-\phi)} = R_{0d} + R_{0i}$ and
$R_1 = \left(\beta - \beta_1 \frac{k_1 K}{p+k_1 K}\right) \frac{\Lambda}{d(\nu+\alpha+d)} + \frac{\eta\phi_1\Lambda}{dL(\nu+\alpha+d)\left(\phi_0-\phi+\Phi\frac{(1-k_1)K}{q+(1-k_1)K}\right)} = R_{1d} + R_{1i}$.

The quantity R_0 is known as basic reproduction number in absence of budget, which is defined as the number of secondary infected individuals produced by an infected individual due to direct contact as well as indirectly through bacteria during his / her whole infectious period in entirely susceptible population [5]. The basic reproduction number (R_0) is an important non-dimensional threshold quantity in epidemiology to predict disease outbreak. If $R_0 > 1$ (or $R_0 < 1$), then on average an infected individual will produce more than (or less than) one secondary infected individuals during his or her whole infectious period in entirely susceptible population and hence disease will persist (or be eradicated) in the population, respectively. The quantity R_{0d} captures the dynamics of simple SIS model with immigration when disease is propagated in the system through direct contact and the quantity R_{0i} captures the dynamics of simple SIS model with immigration when disease spreads in the system indirectly through bacteria present in the environment. In presence of budget to warn people and for sanitation, the basic reproduction number $R_0 = R_{0d} + R_{0i}$ is modified to $R_1 = R_{1d} + R_{1i}$. It is easy to note that $R_1 < R_0$, indicating that the presence of awareness and sanitation coverage lowers the epidemic threshold and reduces the infection risk through budget allocation. The basic reproduction number (R_1) is obtained by applying next generation matrix technique [5], for more detail, see **Appendix B**.
The feasibility of equilibrium E_1 and E_3 is trivial. In the following, we show the feasibility of equilibrium E_2 and E^*.
Feasibility of equilibrium E_2:
In equilibrium $E_2(I_2, N_2, B_2, 0)$, the values of I_2, N_2 and B_2 are obtained by solving the following set of algebraic equations:

$$\beta(N - I)I + \eta\frac{B}{L + B}(N - I) - (\nu + \alpha + d)I = 0, \tag{4.1}$$

$$\Lambda - dN - \alpha I = 0, \tag{4.2}$$

$$\phi_1 I - (\phi_0 - \phi)B = 0. \tag{4.3}$$

Using equations (4.2) and (4.3) in equation (4.1), we obtain following equation in I ($I \neq 0$):

$$F(I) = \left(\beta + \frac{\eta\phi_1}{L(\phi_0 - \phi) + \phi_1 I}\right)\left(\frac{\Lambda - (\alpha + d)I}{d}\right) - (\nu + \alpha + d) = 0. \tag{4.4}$$

From equation (4.4), we may easily note that:
(i) $F(0) = \left(\left(\beta + \frac{\eta\phi_1}{L(\phi_0-\phi)}\right)\frac{\Lambda}{d} - (\nu + \alpha + d)\right) > 0$, if $R_0 > 1$,
(ii) $F\left(\frac{\Lambda}{\alpha+d}\right) = -(\nu + \alpha + d) < 0$,
(iii) $F'(I) = -\left[\frac{\alpha+d}{d}\left(\beta + \frac{\eta\phi_1}{L(\phi_0-\phi)+\phi_1 I}\right) + \frac{\eta\phi_1^2}{(L(\phi_0-\phi)+\phi_1 I)^2}\left(\frac{\Lambda-(\alpha+d)I}{d}\right)\right] < 0$ for $I \in \left(0, \frac{\Lambda}{\alpha+d}\right)$.
Thus, $F(I) = 0$ has a unique positive root $I = I_2$ (say) in $\left(0, \frac{\Lambda}{\alpha+d}\right)$, provided $R_0 > 1$ and for this positive

value of $I = I_2$, from equations (4.2) and (4.3), we get positive values of $N_2 = \frac{\Lambda - \alpha I_2}{d}$ and $B_2 = \frac{\phi_1 I_2}{\phi_0 - \phi}$, respectively. It may be noted that if $R_0 < 1$, the equation (4.4) has no positive solution in $\left(0, \frac{\Lambda}{\alpha + d}\right)$. Thus, the equilibrium $E_2(I_2, N_2, B_2, 0)$ is feasible provided $R_0 > 1$.

Feasibility of equilibrium E^*:
In equilibrium $E^*(I^*, N^*, B^*, M^*)$, the values of I^*, N^*, B^* and M^* are obtained by solving the following set of algebraic equations:

$$\left(\beta - \beta_1 \frac{k_1 M}{p + k_1 M}\right)(N - I)I + \eta \frac{B}{L + B}(N - I) - (\nu + \alpha + d)I = 0, \tag{4.5}$$

$$\Lambda - dN - \alpha I = 0, \tag{4.6}$$

$$\phi_1 I - (\phi_0 - \phi)B - \Phi \frac{(1 - k_1)M}{q + (1 - k_1)M} B = 0, \tag{4.7}$$

$$r\left(1 - \frac{M}{K}\right) + \theta I = 0. \tag{4.8}$$

Using equation (4.8) in equation (4.7), we obtain

$$B = \frac{\phi_1 I(qr + (1 - k_1)K(r + \theta I))}{(\phi_0 - \phi)(qr + (1 - k_1)K(r + \theta I)) + \Phi(1 - k_1)K(r + \theta I)}. \tag{4.9}$$

Further, using equations (4.5), (4.6) and (4.9), we have following equation in I ($I \neq 0$):

$$\begin{aligned} G(I) \;=\; & \left(\beta - \beta_1 \frac{k_1 K(r + \theta I)}{pr + k_1 K(r + \theta I)}\right)\left(\frac{\Lambda - (\alpha + d)I}{d}\right) \\ & + \frac{\eta \phi_1 (qr + (1 - k_1)K(r + \theta I))}{(L(\phi_0 - \phi) + \phi_1 I)(qr + (1 - k_1)K(r + \theta I)) + L\Phi(1 - k_1)K(r + \theta I)} \\ & \times \left(\frac{\Lambda - (\alpha + d)I}{d}\right) - (\nu + \alpha + d) = 0. \end{aligned} \tag{4.10}$$

From equation (4.10), we may easily note that:
(i) $G(0) = \left\{\left(\left(\beta - \beta_1 \frac{k_1 K}{p + k_1 K}\right) + \frac{\eta \phi_1}{L\left(\phi_0 - \phi + \Phi \frac{(1 - k_1)K}{q + (1 - k_1)K}\right)}\right)\frac{\Lambda}{d} - (\nu + \alpha + d)\right\} > 0$, if $R_1 > 1$.
(ii) $G\left(\frac{\Lambda}{\alpha + d}\right) = -(\nu + \alpha + d) < 0$, and (iii) $G'(I) < 0$ in $\left(0, \frac{\Lambda}{\alpha + d}\right)$.
Thus, $G(I) = 0$ has a unique positive root $I = I^*$ (say) in $\left(0, \frac{\Lambda}{\alpha + d}\right)$, provided $R_1 > 1$. Now, substituting the positive value of $I = I^*$ in equations (4.6), (4.8) and (4.9), we obtain positive values of N^*, B^* and M^*, respectively. Thus, the interior equilibrium $E^*(I^*, N^*, B^*, M^*)$ is feasible provided $R_1 > 1$ and $I^* \in \left(0, \frac{\Lambda}{\alpha + d}\right)$.

Remark 1. From equation (4.10), it is easy to note that $\frac{dI^*}{d\theta} < 0$, $\frac{dI^*}{dK} < 0$, $\frac{dI^*}{dp} > 0$ and $\frac{dI^*}{dq} > 0$. This implies that the equilibrium number of infected individuals decreases with the increase in per-capita growth rate of budget allocation due to increase in infected individuals (θ) and it also decreases as the carrying capacity of budget allocation to warn people and for sanitation (K) increases whereas the equilibrium number of infected individuals increases as the values of half saturation point (i.e., p and q) increases. This is because the contact rate of susceptible with infected individuals increases and reduction rate of bacteria from the environment through sanitation coverage decreases as the values of p and q increases.

Remark 2. From equation (4.10), it is easy to see that $\frac{dI^*}{d\beta_1} < 0$ and $\frac{dI^*}{d\Phi} < 0$. This implies that the equilibrium number of infected individuals decreases as efficacy of budget allocation to reduce the contact rate via propagating awareness (β_1) and efficacy of sanitation coverage to reduce bacteria in the environment due to budget allocation (Φ) increases. Further, it is also easy to note that $\frac{dM^*}{d\beta_1} < 0$ and $\frac{dM^*}{d\Phi} < 0$, i.e., equilibrium amount of budget required for awareness and sanitation decreases as the values of β_1 and Φ increases. This is because the increase in efficacy of budget allocation to reduce the contact rate via propagating awareness (β_1) and efficacy of sanitation coverage to reduce the bacteria in the environment (Φ) gives less number of infected individuals and hence less amount of budget will be required to control the infection.

Remark 3. Form equation (4.10), it is easy to note that $\frac{dI^*}{dk_1} < 0$ if the condition $\mathbb{A} - \mathbb{S} = \left\{\frac{\beta_1 p}{(pr+k_1K(r+\theta I^*))^2} - \frac{\eta\phi_1 L\Phi q}{[(L(\phi_0-\phi)+\phi_1 I^*)(qr+(1-k_1)K(r+\theta I^*))+L\Phi(1-k_1)K(r+\theta I^*)]^2}\right\} > 0$ is satisfied. This implies that the equilibrium number of infected individuals decreases as the values of fractional constant of budget allocation to warn people via propagating awareness (k_1) increases. Further, it is easy to see that $\frac{dI^*}{dk_1} > 0$ $\left(i.e., \frac{dI^*}{d(1-k_1)} < 0\right)$ if condition $\mathbb{A} - \mathbb{S} < 0$ holds. This implies that when $\mathbb{A} < \mathbb{S}$, the equilibrium number of infected individuals decreases with the increase in fraction of budget allocation used for sanitation coverage (i.e., $1 - k_1$). The condition tells that when budget should spend on awareness/sanitation to reduce the number of infected individuals.

5. Stability analysis in absence of delay (i.e., $\tau = 0$)

5.1. Local stability analysis

In this section, we present the results of local stability analysis of equilibrium E_1, E_2, E_3 and E^* in absence of delay (i.e., $\tau = 0$) by finding the sign of real part of eigenvalues of Jacobian matrix obtained for model system (2.3) evaluated at equilibrium E_1, E_2, E_3 and E^*. The local stability conditions of equilibrium E_i ($i = 1, 2, 3$) and E^* are stated in the following theorem:

Theorem 5.1. *(i) The disease and budget-free equilibrium E_1 is always feasible and unstable.*
(ii) The budget-free endemic equilibrium E_2 is feasible if $R_0 > 1$ and is unstable.
(iii) The disease-free equilibrium E_3 is always feasible and is locally asymptotically stable whenever $R_1 < 1$, whereas it is unstable with locally unstable manifold either in I-direction or B-direction and locally stable manifold in $N - M$ plane, whenever $R_1 > 1$.
(iv) The interior equilibrium E^* is feasible if $R_1 > 1$ and is locally asymptotically stable provided

$$C_3(C_1C_2 - C_3) - C_1^2C_4 > 0, \tag{5.1}$$

where, C_i's ($i = 1, 2, 3, 4$) are coefficients of the characteristic equation of the Jacobian matrix evaluated at equilibrium E^*, defined in the proof given in **Appendix C**.

