

REVIEW OF STABILITY AND STABILIZATION FOR IMPULSIVE DELAYED SYSTEMS

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(Communicated by Yang Kuang)

ABSTRACT. This paper reviews some recent works on impulsive delayed systems (IDSs). The prime focus is the fundamental results and recent progress in theory and applications. After reviewing the relative literatures, this paper provides a comprehensive and intuitive overview of IDSs. Five aspects of IDSs are surveyed including basic theory, stability analysis, impulsive control, impulsive perturbation, and delayed impulses. Then the research prospect is given, which provides a reference for further study of IDSs theory.

1. Introduction. Time delay systems represent one of the most popular class of systems. Time delay, whether occurs in the system state, the control input, or the measurement, is often inevitable in practical systems and can be a source of instability and poor performance [21, 19, 1, 61, 41, 20, 94, 95, 79, 72]. The future evolution of the system state of a time delay system depends not only on its current value, but also on its past values [124, 15, 88, 87, 55, 121, 111, 81, 34, 14, 117, 46]. Many processes have time delay characteristics in their dynamics. Since time delays often appear in engineering, biological and economical systems, and sometimes they may poorly affect the performance of a system. The problem of stability of IDSs and impulsive stabilization of delay systems have been extensively investigated [119, 108, 104, 96, 36, 98, 18, 24]. For example, [104] studied the stability of a class of nonlinear impulsive switching systems with time-varying delays. Based on the common Lyapunov function method and Razumikhin technique, several stability criteria are established for nonlinear impulsive switching systems with time-varying delays. In [98], by structuring hybrid impulsive and feedback controllers, synchronization problem of the memristive delayed neural networks is proposed. Then, based on differential inclusions, several synchronization criteria for the memristive delayed neural networks are obtained by impulsive control theories, special inequalities and the Lyapunov-type functional. In literatures, the research results concerning time delay systems can be classified into two types. One is delay-independent conditions, the other is delay-dependent conditions. Delay-dependent conditions are less

2010 *Mathematics Subject Classification.* Primary: 58F15, 58F17; Secondary: 53C35.

Key words and phrases. Impulsive delayed systems, stability analysis, impulsive control, impulsive perturbation, delayed impulses.

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conservative compared with the delay-independent conditions because they incorporate the information of time delays. Various techniques have been developed in literatures to derive delay-dependent conditions, such as the Lyapunov-Razumikhin method and the Lyapunov-Krasovskii functional method [18]. In [24], the authors proposed the equivalence between stability conditions for switched systems and the Lyapunov-Krasovskii functional stability conditions for discrete-time delay systems. This provides us another method to investigate time delay systems.

Impulsive effects exist widely in the world. As we know, the state of systems are often subject to instantaneous disturbances and experience abrupt changes at certain instants, which might be caused by frequent changes or other suddenly noises. These systems are called impulsive systems, which are governed by impulsive differential equations or impulsive difference equations [42, 118, 59, 57, 2]. In the past decades, there has been a growing interest in the theory of impulsive dynamical systems because of their applications to various problems arising in communications, control technology, impact mechanics, electrical engineering, medicine, biology, and so on [54, 28, 22, 17, 33, 32, 106, 3, 112, 74, 100, 75, 83, 31]. For example, [54] investigated the p th moment exponential stability of impulsive stochastic functional differential equations. Some sufficient conditions are obtained to ensure the p th moment exponential stability of the equilibrium solution by the Razumikhin method and Lyapunov functions. In [100], the authors proposed a IDS model for insulin therapy for both type 1 and type 2 diabetes mellitus with time delay in insulin production. It is shown that impulsive exogenous insulin infusions can mimic natural pancreatic insulin production. From the viewpoint of impulsive effects, the stability analysis for impulsive time-delay systems can be generally classified into two groups: impulsive stabilization and impulsive perturbation. The case where a given system without impulses is unstable or stable and can be turned into uniformly stable, uniformly asymptotically stable, and even exponentially stable under proper impulsive control, it is regarded as impulsive stabilization problem. Now it has been shown that impulsive stabilization problems can be widely applied to many fields such as orbital transfer of satellite, dosage supply in pharmacokinetics, ecosystems management, and synchronization in chaotic secure communication systems [35, 116, 9]. Alternatively, the case where a given system without impulses is stable and can remain stable under certain impulsive interference, it is regarded as impulsive perturbation problem. Up to now, many interesting results dealing with impulsive perturbations of time-delay systems have been reported [89, 91, 13].

