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Research article

Dynamics of a delay turbidostat system with contois growth rate

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Abstract: In this contribution, the dynamic behaviors of a turbidostat model with Contois growth rate and delay are investigated. The qualitative properties of the system are carried out including the stability of the equilibria and the bifurcations. More concretely, we exhibit the transcritical bifurcation by reducing the system without delay to a 1-dimensional system on a center manifold and find that Hopf bifurcation occurs by choosing the delay as bifurcation parameter. Also, using the normal form theory and the center manifold theorem we determine the direction and stability of the bifurcating periodic solutions induced by the Hopf bifurcation. Finally, numerical simulations are presented to support our theoretical results.

Keywords: turbidostat; contois growth rate; delay; stability; bifurcation

1. Introduction

The chemostat is a laboratory bio-reactor used for the continuous culture of microorganism (Figure 1). The apparatus is of both ecological and mathematical interest since it can be used to represent a lot of microorganism systems, such as the simple lake and the waste-water treatment, and plays an important role in micrology and population [8, 9, 30].

Recently, a large of research devoted to modifying the chemostat for higher economical values by controlling the dilution rate of the chemostat. The chemostat with the feedback control of the dilution rate is established, which is referred to as turbidostat by biologists [14] (Figure 1). In the turbidostat, the optical sensor measures the concentration of the microorganism and the signal feedback control the dilution rate. For example, Flegr [5] analyzed the coexistence of two species in the turbidostat by numerical analysis, and later Leenheer and Smith [13] also demonstrated results of Flegr by theoretical analysis. They showed that a turbidostat with monotone response functions permits a unique coexistence equilibrium and if it is locally asymptotically stable then it is globally attracting. Furthermore, they also obtained that coexistence is not achievable in the turbidostat with more than two competing organisms. Li [14] established a mathematical model of competition in a turbidostat

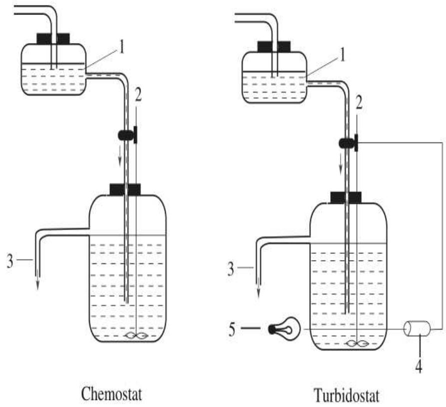

Figure 1. The sketch map of chemostat and turbidostat: 1. Reservoir of sterile medium; 2. Valve controlling flowofmedium; 3. Outletforspentmedium; 4. Photocell; 5. Lightsource.

for an inhibitory nutrient and achieved the stability of the equilibrium and the existence of limit cycles. Li and Chen [16] studied a mathematical model of the turbidostat with impulsive state feedback control and obtained the existence and the stability of periodic solutions of order one.

The uptake function in the model of microorganism continuous culture is the growth rate of the microorganism including many forms, such as, Monod [19], Moser [20] and Contois [4]. The most classical and common uptake function is Monod one, which is the function of concentration of microorganism and the constant half saturation term. However, Contois [4] presented experimental results to show that it is not necessarily a constant. Thus, the uptake function of Contois was introduced, which depends on the ratio substrate to microorganism. Jost [24] explored the relation of the Contois growth rate in microbiology and the ratio-dependent functional response in population ecology introduced by Arditi and Ginzburg [1]. As such, many researchers also call it ratio-dependent growth rate just as in population ecology [10, 11, 17, 34]. Especially, Hu et al. [11] investigated the existence and the stability of the periodic solution of order one for the turbidostat system with ratio-dependent growth rate and impulsive state feedback control. The Contois model gave predictions that were in excellent agreement with experiment measurement. Especially, the Contois model was found to accurately describe the processing of industrial wastewaters. Nelson investigated a series of chemostat models with Contois growth rate. For example, Nelson and Sidhu [21] investigated the Hopf bifurcation and degenerate Hopf bifurcation of the chemostat model with Contois growth rate and the variable yield. Alqahtani et al. [22] studied the chemostat model with variable yield and contois growth kinetics with substrate inhibition and showed the Hopf bifurcation, degenerate Hopf bifurcation and Bogdanov-Takens bifurcation.

Researchers recognized that time delays are natural in the ecological systems [12, 27, 29]. Smith and Waltman [30] showed that there are two obvious sources of delays in the cultivation of microorganism, one of which is the possibility that the microorganism stores the nutrient. Caperon [3] first introduced the delay into the chemostat model by some experiments. Bush and Cook [2] corrected Caperon's model and established a chemostat model with delay in the growth rate of microorganism. The time delays yield some complex impacts to the chemostat model and biologists gave the reasonable explanations for the observations and experimental datas by investigating the chemostat model with delays [18, 15, 23, 26, 31]. For example, Li et al. [15] considered the global dynamics of the chemostat model with two populations of microorganisms competing for two perfectly complementary nutrients when distributed delays are involved. Ruan and Wolkowicz [26] studied the existence and the stability of the Hopf bifurcation of the chemostat model with a distributed delay and found that the periodic solutions become unstable if the dilution rate is increased. Wang and Wolkowicz [31] analyzed the local stability of the equilibria and the globally asymptotical stability of the single species survival equilibrium when n species compete in the chemostat with time delay for a single resource, hence the competitive exclusion principle holds. Yao et al. [32] discussed the delayed turbidostat model with Monod growth rate and obtained the phenomenon of oscillation. In the turbidostat, additional factor causing delay is due to the process that the sensor controls the dilution, for example, Tagashira and Hara [28] and Yuan et al. [33] considered the turbidostat models of the dilution rate with delayed feedback control.

Li and Xu [17] investigated the following chemostat system with Contois growth rate and time delay

$$\begin{cases} \frac{dS(t)}{dt} = d(S_0 - S(t)) - \frac{1}{\alpha+\beta S(t)} \frac{mS(t)x(t)}{ax(t)+S(t)}, \\ \frac{dx(t)}{dt} = x(t)\{\frac{mS(t-\tau)}{ax(t-\tau)+S(t-\tau)} - d\}, \end{cases} \tag{1.1}$$

where $S(t)$ and $x(t)$ present the concentration of substrate and the concentration of microorganism at time t, respectively, $S_0 > 0$ stands for the input concentration of the nutrient, $\alpha + \beta S(t)$ ($\alpha > 0, \beta > 0$) is variable yield, $m > 0$ and $a > 0$ are growth parameters of the microorganism, $\tau > 0$ is the time delay of the growth response of the microorganism, and $d > 0$ presents the flow volume. They considered the local and global stability of the equilibria and Hopf bifurcation.

In this paper, motivated by the works of Bush and Cook ([2]), Contois ([4]) and Li and Xu ([17]), one focuses on the dynamics of a turbidostat model with Contois growth rate and time delay which follows as

$$\begin{cases} \frac{dS(t)}{dt} = (d + kx(t))(S_0 - S(t)) - \frac{1}{\gamma} \frac{mS(t)x(t)}{ax(t)+S(t)}, \\ \frac{dx(t)}{dt} = x(t)\{\frac{mS(t-\tau)}{ax(t-\tau)+S(t-\tau)} - d - kx(t)\}, \end{cases} \tag{1.2}$$

where $S(t)$, $x(t)$ and parameters S_0, m, a, τ are defined as in (1.1), $\gamma > 0$ is yield constant and $d + kx(t)$ ($d > 0, k > 0$) presents the dilution rate of the turbidostat.

The aim of this paper is to investigate the qualitative properties of system (1.2) including the stability of the equilibria and the bifurcations. The outline of this paper is as follows. In Section 2, I analyze the existence and local stability of the equilibria, and the transcritical bifurcation at the boundary equilibrium and the Hopf bifurcation at the positive equilibrium. In Section 3 mostly focuses on the stability and type of the bifurcating periodic solutions induced by Hopf bifurcation to system (1.2). I in Section 4 further illustrate our main results by numerical simulations.

2. Stability of the equilibria and bifurcations

In this section, we investigate the existence and stability of the equilibria and the bifurcations at equilibria including the transcritical bifurcation and Hopf bifurcation of system (1.2).

It is convenient to introduce dimensionless variables. In particular, we set

$$S = \tilde{S} S_0, \quad x = \tilde{x}\gamma S_0, \quad k = \frac{d\tilde{k}}{\gamma S_0}, \quad a = \frac{\tilde{a}}{\gamma}, \quad m = d\tilde{m}, \quad t = \frac{\tilde{t}}{d}.$$

System (1.2) becomes

$$\begin{cases} \frac{dS(t)}{dt} = (1 + kx(t))(1 - S(t)) - \frac{mS(t)x(t)}{ax(t)+S(t)}, \\ \frac{dx(t)}{dt} = x(t)(\frac{mS(t-\tau)}{ax(t-\tau)+S(t-\tau)} - 1 - kx(t)), \end{cases} \tag{2.1}$$

where we still denote $\tilde{S}, \tilde{x}, \tilde{k}, \tilde{a}, \tilde{m}, \tilde{t}$ with S, x, k, a, m, t, respectively. First of all, we discuss the existence of the equilibria of system (2.1) and have the following result.

Lemma 2.1. *System (2.1) always has a boundary equilibrium $E_0 : (1, 0)$; If $m > 1$, then system (2.1) has a unique positive equilibrium $E_1 : (S^*, x^*)$, where $S^* = \frac{1+k}{m+k}$ when $a = 1$,*

$$S^* = \frac{-(2ak + a - k + m - 1) + \sqrt{(2ak + a - k + m - 1)^2 + 4ak(1 - a)(1 + k)}}{2k(1 - a)},$$

when $a \neq 1$ and $x^* = 1 - S^*$.

Proof. From the right part of system (2.1), we can obtain the boundary equilibrium easily, which we denote as $E_0 : (1, 0)$. In order to get the positive equilibrium, we need to discuss the roots of the equation $\frac{mS}{a+(1-a)S} - 1 - k(1 - S) = 0$ on the interval $(0, 1)$. We define

$$f(S) := (1 - a)kS^2 + (2ak + a - k + m - 1)S - a(1 + k). \tag{2.2}$$

Therefore, we just need to discuss the zeros of the function $f(S)$ on the interval $(0, 1)$, which need to be discussed in the following three cases.

(i). If $a = 1$, then we have $f(S) = (k + m)S - 1 - K$ and $f(S)$ has a zero $S = \frac{1+k}{m+k}$ on the interval $(0, 1)$ when $m > 1$, and $f(S)$ has no zero on the interval $(0, 1)$ when $m \leq 1$. Thus, we obtain $S^* = \frac{1+k}{m+k}$ when $a = 1$ and $m > 1$.