The above theorem tells that if E^* is feasible, then E_3 is saddle point with locally unstable manifold

either in I-direction or B-direction and locally stable manifold in $N-M$ plane. Thus, E_3 is unstable whenever E^* is feasible.
The local stability analysis of the interior equilibrium tells that if the initial values of any trajectory are near to equilibrium $E^*(I^*, N^*, B^*, M^*)$, then the trajectory approaches to the equilibrium E^* under the condition (5.1). Thus, the initial values of the state variables I, N, B and M are near to the corresponding equilibrium levels, the equilibrium number of infected individuals get stabilized provided condition (5.1) is satisfied.

5.2. Global stability analysis

In this section, first we present the results of global stability analysis of equilibrium E_3 ($\tau = 0$). Further, we present the results of global stability analysis of equilibrium E^* in absence of delay ($\tau = 0$) using Lyapunov's direct method by defining a suitable scalar-valued function, called the Lyapunov function and thus we have the following results regarding the global stability of equilibrium E_3 and E^*.

Theorem 5.2. *The disease-free equilibrium E_3 of model system (2.3) is globally asymptotically stable in the region $\mathbb{R}^2_+$ of $I-B$ plane if $R_1 < 1$.*

Proof. From model system (2.3), we have

$$\frac{dI}{dt} = \left(\beta - \beta_1 \frac{k_1 M}{p + k_1 M}\right)(N - I)I + \eta \frac{B}{L + B}(N - I) - (\nu + \alpha + d)I := f_1,$$
$$\frac{dB}{dt} = \phi_1 I - (\phi_0 - \phi)B - \Phi \frac{(1 - k_1)M}{q + (1 - k_1)M} B := f_2.$$

Consider, $h(I, B) = \frac{1}{IB}$ and $\triangle_{E_3}(I, B) = \frac{\partial}{\partial I}(hf_1) + \frac{\partial}{\partial B}(hf_2)$.
Since $h(I, B) = \frac{1}{IB} > 0$ for all $I, B > 0$ and so we obtain
$\triangle_{E_3}(I, B) = \frac{\partial}{\partial I}(hf_1) + \frac{\partial}{\partial B}(hf_2) = -\left[\frac{1}{B}\left(\beta - \beta_1 \frac{k_1 M}{p+k_1 M}\right) + \frac{\eta}{L+B} + \frac{\phi_1}{B^2}\right] < 0.$
Clearly, $\triangle_{E_3}(I, B)$ does not change its sign and also it is not identically zero in positive quadrant of $I-B$ plane. By Bendixson-Dulac criteria, the system has no limit cycle in positive quadrant of $I-B$ plane. As disease-free equilibrium E_3 is locally asymptotically stable whenever $R_1 < 1$ and so it will be globally asymptotically stable in the region $\mathbb{R}^2_+$ of $I-B$ plane, if $R_1 < 1$. □

Theorem 5.3. *The interior equilibrium E^*, if feasible, is globally asymptotically stable in Ω provided the following inequalities are satisfied:*

$$\left(\frac{\eta}{I^*}\right)^2 < \frac{4}{5}\left(\frac{d}{\alpha}\right)\left(\beta - \beta_1 \frac{k_1 M^*}{p + k_1 M^*}\right)^2, \tag{5.2}$$

$$\left(\frac{\beta_1 k_1 \Lambda}{d(p + k_1 M^*)}\right) < \frac{4}{15}\left(\frac{r}{K\theta}\right)\left(\beta - \beta_1 \frac{k_1 M^*}{p + k_1 M^*}\right), \tag{5.3}$$

$$G_{11} < G_{12} \min\{G_{13}, G_{14}\}, \tag{5.4}$$

where,

$G_{11} = \frac{15}{4} \frac{\left(\frac{\eta(N^*-I^*)}{I^*(L+B^*)}\right)^2}{\left(\beta - \beta_1 \frac{k_1 M^*}{p+k_1 M^*}\right)\left(\phi_0 - \phi + \Phi \frac{(1-k_1)M^*}{q+(1-k_1)M^*}\right)}$, $G_{12} = \frac{4}{3}\left(\phi_0 - \phi + \Phi \frac{(1-k_1)M^*}{q+(1-k_1)M^*}\right)$,
$G_{13} = \frac{1}{5\phi_1^2}\left(\beta - \beta_1 \frac{k_1 M^*}{p+k_1 M^*}\right)$ and $G_{14} = \left(\frac{m_3 r}{3K}\right)\left(\frac{d(\phi_0-\phi)(q+(1-k_1)M^*)}{\phi_1 \Lambda \Phi(1-k_1)}\right)^2$.

The inequality (5.3) is the condition for determining the positive value of m_3. For the proof of this theorem, see **Appendix D**.

6. Stability analysis in presence of delay (i.e., $\tau > 0$)

In previous sections, we have summarized the results of qualitative analysis of model system (2.3), regarding the feasibility of equilibrium and their stability properties in absence of delay (i.e., $\tau = 0$). In this section, we present the stability of interior equilibrium E^* of model system (2.3) in presence of delay and also explore the possibility of Hopf-bifurcation at the interior equilibrium E^* by taking time delay τ as a bifurcation parameter. First, we linearize the model system (2.3) about the equilibrium $E^*(I^*, N^*, B^*, M^*)$ by using the following transformations: $I(t) = I^* + i(t)$, $N(t) = N^* + n(t)$, $B(t) = B^* + b(t)$ and $M(t) = M^* + m(t)$. The linear system of model system (2.3) about the equilibrium E^* is given as follows:

$$\frac{d\upsilon}{dt} = P_1\upsilon(t) + P_2\upsilon(t-\tau), \tag{6.1}$$

where, $\upsilon(t) = [i(t), n(t), b(t), m(t)]^T$,

$$P_1 = \begin{pmatrix} -J_{11}^* & J_{12}^* & J_{13}^* & -J_{14}^* \\ -\alpha & -d & 0 & 0 \\ \phi_1 & 0 & -J_{33}^* & -J_{34}^* \\ 0 & 0 & 0 & -\frac{r}{K}M^* \end{pmatrix} \text{ and } P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \theta M^* & 0 & 0 & 0 \end{pmatrix}.$$

In the above system $i(t)$, $n(t)$, $b(t)$ and $m(t)$ are small perturbations around the equilibrium E^*. The characteristic equation for the linearized system (6.1) is obtained as:

$$\chi^4 + \rho_1\chi^3 + \rho_2\chi^2 + \rho_3\chi + \rho_4 + (\sigma_1\chi^2 + \sigma_2\chi + \sigma_3)e^{-\chi\tau} = 0, \tag{6.2}$$

where, $\rho_1 = d + \frac{r}{K}M^* + J_{11}^* + J_{33}^*$,
$\rho_2 = \alpha J_{12}^* + \frac{dr}{K}M^* + \left(d + \frac{r}{K}M^*\right)(J_{11}^* + J_{33}^*) + \beta(M^*)I^*J_{33}^* + \frac{\eta\phi_1(N^*B^*+LI^*)}{(L+B^*)^2}$,
$\rho_3 = \alpha J_{12}^*\left(\frac{rM^*}{K} + J_{33}^*\right) + \left(d + \frac{rM^*}{K}\right)\left(\beta(M^*)I^*J_{33}^* + \frac{\eta\phi_1(N^*B^*+LI^*)}{(L+B^*)^2}\right)$
$+ \frac{drM^*}{K}(J_{11}^* + J_{33}^*)$,
$\rho_4 = \frac{\alpha r}{K}M^*J_{12}^*J_{33}^* + \frac{dr}{K}M^*\left(\beta(M^*)I^*J_{33}^* + \frac{\eta\phi_1(N^*B^*+LI^*)}{(L+B^*)^2}\right)$,
$\sigma_1 = \theta M^*J_{14}^*$, $\sigma_2 = \theta M^*(J_{13}^*J_{34}^* + J_{14}^*J_{33}^* + dJ_{14}^*)$, $\sigma_3 = d\theta M^*(J_{13}^*J_{34}^* + J_{14}^*J_{33}^*)$, $\beta(M^*) = \left(\beta - \beta_1\frac{k_1M^*}{p+k_1M^*}\right)$,
$J_{11}^* = \beta(M^*)I^* + \eta\frac{B^*}{L+B^*}\left(\frac{N^*}{I^*}\right)$, $J_{12}^* = \beta(M^*)I^* + \eta\frac{B^*}{L+B^*}$, $J_{13}^* = \frac{\eta L}{(L+B^*)^2}(N^* - I^*)$, $J_{14}^* = \frac{\beta_1k_1p}{(p+k_1M^*)^2}(N^* - I^*)I^*$,
$J_{33}^* = \left(\phi_0 - \phi + \Phi\frac{(1-k_1)M^*}{q+(1-k_1)M^*}\right)$, and $J_{34}^* = \frac{\Phi(1-k_1)qB^*}{(q+(1-k_1)M^*)^2}$.
In writing the above values of $\rho_i's (i = 1, 2, 3, 4)$, we have used the fact $J_{11}^*J_{33}^* - J_{13}^*\phi_1 = \beta(M^*)I^*J_{33}^* + \frac{\eta\phi_1(N^*B^*+LI^*)}{(L+B^*)^2}$.
Now, in order to show the existence of Hopf-bifurcation, we must have a pair of purely imaginary roots of the characteristic equation (6.2). For this purpose, we substitute, $\chi = i\,\omega$ $(\omega > 0)$ in equation (6.2) and separate real and imaginary parts, we have the following transcendental equations:

$$\omega^4 - \rho_2\omega^2 + \rho_4 = -\sigma_2\omega\sin(\omega\tau) - (\sigma_3 - \sigma_1\omega^2)\cos(\omega\tau), \tag{6.3}$$

$$\rho_1\omega^3 - \rho_3\omega = \sigma_2\omega\cos(\omega\tau) - (\sigma_3 - \sigma_1\omega^2)\sin(\omega\tau). \tag{6.4}$$

On squaring and adding equations (6.3) and (6.4) and substituting $\omega^2 = \varphi$, we obtain the fourth degree polynomial equation in φ as follows:

$$H(\varphi) = \varphi^4 + D_1\varphi^3 + D_2\varphi^2 + D_3\varphi + D_4 = 0, \tag{6.5}$$

where, $D_1 = \rho_1^2 - 2\rho_2$, $D_2 = \rho_2^2 - 2\rho_1\rho_3 + 2\rho_4 - \sigma_1^2$, $D_3 = \rho_3^2 - 2\rho_2\rho_4 - \sigma_2^2 + 2\sigma_1\sigma_3$, $D_4 = \rho_4^2 - \sigma_3^2$.
Now, we discuss the nature of roots of fourth degree polynomial equation (6.5) in the following cases:
(H_1): If all the coefficients D_i's $(i = 1, 2, 3, 4)$ in $H(\varphi)$ are positive then by Descartes' rule of signs, the equation (6.5) has no positive real root and thus the characteristic equation (6.2) has no pair of purely imaginary roots for any $\tau > 0$. Thus, all the roots of equation (6.2) are either negative or with negative real part in presence of delay (i.e., $\tau > 0$). In this case, we have following theorem:

Theorem 6.1. *If condition (H_1) is satisfied, then the interior equilibrium E^*, if feasible, is locally asymptotically stable for all $\tau > 0$, provided it is stable in absence of delay.*