By analyzing related literatures, this paper provides a comprehensive and intuitive overview for IDSs, which include the basic theory of IDSs, stability analysis with impulsive control and impulsive perturbation, and delayed impulses. Essentially, it provides an overview on the progress of stability and stabilization problem of IDSs. The rest of this paper is organized as follows. In Section 2, some notations and definitions of stability are presented. Section 3 covers the effects of impulses for IDSs. Section 4 considers the impulsive control problem. Section 5 considers the impulsive perturbation problem. Section 6 covers the delayed impulses. Section 7 concludes the paper and discuss the future research direction on this topic.

2. Impulsive delayed systems.

Notations. Let \mathbb{R} denote the set of real numbers, \mathbb{R}_+ the set of positive numbers, \mathbb{Z}_+ the set of positive integer, \mathbb{N} the set of nonnegative integer. \mathbb{R}^n is the n -dimensional real spaces equipped with the Euclidean norm $\|\cdot\|$. $\mathbb{R}^{n \times m}$ denotes

the $n \times m$ -dimensional real spaces. I denotes the identity matrix with appropriate dimensions. For any interval $J \subseteq \mathbb{R}$, set $S \subseteq \mathbb{R}^k (1 \leq k \leq n)$, $C(J, S) = \{\varphi : J \rightarrow S \text{ is continuous}\}$ and $C^1(J, S) = \{\varphi : J \rightarrow S \text{ is continuously differentiable}\}$. $PC(J, S) = \{\varphi : J \rightarrow S \text{ is continuous everywhere except at finite number of points } t, \text{ at which } \varphi(t^+), \varphi(t^-) \text{ exist and } \varphi(t^+) = \varphi(t)\}$. In particular, for given $\tau > 0$, set $C_\tau = C([- \tau, 0], \mathbb{R}^n)$ and $PC_\tau = PC([- \tau, 0], \mathbb{R}^n)$ with the norm $\|\cdot\|_\tau$ defined by $\|\varphi\|_\tau = \sup\{\|\varphi(s)\| : s \in [- \tau, 0]\}$. $\Lambda = \{1, 2, \dots, n\}$. $S(\rho) = \{x \in \mathbb{R}^n : |x| < \rho\}$. A function $\alpha : [0, \infty) \rightarrow [0, \infty)$ is of class \mathcal{K} if α is continuous, strictly increasing, and $\alpha(0) = 0$. In addition, if α is unbounded, then it is of class \mathcal{K}_∞ .

In recent years, there are many results on impulsive delayed systems (IDSs). Roughly speaking, an impulsive dynamical system consists of three elements: a continuous-time dynamical equation, which governs the evolution of the system between reset (impulsive) events; a difference equation, which describes the way the system states are instantaneously changed; and finally a criterion for determining when the states of the system are to be reset. In addition, it is well known that time-delays phenomena frequently appear in many practical problems, such as biological systems, mechanical, transmissions, fluid transmissions, networked control systems [122, 47, 123, 86, 80, 25, 45]. Therefore, it is not surprising that IDSs with time delays have become an attractive research field. In the following, consider the impulsive functional differential equation

$$\begin{cases} \dot{x}(t) = f(t, x_t), & t > t_0, t \neq t_k, \\ \Delta x = I_k(t, x(t^-)), & t = t_k, k \in \mathbb{N}, \end{cases} \tag{1}$$

where $f, I_k : \mathbb{R}_+ \times PC_\tau \mapsto \mathbb{R}^n$, $\Delta x = x(t_k) - x(t_k^-)$. For each $t \geq t_0$, $x_t \in PC_\tau$ is defined by $x_t(s) = x(t + s), s \in [- \tau, 0]$. The impulse times $\{t_k\}$ satisfy $0 \leq t_0 < \dots < t_k < \dots, \lim_{k \rightarrow \infty} t_k = \infty$. With Eq. (1), one associates an initial condition of the form

$$x_{t_0} = \phi, \tag{2}$$

where $t_0 \in \mathbb{R}_+$ and $\phi \in PC_\tau$.