(ii). If $0 < a < 1$, then $f(S)$ is a quadratic function of S. Since the coefficient of quadratic term $k(1 - a) > 0$ and $f(0) = -a(1 + k) < 0$, $f(S) = 0$ has two roots

$$\begin{aligned} S_1 &:= \frac{-(2ak+a-k+m-1) - \sqrt{(2ak+a-k+m-1)^2+4ak(1-a)(1+k)}}{2k(1-a)}, \\ S_2 &:= \frac{-(2ak+a-k+m-1) + \sqrt{(2ak+a-k+m-1)^2+4ak(1-a)(1+k)}}{2k(1-a)}, \end{aligned} \tag{2.3}$$

and it is obvious that $S_1 < 0, S_2 > 0$. By simplifying the inequality $S_2 < 1$, we can get that $S_2 < 1$ if and only if $m > 1$. Thus, we have $S^* = S_2$ when $0 < a < 1$ and $m > 1$.

(iii). If $a > 1$, then $f(S)$ still is a quadratic function of S with the coefficient of quadratic term $k(1 - a) < 0$ and $f(0) = -a(1 + k) < 0$. The discriminant of $f(S)$ is

$$\Delta := a^2 + 2(2km - k + m - 1)a + (k - m + 1)^2.$$

We can take Δ as the function of a. Therefore, the discriminant of Δ is $\tilde{\Delta} := km(m - 1)(1 + k)$. For this case we need further discussion in the following three cases.

(a). If $m = 1$, then $\tilde{\Delta} = 0$ and the double root of $\Delta = 0$ is $-k$. Therefore, $\Delta > 0$ holds when $a > 1$. We have $f(S) = 0$ has two roots $S_1 = \frac{a(1+k)}{k(a-1)} > 1$ and $S_2 = 1$. Thus, $f(S)$ has no zero on the interval $(0, 1)$ when $a > 1$ and $m = 1$.

(b). If $m < 1$, then $\tilde{\Delta} < 0$, which implies $\Delta > 0$. We have $f(S)$ has two roots S_1 and S_2 presented in (2.3) and $S_2 < S_1$. By simplifying $S_2 > 1$, we can get that $S_2 > 1$ if and only if $m < 1$. Thus, $f(S)$ has no zero on the interval $(0, 1)$ when $a > 1$ and $m < 1$.

(c). If $m > 1$, then $\tilde{\Delta} > 0$. Since $4km - 2k + 2m - 2 > 0$ and $(k - m + 1)^2 \geq 0$, the two roots of $\Delta = 0$ denoted by a_1 and a_2 satisfy $a_1 < a_2 \leq 0$. Therefore, $\Delta > 0$ holds when $a > 1$ and $m > 1$. We obtain $f(S)$ has two roots presented in (2.3) and $S_1 > S_2$. Further, we can get $0 < S_2 < 1$ if and only if $m > 1$ and $a > 1$. Suppose that $S_1 < 1$, then we get $m > 1$, which is a contradiction. Thus, we have $S^* = S_2$ when $a > 1$ and $m > 1$.

This completes the proof. □

From the above analysis we have the existence of equilibria of system (2.1). In the following, we consider qualitative properties of system (2.1) including stability of the equilibria and the bifurcations. We discuss system (2.1) in two cases: (i) $\tau = 0$ and (ii) $\tau > 0$.

Case (i): $\tau = 0$, i.e. there is no time delay in system (2.1), then system (2.1) becomes

$$\begin{cases} \frac{dS(t)}{dt} = (1 + kx(t))(1 - S(t)) - \frac{mS(t)x(t)}{ax(t)+S(t)}, \\ \frac{dx(t)}{dt} = x(t)(\frac{mS(t)}{ax(t)+S(t)} - 1 - kx(t)). \end{cases} \tag{2.4}$$

We first investigate the stability of the equilibria E_0 and E_1 and have the following result.

Theorem 2.2. *If $m < 1$, then E_0 is a stable node; If $m > 1$, then E_0 is a saddle and E_1 is a stable node; If $m = 1$, then E_0 is a saddle-node.*

Proof. The Jacobian matrix at E_0 is

$$J_0 := \begin{pmatrix} -1 & -m \\ 0 & m-1 \end{pmatrix}.$$

The determinant, trace and discriminant are respectively

$$D_0 := 1 - m, \quad T_0 := m - 2, \quad \Delta_0 := T_0^2 - 4D_0 = m^2 > 0.$$

If $m < 1$, then $D_0 > 0$ and $T_0 < 0$. Thus, E_0 is a stable node. If $m > 1$, then $D_0 < 0$. Thus, E_0 is a saddle. If $m = 1$, then $D_0 = 0$ and $T_0 = -1$. This is the degenerate case, which needs further discussion.

Translating the equilibrium E_0 to the origin by $\tilde{S} = S - 1$, system (2.4) becomes the following system

$$\begin{cases} \frac{dS(t)}{dt} = -S(t)(1 + kx(t)) - \frac{(S(t)+1)x(t)}{ax(t)+S(t)+1}, \\ \frac{dx(t)}{dt} = x(t)(\frac{S(t)+1}{ax(t)+S(t)+1} - 1 - kx(t)), \end{cases} \tag{2.5}$$

where we still use S to present $\tilde{S}$. Using the linear transformation $u = x$, $v = S + x$ and time-rescaling $t_1 = -t$ to normalize the linear part of system (2.5), we can change system (2.5) into the following

$$\begin{cases} \frac{du}{dt} = (a + k)u^2 + a(1 - a)u^3 - au^2v + \cdots := \Phi(u, v), \\ \frac{dv}{dt} = v + kuv := \Psi(u, v), \end{cases} \tag{2.6}$$

where we still denote t_1 as t. By the implicit function theorem, there is a unique function $v = \phi(u)$ such that $\Psi(u, v) = 0$. We can obtain $v = \phi(u) = 0$ by solving $\Psi(u, v) = 0$. Substituting $v = 0$ into $\Phi(u, v) = 0$, we get

$$\Phi(u, v) = (a + k)u^2 + a(1 - a)u^3 + \cdots .$$

Thus E_0 is a saddle-node when $m = 1$ by Theorem 7.1 in Zhang et al. ([35]). Moreover, the parabolic sector of the saddle-node lies on the right-hand plane in the (u, v)-coordinates and the two hyperbolic sectors lie on the left-hand plane since $a + k > 0$.

We continue to prove the properties of E_1.

The Jacobian matrix at E_1 is

$$J_1 := \begin{pmatrix} -1 - kx^* - \frac{ma(x^*)^2}{(ax^*+S^*)^2} & kx^* - \frac{m(S^*)^2}{(ax^*+S^*)^2} \\ \frac{ma(x^*)^2}{(ax^*+S^*)^2} & -1 - 2kx^* + \frac{m(S^*)^2}{(ax^*+S^*)^2} \end{pmatrix}.$$

The determinant, trace and discriminant of J_1 are respectively

$$\begin{aligned} D_1 &:= (1 + kx^*)\{\tfrac{maS^*x^*+kx^*(ax^*+S^*)^2+max^*}{(ax^*+S^*)^2}\} > 0, \\ T_1 &:= -2 - 3kx^* + \tfrac{m(S^*)^2}{(ax^*+S^*)^2} - \tfrac{ma(x^*)^2}{(ax^*+S^*)^2} \\ &= -\{1 + 2k^* + \tfrac{maS^*x^*}{(ax^*+S^*)^2} + \tfrac{ma(x^*)^2}{(ax^*+S^*)^2}\} < 0, \\ \Delta_1 &:= T_1^2 - 4D_1 = \{-kx^* + \tfrac{m(S^*)^2}{(ax^*+S^*)^2} - \tfrac{ma(x^*)^2}{(ax^*+S^*)^2}\}^2 \geq 0. \end{aligned}$$

Thus, E_1 is a stable node if it exists.

We complete the proof. □

It is indicated in Lemma 2.1 that system (2.4) has either exact one equilibrium E_0 when $m \leq 1$ or exact two equilibria E_0 and E_1 when $m > 1$. In the following we reduce the system to a 1-dimensional system on a center manifold and display the mechanism for E_1 to arise.

Theorem 2.3. *System (2.4) experiences a transcritical bifurcation at E_0 when $m = 1$.*

Proof. Let $\varepsilon = m - 1$ and translate the equilibrium E_0 to the origin by $\tilde{S} = S - 1$, system (2.4) becomes the following system

$$\begin{cases} \frac{dS}{dt} = -(1 + kx)S - \frac{(1+\varepsilon)(S+1)}{ax+S+1}x, \\ \frac{dx}{dt} = x\{\frac{(1+\varepsilon)(S+1)}{ax+S+1} - 1 - kx\}, \end{cases} \tag{2.7}$$

where we still denote $\tilde{S}$ as S.

Applying the transformation $S = -u + v$ and $x = u$ to diagonalize the linear part of system (2.7), we can change system (2.7) into the suspended system

$$\begin{cases} \frac{du}{dt} = \varepsilon u - (a + k)u^2 - a\varepsilon u^2 + au^2v + a(a - 1)u^3 + \cdots , \\ \frac{dv}{dt} = -v - kuv, \\ \frac{d\varepsilon}{dt} = 0. \end{cases} \tag{2.8}$$

By the center manifold theory, system (2.8) has a two-dimensional center manifold $W^c : v = h(u, \varepsilon)$ near O, which is tangent to the plane $v = 0$ at O in the (u, v, ε)-space. In order to obtain the second-order approximation of function h, we set

$$v = h(u, \varepsilon) = a_1u^2 + b_1u\varepsilon + c_1\varepsilon^2 + o(|u, \varepsilon|^2). \tag{2.9}$$

Substituting (2.9) into the equality $\dot{v} = h_u \dot{u}$ and comparing the coefficient of u^2, ε^2 and $u\varepsilon$, we have $a_1 = b_1 = c_1 = 0$. Hence the center manifold is $v = o(|u, \varepsilon|^2)$ and the restricted system of (2.8) on the center manifold (2.9) is

$$\frac{du}{dt} = \varepsilon u - (a + k + a\varepsilon)u^2 + a(a-1)u^3 + \cdots . \tag{2.10}$$

The expression (2.10) shows that a transcritical bifurcation occurs at E_0 as ε varies through the bifurcation value $\varepsilon = 0$ ([6]). More concretely, when $\varepsilon < 0$, E_0 is stable and the other equilibrium appears on the negative u-axis; when $\varepsilon = 0$, the two equilibria coincide at E_0, which is a saddle-node; when $\varepsilon > 0$, E_0 remains an equilibrium but is unstable while a stable equilibrium E_1 arises.