(H_2): If not all the coefficients D_i's $(i = 1, 2, 3, 4)$ in equation (6.5) are positive. Using Descartes' rule of signs, we have following conditions in which the equation (6.5) has exactly one positive real root:
$(B1)$ $D_1 > 0$, $D_2 > 0$, $D_3 > 0$, $D_4 < 0$. $(B2)$ $D_1 > 0$, $D_2 > 0$, $D_3 < 0$, $D_4 < 0$.
$(B3)$ $D_1 > 0$, $D_2 < 0$, $D_3 < 0$, $D_4 < 0$. $(B4)$ $D_1 < 0$, $D_2 < 0$, $D_3 < 0$, $D_4 < 0$.
If any of the conditions (Bi) $(i = 1, 2, 3, 4)$ are satisfied then equation (6.2) has a pair of purely imaginary root $\pm i\,\omega_0$.
Further, from transcendental equations (6.3) and (6.4) corresponding to positive value of ω_0, we have

$$\tan(\omega_0\tau) = \frac{\Theta_1}{\Theta_2}, \tag{6.6}$$

where, $\Theta_1 = \sigma_2\omega_0(\omega_0^4 - \rho_2\omega_0^2 + \rho_4) + (\sigma_3 - \sigma_1\omega_0^2)(\rho_1\omega_0^3 - \rho_3\omega_0)$ and $\Theta_2 = (\sigma_3 - \sigma_1\omega_0^2)(\omega_0^4 - \rho_2\omega_0^2 + \rho_4) - \sigma_2\omega_0(\rho_1\omega_0^3 - \rho_3\omega_0)$. Thus, the value of τ_k corresponding to positive value of ω_0 may be obtained as follows:

$$\tau_k = \frac{k\pi}{\omega_0} + \frac{1}{\omega_0}\tan^{-1}\left(\frac{\Theta_1}{\Theta_2}\right), \tag{6.7}$$

for $k = 0, 1, 2, 3, ...$
Using Butler's Lemma [8], we can say that the interior equilibrium E^* of the model system (2.3) remains stable for $\tau < \tau_0$. Now, we explore, whether the Hopf-bifurcation occurs or not as τ increases through τ_0. For this we need to prove the following Lemma:

Lemma 6.2. *If condition (H_2) is satisfied, then the following transversality condition is satisfied:*

$$sgn\left[\frac{d(\Re(\chi))}{d\tau}\right]_{\tau=\tau_0} > 0. \tag{6.8}$$

Proof. Differentiating equation (6.2), with respect to τ, we have

$$\frac{d\chi}{d\tau} = \frac{(\sigma_1\chi^2 + \sigma_2\chi + \sigma_3)\chi e^{-\chi\tau}}{(4\chi^3 + 3\rho_1\chi^2 + 2\rho_2\chi + \rho_3) + (2\sigma_1\chi + \sigma_2)e^{-\chi\tau} - (\sigma_1\chi^2 + \sigma_2\chi + \sigma_3)\tau e^{-\chi\tau}}.$$

This gives

$$\left(\frac{d\chi}{d\tau}\right)^{-1} = \frac{(4\chi^3 + 3\rho_1\chi^2 + 2\rho_2\chi + \rho_3) + (2\sigma_1\chi + \sigma_2)e^{-\chi\tau}}{\chi(\sigma_1\chi^2 + \sigma_2\chi + \sigma_3)e^{-\chi\tau}} - \frac{\tau}{\chi}. \tag{6.9}$$

This implies that

$$sgn\left[\frac{d(\Re(\chi))}{d\tau}\right]_{\tau=\tau_0} = sgn\left[\frac{d(\Re(\chi))}{d\tau}\right]^{-1}_{\tau=\tau_0} \tag{6.10}$$

$$= sgn\left[\Re\frac{d(\chi)}{d\tau}\right]^{-1}_{\chi=i\omega_0} \tag{6.11}$$

$$= sgn\left[\frac{4\omega_0^6 + 3D_1\omega_0^4 + 2D_2\omega_0^2 + D_3}{\sigma_2^2\omega_0^2 + (\sigma_3 - \sigma_1\omega_0^2)^2}\right] \tag{6.12}$$

$$= sgn\left[\frac{H'(\omega_0^2)}{\sigma_2^2\omega_0^2 + (\sigma_3 - \sigma_1\omega_0^2)^2}\right]. \tag{6.13}$$

Thus, it may be noted that $H'(\omega_0^2) > 0$ if one of the conditions $(Bi)(i = 1, 2, 3, 4)$ is satisfied. This proves the Lemma 6.2.

Thus, we have the following result, which is stated in the form of a theorem as follows:

Theorem 6.3. *If condition (5.1) holds and any one of the conditions* $(Bi)(i = 1, 2, 3, 4)$ *is satisfied then the interior equilibrium* E^* *is locally asymptotically stable for all* $\tau \in [0, \tau_0)$ *and becomes unstable for* $\tau > \tau_0$. *The model system (2.3) undergoes a supercritical Hopf-bifurcation when* $\tau = \tau_0$, *yielding a family of periodic solutions bifurcating from equilibrium* E^* *as* τ *passes through the critical value* τ_0 *[10].*

Remark 4. Further, if none of the conditions $(Bi)(i = 1, 2, 3, 4)$ is satisfied, then the equation (6.5) may have more than one positive roots. As a result equation (6.2) may have more than one pair of purely imaginary roots and the system may possess the finite number of stability switches as the delay parameter τ increases.

(H_3): Analytically, it is not easy to find the exact condition in which equation (6.5) possess two positive real roots, i.e., equation (6.2) has two pair of purely imaginary roots. So we discuss the result numerically in which equation (6.5) has two positive real roots. Numerically, it is obtained that the equation (6.5) has two positive real roots φ_+ (corresponding to ω_+^2) and φ_- (corresponding to ω_-^2) where $(\varphi_+ > \varphi_-)$, i.e., the characteristic equation (6.2) has two pairs of purely imaginary roots $\pm i\omega_\pm$. For these positive values of $\omega_\pm$, from equations (6.3) and (6.4) we can obtain the positive value of $\tau_k^\pm$ as follows:

$$\tau_k^\pm = \frac{1}{\omega_\pm}\tan^{-1}\left[\frac{\sigma_2\omega_\pm(\omega_\pm^4 - \rho_2\omega_\pm^2 + \rho_4) + (\sigma_3 - \sigma_1\omega_\pm^2)(\rho_1\omega_\pm^3 - \rho_3\omega_\pm)}{(\sigma_3 - \sigma_1\omega_\pm^2)(\omega_\pm^4 - \rho_2\omega_\pm^2 + \rho_4) - \rho_2\omega_\pm(\rho_1\omega_\pm^3 - \rho_3\omega_\pm)}\right] + \frac{(2k+1)\pi}{\omega_\pm}, \tag{6.14}$$

for $k = 0, 1, 2, 3, 4, ...$

Using Butler's lemma [8], we can say that the equilibrium E^* of the model system (2.3) remains stable for $\tau < \tau_k^+$ and unstable for $\tau < \tau_k^-$.

Now, we explore whether Hopf-bifurcation occurs or not as τ passes through $\tau_k^\pm$. For this we need to prove the following lemma:

Lemma 6.4. *If equation (6.5) has two positive real roots, then the following transversality conditions are satisfied:*

$$sgn\left[\Re\frac{d(\chi)}{d\tau}\right]_{\tau=\tau_k^+} > 0 \text{ and } sgn\left[\Re\frac{d(\chi)}{d\tau}\right]_{\tau=\tau_k^-} < 0. \tag{6.15}$$

Proof. Differentiating (6.2) with respect to τ, we obtain

$$sgn\left[\frac{d(\Re(\chi))}{d\tau}\right]_{\tau=\tau_k^\pm} = sgn\left[\frac{d(\Re(\chi))}{d\tau}\right]^{-1}_{\tau=\tau_k^\pm} \tag{6.16}$$

$$= sgn\left[\Re\frac{d(\chi)}{d\tau}\right]^{-1}_{\chi=i\ \omega_\pm} \tag{6.17}$$

$$= sgn\left[\frac{4\omega_\pm^6 + 3D_1\omega_\pm^4 + 2D_2\omega_\pm^2 + D_3}{\sigma_2^2\omega_\pm^2 + (\sigma_3 - \sigma_1\omega_\pm^2)^2}\right] \tag{6.18}$$

$$= sgn\left[\frac{H'(\omega_\pm^2)}{\sigma_2^2\omega_\pm^2 + (\sigma_3 - \sigma_1\omega_\pm^2)^2}\right]. \tag{6.19}$$

Hence, if equation (6.5) has two positive real roots then $H'(\omega_+^2) > 0$ and $H'(\omega_-^2) < 0$ and therefore the transversality conditions hold. This proves the Lemma 6.4. □

Thus, we have the following result regarding the Hopf-bifurcation theorem of functional differential equation [10, 14], which is stated as follows:

Theorem 6.5. *If equation (6.5) has two positive real roots and condition (5.1) is satisfied, then there exists a positive integer m such that there are m switches from stability to instability and eventually the system becomes unstable. More precisely, when* $\tau \in [0, \tau_0^+), (\tau_0^-, \tau_1^+), ..., (\tau_{m-1}^-, \tau_m^+)$*, the equilibrium* E^* *is stable while it is unstable when* $\tau \in (\tau_0^+, \tau_0^-), (\tau_1^+, \tau_1^-), ..., (\tau_{m-1}^+, \tau_{m-1}^-)$.

□

7. Numerical simulation

In previous sections, we have discussed the qualitative behavior of nonlinear system around the equilibrium to get insight regarding the disease dynamics and obtained results for the feasibility of equilibria, its stability properties in absence as well as in presence of delay and Hopf-bifurcation of interior equilibrium in presence of delay. In this section, to validate our analytical findings, we go for numerical simulations using MATLAB. We have chosen the following set of parameter values to validate our analytical results, which are given in Table 2.

Most of parameter values are taken from Misra et al. [27] and rest are assumed. For this set of parameter values, it may be checked that the condition for the feasibility of equilibrium E^* (i.e., $R_1 > 1$) is satisfied. The components of interior equilibrium $E^*(I^*, N^*, B^*, M^*)$ for this data are obtained as:

$$I^* = 412 \text{ persons}, \quad N^* = 39176 \text{ persons}, \quad B^* = 2472\ cells/mm^3, \quad M^* = 512.$$

The value of basic reproduction number (R_0) in absence of budget is found to be 4.42 and value of modified basic reproduction number (R_1) in presence of budget is found to be 1.81. To demonstrate the impacts of budget allocation on epidemic threshold (i.e., R_1), we have drawn contour plots of R_1 for

Table 2. Parameter values for simulation.