Definition 2.1. [85]. A function $x(t)$ is called a solution of the initial value problem (1) and (2) if $x : [t_0 - \tau, \beta) \mapsto \mathbb{R}^n$, for some $\beta (t_0 \leq \beta < \infty)$, is continuous for $t \in [t_0 - \tau, \beta) \setminus \{t_k, k \in \Lambda\}$, $x(t_k^-)$ and $x(t_k^+)$ exist, and $x(t_k^+) = x(t_k)$ for $t_k \in [t_0 - \tau, \beta)$, and satisfies (1) and (2).

Under the following hypotheses $(H_1) - (H_4)$, the initial value problem (1) and (2) exists with a unique solution which will be written in the form $x(t, t_0, \phi)$, see [85] for detailed information.

(H_1) f is continuous on $[t_{k-1}, t_k) \times PC_\tau$ for each $k \in \mathbb{Z}_+$, and for all $\varphi \in PC_\tau$ and $k \in \mathbb{Z}_+$, the limit $\lim_{(t, \phi) \rightarrow (t_k^-, \varphi)} f(t, \phi) = f(t_k^-, \varphi)$ exists.

(H_2) f is locally Lipschitzian in ϕ in each compact set in PC_τ .

(H_3) For each $k \in \mathbb{Z}_+$, $I_k(t, x) \in C(\mathbb{R}_+ \times S(\rho), \mathbb{R}^n)$.

(H_4) There exists a $\rho_1 > 0 (\rho_1 < \rho)$ such that $x \in S(\rho_1)$ implies that $x + I_k(t_k, x) \in S(\rho)$ for all $k \in \mathbb{Z}_+$.

Assume that conditions $(H_1) - (H_4)$ hold and moreover, $f(t, 0) \equiv 0$ and $I_k(t_k, 0) \equiv 0$, then $x(t) \equiv 0$ is the solution of (1), which is called the trivial solution.

Definition 2.2. [56]. The trivial solution of system (1) is said to be

(S_1) stable if for any $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, there exists some $\delta = \delta(\varepsilon, t_0) > 0$, such that $\|\phi\|_\tau < \delta$ implies $\|x(t, t_0, \phi)\| < \varepsilon$, for all $t > t_0$;

(S_2) uniformly stable if the δ in (S_1) is independent of t_0 ;

(S_3) asymptotically stable if (S_1) holds and for any $t_0 \in \mathbb{R}_+$, there exists some $\eta = \eta(t_0) > 0$ such that if $\phi \in PC_\tau$ with $\|\phi\|_\tau < \eta$, then $\lim_{t \rightarrow \infty} x(t, t_0, \phi) = 0$;

(S_4) uniformly asymptotically stable if (S_2) holds and there exists some $\eta > 0$ such that for every $\gamma > 0$, there exists some $T = T(\eta, \gamma) > 0$ such that if $\phi \in PC_\tau$ with $\|\phi\|_\tau < \eta$, then $\|x(t, t_0, \phi)\| < \gamma$ for $t \geq t_0 + T$;

(S_5) globally stable (globally exponentially stable) if (S_1) holds and for any δ , there exist $\lambda > 0$ and $k(\delta) > 0$, when $\|\phi\|_\tau < \delta$ we have

$$\|x(t, t_0, \phi)\| \leq k(\delta)e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0.$$

Definition 2.3. [38]. A map $x : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is said to be an ω -periodic solution of the system (1) provided:

(i) x satisfies (1) and is a piecewise continuous map with first-class discontinuity points;

(ii) x satisfies $x(t + \omega) = x(t)$ for $t \neq \tau_k$ and $x(\tau_k + \omega^+) = x(\tau_k^+)$ for $k \in \mathbb{N}$.

Definition 2.4. [38]. Let $x^* = x^*(t, t_0, \phi^*)$ be an ω -periodic solution of the system (1) with initial condition (t_0, ϕ^*) . Then x^* is said to be globally attractive if for any solution $x = x(\cdot, t_0, \phi)$ of the system (1) through (t_0, ϕ) , $|x - x^*| \rightarrow 0$ as $t \rightarrow \infty$.

Definition 2.5. [109]. The function $V : [T_0 - \tau, \infty) \times \mathbb{R}^n \mapsto \mathbb{R}_+$ belongs to class v_0 if

(1) V is continuous on each of the sets $[t_{k-1}, t_k) \times \mathbb{R}_+$ and $\lim_{(t,u) \rightarrow (t_k^-, v)} V(t, u) = V(t_k^-, v)$;

(2) $V(t, x)$ is locally Lipschitzian in x and $V(t, 0) \equiv 0, \forall t \geq t_0$.