We complete the proof. □

Case (ii): $\tau > 0$.

In this case we will investigate the effect of time delay on the system. The time delay can cause the loss of stability of E_1 and can induce periodic solutions. Note that if $m > 1$, then system (2.1) has a unique positive equilibrium E_1. To further consider the local stability of E_1 and the Hopf bifurcations induced by the delay, we set $\tilde{x}(t) = x(t) - x^*$, $\tilde{S}(t) = S(t) - S^*$ and still denote $\tilde{x}, \tilde{S}$ as x, S. Then system (2.1) can be changed into the following system

$$\begin{cases} \frac{dS(t)}{dt} = a_{11}S(t) + a_{12}x(t) + a_{13}S^2(t) + a_{14}x^2(t) + a_{15}S(t)x(t) \\ \qquad + a_{16}x^3(t) + a_{17}S(t)x^2(t) + a_{18}S^2(t)x(t) + a_{19}S^3(t) + \cdots , \\ \dfrac{dx(t)}{dt} = b_{11}S(t-\tau) + b_{12}x(t) + b_{13}x(t-\tau) + b_{14}S^2(t-\tau) + b_{15}x^2(t) \\ \qquad + b_{16}x^2(t-\tau) + b_{17}S(t-\tau)x(t) + b_{18}x(t)x(t-\tau) \\ \qquad + b_{19}S(t-\tau)x(t-\tau) + b_{20}x(t)x^2(t-\tau) + b_{21}x^3(t-\tau) \\ \qquad + b_{22}x(t)x(t-\tau)S(t-\tau) + b_{23}S(t-\tau)x^2(t-\tau) \\ \qquad + b_{24}S^2(t-\tau)x(t) + b_{25}x(t-\tau)S^2(t-\tau) + b_{26}S^3(t-\tau) + \cdots , \end{cases} \tag{2.11}$$

where

$$\begin{aligned} &a_{11} = -1 - kx^* - \tfrac{ma(x^*)^2}{(ax^*+S^*)^2}, a_{12} = kx^* - \tfrac{m(S^*)^2}{(ax^*+S^*)^2}, a_{13} = \tfrac{ma(x^*)^2}{(ax^*+S^*)^3}, \\ &a_{14} = \tfrac{ma(S^*)^2}{(ax^*+S^*)^3}, a_{15} = -k - \tfrac{2maS^*x^*}{(ax^*+S^*)^3}, a_{16} = -\tfrac{ma^2(S^*)^2}{(ax^*+S^*)^4}, \\ &a_{17} = \tfrac{maS^*(2ax^*-S^*)}{(ax^*+S^*)^4}, a_{18} = -\tfrac{max^*(ax^*-2S^*)}{(ax^*+S^*)^4}, a_{19} = -\tfrac{ma(x^*)^2}{(ax^*+S^*)^4}, \\ &b_{11} = \tfrac{ma(x^*)^2}{(ax^*+S^*)^2}, b_{12} = -1 - 2kx^* + \tfrac{mS^*}{ax^*+S^*}, b_{13} = -\tfrac{maS^*x^*}{(ax^*+S^*)^2}, \\ &b_{14} = -\tfrac{ma(x^*)^2}{(ax^*+S^*)^3}, b_{15} = -k, b_{16} = \tfrac{ma^2S^*x^*}{(ax^*+S^*)^2}, b_{17} = \tfrac{max^*}{(ax^*+S^*)^2}, \\ &b_{18} = -\tfrac{maS^*}{(ax^*+S^*)^2}, b_{19} = -\tfrac{max^*(ax^*-S^*)}{(ax^*+S^*)^3}, b_{20} = \tfrac{ma^2S^*}{(ax^*+S^*)^3}, \\ &b_{21} = -\tfrac{ma^3S^*x^*}{(ax^*+S^*)^4}, b_{22} = -\tfrac{ma(ax^*-S^*)}{(ax^*+S^*)^3}, b_{23} = \tfrac{ma^2x^*(ax^*-2S^*)}{(ax^*+S^*)^4}, \\ &b_{24} = -\tfrac{max^*}{(ax^*+S^*)^3}, b_{25} = \tfrac{max^*(2ax^*-S^*)}{(ax^*+S^*)^4}, b_{26} = \tfrac{ma(x^*)^2}{(ax^*+S^*)^4}. \end{aligned}$$

The following characteristic equation can be achieved from the linear system of system (2.11)

$$\lambda^2 - (a_{11} + b_{12})\lambda + e^{-\lambda\tau}(-b_{13}\lambda + a_{11}b_{13} - a_{12}b_{11}) + a_{11}b_{12} = 0. \tag{2.12}$$

To investigate the stability of the equilibrium E_1 and Hopf bifurcation of (2.1), we must study the distribution of the roots of (2.12).

Lemma 2.4. *If $a_{11}b_{12} + a_{12}b_{11} - a_{11}b_{13} < 0$, then $\pm iw_0 (w_0 > 0)$ are the eigenvalues of (2.12) when $\tau = \tau_j$. The values of w_0 and τ_j can be presented as follows*

$$
\begin{aligned}
w_0 &= \sqrt{\frac{2a_{11}b_{12}+b_{13}^2-(a_{11}+b_{12})^2+\sqrt{\{-2a_{11}b_{12}-b_{13}^2+(a_{11}+b_{12})^2\}^2-4\{a_{11}^2b_{12}^2-(a_{11}b_{13}-a_{12}b_{11})^2\}}}{2}},\\
\tau_j &= \frac{1}{w_0}\{\arccos(\frac{(w_0^2-a_{11}b_{12})(a_{11}b_{13}-a_{12}b_{11})-b_{13}(a_{11}+b_{12})w_0^2}{(a_{11}b_{13}-a_{12}b_{11})^2+b_{13}^2w_0^2}) + 2j\pi\}, \quad j = 0, 1, 2\cdots.
\end{aligned}
$$

Proof. If $\lambda = iw$ $(w > 0)$ is a root of (2.12), then we have

$$
-w^2 - iw(a_{11} + b_{12}) + (\cos w\tau - i\sin w\tau)(-ib_{13}w + a_{11}b_{13} - a_{12}b_{11}) + a_{11}b_{12} = 0.
$$

Separating the real and imaginary parts, we obtain

$$
\begin{aligned}
(a_{11}b_{13} - a_{12}b_{11})\cos w\tau - b_{13}w\sin w\tau &= w^2 - a_{11}b_{12},\\
(a_{11}b_{13} - a_{12}b_{11})\sin w\tau + b_{13}w\cos w\tau &= -w(a_{11} + b_{12}),
\end{aligned} \tag{2.13}
$$

which implies the following equation

$$
w^4 + \{-2a_{11}b_{12} + (a_{11} + b_{12})^2 - b_{13}^2\}w^2 + a_{11}^2b_{12}^2 - (a_{11}b_{13} - a_{12}b_{11})^2 = 0.
$$

Denote

$$
h(w) := w^4 + (a_{11}^2 + b_{12}^2 - b_{13}^2)w^2 + a_{11}^2b_{12}^2 - (a_{11}b_{13} - a_{12}b_{11})^2.
$$

Since

$$
a_{11}b_{12} + a_{11}b_{13} - a_{12}b_{11} = (1 + kx^*)\{kx^* + \frac{maS^*x^*}{(ax^*+S^*)^2}\} + \frac{m^2aS^*(x^*)^2(ax^*+S^*)}{(ax^*+S^*)^4} > 0
$$

and the condition $a_{11}b_{12}+a_{12}b_{11}-a_{11}b_{13} < 0$, we have the constant term of $h(w)$ is negative. In addition, we can obtain the coefficient of quadratic term of $h(w)$ is positive. In fact,

$$
\begin{aligned}
a_{11}^2 + b_{12}^2 - b_{13}^2 &= \{1 + kx^* + \frac{ma(x^*)^2}{(ax^*+S^*)^2}\}^2 + (1 + 2kx^* - \frac{mS^*}{ax^*+S^*})^2 - \frac{(maS^*x^*)^2}{(ax^*+S^*)^4}\\
&= \{\frac{mS^*}{ax^*+S^*} + \frac{ma(x^*)^2}{(ax^*+S^*)^2}\}^2 + (kx^*)^2 - \frac{(maS^*x^*)^2}{(ax^*+S^*)^4}\\
&= \frac{m^2\{(S^*)^2+a(x^*)^2\}^2+2m^2aS^*x^*\{(S^*)^2+a(x^*)^2\}}{(ax^*+S^*)^4} + (kx^*)^2 > 0.
\end{aligned}
$$

Thus, $h(w)$ has a unique positive real root

$$
\begin{aligned}
w_0 = \frac{\sqrt{2}}{2}\{&2a_{11}b_{12} + b_{13}^2 - (a_{11} + b_{12})^2\\
&+ \sqrt{(-2a_{11}b_{12} - b_{13}^2 + (a_{11} + b_{12})^2)^2 - 4(a_{11}^2b_{12}^2 - (a_{11}b_{13} - a_{12}b_{11})^2)}\}^{\frac{1}{2}}.
\end{aligned} \tag{2.14}
$$

Substituting w_0 into (2.13), we conclude that

$$
\begin{aligned}
\tau_j = \frac{1}{w_0}&\{\arccos(\frac{(w_0^2-a_{11}b_{12})(a_{11}b_{13}-a_{12}b_{11})-b_{13}(a_{11}+b_{12})w_0^2}{(a_{11}b_{13}-a_{12}b_{11})^2+b_{13}^2w_0^2}) + 2j\pi\},\\
&j = 0, 1, 2\cdots.
\end{aligned} \tag{2.15}
$$

Hence, (2.12) has a pair of purely imaginary roots $\pm iw_0$ as $\tau = \tau_j$, $j = 0, 1, 2\cdots$.
This completes the proof. □

Let $\lambda(\tau) = \alpha(\tau) + i\beta(\tau)$ denote the root of (2.12) near $\tau = \tau_j$ satisfying $\alpha(\tau_j) = 0$ and $\beta(\tau_j) = w_0$. Then we have the following transversality condition.