Parameters	Values (Units)	Parameters	Values (Units)
Λ	400 $person\ day^{-1}$	K	100
β	0.000028 $person^{-1}day^{-1}$	k_1	0.2
β_1	0.000020 $person^{-1}day^{-1}$	p	60
η	0.0005 day^{-1}	L	1000 $cells/mm^3$
ν	0.6 day^{-1}	α	0.02 day^{-1}
d	0.01 day^{-1}	ϕ_0	0.08 day^{-1}
ϕ_1	5 $cells\ (mm^3\ person\ day)^{-1}$	ϕ	0.02 day^{-1}
Φ	1 day^{-1}	q	120
r	0.005 day^{-1}	θ	0.00005 $(person\ day)^{-1}$

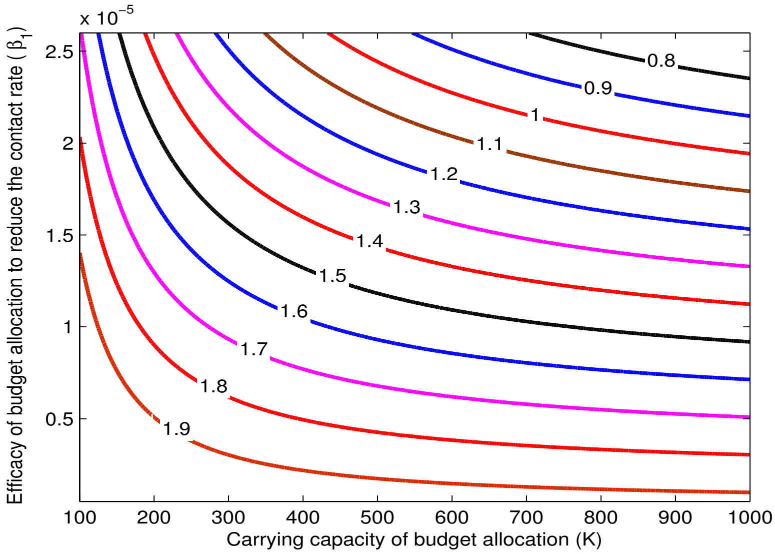

Figure 1. Effect of changing the values of K and β_1 on R_1 in absence of delay ($\tau = 0$), for the parameter values given in Table 2.

different values of carrying capacity of budget allocation (K), efficacy of budget allocation to reduce contact rate via propagating awareness (β_1) and efficacy of sanitation coverage to reduce bacteria in the environment due to budget allocation (Φ) in Figs. 1 and 2, respectively.

From these figures, it is observed that the value of R_1 is higher, for small values of K, β_1 and Φ. However, on increasing the values of these parameters, R_1 becomes less than unity and ceases the feasibility of interior equilibrium E^*, showing the importance of budget allocation to warn people and for sanitation to control the spread of infection. In Fig. 3, we have shown the effect of changing the values of k_1 and $1 - k_1$ on epidemic threshold (R_1). From this figure, it is clear that the epidemic threshold decreases on increasing the values of fractional constant of budget allocation used to warn people via propagating awareness (k_1) up to a threshold value (i.e., $k_1 < 0.53$) and above which the fraction of budget allocation used for sanitation coverage is responsible to reduce the epidemic threshold.

To see the effect of sanitation and awareness programs, we have made comparison plot of infected population $I(t)$ in absence and presence of budget allocation, which is shown in Fig 4. This figure shows that sanitation and awareness are beneficial to reduce the disease burden and thus disease can be controlled. In Figs. 5 (a),(b) (c) and (d), we have drawn the variation of infected population $I(t)$, with

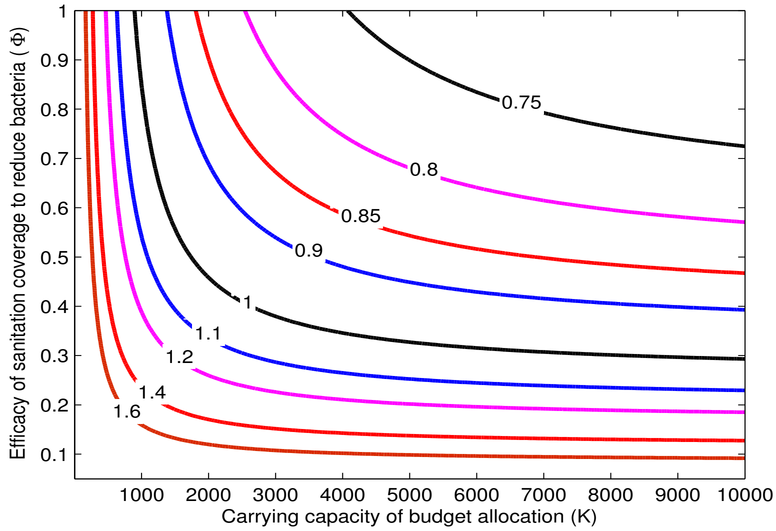

Figure 2. Effect of changing the values of K and Φ on R_1 in absence of delay ($\tau = 0$), for the parameter values given in Table 2.

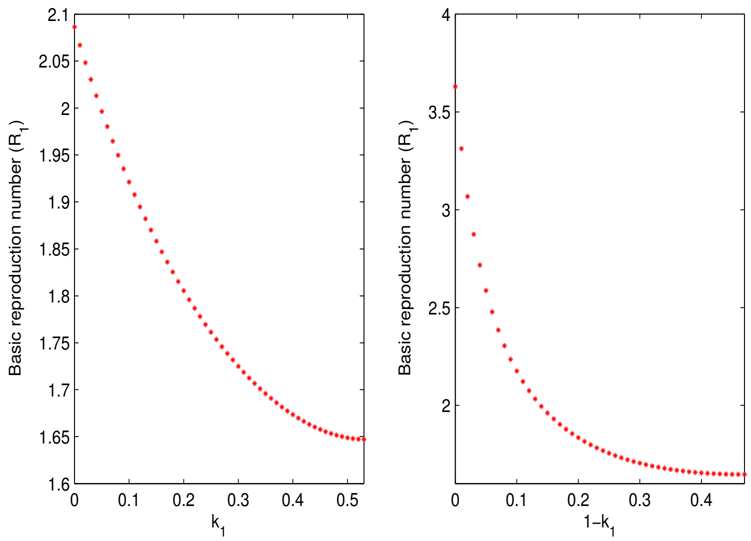

Figure 3. Effect of changing the values of k_1 and $1 - k_1$ on R_1 in absence of delay ($\tau = 0$), for the parameter values given in Table 2, which shows that epidemic threshold decreases with the increase in values of k_1 up to a threshold value ($k_1 < 0.53$) and above which the fraction of budget allocation used for sanitation coverage is responsible to reduce the epidemic threshold.

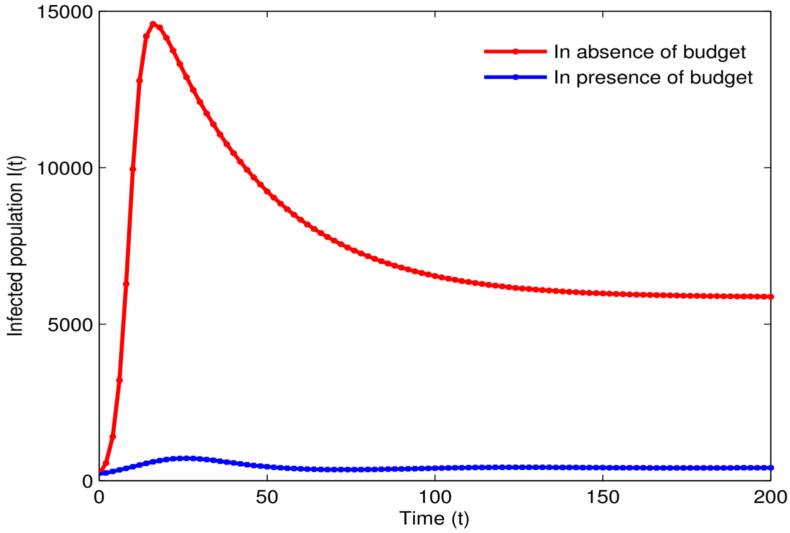

Figure 4. Comparison of infected individuals $I(t)$ with respect to time t in absence and presence of budget allocation, for the parameter values given in Table 2 (for $\tau = 0$).

respect to time t, for different values of per-capita growth rate of budget allocation due to increase in infected individuals (θ) and carrying capacity of budget allocation (K) and half saturation point p and q, respectively. From these figures, it is apparent that as the values of θ or K increases, the equilibrium number of infected individuals decreases and also reduces its epidemic peak whereas the equilibrium number of infected individuals increases with the increase in values of p and q, which supports remark 1.

In remark 2, it is shown that the equilibrium number of infected individuals and equilibrium amount of budget required for awareness and sanitation decreases as the values of efficacy of budget allocation to reduce the contact rate via propagating awareness (β_1) and efficacy of sanitation coverage to reduce bacteria in the environment due to budget allocation (Φ) increases. For this, we have plotted the variation of infected individuals $I(t)$ and budget allocation $M(t)$, with respect to time t, for different values of β_1 and Φ, shown in Figs. 6 (a), (b), (c) and (d), respectively, which supports remark 2. This is because the increase in efficacy of budget allocation to reduce the contact rate (β_1) and efficacy of sanitation coverage to reduce the bacteria from the environment (Φ) gives less number of infected individual and hence less amount of budget will be requisite to control the spread of infection.
In Figs. 7 (a) and (b), we have plotted the variation of $I(t)$ with respect to time t for different values of k_1 for $\beta_1 = 0.000014$ $person^{-1}$ day^{-1}, $p = 600$, $q = 1200$, $\eta = 0.5$ day^{-1}, $L = 5000$ $cells/mm^3$ and $\phi_1 = 1$ $cells$ $(mm^3$ $person$ $day)^{-1}$, keeping the rest of the parameter values same as given in Table 2, when conditions $\mathbb{A} > \mathbb{S}$ ($k_1 < k_{1_c} = 0.4367$) and $\mathbb{A} < \mathbb{S}$ ($k_1 > k_{1_c}$) are satisfied, respectively. From these figures, it is observed that the equilibrium number of infected individuals decreases on increasing the values of fractional constant of budget allocation used to warn people via propagating awareness (k_1) up to a threshold value (i.e., $k_1 < k_{1_c}$). However, further increase in values of k_1 above a threshold value ($k_1 > k_{1_c}$), the fraction of budget allocation used for sanitation coverage (i.e., $1 - k_1$) is responsible to reduce the equilibrium number of infected individuals, which supports remark 3. For the given data, it is observed that up to 43.67% of budget is used for awareness and the remaining 56.33% is used for

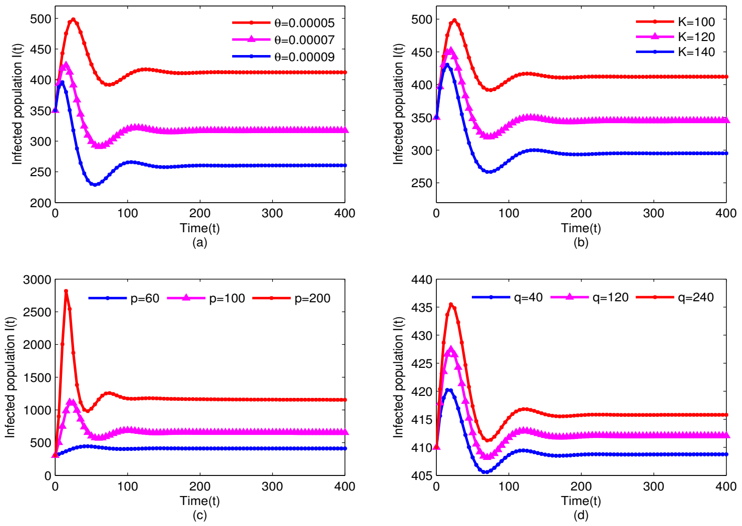

Figure 5. Variation of $I(t)$ with respect to time t for different values of θ, K, p and q in absence of delay ($\tau = 0$), shown in Figs. (a), (b), (c) and (d), respectively, for the parameter values given in Table 2.