Definition 2.6. [109]. Let $V \in v_0$, the upper right-hand derivative of V with respect to system (1) is defined by

$$D^+V(t, \psi(0)) = \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(t+h, \psi(0) + hf(t, \psi)) - V(t, \psi(0))],$$

for $(t, \psi) \in [t_{k-1}, t_k) \times PC_\tau$.

3. Effects of impulses. Generally speaking, existing results on stability for IDSs can be classified into two groups: impulsive stabilization and impulsive perturbation. In the case where a given equation without impulses is unstable or stable, it can be tended to uniformly stable, uniformly asymptotically stable even exponentially stable under proper impulsive control. Such case is regarded as impulsive stabilization. In the case where a given equation without impulses is stable, and it can remain the stability behavior under certain impulsive interference, it is regarded as impulsive perturbation. At each discontinuous point t_k , suppose that $V(t_k) \leq \mu V(t_k^-)$, where the constant μ represents the impulsive strength. There are three kinds of impulsive strength μ for IDSs:

(1) When $|\mu| < 1$, the impulses are beneficial for the stability of the system.

Hence we call them beneficial impulses when $|\mu| < 1$, which can be categorized as impulsive controllers that would enhance the stability of the system.

(2) When $|\mu| = 1$, the impulses are neither harmful nor beneficial for the stability of IDSs since the absolute value of the differences between the nodes states remains unchanged. Hence, the impulses with $|\mu| = 1$ are named as inactive impulses.

(3) When $|\mu| > 1$, the impulses might destroy the stability of the system. Thus we call them harmful impulses if $|\mu| > 1$, which can be regarded as the perturbations that would suppress the stability of the system.

In the following, two examples are given to illustrate the effects of impulses for the IDSs.

Example 3.1. Consider a simple IDS:

$$\begin{cases} \dot{x}(t) = 0.5x(t) + 0.6x(t - 1), & t \geq 0, \\ x(k) = \mu x(k^-), & k \in \mathbb{N}, \end{cases} \tag{3}$$

where $\mu = 0.4$. Figs. 3.1 (a) and (b) show the time response of states $x(t)$ of (3) with or without impulsive control, respectively. Obviously, it can be seen from the figures that the system (3) is unstable when there is no impulsive control and becomes stable under proper impulsive control, which can be obtained by the results in [48]. It implies that impulses contribute to system dynamics.

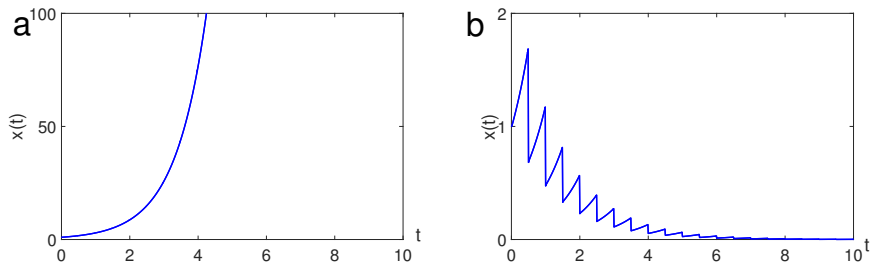


Fig.3.1 (a) State trajectory of system (3) without impulse control. (b) State trajectory of system (3) with impulsive stabilization.

Example 3.2. Consider another IDS:

$$\begin{cases} \dot{x}(t) = -0.7x(t) + 0.35x(t - 1), & t \geq 0, \\ x(k) = \mu x(k^-), & k \in \mathbb{N}, \end{cases} \tag{4}$$

where $\mu = 1.3$. Figs. 3.2 (a) and (b) show the time response of states $x(t)$ of (4) with or without impulsive perturbation, respectively. It can be seen from the figures that the system (4) is stable when there is no impulsive perturbation and remains stable under proper impulsive perturbation.