Lemma 2.5. *The following transversality condition holds*

$$\frac{dRe(\lambda(\tau_j))}{d\tau} > 0, \quad j = 0, 1, 2 \cdots .$$

Proof. Differentiating (2.12) with respect to τ yields that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{(2\lambda - a_{11} - b_{12})e^{\lambda\tau} - b_{13}}{\lambda(-b_{13}\lambda + a_{11}b_{13} - a_{12}b_{11})} - \frac{\tau}{\lambda}.$$

Thus,

$$\begin{aligned} Re\left\{\left(\frac{d\lambda(\tau_j)}{d\tau}\right)^{-1}\right\} &= Re\left\{\frac{(2iw_0 - a_{11} - b_{12})(\cos w_0\tau + i\sin w_0\tau) - b_{13}}{w_0^2 b_{13} + iw_0(a_{11}b_{13} - a_{12}b_{11})}\right\} \\ &= \frac{h'(w_0)}{2w_0^3 b_{13}^2 + 2w_0(a_{11}b_{13} - a_{12}b_{11})^2}. \end{aligned}$$

Since $h(w)$ is a quartic function of w and the leading coefficient of which is positive, furthermore, from the analysis of Lemma 2.4, we obtain that the four roots of equation $h(w) = 0$ are a pair of conjugate complex roots, a negative real root and a positive real root w_0, respectively. Hence $h(w)$ is monotonically increasing at w_0, i.e., $h'(w_0) > 0$, which implies $h'(w_0)/\{2w_0^3 b_{13}^2 + 2w_0(a_{11}b_{13} - a_{12}b_{11})^2\} > 0$. Thus, we have

$$sign\left\{Re\left(\frac{d\lambda(\tau_j)}{d\tau}\right)\right\} = sign\left\{Re\left(\frac{d\lambda(\tau_j)}{d\tau}\right)^{-1}\right\} = 1.$$

The proof is completed. □

Now we have the following result about the distribution of the roots of the exponential polynomial (2.12) by Corollary 2.4 in [25].

Lemma 2.6. *Suppose that w_0 and τ_j ($j = 0, 1, 2 \cdots$) are defined by (2.14) and (2.15), respectively. We have the following results*

(i) If $a_{11}b_{12} + a_{12}b_{11} - a_{11}b_{13} \geq 0$, then all the roots of (2.12) have negative real parts for all $\tau > 0$.

(ii) If $a_{11}b_{12} + a_{12}b_{11} - a_{11}b_{13} < 0$ and $\tau = \tau_j$, then (2.12) has a pair of simple imaginary roots $\pm iw_0$. Furthermore, if $\tau \in [0, \tau_0)$, then all the roots of (2.12) have negative real parts; if $\tau \in (\tau_j, \tau_{j+1})$, $j = 0, 1, 2 \cdots$, then (2.12) has at least one root with positive real parts.

From Lemma 2.4, Lemma 2.5, Lemma 2.6 and the Hopf bifurcation theorem, we have the following result of E_1.

Theorem 2.7. *For system (2.1), we have*

(i) if $a_{11}b_{12} + a_{12}b_{11} - a_{11}b_{13} \geq 0$, then the unique positive equilibrium E_1 is asymptotically stable for all $\tau \geq 0$.

(ii) if $a_{11}b_{12} + a_{12}b_{11} - a_{11}b_{13} < 0$, then the equilibrium E_1 is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$. Hopf bifurcation occurs when $\tau = \tau_j$, $j = 0, 1, 2 \cdots$.

3. Direction and stability of the hopf bifurcation

In this section, we consider the direction and stability of the bifurcating periodic solutions of system (2.1) induced by the Hopf bifurcation by using the normal form theory and the center manifold theorem by Hassard et al. ([7]). We compute (see Appendix for details of the computation)

$$g(z,\bar{z}) = g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \cdots,$$

where the first four coefficients g_{20}, g_{11}, g_{02} and g_{21} that we need for determining the properties of the Hopf bifurcation are of the following forms

$$\begin{aligned}
g_{20} =& 2\tau_j\bar{D}\{a_{14}q_1^2 + a_{15}q_1 + a_{13} + \bar{q}_1^*(b_{15}q_1^2 + (b_{18}q_1^2 + b_{17}q_1)e^{-iw_0\tau_j} + (b_{16}q_1^2 \\
&+ b_{19}q_1 + b_{14})e^{-2iw_0\tau_j})\},\\
g_{11} =& \tau_j\bar{D}\{a_{15}q_1 + a_{15}\bar{q}_1 + 2a_{13} + 2a_{14}\bar{q}_1q_1 + \bar{q}_1^*(2b_{14} + b_{19}q_1 + b_{19}\bar{q}_1 + 2b_{15}\bar{q}_1q_1 \\
&+ 2b_{16}\bar{q}_1q_1 + (b_{17}\bar{q}_1 + b_{18}q_1\bar{q}_1)e^{-iw_0\tau_j} + (b_{17}q_1 + b_{18}\bar{q}_1q_1)e^{iw_0\tau_j})\},\\
g_{02} =& 2\tau_j\bar{D}\{a_{13} + a_{14}\bar{q}_1^2 + a_{15}\bar{q}_1 + \bar{q}_1^*(b_{15}\bar{q}_1^2 + (b_{16}\bar{q}_1^2 + b_{19}\bar{q}_1 + b_{14})e^{2iw_0\tau_j} + (b_{18}\bar{q}_1^2 \\
&+ b_{17}\bar{q}_1)e^{iw_0\tau_j})\},\\
g_{21} =& \tau_j\bar{D}\{3a_{16}q_1^2\bar{q}_1 + a_{17}q_1^2 + 2a_{17}q_1\bar{q}_1 + 2a_{18}q_1 + a_{18}\bar{q}_1 + 3a_{19} + (\tfrac{1}{2}a_{15} \\
&+ a_{14}\bar{q}_1)W_{20}^{(2)}(0) + (a_{15} + 2a_{14}q_1)W_{11}^{(2)}(0) + (a_{15}q_1 + 2a_{13})W_{11}^{(1)}(0) + (\tfrac{1}{2}a_{15}\bar{q}_1 \\
&+ a_{13})W_{20}^{(1)}(0) + \bar{q}_1^*(2b_{24}q_1 + (b_{22}q_1 + b_{22}\bar{q}_1)q_1 + (3b_{26} + 2b_{23}\bar{q}_1q_1 + 3b_{21}\bar{q}_1q_1^2 \\
&+ b_{23}q_1^2 + b_{25}\bar{q}_1 + 2b_{25}q_1)e^{-iw_0\tau_j} + (b_{24}\bar{q}_1 + b_{20}q_1^2\bar{q}_1 + b_{22}q_1\bar{q}_1)e^{-2iw_0\tau_j} \\
&+ 2b_{20}\bar{q}_1q_1^2 + ((b_{19}q_1 + 2b_{14})e^{-iw_0\tau_j} + b_{17}q_1)W_{11}^{(1)}(-1) + ((\tfrac{1}{2}b_{19}\bar{q}_1 + b_{14})e^{iw_0\tau_j} \\
&+ \tfrac{1}{2}b_{17}\bar{q}_1)W_{20}^{(1)}(-1) + ((b_{18}q_1 + b_{17})e^{-iw_0\tau_j} + 2b_{15}q_1)W_{11}^{(2)}(0) + (\tfrac{1}{2}(b_{18}\bar{q}_1 \\
&+ b_{17})e^{iw_0\tau_j} + b_{15}\bar{q}_1)W_{20}^{(2)}(0) + ((2b_{16}q_1 + b_{19})e^{-iw_0\tau_j} + b_{18}q_1)W_{11}^{(2)}(-1) \\
&+ ((b_{16}\bar{q}_1 + \tfrac{1}{2}b_{19})e^{iw_0\tau_j} + \tfrac{1}{2}b_{18}\bar{q}_1)W_{20}^{(2)}(-1))\},
\end{aligned}$$

in which the terms $W_{11}^{(1)}(0)$, $W_{11}^{(2)}(0)$, $W_{11}^{(1)}(-1)$, $W_{11}^{(2)}(-1)$, $W_{20}^{(1)}(0)$, $W_{20}^{(2)}(0)$, $W_{20}^{(1)}(-1)$, $W_{20}^{(2)}(-1)$, q_1, q_1^* and $\bar{D}$ are calculated in Appendix.

Now using the four coefficients we obtain the values of the parameters μ_2, β_2 and T_2

$$\begin{aligned}
&c_1(0) = \frac{i}{2w_0\tau_j}(g_{11}g_{20} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}, \quad \mu_2 = \frac{-Re(c_1(0))}{Re(\lambda'(\tau_j))},\\
&\beta_2 = 2Re(c_1(0)), \quad T_2 = -\frac{Im(c_1(0)) + \mu_2 Im(\lambda'(\tau_j))}{w_0\tau_j}.
\end{aligned}$$

Thus, using the quantities above, the properties of the Hopf bifurcation are determined by the following theorem.

Theorem 3.1. *The properties of the Hopf bifurcation are determined by the parameters μ_2, β_2 and T_2, where μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ (< 0), then the Hopf bifurcation is supercritical (subcritical); β_2 determines the stability of the bifurcating periodic solutions: if $\beta_2 < 0$ (> 0), the bifurcating periodic solutions are stable (unstable); and T_2 determines the period of the bifurcating periodic solutions: if $T_2 > 0$ (< 0), the period increases (decreases).*

4. Numerical simulation and discussion

We discussed a turbidostat system with Contois growth rate and time delay in this paper. We investigated the qualitative properties of system (1.2) including the existence and the stability of the equilibria and the bifurcations. In the case of no time delay, the boundary equilibrium E_0 is globally asymptotically stable if it is the unique equilibrium, and the positive equilibrium E_1 arising from a transcritical bifurcation is globally asymptotically stable if it exists. In the case of system with delay, the stability of E_1 is changed and the Hopf bifurcation occurs by choosing the delay as the bifurcation parameter. Furthermore, the stability and direction of the bifurcating periodic solutions are discussed by using the normal form and center manifold theorem. More concretely, when the delay is greater than the critical value τ_0, the positive equilibrium E_1 loses its stability and the stable periodic solution appears. The qualitative analysis and theoretical results show that the delay can change the topological structures of the system and produces more complicated dynamic behaviors.