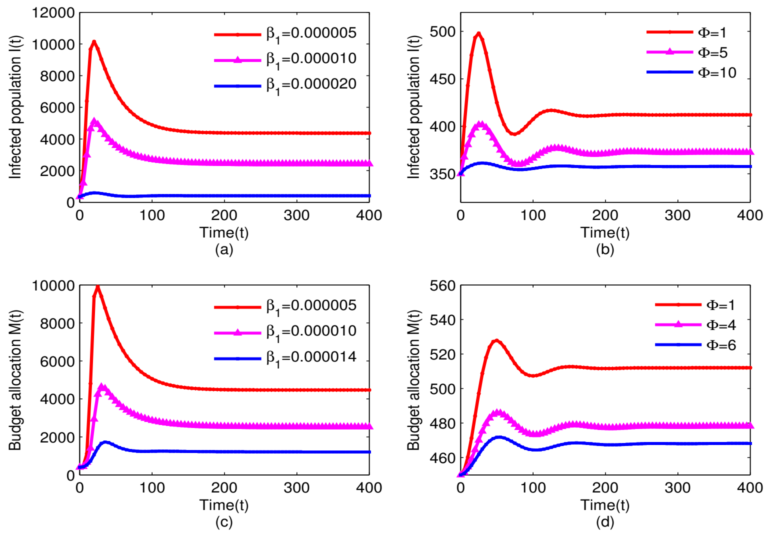

Figure 6. Variation of $I(t)$ and $M(t)$ with respect to time t for different values of β_1 and Φ, in absence of delay ($\tau = 0$), shown in Figs. (a), (b), (c) and (d), respectively, for the parameter values given in Table 2.

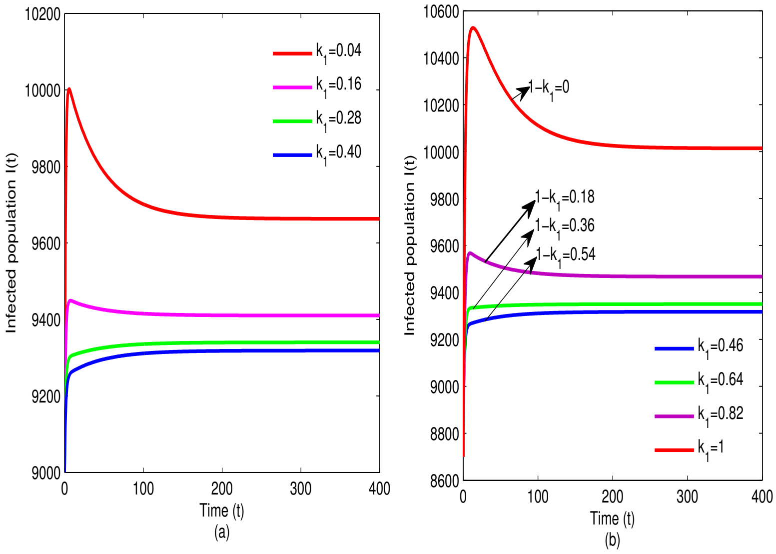

Figure 7. Variation of $I(t)$ with respect to time t for different values of k_1 in absence of delay ($\tau = 0$) (when conditions $\mathbb{A} > \mathbb{S}$ ($k_1 < k_{1_c} = 0.4367$) and $\mathbb{A} < \mathbb{S}$ ($k_1 > k_{1_c}$) are satisfied), which is shown in Figs. (a) and (b), respectively for $\beta_1 = 0.000014$ $person^{-1}$ day^{-1}, $p = 600$, $q = 1200$, $\eta = 0.5$ day^{-1}, $L = 5000$ $cells/mm^3$, $\phi_1 = 1$ $cells$ $(mm^3$ $person$ $day)^{-1}$, keeping rest of the parameter values same as given in Table 2, showing that up to 43.67% of budget is used for awareness and the remaining 56.33% is used for sanitation to control the spread of infection.

sanitation is beneficial to control the spread of infection.

The eigenvalues of the Jacobian matrix corresponding to equilibrium E^* for the model system (2.3) in absence of delay are obtained as -0.0105, -0.84344, $-0.02765+0.06194\,i$ and $-0.02765-0.06194\,i$. It may be noted here that two eigenvalues of the Jacobian matrix corresponding to the equilibrium E^* are negative and other two eigenvalues are with negative real part, which shows that the equilibrium E^* is locally asymptotically stable in absence of delay. For the set of parameter values given in Table 2, local stability condition (5.1) stated in Theorem 5.1 in absence of delay is also satisfied. For the parameter values $\beta_1 = 0.0000012$ $person^{-1}$ day^{-1}, $\Phi = 0.8$ day^{-1}, $p = 600$, $q = 1200$, $\eta = 0.00005$ day^{-1}, $\phi_1 = 10$ $cells$ $(mm^3$ $person$ $day)^{-1}$, $\phi_0 = 0.008$ day^{-1} and $\phi = 0.002$ day^{-1}, keeping rest of the parameter values same as given in Table 2. It is observed that the global stability conditions for the model system (2.3) in absence of delay, stated in Theorem 5.3 are satisfied for this data. In Fig. 8, we have also made a plot in $I - B - M$ space to demonstrate the nonlinear stability behavior of the model system (2.3), which shows that the equilibrium E^* is globally asymptotically stable inside the region of attraction Ω, showing that the solution trajectories starting inside the region of attraction approach towards its equilibrium E^*.

This shows that the disease remains endemic in the region for smaller values of efficacy of budget allocation to reduce contact rate via propagating awareness (β_1), efficacy of sanitation coverage to reduce bacteria in the environment due to budget allocation (Φ) and for higher values of growth-rate of bacteria due to increase in infected individuals (ϕ_1).
The introduction of time delay in per-capita growth rate of budget allocation due to increase in infected

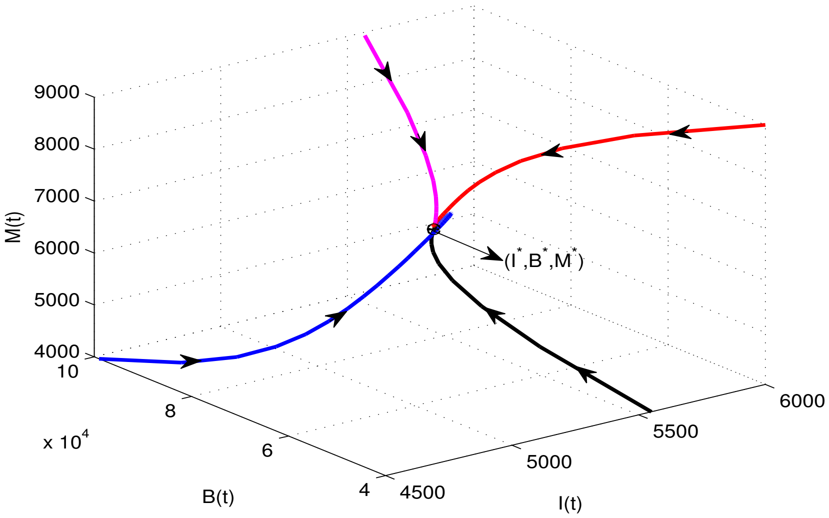

Figure 8. Nonlinear stability behavior in $I - B - M$ space in absence of delay ($\tau = 0$) for $\beta_1 = 0.0000012$ $person^{-1}$ day^{-1}, $\Phi = 0.8$ day^{-1}, $p = 600$, $q = 1200$, $\eta = 0.00005$ day^{-1}, $\phi_1 = 10$ $cells$ $(mm^3$ $person$ $day)^{-1}$, $\phi_0 = 0.008$ day^{-1} and $\phi = 0.002$ day^{-1}, keeping rest of the parameter values same as given in Table 2, which shows that all solution trajectories attain their equilibrium E^* inside the region of attraction Ω.

individuals changes the dynamics of the system. To assess the effect of time delay in per-capita growth rate of budget allocation, we consider two scenarios.

Scenario (a): For the set of parameter values given in Table 2, it is observed that the equation (6.5) has exactly one positive real root, i.e., the characteristic equation (6.2) has a pair of purely imaginary roots and we have computed the critical value of time delay τ_0, which is found to be 17.37 days. In Fig. 9, we have plotted the variation of infected population $I(t)$, total population $N(t)$, density of bacteria $B(t)$ and budget allocation by the government to warn people and for sanitation $M(t)$ with respect to time t for $\tau = 15$ days ($< \tau_0$). From these diagrams, it is clear that for $\tau = 15$ days ($< \tau_0$) , all the variables attain their equilibrium values and we have damped oscillations. In Fig. 10, we draw a phase portrait in $I - B - M$ space for $\tau = 15$ days ($< \tau_0$). This figure shows that the solution trajectory starting from outside, approaches towards its equilibrium E^* for $\tau = 15$ days ($< \tau_0$), i.e., the interior equilibrium E^* is stable for $\tau \in [0, \tau_0)$.

Further, we plot the variation of $I(t)$, $N(t)$, $B(t)$ and $M(t)$ with respect time t for $\tau = 19$ days ($> \tau_0$), which is shown in Fig. 11. These diagrams reveal that as the values of τ exceed its critical value τ_0 (i.e., $\tau > \tau_0$), all the variables show oscillatory behavior for $\tau = 19$ days ($> \tau_0$) and we have undamped sustained oscillations. Now we plot a phase portrait in $I - B - M$ space for $\tau = 19$ days ($> \tau_0$), which is shown in Fig. 12.

This diagram exhibits that the bifurcating periodic solution is orbitally stable, i.e., two solution trajectories one initiating from inside and other initiating from outside the limit cycle approach towards the limit cycle. To get the better understanding of the effect of time delay in per-capita growth rate of budget allocation due to increase in infected individuals, we have made bifurcation diagram by taking time delay τ as a bifurcation parameter, shown in Fig. 13. This figure demonstrates that all the

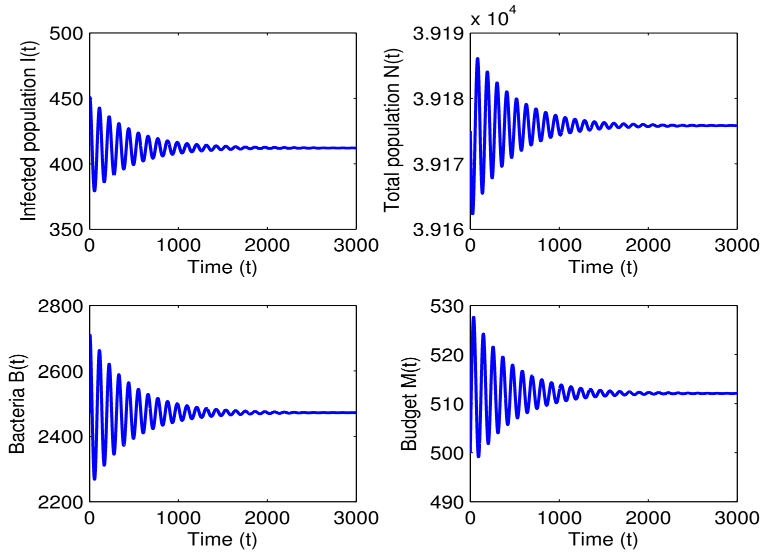

Figure 9. Variation of $I(t)$, $N(t)$, $B(t)$ and $M(t)$ with respect to t for $\tau = 15$ days $< \tau_0 = 17.37$ days, for the parameter values given in Table 2, we have damped oscillation.