The above two examples fully illustrate the different effects of impulse for stability on the IDSs. In recent years, there are many researches on stability analysis for IDSs. For example, in [63], criteria on uniform asymptotic stability were established for impulsive delay differential equations by using Lyapunov functions and Raxumikhin techniques. [44] presented some sufficient conditions for global exponential stability for a class of delay difference equations with impulses by means of constructing an

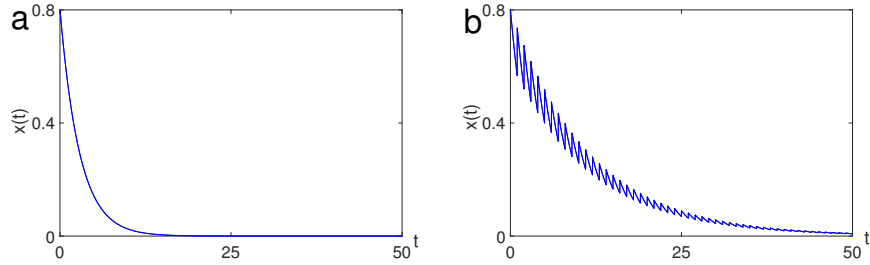


Fig.3.2 (a) State trajectory of system (4) without impulsive perturbation. (b) State trajectory of system (4) with impulsive perturbation.

extended impulsive delay difference inequality. In [39], authors addressed the impulsive systems with unbounded time-varying delay and introduced a new impulsive delay inequality that involves unbounded and non-differentiable time-varying delay. Some sufficient conditions ensuring stability and stabilization of impulsive time-invariant and time-varying systems are derived, respectively. [115] investigated the synchronization problem of coupled switched neural networks (SNNs) with mode-dependent impulsive effects and time delays. The impulses considered here include those that suppress synchronization or enhance synchronization. Based on switching analysis techniques and the comparison principle, the exponential synchronization criteria are derived for coupled delayed SNNs with mode-dependent impulsive effects. In addition, the concept of “average impulsive interval” was introduced in [68] by referring to the concept of average dwell time [23] to characterize how often or how seldom impulses occur.

Definition 3.3. [68]. The average impulsive interval (AII) of the impulsive sequence $\xi = \{t_1, t_2, \dots\}$ is equal to τ_α , if there exist positive integer N_0 and positive number τ_α such that

$$\frac{T-t}{\tau_\alpha} - N_0 \leq N_\xi(T, t) \leq \frac{T-t}{\tau_\alpha} + N_0, \quad \forall T \geq t \geq 0, \quad (5)$$

where $N_\xi(T, t)$ denotes the number of impulsive times of the impulsive sequence ξ on the interval $[t, T)$. The idea behind it is that there may exist some consecutive impulse signals separated by less than or greater than τ_α , but the average interval between consecutive impulse signals is τ_α .

For most impulsive signals, the occurrence of impulses is not uniformly distributed. Fig.3.3 presents a specific form of a non-uniformly distributed impulsive sequence. One may observe from Fig. 3.3 that the impulses seldom occur in some time intervals, but frequently occur in some other intervals. For such impulsive signals, it is possible that the lower bound of the impulsive intervals is small or the upper bound is quite large. Hence, many previous results cannot be effectively applied to dynamical systems with the impulsive signal shown in Fig. 3.3.

In addition, note that inequality (5) can be rewritten as follows:

$$N_\xi(T, t) \geq \frac{T-t}{\tau_\alpha} - N_0, \quad \forall T \geq t \geq 0. \quad (6)$$

$$N_\xi(T, t) \leq \frac{T-t}{\tau_\alpha} + N_0, \quad \forall T \geq t \geq 0. \quad (7)$$

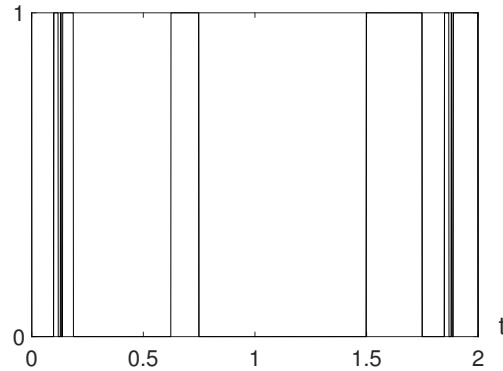


Fig.3.3 Impulsive sequence.