We next offer an example to illustrate the feasibility of our results. When system (2.1) without delay, i.e. $\tau = 0$, we set $a = 1$ and $k = 0.6$, the system has the unique equilibrium E_0, which is a stable node as $m = 0.5$; the system has a saddle E_0 and a stable node $E_1 = (0.76, 0.24)$ as $m = 1.5$; the system has the unique equilibrium E_0, which is a saddle-node as $m = 1$ (Figure 2). However, when $m = 1.5$, $a = 1$ and $k = 0.6$, by choosing the time delay τ as the bifurcation parameter, we obtained the critical value of bifurcation $\tau_0 \approx 7$. Thus, the positive equilibrium E_1 is asymptotically stable when $\tau = 6 < \tau_0$, which is supported by Figure 3. The positive equilibrium E_1 is unstable and a stable bifurcating periodic solution occurs from E_1 when $\tau = 8 > \tau_0$, which can be seen clearly in Figure 4. According to (4.10) we can compute

$$g_{20} \approx -4.25 - 9.16i, \quad g_{11} \approx 0.21 + 1.3i, \quad g_{02} \approx 11.79 - 0.01i, \quad g_{21} \approx -47.16 - 5.86i.$$

Further, we can get

$$c_1(0) \approx -21.71 - 12.63i.$$

Then, in accordance with (4.22), we can obtain

$$\mu_2 \approx 2131.70 > 0, \quad \beta_2 \approx -43.42 < 0, \quad T_2 \approx 38.7 > 0.$$

Therefore, when $\tau = 8$, $\mu_2 > 0$ and $\beta_2 < 0$, then the Hopf bifurcation for system (2.1) is supercritical and the stable bifurcating periodic solutions occur from the positive equilibrium E_1. From Figure 3 and Figure 4, we can find that the dynamics of system (2.1) change when τ locates near τ_0. The bifurcation diagram of $x-\tau$ is presented in Figure 5, from which we find that even for parameter values not chosen, the stable periodic solutions occur in a large region of time delay.

Remark 1. This is a remark about Figure 2. In fact, considering the biological background, the set $\Omega = \{(S, x) : 0 \leq S \leq 1, x \geq 0\}$ is positively invariant with respect to system (2.4). Further, E_0 is globally asymptotically stable with respect to Ω if $m < 1$; E_0 is globally attractive with respect to Ω if $m = 1$; and E_1 is globally asymptotically stable with respect to Ω if $m > 1$. These results can be proved easily by Liapunov-LaSalle invariant principle and Poincare-Bendixson Theorem respectively.

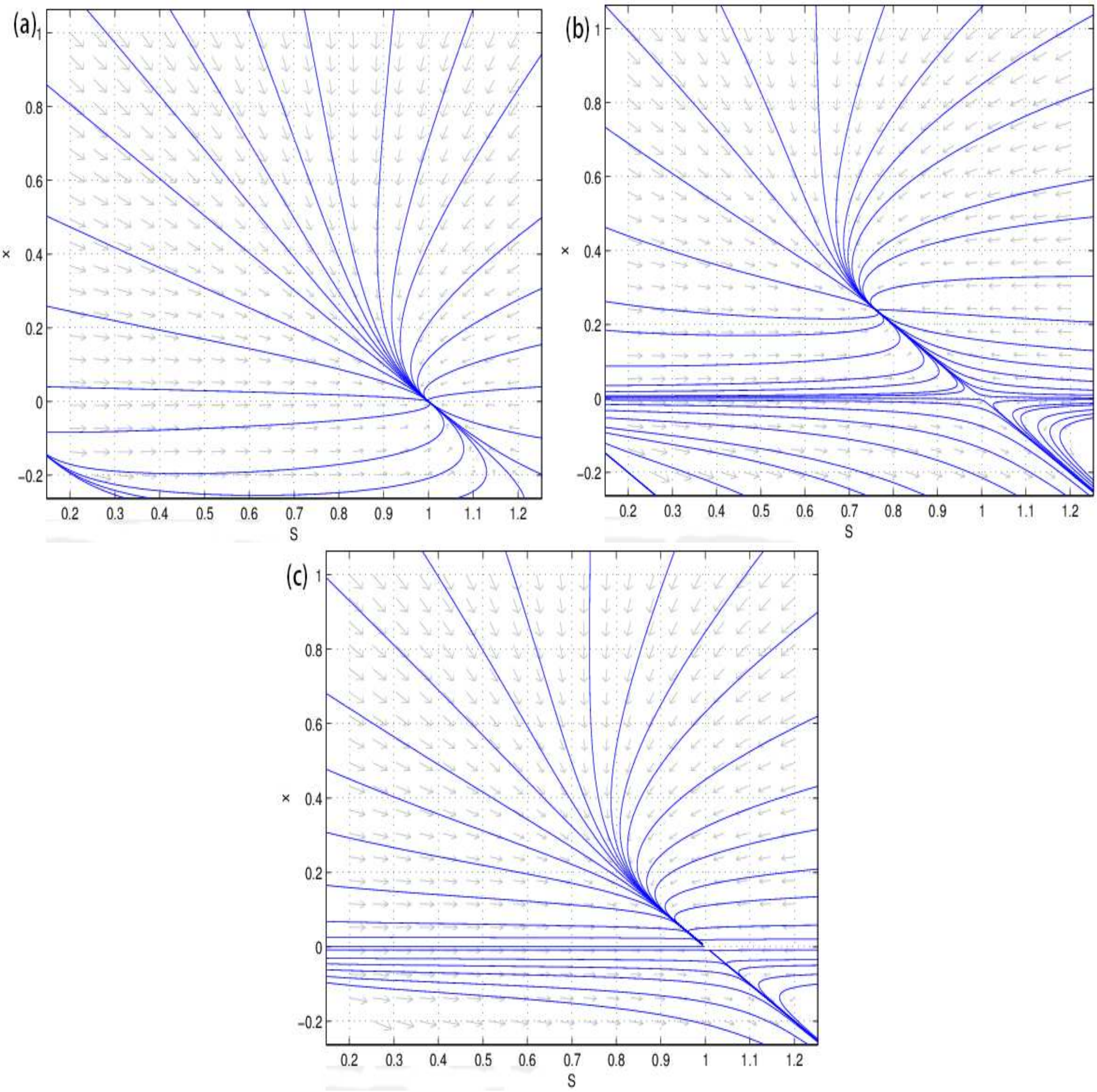

Figure 2. (a) When $m = 0.5$, $a = 1$ and $k = 0.6$, E_0 is the unique equilibrium and is a stable node; (b) When $m = 1.5$, $a = 1$ and $k = 0.6$, E_0 is a saddle and $E_1 = (0.76, 0.24)$ is a stable node; (c) When $m = 1$, $a = 1$ and $k = 0.6$, E_0 is the unique equilibrium and is a saddle-node.

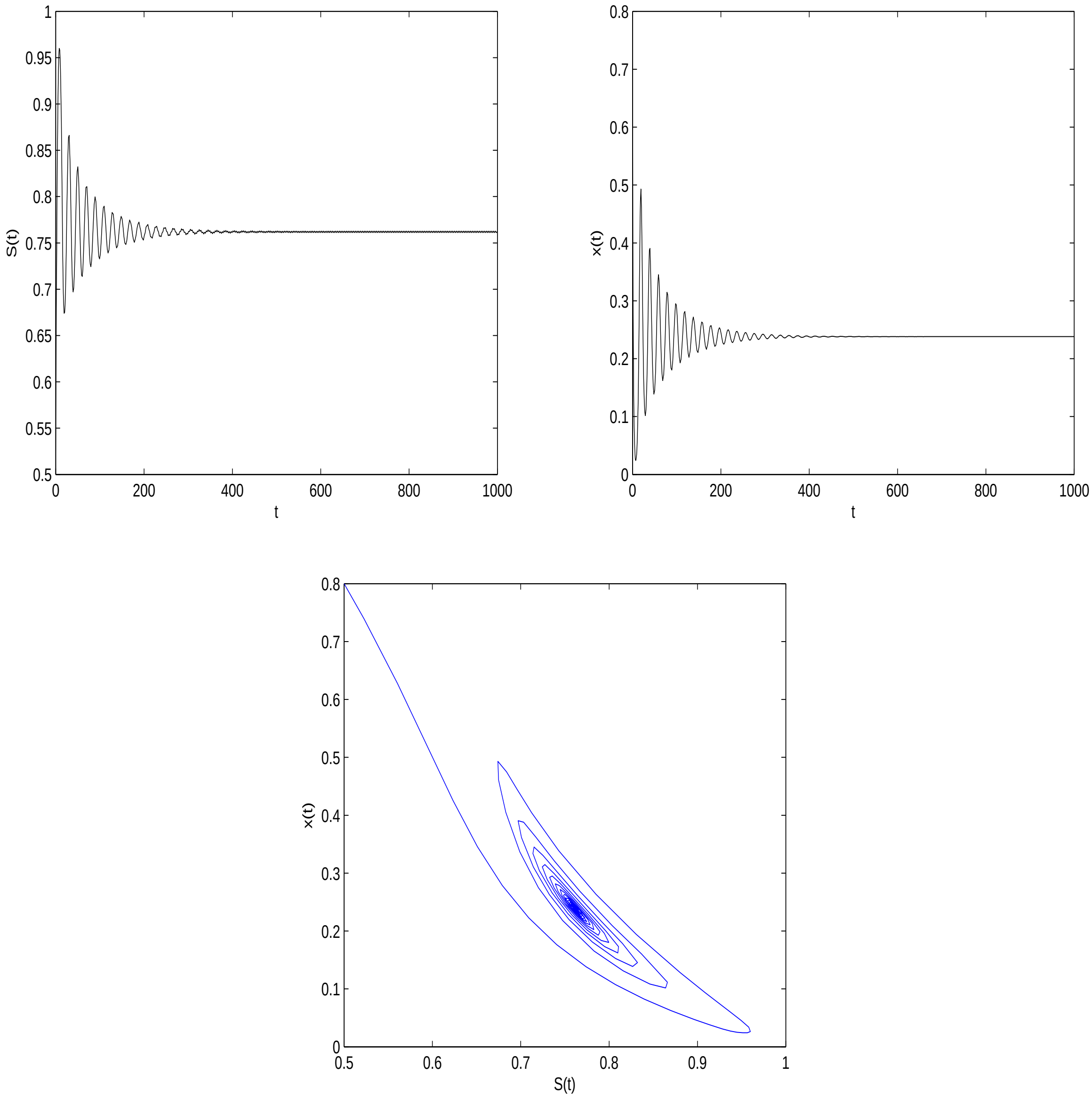

Figure 3. The positive equilibrium $E_1 = (0.76, 0.24)$ of system (2.1) is asymptotically stable when $m = 1.5$, $a = 1$, $k = 0.6$ and $\tau = 6 < \tau_0 \approx 7$. Here $(S(0), x(0)) = (0.5, 0.8)$.