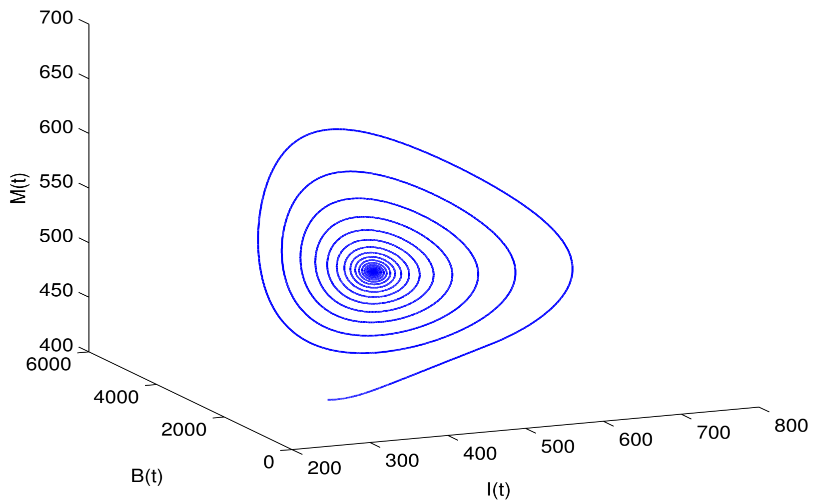

Figure 10. Phase portrait in $I - B - M$ space for $\tau = 15$ days $< \tau_0 = 17.37$ days , for the parameter values given in Table 2, which shows that the equilibrium E^* is stable.

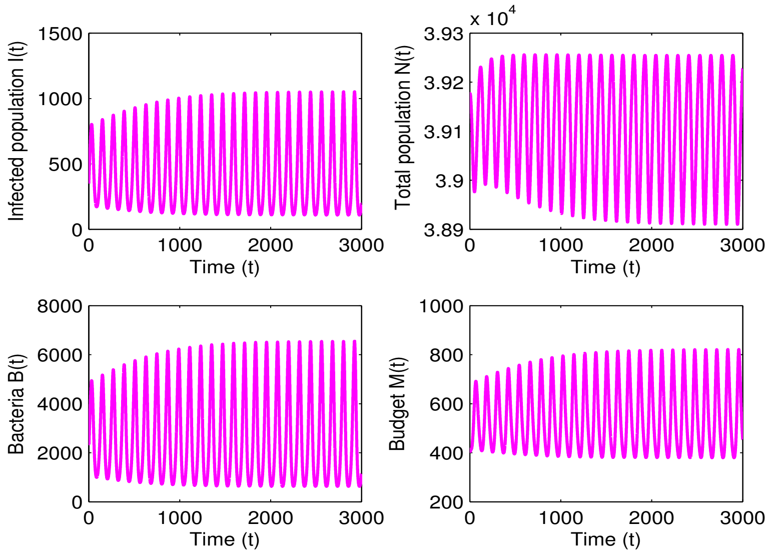

Figure 11. Variation of $I(t)$, $N(t)$, $B(t)$ and $M(t)$ with respect to t for $\tau = 19$ days $> \tau_0 = 17.37$ days, for the parameter values given in Table 2, we have undamped sustained oscillation.

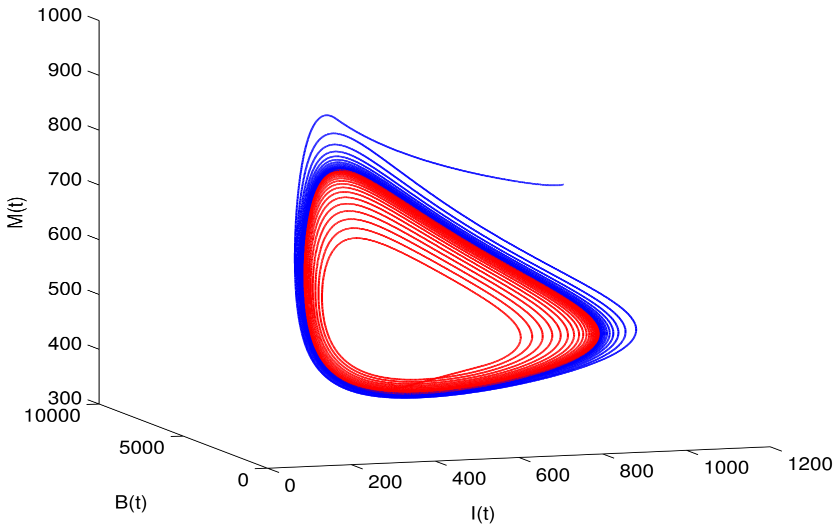

Figure 12. Appearance of limit cycle in $I - B - M$ space for $\tau = 19$ days $> \tau_0 = 17.37$ days along with the solution trajectory starting from outside with initial conditions $(1000, 7300, 700)$ and the solution trajectory initiating from inside with initial conditions $(350, 2400, 400)$ approach towards the limit cycle, for the parameter values given in Table 2, which shows that the equilibrium E^* is unstable.

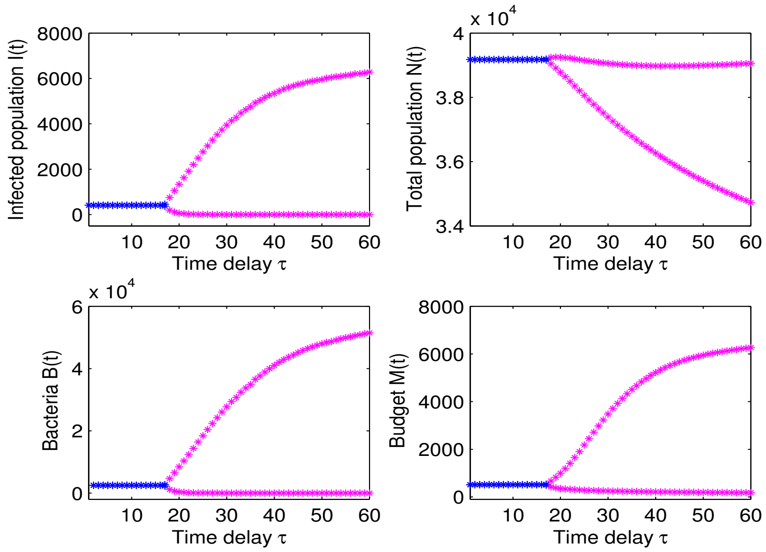

Figure 13. Bifurcation diagram of $I(t)$, $N(t)$, $B(t)$ and $M(t)$ for different values of time delay τ, for the parameter values given in Table 2, which shows that interior equilibrium E^* is stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0 = 17.37$ days.

variables settle down to their equilibrium values for $\tau \in [0, \tau_0)$. However, on increasing the values of τ above a threshold value (i.e., $\tau > \tau_0$), periodic oscillations with increasing amplitude are observed and the amplitude of these oscillation increases with the increase in values of time delay $\tau > \tau_0$. Thus, the interior equilibrium E^* of model system (2.3) is stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. The epidemiological meaning of above discussion is that if the number of reported cases of infected individuals known to government is older than 17.37 days, i.e., delay in providing funds exceeds a threshold value ($\tau > \tau_0$). In this case, sometimes the number of infected individuals and density of bacteria will be high and sometimes low. Thus, it may be difficult to make the prediction about the actual size and severity of epidemic outbreak. If one wants to predict actual size of epidemic then the number of reported cases of infected individuals should not be older than 17.37 days for this data.

Scenario (b): Further, we decrease the values of efficacy of budget allocation to reduce the contact via propagating awareness (β_1), efficacy of sanitation coverage to reduce bacteria in the environment due to budget allocation (Φ) and increase the values of half saturation point p and q, i.e., for $\beta_1 = 0.000012\ person^{-1}\ day^{-1}$, $p = 300$, $\Phi = 0.8\ day^{-1}$ and $q = 500$, keeping rest of the parameter values same as given in Table 2. It is observed that the equation (6.5) has exactly two positive real roots $\varphi_+ = 0.00303$ and $\varphi_- = 0.000696$ (where $\varphi_+ > \varphi_-$), i.e., the characteristic equation (6.2) has two pairs of purely imaginary roots $\pm i\omega_\pm$. For these positive values of $\omega_\pm$, from equation (6.14), we have found the positive values of $\tau_k^\pm (k = 0, 1, 2, ...)$ as follows:

$\tau_0^+ = 42$ days	$\tau_1^+ = 156$ days	$\tau_2^+ = 270$ days	$\tau_3^+ = 384$ days,...
$\tau_0^- = 119$ days	$\tau_1^- = 357$ days	$\tau_2^- = 595$ days	$\tau_3^- = 833$ days,...

Using Theorem 6.5, we can say that for given set of parameter values, the interior equilibrium E^* is stable for $\tau \in [0, 42) \cup (119, 156)$ and is unstable for $\tau \in (42, 119) \cup (156, 357)$. To get better

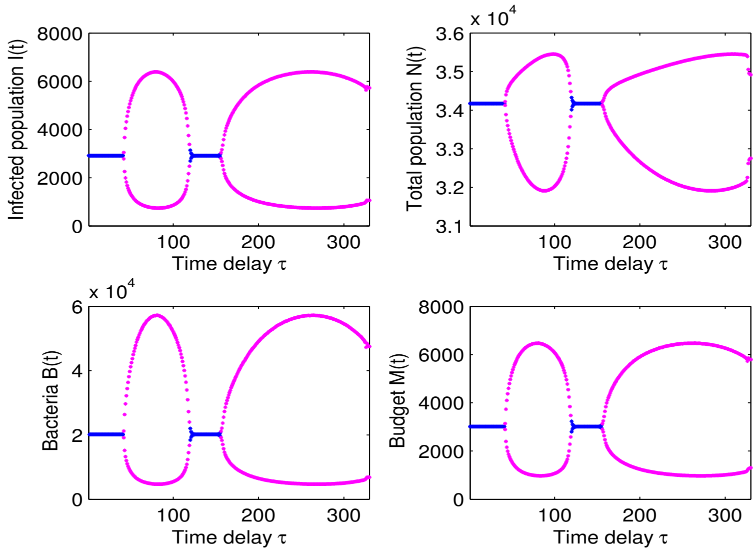

Figure 14. Bifurcation diagram of $I(t)$, $N(t)$, $B(t)$ and $M(t)$ with respect to time t for different values of time delay τ for $\beta_1 = 0.000012$ $person^{-1}$ day^{-1}, $p = 300$, $\Phi = 0.8$ day^{-1} and $q = 500$, keeping rest of the parameter values same as given in Table 2, which shows that the interior equilibrium is stable for $\tau \in [0, 42) \cup (119, 156)$ and unstable for $\tau \in (42, 119) \cup (156, 357)$.

understanding of effect of time delay on the dynamics of the system. We have made bifurcation diagrams of all dynamical variables with respect to time delay τ, which are shown in Fig. 14. From these bifurcation diagrams, it is observed that for $\tau \in [0, 42) \cup (119, 156)$, all the dynamical variables attain their equilibrium values, i.e., the interior equilibrium E^* shows stable character. However, for $\tau \in (42, 119) \cup (156, 357)$, all the dynamical variables show the oscillatory behavior whose maximum and minimum values are plotted in Fig. 14, i.e., the interior equilibrium E^* demonstrates the unstable character and Hopf-bifurcation occurs at $\tau = 42, 119, 156$. This reveals that the interior equilibrium of model system (2.3) switches from stability to instability and eventually the system becomes unstable.