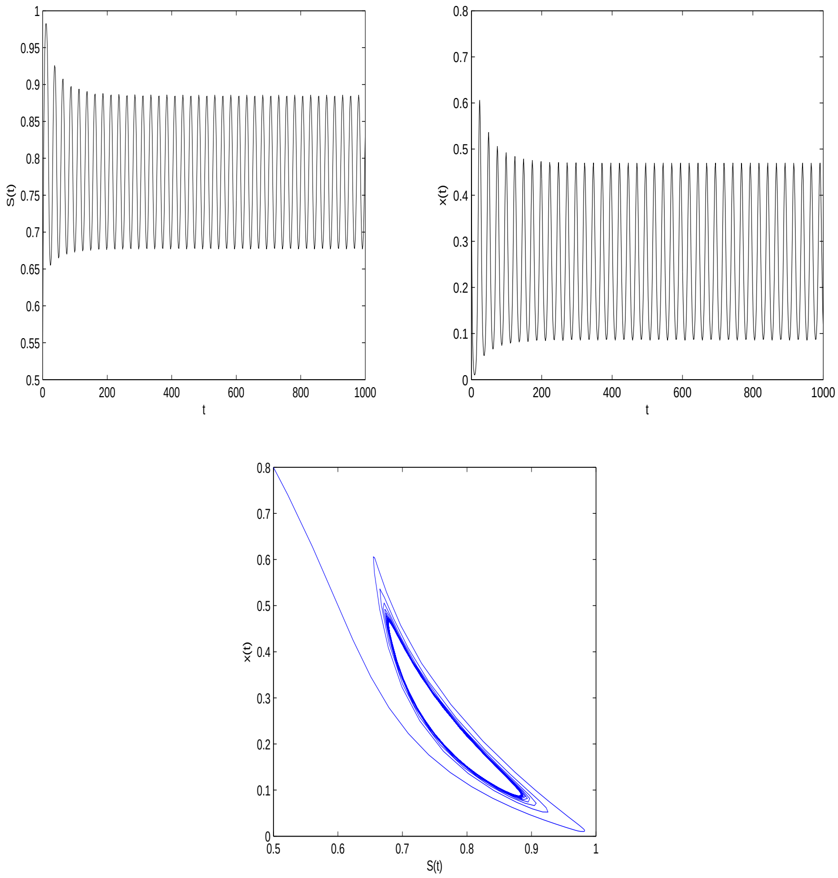

Figure 4. The positive equilibrium $E_1 = (0.76, 0.24)$ of system (2.1) is unstable and a bifurcating stable periodic solution occurs from E_1 when $m = 1.5$, $a = 1$, $k = 0.6$ and $\tau = 8 > \tau_0 \approx 7$. Here $(S(0), x(0)) = (0.5, 0.8)$.

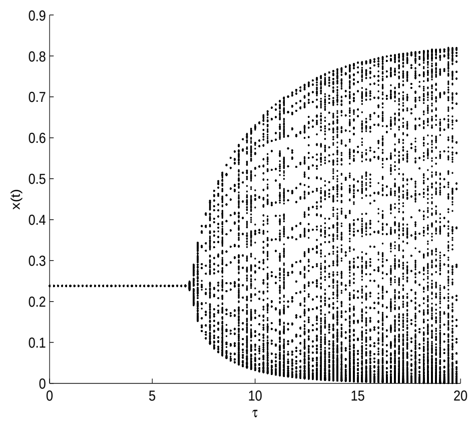

Figure 5. The bifurcation diagram of $x - \tau$ with $m = 1.5$, $a = 1$, $k = 0.6$ and $(S(0), x(0)) = (0.5, 0.8)$.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. R. Arditi and L.R. Ginzburg, Coupling in predator-prey dynamics: ratio-dependence, *J. Theor. Biol.*, **139** (1989), 311–332.
2. A.W. Bush and A.E. Cook, The effect of time delay and growth rate inhibition in the bacterial treatment of wastewater, *J. Theoret. Biol.*, **63** (1975), 385–396.
3. J. Caperon, Time lag in population growth response of isochrysis galbana to a variable nitrate environment, *Ecology*, **50** (1969), 188–192.
4. D.E. Contois, Kinetics of bacterial growth: relationship between population density and specific growth rate of continuous cultures, *J. Gen. Microbiol.*, **21** (1959), 40–50.
5. J. Flegr, Two distinct types of natural selection in turbidostat-like and chemostat-like ecosystems, *J. Theor. Biol.*, **188** (1997), 121–126.
6. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer, New York, 1983.
7. B.D. Hassard, N.D. Kazarinoff and Y.H. Wan, *Theory and applications of Hopf bifurcation*, Cambridge University Press, Cambridge, 1981.
8. D. Herbert, R. Elsworth and B.C. Telling, The continuous culture of bacteria; a theoretical and experimental study, *J. Gen. Microbiol.*, **14** (1956), 601–622.
9. S.B. Hsu, S. Hubbell and P. Waltman, A mathematical theory for single-nutrient competition in continuous culture of microorganism, *SIAM J. Appl. Math.*, **32** (1997), 366–383.
10. Z.X. Hu, G.K. Gao and W.B. Ma, Dynamics of a three-species ratio-dependent diffusive model, *Nonlinear Anal. Real*, **11** (2010), 2106–2114.
11. X.Y. Hu, Z.X. Li and X.G. Xiang, Feedback control for a turbidostat model with ratio-dependent growth rate, *J. Appl. Math. Inform.*, **31** (2013), 385–398.
12. Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, New York, 1993.
13. P.D. Leenheer and H. Smith, Feedback control for the chemostat, *J. Math. Biol.*, **46** (2003), 48–70.
14. B.T. Li, Competition in a turbidostat for an inhibitory nutrient, *J. Biol. Dynam.*, **2** (2008), 208–220.
15. B.T. Li, G.S.K. Wolkowicz and Y. Kuang, Global asymptotic behavior of a chemostat model with two perfectly complementary resources and distributed delay, *SIAM J. Appl. Math.*, **60** (2000), 2058–2086.

16. Z.X. Li and L.S. Chen, Periodic solution of a turbidostat model with impulsive state feedback control, *Nonlinear Dynam.*, **58** (2009), 525–538.
17. Z. Li and R. Xu, Stability analysis of a ratio-dependent chemostat model with time delay and variable yield, *Int. J. Biomath.*, **3** (2010), 243–253.
18. N. MacDonald, Time lag in simple chemostat models, *Biot. echnol. Bioeng.*, **18** (1976), 805–812.
19. J. Monod, The growth of bacterial culture, *Annu. Rev. Microbiol.*, **3** (1949), 371–394.
20. H. Moser, The dynamics of bacterial populations maintained in the chemostat, *Cold Spring Harb. Sym.*, **22** (1957), 121.
21. M.I. Nelson and H.S. Sidhu, Reducing the emission of pollutants in food processing wastewaters, *Chem. Eng. Process.*, **46** (2007), 429–436.
22. R.T. Alqahtani, M.I. Nelson and A.L. Worthy, Analysis of a chemostat model with variable yield coefficient and substrate inhibition: contois growth kinetics, *Chem. Eng. Process.*, **202** (2015), 332–344.
23. Z.C. Jiang and W.B. Ma, Delayed feedback control and bifurcation analysis in a chaotic chemostat system, *Int. J. Bifurcat. Chaos*, **25** (2015), 1550087.
24. C. Jost, Predator-prey theory: hidden twins in ecology and microbiology, *Oikos*, **90** (2000), 202–208.
25. S.G. Ruan and J.J. Wei, On the zeros of transcendental functions with applications to stability of delay differential equations with two delays, *Dynam. Cont. Discrete Impul. syst. Ser. A.*, **10** (2003), 863–874.
26. S.G. Ruan and G.S.K. Wolkowicz, Bifurcation analysis of a chemostat model with a distributed delay, *J. Math. Anal. Appl.*, **204** (1996), 786–812.
27. H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Springe, New York, 2010.
28. O. Tagashira and T. Hara, Delayed feedback control for a chemostat model, *Math. Biosci.*, **201** (2006), 101–112.
29. Y. Tian, Y. Bai and P. Yu, Impact of delay on HIV-1 dynamics of fighting a virus with another virus, *Math. Biosci. Eng.*, **11** (2014), 1181–1198.
30. L. Smith and P. Waltman, *The Theory of the Chemostat: Dynamics of Microbial Competition*, Cambridge University Press, UK, 1995.
31. L. Wang and G.S.K. Wolkowicz, A delayed chemostat model with general nonmonotone response functions and differential removal rates, *J. Math. Anal. Appl.*, **321** (2006), 452–468.
32. Y. Yao, Z.X. Li and Z.J. Liu, Hopf bifurcation analysis of a turbidostat model with discrete delay, *Appl. Math. Comput.*, **262** (2015), 267–281.
33. S.L. Yuan, P. Li and Y.L. Song, Delay induced oscillations in a turbidostat with feedback control, *J. Math. Chem.*, **49** (2011), 1646–1666.
34. Z. Zhao and X.Y. Song, Bifurcation and complexity in a ratio-dependent predator-prey chemostat with pulsed input, *Appl. Mat. Ser. B*, **22** (2007), 379–387.

35. Z.F. Zhang, T.R. Ding, W.Z. Huang and Z.X. Dong, *Qualitative Theory of Differential Equations*, American Mathematical Society, Providence, RI., 1992.

Appendix

We obtained in Section 2 that system (2.1) undergoes the Hopf bifurcation at the positive equilibrium E_1 when $\tau = \tau_j$. In the following the properties of the Hopf bifurcation are determined. We first consider system (2.1) by the transformation $\tilde{y}_1(t) = S(\tau t)$, $\tilde{y}_2(t) = x(\tau t)$, $\tau = \tau_j + \mu$ and still denote $\tilde{y}_1(t), \tilde{y}_2(t)$ as $y_1(t), y_2(t)$. Then system (2.1) is equivalent to a functional differential equation defined in $C = C([-1, 0], R^2)$

$$\dot{y}(t) = L_\mu(y_t) + h(\mu, y_t), \tag{4.1}$$

where $y(t) = (y_1(t), y_2(t))^T \in R^2$, and $L_\mu : C \to R^2$ and $h : R \times C \to R^2$ are given respectively by

$$L_\mu\varphi = (\tau_j + \mu)\begin{pmatrix} a_{11} & a_{12} \\ 0 & b_{12} \end{pmatrix}\begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \end{pmatrix} + (\tau_j + \mu)\begin{pmatrix} 0 & 0 \\ b_{11} & b_{13} \end{pmatrix}\begin{pmatrix} \varphi_1(-1) \\ \varphi_2(-1) \end{pmatrix}$$

and

$$h(\mu, \varphi) = (\tau_j + \mu)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

where

$$\begin{aligned} h_1 =& a_{13}\varphi_1^2(0) + a_{14}\varphi_2^2(0) + a_{15}\varphi_1(0)\varphi_2(0) + a_{16}\varphi_2^3(0) + a_{17}\varphi_1(0)\varphi_2^2(0) \\ &+ a_{18}\varphi_1^2(0)\varphi_2(0) + a_{19}\varphi_1^3(0) + \cdots, \\ h_2 =& b_{14}\varphi_1^2(-1) + b_{15}\varphi_2^2(0) + b_{16}\varphi_2^2(-1) + b_{17}\varphi_1(-1)\varphi_2(0) + b_{18}\varphi_2(0)\varphi_2(-1) \\ &+ b_{19}\varphi_1(-1)\varphi_2(-1) + b_{20}\varphi_2^2(-1)\varphi_2(0) + b_{21}\varphi_2^3(-1) \\ &+ b_{22}\varphi_2(0)\varphi_2(-1)\varphi_1(-1) + b_{23}\varphi_1(-1)\varphi_2^2(-1) + b_{24}\varphi_1^2(-1)\varphi_2(0) \\ &+ b_{25}\varphi_1^2(-1)\varphi_2(-1) + b_{26}\varphi_1^3(-1) + \cdots, \end{aligned}$$

and $\varphi = (\varphi_1, \varphi_2)^T \in C$.

By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $\eta(\theta, \mu)$ for $\theta \in [-1, 0]$ such that

$$L_\mu\varphi = \int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta) \quad \text{for} \quad \varphi \in C.$$

Indeed, we may choose

$$\eta(\theta, \mu) = (\tau_j + \mu)\begin{pmatrix} a_{11} & a_{12} \\ 0 & b_{12} \end{pmatrix}\delta(\theta) - (\tau_j + \mu)\begin{pmatrix} 0 & 0 \\ b_{11} & b_{13} \end{pmatrix}\delta(\theta + 1),$$

where δ is the Dirac delta function.

For $\varphi \in C^1([-1, 0], R^2)$, we further define the operators A and B as

$$A(\mu)\varphi = \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^{0} d\eta(\theta, \mu)\varphi(\theta), & \theta = 0, \end{cases} \qquad B(\mu)\varphi = \begin{cases} 0, & \theta \in [-1, 0), \\ h(\mu, \varphi), & \theta = 0. \end{cases}$$

Then system (4.1) is equivalent to

$$\dot{y}_t = A(\mu)y_t + B(\mu)y_t, \tag{4.2}$$

where $y_t(\theta) = y(t+\theta)$ for $\theta \in [-1, 0]$.

For $\psi \in C^1([0,1],(R^2)^*)$, we define A^* of A as

$$A^*\psi = \begin{cases} \dfrac{-d\psi(s)}{ds}, & s \in (0,1], \\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0. \end{cases}$$

In order to normalize the eigenvectors of A and A^*, we define a bilinear inner product

$$\langle \psi(s), \varphi(\theta)\rangle = \bar{\psi}^T(0)\varphi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}^T(\xi-\theta)d\eta(\theta)\varphi(\xi)d\xi, \tag{4.3}$$

where $\eta(\theta) = \eta(\theta, 0)$.

From the above discussion we assume $q(\theta)$ and $q^*(s)$ are eigenvectors of A and A^* corresponding to $iw_0\tau_j$ and $-iw_0\tau_j$. Suppose that $q(\theta) = (1, q_1)^T e^{iw_0\tau_j\theta}$ is the eigenvector of $A(0)$ corresponding to $iw_0\tau_j$, then $Aq(0) = iw_0\tau_j q(0)$. On the basis of the definitions of $A(0)$, $L_\mu\varphi$ and $\eta(\theta,\mu)$, the following conclusion can be achieved

$$\tau_j \begin{pmatrix} a_{11} - iw_0 & a_{12} \\ b_{11}e^{-iw_0\tau_j} & b_{12} - b_{13}e^{-iw_0\tau_j} - iw_0 \end{pmatrix} \begin{pmatrix} 1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We can get

$$q_1 = \frac{iw_0 - a_{11}}{a_{12}}. \tag{4.4}$$

Similarly, by the definition of A^*, we have

$$\tau_j \begin{pmatrix} a_{11} + iw_0 & b_{11}e^{iw_0\tau_j} \\ a_{12} & b_{12} + b_{13}e^{iw_0\tau_j} + iw_0 \end{pmatrix} \begin{pmatrix} 1 \\ q_1^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$q_1^* = \frac{iw_0 - a_{11}}{b_{11}e^{iw_0\tau_j}}. \tag{4.5}$$

We can then obtain the value of D by $\langle q^*, q\rangle = 1$. Further it holds from (4.3) that

$$\begin{aligned} \langle q^*(s), q(\theta)\rangle &= \bar{D}(1, \bar{q}_1^*)(1, q_1)^T \\ &\quad - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}(1, \bar{q}_1^*)e^{-iw_0\tau_j(\xi-\theta)}d\eta(\theta)(1, q_1)^T e^{i\xi w_0\tau_j}d\xi \\ &= \bar{D}\{1 + \bar{q}_1^* q_1 - \int_{-1}^{0}(1, \bar{q}_1^*)\theta e^{i\theta w_0\tau_j}d\eta(\theta)(1, q_1)^T\} \\ &= \bar{D}\{1 + \bar{q}_1^* q_1 + \tau_j e^{-iw_0\tau_j}\bar{q}_1^*(b_{11} + b_{13}q_1)\}. \end{aligned}$$

Hence

$$D = \frac{1}{1 + q_1^*\bar{q}_1 + \tau_j e^{iw_0\tau_j} q_1^*(b_{11} + b_{13}\bar{q}_1)}. \tag{4.6}$$

In the following part, we compute the coordinates describing center manifold C_0 at $\mu = 0$ by using the theory in [7]. Define

$$z(t) = \langle q^*(s), y_t(\theta)\rangle, \quad W(t,\theta) = y_t(\theta) - 2Re\{z(t)q(\theta)\}. \tag{4.7}$$

Then on the center manifold C_0, we have

$$W(t,\theta) = W(z(t),\bar{z}(t),\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots,$$

where z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. It is easy to see that W is real if y_t is real. Thereby, we next only consider real solutions of (4.2). If $y_t \in C_0$ is the solution of (4.2), since $\mu = 0$, which implies that

$$\begin{aligned}\dot{z}(t) &=< q^*, \dot{y}_t >=< q^*, A\dot{y}_t + B\dot{y}_t >=< A^*q^*, \dot{y}_t > + < q^*, B\dot{y}_t > \\ &= iw_0\tau_j z + \bar{q}^*(0)h(0, W(z,\bar{z},0) + 2Re\{zq(0)\}) \\ &\overset{\text{def}}{=} iw_0\tau_j z + \bar{q}^*(0)h_0(z,\bar{z}) = iw_0\tau_j z + g(z,\bar{z}),\end{aligned}$$

where

$$g(z,\bar{z}) = \bar{q}^*(0)h_0(z,\bar{z}) = g_{20}\tfrac{z^2}{2} + g_{11}z\bar{z} + g_{02}\tfrac{\bar{z}^2}{2} + g_{21}\tfrac{z^2\bar{z}}{2} + \cdots. \tag{4.8}$$

In the following we need to compute the coefficients g_{20}, g_{11}, g_{02} and g_{21}. Note that $y_t(\theta) = (y_{1t}(\theta), y_{2t}(\theta))^T = W(t,\theta) + zq(\theta) + \overline{zq(\theta)}$ and $q(\theta) = (1, q_1)^T e^{iw_0\tau_j\theta}$. It then follows that

$$\begin{cases} y_{1t}(0) = z + \bar{z} + W_{20}^{(1)}(0)\dfrac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\dfrac{\bar{z}^2}{2} \\ \qquad\quad + O(|(z,\bar{z})|^3), \\ y_{2t}(0) = q_1 z + \bar{q}_1\bar{z} + W_{20}^{(2)}(0)\dfrac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\dfrac{\bar{z}^2}{2} \\ \qquad\quad + O(|(z,\bar{z})|^3), \\ y_{1t}(-1) = ze^{-iw_0\tau_j} + \bar{z}e^{iw_0\tau_j} + W_{20}^{(1)}(-1)\dfrac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} \\ \qquad\quad + W_{02}^{(1)}(-1)\dfrac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3), \\ y_{2t}(-1) = q_1 ze^{-iw_0\tau_j} + \bar{q}_1\bar{z}e^{iw_0\tau_j} + W_{20}^{(2)}(-1)\dfrac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} \\ \qquad\quad + W_{02}^{(2)}(-1)\dfrac{\bar{z}^2}{2} + O(|(z,\bar{z})|^3). \end{cases} \tag{4.9}$$

It follows together with the definition $h(\mu,\varphi)$ that

$$\begin{aligned} g(z,\bar{z}) &= \bar{q}^*(0)h_0(z,\bar{z}) \\ &= \tau_j\bar{D}\{a_{13}\varphi_1^2(0) + a_{14}\varphi_2^2(0) + a_{15}\varphi_1(0)\varphi_2(0) + a_{16}\varphi_2^3(0) + a_{17}\varphi_1(0)\varphi_2^2(0) \\ &\quad + a_{18}\varphi_1^2(0)\varphi_2(0) + a_{19}\varphi_1^3(0) + \cdots + \bar{q}_1^*(b_{14}\varphi_1^2(-1) + b_{15}\varphi_2^2(0) \\ &\quad + b_{16}\varphi_2^2(-1) + b_{17}\varphi_1(-1)\varphi_2(0) + b_{18}\varphi_2(0)\varphi_2(-1) + b_{19}\varphi_1(-1)\varphi_2(-1) \\ &\quad + b_{20}\varphi_2^2(-1)\varphi_2(0) + b_{21}\varphi_2^3(-1) + b_{22}\varphi_2(0)\varphi_2(-1)\varphi_1(-1) \\ &\quad + b_{23}\varphi_1(-1)\varphi_2^2(-1) + b_{24}\varphi_1^2(-1)\varphi_2(0) + b_{25}\varphi_1^2(-1)\varphi_2(-1) \\ &\quad + b_{26}\varphi_1^3(-1) + \cdots)\}. \end{aligned}$$