8. Conclusion

Healthy sanitation practices and awareness among the individuals regarding the control mechanisms can substantially reduce the density of bacteria shed in the environment as well as the individuals change their behavior towards the disease and reduce their contacts with infected individuals as they use precautionary measures during the infection period such as improved sanitation, safe drinking water, vaccination, adequate medical care, voluntary quarantine etc., which are crucial for public health. Keeping this in view, in this paper, we have proposed and analyzed a nonlinear mathematical model to assess the impacts of sanitation and information campaigns on controlling the transmission dynamics of communicable diseases, such as typhoid fever in the community. In the modeling process, it is assumed that the susceptible individuals contract infection via the direct contact with infected individuals as well as indirectly through bacteria present in the environment. It is also assumed that the growth rate of budget required for sanitation and awareness increases logistically and its per-capita growth rate increases due to increase in infected individuals. Further, it is assumed that a fraction of budget is used to warn people via propagating awareness whereas the remaining part is used for sanitation to reduce the density of bacteria shed in the environment.

First, the deterministic mathematical model is analyzed qualitatively. The three boundary equilibria E_1, E_2 and E_3 and an interior equilibrium E^* are feasible. The positivity of solutions, boundedness and expressions for basic reproduction number in absence as well as in presence of budget are obtained. It is found that the efficacy of budget allocation to reduce the contact rate via propagating awareness (β_1), efficacy of sanitation coverage to reduce bacteria in the environment due to budget allocation (Φ) and carrying capacity of budget allocation for sanitation and awareness programs (K) have potential to reduce the epidemic threshold (i.e., R_1), and thus control the spread of infection. It is also observed that on increasing the per-capita growth rate of budget allocation due to increase in infected individuals (θ), carrying capacity of budget allocation (K), efficacy of budget allocation to reduce the contact rate via propagating awareness (β_1) and efficacy of sanitation coverage to reduce bacteria in the environment due to budget allocation (Φ), the equilibrium number of infected individuals decreases and also reduces its epidemic peak. The condition when budget should spend on sanitation/awareness to reduce the number of infected individuals has been obtained. For the given data, it is observed that up to 43.67% of budget is used for awareness and the remaining 56.33% is used for sanitation is beneficial to control the spread of infection.

The linear and nonlinear stability analysis of equilibrium in absence of delay are discussed, stated in theorems 5.1- 5.3, respectively. The study reveals that disease remains endemic in the region if efficacy of budget allocation to reduce contact rate via propagating awareness (β_1) and efficacy of sanitation coverage to reduce bacteria in the environment due to budget allocation (Φ) are not strong enough.

Further, to predict more realistic situation over the dynamics of the system, we have introduced time delay in per-capita growth of budget allocation due to increase in infected individuals, in the presence of delay, the stability analysis of interior equilibrium E^* and existence of Hopf-bifurcation are discussed. It is observed that the introduction of time delay changes the dynamics of the system as delay parameter crosses a critical threshold. For one set of parameter values, it is shown that the interior equilibrium E^* of the model system (2.3) is stable for suitably small values of τ (i.e., $\tau \in [0, \tau_0)$). However, on increasing the values of time delay above a threshold value (i.e., $\tau > \tau_0$), the periodic oscillations with increasing amplitude are observed, i.e., the interior equilibrium E^* becomes unstable. For another set of parameter values, it is found that the interior equilibrium of model system (2.3) switches from stability to instability, instability to stability and eventually the system becomes unstable. The study reveals that the combined effect of sanitation and awareness through budget allocation are much beneficial to control the spread of disease. However, delay in providing funds destabilizes the system and may cause stability switches through Hopf-bifurcation, which brings challenges to predict and control the spread of infection and have possibility of multiple epidemic outbreaks. Thus, it is observed that timely and continuous allocation of funds regarding the sanitation and awareness are essential for the control of infectious diseases.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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Appendix

A.

A.1. Positivity of solutions

Here, we show that all the variables of model system (2.1) are non-negative for all time t. Since the positivity of $S(t)$ relies on positivity of $I(t)$, first we prove the positivity of $I(t)$. From the second equation of model system (2.1), we have

$$\frac{dI}{dt} = \left(\left(\beta - \beta_1 \frac{k_1 M}{p + k_1 M}\right) S + \eta \frac{BS}{(L+B)I} - (\nu + \alpha + d)\right) I.$$

The above equation can be written as,

$$\frac{dI(t)}{dt}\exp\left(-\int_0^t f_1(s)ds\right)-f_1(t)I(t)\exp\left(-\int_0^t f_1(s)ds\right)=0,$$

where, $f_1(s)=\left(\left(\beta-\beta_1\frac{k_1M(s)}{p+k_1M(s)}\right)S(s)+\eta\frac{B(s)S(s)}{(L+B(s))I(s)}-(\nu+\alpha+d)\right)$.
This can be rewritten as,

$$\frac{d}{dt}\left(I(t)\exp\left(-\int_0^t f_1(s)ds\right)\right)=0.$$

And thus we have

$$I(t)=I(0)\left(\exp\left(\int_0^t f_1(s)ds\right)\right).$$

This shows that $I(t)\geq 0$ for all $t\geq 0$.
Further, from the first equation of model system (2.1), we have

$$\frac{dS(t)}{dt} = \Lambda+\nu I-\left(\left(\beta-\beta_1\frac{k_1M(t)}{p+k_1M(t)}\right)I(t)+\eta\frac{B(t)}{L+B(t)}+d\right)S(t).$$

The above equation can be written as

$$\frac{dS(t)}{dt}\exp\left(\int_0^t f_2(s)ds\right)+f_2(t)S(t)\exp\left(\int_0^t f_2(s)ds\right) = (\Lambda+\nu I(t))\exp\left(\int_0^t f_2(s)ds\right),$$

where, $f_2(s)=\left(\left(\beta-\beta_1\frac{k_1M(s)}{p+k_1M(s)}\right)I(s)+\eta\frac{B(s)}{L+B(s)}+d\right)$.
This implies that

$$\frac{d}{dt}\left(S(t)\exp\left(\int_0^t f_2(s)ds\right)\right)=(\Lambda+\nu I(t))\exp\left(\int_0^t f_2(s)ds\right).$$

And hence we obtain

$$S(t)=S(0)\exp\left(-\int_0^t f_2(s)ds\right) +\exp\left(-\int_0^t f_2(s)ds\right)\int_0^t(\Lambda+\nu I(s))\exp\left(\int_0^s f_2(u)du\right)ds.$$

This shows that $S(t)>0$ for all $t>0$. Similarly, it is easy to show that $B(t)>0$ and $M(t)>0$ for all $t>0$. Thus, the solution $S(t)$, $I(t)$, $B(t)$ and $M(t)$ of model system (2.1) with initial conditions $S(0)>0$, $I(0)\geq 0$, $B(0)\geq 0$ and $M(0)\geq 0$ are positive for all $t>0$. Hence the proof.

A.2.

Boundedness From the second equation of model system (2.3), we have

$$\frac{dN(t)}{dt} \leq \Lambda - dN(t).$$

The above equation can be written as

$$\frac{d}{dt}\left(N(t)e^{dt}\right) \leq \Lambda e^{dt}.$$

Now, integrating above equation from 0 to t, we obtain

$$N(t) \leq N_0 e^{-dt} + \frac{\Lambda}{d}.$$

Therefore, by using the theory of differential inequality [17], we have $\lim_{t\to\infty} \sup N(t) \leq \frac{\Lambda}{d}$, and thus $0 \leq N(t) \leq \frac{\Lambda}{d}$ for large $t > 0$. Now $S(t) = N(t) - I(t) \geq 0$, yields that $0 \leq I(t) \leq N(t) \leq \frac{\Lambda}{d}$ for large $t > 0$.

From the third equation of model system (2.3), and using the fact that $I(t) \leq \frac{\Lambda}{d}$ for large $t > 0$, we have

$$\frac{dB(t)}{dt} + (\phi_0 - \phi)B(t) \leq \phi_1 \frac{\Lambda}{d}.$$

From the theory of differential inequality, we obtain

$$\lim_{t\to\infty} \sup B(t) \leq \frac{\phi_1 \Lambda}{d(\phi_0 - \phi)} = B_m(\text{say}).$$

This implies that $0 \leq B(t) \leq B_m$ for large $t > 0$.

Further, from the fourth equation of model system (2.3), and using the fact that $I(t) \leq \frac{\Lambda}{d}$ for large $t > 0$, we obtain

$$\frac{dM(t)}{dt} \leq \left(r + \theta\frac{\Lambda}{d}\right)M(t) - \frac{r}{K}M(t)^2.$$

From the theory of differential inequality, we have

$$\lim_{t\to\infty} \sup M(t) \leq \frac{K}{r}\left(r + \theta\frac{\Lambda}{d}\right) = M_m(\text{say}).$$

This implies that $0 \leq M(t) \leq M_m$ for large $t > 0$.

B.

Basic reproduction number (R_1) Here, we obtain the basic reproduction number (R_1) of model system (2.3) using next generation matrix approach. We have the matrix of new infection $\mathcal{H}(x)$ and the matrix of transition $\mathcal{G}(x)$. Consider $x = (I, N, B, M)^T$, the model system (2.3) (for $\tau = 0$) can be rewritten as:

$$\frac{dx}{dt} = \mathcal{H}(x) - \mathcal{G}(x),$$

where,

$$\mathcal{H}(x) = \begin{pmatrix} \left(\beta - \beta_1 \frac{k_1 M}{p+k_1 M}\right)(N-I)I + \eta \frac{B}{L+B}(N-I) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\mathcal{G}(x) = \begin{pmatrix} (\nu+\alpha+d)I \\ -\Lambda + dN + \alpha I \\ -\phi_1 I + (\phi_0 - \phi)B + \Phi \frac{(1-k_1)M}{q+(1-k_1)M} B \\ -r\left(1 - \frac{M}{K}\right)M - \theta IM \end{pmatrix}.$$

The Jacobian matrix of $\mathcal{H}(x)$ and $\mathcal{G}(x)$ evaluated at disease-free equilibrium $E_3\left(0, \frac{\Lambda}{d}, 0, K\right)$ are

$$J\mathcal{H}(E_3) = \begin{pmatrix} \left(\beta - \beta_1 \frac{k_1 K}{p+k_1 K}\right)\frac{\Lambda}{d} & 0 & \frac{\eta\Lambda}{dL} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$J\mathcal{G}(E_3) = \begin{pmatrix} (\nu+\alpha+d) & 0 & 0 & 0 \\ \alpha & d & 0 & 0 \\ -\phi_1 & 0 & \left(\phi_0 - \phi + \Phi \frac{(1-k_1)K}{q+(1-k_1)K}\right) & 0 \\ -\theta K & 0 & 0 & r \end{pmatrix}.$$

The next generation matrix $\mathcal{K} = J\mathcal{H}(E_3)(J\mathcal{G}(E_3))^{-1}$ is given by

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}_{11} & 0 & \frac{\eta\Lambda}{dL\left(\phi_0 - \phi + \Phi \frac{(1-k_1)K}{q+(1-k_1)K}\right)} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where, $\mathcal{K}_{11} = \left(\beta - \beta_1 \frac{k_1 K}{p+k_1 K}\right)\frac{\Lambda}{d(\nu+\alpha+d)} + \frac{\eta\phi_1\Lambda}{dL(\nu+\alpha+d)\left(\phi_0-\phi+\Phi\frac{(1-k_1)K}{q+(1-k_1)K}\right)}$.
Therefore, the basic reproduction number $R_1 = \rho(\mathcal{K}) = \max\{|\psi| : \psi \in \rho(\mathcal{K})\}$ is spectral radius of the matrix $\mathcal{K}$ and is obtained as.

$$R_1 = \left(\beta - \beta_1 \frac{k_1 K}{p + k_1 K}\right)\frac{\Lambda}{d(\nu+\alpha+d)} + \frac{\eta\phi_1\Lambda}{dL(\nu+\alpha+d)\left(\phi_0 - \phi + \Phi\frac{(1-k_1)K}{q+(1-k_1)K}\right)}.$$

C.