By substituting (4.9) into the above equation and comparing the coefficients with (4.8) we have

$$\begin{cases} g_{20} = 2\tau_j\bar{D}\{a_{14}q_1^2 + a_{15}q_1 + a_{13} + \bar{q}_1^*(b_{15}q_1^2 + (b_{18}q_1^2 + b_{17}q_1)e^{-iw_0\tau_j} \\ \qquad + (b_{16}q_1^2 + b_{19}q_1 + b_{14})e^{-2iw_0\tau_j})\}, \\ g_{11} = \tau_j\bar{D}\{a_{15}q_1 + a_{15}\bar{q}_1 + 2a_{13} + 2a_{14}\bar{q}_1q_1 + \bar{q}_1^*(2b_{14} + b_{19}q_1 + b_{19}\bar{q}_1 \\ \qquad + 2b_{15}\bar{q}_1q_1 + 2b_{16}\bar{q}_1q_1 + (b_{17}\bar{q}_1 + b_{18}q_1\bar{q}_1)e^{-iw_0\tau_j} \\ \qquad + (b_{17}q_1 + b_{18}\bar{q}_1q_1)e^{iw_0\tau_j})\}, \\ g_{02} = 2\tau_j\bar{D}\{a_{13} + a_{14}\bar{q}_1^2 + a_{15}\bar{q}_1 + \bar{q}_1^*(b_{15}\bar{q}_1^2 + (b_{16}\bar{q}_1^2 + b_{19}\bar{q}_1 \\ \qquad + b_{14})e^{2iw_0\tau_j} + (b_{18}\bar{q}_1^2 + b_{17}\bar{q}_1)e^{iw_0\tau_j})\}, \\ g_{21} = \tau_j\bar{D}\{3a_{16}q_1^2\bar{q}_1 + a_{17}q_1^2 + 2a_{17}q_1\bar{q}_1 + 2a_{18}q_1 + a_{18}\bar{q}_1 + 3a_{19} \\ \qquad + (\frac{1}{2}a_{15} + a_{14}\bar{q}_1)W_{20}^{(2)}(0) + (a_{15} + 2a_{14}q_1)W_{11}^{(2)}(0) \\ \qquad + (a_{15}q_1 + 2a_{13})W_{11}^{(1)}(0) + (\frac{1}{2}a_{15}\bar{q}_1 + a_{13})W_{20}^{(1)}(0) \\ \qquad + \bar{q}_1^*(2b_{24}q_1 + (b_{22}q_1 + b_{22}\bar{q}_1)q_1 + (3b_{26} + 2b_{23}\bar{q}_1q_1 + 3b_{21}\bar{q}_1q_1^2 \\ \qquad + b_{23}q_1^2 + b_{25}\bar{q}_1 + 2b_{25}q_1)e^{-iw_0\tau_j} + (b_{24}\bar{q}_1 + b_{20}q_1^2\bar{q}_1 \\ \qquad + b_{22}q_1\bar{q}_1)e^{-2iw_0\tau_j} + 2b_{20}\bar{q}_1q_1^2 + ((b_{19}q_1 + 2b_{14})e^{-iw_0\tau_j} \\ \qquad + b_{17}q_1)W_{11}^{(1)}(-1) + ((\frac{1}{2}b_{19}\bar{q}_1 + b_{14})e^{iw_0\tau_j} + \frac{1}{2}b_{17}\bar{q}_1)W_{20}^{(1)}(-1) \\ \qquad + ((b_{18}q_1 + b_{17})e^{-iw_0\tau_j} + 2b_{15}q_1)W_{11}^{(2)}(0) + (\frac{1}{2}(b_{18}\bar{q}_1 + b_{17})e^{iw_0\tau_j} \\ \qquad + b_{15}\bar{q}_1)W_{20}^{(2)}(0) + ((2b_{16}q_1 + b_{19})e^{-iw_0\tau_j} + b_{18}q_1)W_{11}^{(2)}(-1) \\ \qquad + ((b_{16}\bar{q}_1 + \frac{1}{2}b_{19})e^{iw_0\tau_j} + \frac{1}{2}b_{18}\bar{q}_1)W_{20}^{(2)}(-1))\}. \end{cases} \tag{4.10}$$

In order to determine g_{21} we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. In the light of (4.2) and (4.7), we obtain

$$\begin{aligned} \dot{W} &= \dot{y}_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= \begin{cases} AW - 2Re\{\bar{q}^*(0)h_0(z,\bar{z})q(\theta)\}, & \theta \in [-1,0), \\ AW - 2Re\{\bar{q}^*(0)h_0(z,\bar{z})q(\theta)\} + h_0(z,\bar{z}), & \theta = 0, \end{cases} \\ &\overset{\text{def}}{=} AW + H(z,\bar{z},\theta), \end{aligned} \tag{4.11}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + H_{20}(\theta)\frac{z^3}{6} + \cdots . \tag{4.12}$$

Expanding the previous series and comparing the coefficients, we obtain that

$$\begin{cases} (A - 2iw_0\tau_jI)W_{20}(\theta) = -H_{20}(\theta), \\ AW_{11}(\theta) = -H_{11}(\theta), \cdots . \end{cases} \tag{4.13}$$

Case 1: We first consider the case $\theta \in [-1,0)$, it follows from (4.11) that

$$\begin{aligned} H(z,\bar{z},\theta) &= -\bar{q}^*(0)h_0(z,\bar{z})q(\theta) - q^*(0)\bar{h}_0(z,\bar{z})\bar{q}(\theta) \\ &= -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta). \end{aligned} \tag{4.14}$$

Comparing the coefficients with (4.8), we obtain

$$\begin{cases} H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta). \end{cases} \tag{4.15}$$

It follows from (4.13), (4.15) and the definition of A that

$$\begin{cases} \dot{W}_{20}(\theta) = 2iw_0\tau_j W_{20}(\theta) + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta), \\ \dot{W}_{11}(\theta) = g_{11}q(\theta) + \bar{g}_{11}\bar{q}(\theta). \end{cases} \tag{4.16}$$

Note that $q(\theta) = (1, q_1)^T e^{iw_0\tau_j\theta}$, then we have

$$\begin{cases} W_{20}(\theta) = \dfrac{ig_{20}}{w_0\tau_j}q(0)e^{iw_0\tau_j\theta} + \dfrac{i\bar{g}_{02}}{3w_0\tau_j}\bar{q}(0)e^{-iw_0\tau_j\theta} + e^{2iw_0\tau_j\theta}E, \\ W_{11}(\theta) = \dfrac{-ig_{11}}{w_0\tau_j}q(0)e^{iw_0\tau_j\theta} + \dfrac{i\bar{g}_{11}}{w_0\tau_j}\bar{q}(0)e^{-iw_0\tau_j\theta} + F, \end{cases} \tag{4.17}$$

where $E = (E^{(1)}, E^{(2)})^T \in R^2$ and $F = (F^{(1)}, F^{(2)})^T \in R^2$ are all constant vectors. In what follows, we will seek appropriate E and F.

Case 2: We now consider the case $\theta = 0$. From (4.11), we have

$$H(z, \bar{z}, \theta) = -2Re\{\bar{q}^*(0)h_0(z, \bar{z})q(\theta)\} + h_0(z, \bar{z}).$$

It then follows from $H(z, \bar{z}, \theta)$ and (4.12) that

$$\begin{cases} H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + \tau_j \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \\ H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_j \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}, \end{cases} \tag{4.18}$$

where

$$\begin{aligned} d_1 &= 2(a_{13} + a_{14}q_1^2 + a_{15}q_1), \\ d_2 &= 2\{b_{15}q_1^2 + (b_{14} + b_{16}q_1^2 + b_{19}q_1)e^{-2iw_0\tau_j} + (b_{17}q_1 + b_{18}q_1^2)e^{-iw_0\tau_j}\}, \\ d_3 &= 2a_{13} + 2a_{14}q_1\bar{q}_1 + a_{15}(q_1 + \bar{q}_1), \\ d_4 &= 2b_{14} + 2q_1\bar{q}_1(a_{15} + b_{16}) + b_{19}(q_1 + \bar{q}_1) + b_{17}(q_1e^{iw_0\tau_j} + \bar{q}_1e^{-iw_0\tau_j}) \\ &\quad + b_{18}q_1\bar{q}_1(e^{iw_0\tau_j} + e^{-iw_0\tau_j}). \end{aligned}$$

From (4.13) and the definition of A, we obtain

$$\begin{cases} \int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2iw_0\tau_j W_{20}(0) - H_{20}(0), \\ \int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \end{cases} \tag{4.19}$$

where $\eta(\theta) = \eta(0, \theta)$.

Substituting (4.17), (4.18) into (4.19) and noting that

$$(iw_0\tau_j I - \int_{-1}^{0} e^{iw_0\tau_j\theta}d\eta(\theta))q(0) = 0,$$

and

$$(- iw_0\tau_j I - \int_{-1}^{0} e^{-iw_0\tau_j\theta}d\eta(\theta))\bar{q}(0) = 0.$$

We conclude that

$$(2iw_0\tau_j I - \int_{-1}^{0} e^{2iw_0\tau_j\theta}d\eta(\theta))E_1 = \tau_j \begin{pmatrix} d_1 \\ d_2 \end{pmatrix},$$

that is

$$\begin{pmatrix} 2iw_0 - a_{11} & -a_{12} \\ -b_{11}e^{-2iw_0\tau_j} & 2iw_0 - b_{12} - b_{13}e^{-2iw_0\tau_j} \end{pmatrix} E_1 = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$

Thus

$$E_1 = \begin{pmatrix} 2iw_0 - a_{11} & -a_{12} \\ -b_{11}e^{-2iw_0\tau_j} & 2iw_0 - b_{12} - b_{13}e^{-2iw_0\tau_j} \end{pmatrix}^{-1} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}. \tag{4.20}$$

Similarly, we have

$$\begin{pmatrix} a_{11} & a_{12} \\ b_{11} & b_{12} + b_{13} \end{pmatrix} E_2 = -\begin{pmatrix} d_3 \\ d_4 \end{pmatrix},$$

which enables us to assert that

$$E_2 = -\begin{pmatrix} a_{11} & a_{12} \\ b_{11} & b_{12} + b_{13} \end{pmatrix}^{-1} \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}. \tag{4.21}$$

Still now we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (4.17). Therefore, all g_{ij} in (4.10) can be determined. Furthermore, we can compute the following values

$$\begin{cases} c_1(0) = \dfrac{i}{2w_0\tau_j}(g_{11}g_{20} - 2|g_{11}|^2 - \dfrac{1}{3}|g_{02}|^2) + \dfrac{g_{21}}{2}, \\ \mu_2 = \dfrac{-Re(c_1(0))}{Re(\lambda'(\tau_j))}, \\ \beta_2 = 2Re(c_1(0)), \\ T_2 = -\dfrac{Im(c_1(0)) + \mu_2 Im(\lambda'(\tau_j))}{w_0\tau_j}. \end{cases} \tag{4.22}$$

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