Proof of Theorem 5.1 Here, we establish the local stability results of the equilibrium E_1, E_2, E_3 and E^* in absence of delay (i.e., $\tau = 0$) by finding the sign of real part of eigenvalues of the Jacobian matrix $J(E_i)$ $(i = 1, 2, 3)$ and $J(E^*)$ evaluated at equilibrium E_i $(i = 1, 2, 3)$ and E^*. The general

Jacobian matrix of model system (2.3) at any equilibrium (I, N, B, M) is as follows:

$$J = \begin{pmatrix} J_{11} & J_{12} & \frac{\eta L(N-I)}{(L+B)^2} & -\frac{\beta_1 k_1 p(N-I)I}{(p+k_1 M)^2} \\ -\alpha & -d & 0 & 0 \\ \phi_1 & 0 & -\left(\phi_0 - \phi + \Phi\frac{(1-k_1)M}{q+(1-k_1)M}\right) & -\frac{\Phi(1-k_1)qB}{(q+(1-k_1)M)^2} \\ \theta M & 0 & 0 & r\left(1-\frac{2M}{K}\right)+\theta I \end{pmatrix}$$

where, $J_{11} = \left(\beta - \beta_1 \frac{k_1 M}{p+k_1 M}\right)(N-2I) - \eta\frac{B}{L+B} - (\nu + \alpha + d)$,
and $J_{12} = \left(\beta - \beta_1 \frac{k_1 M}{p+k_1 M}\right) I + \eta \frac{B}{L+B}$.

(i) From the Jacobian matrix $J(E_1)$, it is easy to see that the one of the eigenvalue of $J(E_1)$ is r, which is positive and rest of the eigenvalues lie in left-half of the complex plane if $R_0 < 1$. Thus, the equilibrium E_1 is unstable with unstable manifold locally in M-direction, whereas it has locally stable manifold in $I - N - B$ space if $R_0 < 1$. Further, the equilibrium E_1 is locally unstable manifold either in $I - M$ plane or $B - M$ plane and has locally stable manifold in N-direction whenever E_2 is feasible (i.e., $R_0 > 1$).
(ii) Further, from the Jacobian matrix $J(E_2)$, it is easy to see that one of the eigenvalue of $J(E_2)$ is $(r + \theta I_2)$, which is positive and other three eigenvalues lie in left-half of complex plane. Thus, the equilibrium E_2 is unstable with unstable manifold locally in M-direction and locally stable manifold in $I - N - B$ space.
(iii) From the Jacobian matrix $J(E_3)$, it is easy to note that two eigenvalues of $J(E_3)$ are $-r$ and $-d$ and other two eigenvalues are either negative or with negative real part if $R_1 < 1$. Thus, the disease-free equilibrium E_3 is locally asymptotically stable if $R_1 < 1$. It is observed that whenever E^* is feasible (i.e., $R_1 > 1$), E_3 is saddle point with locally unstable manifold either in I-direction or B-direction and locally stable manifold in $N - M$ plane. Thus, E_3 is unstable whenever E^* is feasible.
(iv) To investigate the local stability analysis of interior equilibrium E^*, we use the Routh-Hurwitz criterion. The characteristic equation for the Jacobian matrix $J(E^*)$ is obtained as:

$$\psi^4 + C_1\psi^3 + C_2\psi^2 + C_3\psi + C_4 = 0. \tag{C.1}$$

where, $C_1 = d + \frac{r}{K}M^* + J^*_{11} + J^*_{33}$,
$C_2 = \alpha J^*_{12} + \left(d + \frac{r}{K}M^*\right)(J^*_{11} + J^*_{33}) + \frac{dr}{K}M^* + \beta(M^*)I^* J^*_{33} + \frac{\eta\phi_1(N^*B^*+LI^*)}{(L+B^*)^2} + \theta M^* J^*_{14}$,
$C_3 = \alpha J^*_{12}\left(\frac{r}{K}M^* + J^*_{33}\right) + \frac{dr}{K}M^*(J^*_{11} + J^*_{33})$
$+ \left(d + \frac{r}{K}M^*\right)\left(\beta(M^*)I^*J^*_{33} + \frac{\eta\phi_1(N^*B^*+LI^*)}{(L+B^*)^2}\right) + \theta M^*(J^*_{13}J^*_{34} + J^*_{14}J^*_{33} + dJ^*_{14})$,
$C_4 = \frac{\alpha r}{K}M^* J^*_{12}J^*_{33} + \frac{dr}{K}M^*\left(\beta(M^*)I^*J^*_{33} + \frac{\eta\phi_1(N^*B^*+LI^*)}{(L+B^*)^2}\right) + d\theta M^*(J^*_{13}J^*_{34} + J^*_{14}J^*_{33})$,
$\beta(M^*) = \left(\beta - \beta_1\frac{k_1 M^*}{p+k_1 M^*}\right)$, $J^*_{11} = \beta(M^*)I^* + \eta\frac{B^*}{L+B^*}\left(\frac{N^*}{I^*}\right)$, $J^*_{12} = \beta(M^*)I^* + \eta\frac{B^*}{L+B^*}$, $J^*_{13} = \frac{\eta L(N^*-I^*)}{(L+B^*)^2}$, $J^*_{14} = \frac{\beta_1 k_1 p(N^*-I^*)I^*}{(p+k_1M^*)^2}$, $J^*_{33} = \left(\phi_0 - \phi + \Phi\frac{(1-k_1)M^*}{q+(1-k_1)M^*}\right)$ and $J^*_{34} = \Phi\frac{(1-k_1)qB^*}{(q+(1-k_1)M^*)^2}$.
In writing the above values of $C_i's(i = 1, 2, 3, 4)$, we have used the fact $J^*_{11}J^*_{33} - J^*_{13}\phi_1 = \beta(M^*)I^*J^*_{33} + \frac{\eta\phi_1(N^*B^*+LI^*)}{(L+B^*)^2}$. Here, it can be easily noted that C_1, C_2, C_3 and C_4 all are positive. Using Routh-Hurwitz criterion, we may say that all the eigenvalues of the Jacobin matrix $J(E^*)$ will be lie in left-half of complex plan provided the following condition holds:

$$C_3(C_1C_2 - C_3) - C_1^2C_4 > 0.$$

Thus, the interior equilibrium E^* is locally asymptotically stable provided the above condition is satisfied.

D.

Proof of Theorem 5.3 Here, we establish the nonlinear stability analysis of equilibrium $E^*(I^*, N^*, B^*, M^*)$ in absence of delay (i.e., $\tau = 0$). Consider a suitable scalar-valued positive definite function corresponding to reduced model system (2.3) about the interior equilibrium E^*.

$$\begin{aligned} G &= I - I^* - I^* \ln\left(\frac{I}{I^*}\right) + \frac{1}{2}m_1(N - N^*)^2 + \frac{1}{2}m_2(B - B^*)^2 \\ &\quad + m_3\left(M - M^* - M^* \ln\left(\frac{M}{M^*}\right)\right), \end{aligned} \tag{D.1}$$

(where, m_1, m_2 and m_3 are positive constant to be chosen appropriately.)
It can be easily checked that the Lyapunov's function is zero at the equilibrium $E^*(I^*, N^*, B^*, M^*)$ and is positive for all other positive values of I, N, B and M. Differentiating equation (D.1) with respect to time 't' along the solution of model system (2.3) and choosing $m_1 = \frac{1}{\alpha}\left(\beta - \beta_1 \frac{k_1 M^*}{p + k_1 M^*}\right)$, after rearranging the terms, we obtain

$$\begin{aligned} \frac{dG}{dt} &= -\left(\left(\beta - \beta_1 \frac{k_1 M^*}{p + k_1 M^*}\right) + \frac{\eta BN}{(L + B)II^*}\right)(I - I^*)^2 \\ &\quad - \frac{d}{\alpha}\left(\beta - \beta_1 \frac{k_1 M^*}{p + k_1 M^*}\right)(N - N^*)^2 \\ &\quad - m_2\left(\phi_0 - \phi + \Phi \frac{(1 - k_1)M^*}{q + (1 - k_1)M^*}\right)(B - B^*)^2 - m_3\left(\frac{r}{K}\right)(M - M^*)^2 \\ &\quad + \left(\frac{\eta B}{(L + B)I^*}\right)(I - I^*)(N - N^*) + \frac{\eta L(N^* - I^*)}{(L + B)(L + B^*)I^*}(I - I^*)(B - B^*) \\ &\quad + m_2\phi_1(I - I^*)(B - B^*) + m_3\theta(I - I^*)(M - M^*) \\ &\quad - \frac{\beta_1 k_1 p(N - I)}{(p + k_1 M)(p + k_1 M^*)}(I - I^*)(M - M^*) \\ &\quad - m_2 \frac{\Phi(1 - k_1)qB}{(q + (1 - k_1)M)(q + (1 - k_1)M^*)}(B - B^*)(M - M^*). \end{aligned}$$

Now, $\frac{dG}{dt}$ will be negative definite inside the region of attraction Ω provided the following inequalities are satisfied:

$$\left(\frac{\eta}{I^*}\right)^2 < \frac{4}{5}\left(\frac{d}{\alpha}\right)\left(\beta - \beta_1 \frac{k_1 M^*}{p + k_1 M^*}\right)^2, \tag{D.2}$$

$$m_3\theta^2 < \frac{4}{15}\left(\frac{r}{K}\right)\left(\beta - \beta_1 \frac{k_1 M^*}{p + k_1 M^*}\right), \tag{D.3}$$

$$\left(\frac{\beta_1 k_1 \Lambda}{d(p + k_1 M^*)}\right)^2 < \frac{4}{15}\left(\frac{m_3 r}{K}\right)\left(\beta - \beta_1 \frac{k_1 M^*}{p + k_1 M^*}\right), \tag{D.4}$$

$$m_2\phi_1^2 < \frac{4}{15}J_{33}^*\left(\beta - \beta_1 \frac{k_1 M^*}{p + k_1 M^*}\right), \tag{D.5}$$

$$m_2\left(\frac{\phi_1 \Lambda \Phi(1 - k_1)}{d(\phi_0 - \phi)(q + (1 - k_1)M^*)}\right)^2 < \frac{4}{9}\left(\frac{m_3 r}{K}\right)J_{33}^*, \tag{D.6}$$

$$\left(\frac{\eta(N^* - I^*)}{I^*(L + B^*)}\right)^2 \quad < \quad \frac{4m_2}{15} J_{33}^* \left(\beta - \beta_1 \frac{k_1 M^*}{p + k_1 M^*}\right), \tag{D.7}$$

where, $J_{33}^* = \left(\phi_0 - \phi + \Phi \frac{(1-k_1)M^*}{q+(1-k_1)M^*}\right)$.
From inequalities (D.3) and (D.4), we may easily choose the positive value of m_3 provided the inequality (5.3) is satisfied. Further, after choosing the positive value of m_3, from inequalities (D.5)-(D.7), we may easily choose the positive value of m_2 provided the inequality (5.4) is satisfied.
Thus, $\frac{dG}{dt}$ will be negative definite inside the region of attraction Ω, provided the inequalities (5.2)-(5.4) are satisfied.

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