When the original system without impulsive perturbation is stable, and the impulsive effects are harmful, in order to guarantee the stability, the impulses should not occur frequently. That is, there should always exist a requirement that $t_k - t_{k-1} > h_1$, where the quantity h_1 can be regarded as a measure to ensure that the harmful impulses do not occur too frequently. Thus, condition (7) enforces an upper bound on the number of impulses. Conversely, when the original system without impulsive effects is unstable and the impulsive effects are beneficial, in order to ensure the stability of the system, it is usually assumed that the frequency of impulses should not be too low. Therefore, there should always exist a requirement that $t_k - t_{k-1} < h_2$, where the quantity h_2 is chosen to guarantee that there will be no overly long impulsive intervals. Thus condition (6) enforces a lower bound on the number of impulses.

Hence, the concept of AII is suitable for characterizing a wide range of impulsive signals. In recent years, there are many results using the concept of AII [110, 7, 71, 68, 32]. For instance, the global exponential synchronization of delayed complex dynamical networks with nonidentical nodes and stochastic perturbations was studied in [110]. By combining adaptive control and impulsive control schemes, the considered network can be synchronized onto any given goal dynamics. With respect to impulsive control, the concept named AII with “elasticity number” of impulsive sequence is utilized to get less conservative synchronization criterion. [7] investigated the problems of impulsive stabilization and impulsive synchronization of discrete-time delayed neural networks (DDNNs), where two types of DDNNs with stabilizing impulses were studied. [71] established finite time stability (FTS) criteria for the nonlinear impulsive systems, where by using AII method, less conservative conditions were obtained for the FTS problem on the impulsive systems.

4. Impulsive stabilization on IDSs. Impulsive control is to change the state of a system by discontinuous control input at certain time instances. From the control point of view, impulsive control is of distinctive advantage, since control gains are only needed at discrete instances. Thus, there are many interesting results on impulsive stabilization of IDSs. Considering the IDSs (1), authors in [63] established some criteria on uniform asymptotic stability by using Lyapunov functions and Razumikhin techniques, which is given as follows:

Theorem 4.1. [63]. Assume that there exist functions $a, b, c \in \mathbb{K}$, $p \in PC(\mathbb{R}_+, \mathbb{R}_+)$, $g, \hat{g} \in \mathbb{K}$, and $V \in v_0$, such that the following conditions hold:

- (i) $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$, for all $(t, x) \in [t_0 - \tau, +\infty) \times \mathbb{R}^n$;
- (ii) $D^+V(t, \psi(0)) \leq p(t)c(V(t, \psi(0)))$, for all $t \neq t_k$ in \mathbb{R}_+ and $\psi \in PC_\tau$ whenever $V(t, \psi(0)) \geq g(V(t+s, \psi(s)))$ for $s \in [-\tau, 0]$;
- (iii) $V(t_k, \psi(0) + I_k(t_k, \psi)) \leq g(V(t_k^-, \psi(0)))$, for all $(t_k, \psi) \in \mathbb{R}_+ \times PC_\tau$ for which $\psi(0^-) = \psi(0)$;

- (iv) $\tau_0 = \sup_{k \in \mathbb{Z}} \{t_k - t_{k-1}\} < \infty$, $M_1 = \sup_{t \geq 0} \int_t^{t+\tau} p(s)ds < \infty$, $M_2 = \inf_{q>0} \int_{g(q)}^q \frac{ds}{c(s)} > M_1$.

Then the trivial solution of (1) is uniformly asymptotically stable.

It follows from the definitions M_1 and M_2 in (iv) that the value of M_2 is greater than M_1 , which indicates that there may be some increasing on V between impulses, but the decreasing on V at impulses are needed to guarantee the stability. Notice from (iv) that there is a restriction on upper bound of impulse intervals due to the effect of stabilizing impulses. In other words, the time interval without impulsive control cannot be too long. Similar results on stability without time delay can be found in [62].

Based on the ideas given in [63], authors in [105] and [103] further investigated exponential stability and global exponential stability of solutions for IDSs (1), respectively, which play important effects on exponential stability analysis of impulsive time-delay systems.

Theorem 4.2. [103]. Assume that there exist a function $V \in v_0$ and constants $p, c, c_1, c_2 > 0$ and $\alpha > \tau$, $\lambda > c$ such that

- (i) $c_1\|x\|^p \leq V(t, x) \leq c_2\|x\|^p$, for any $t \in \mathbb{R}_+$ $x \in \mathbb{R}^n$;
- (ii) $D^+V(t, \psi(0)) \leq cV(t, \psi(0))$, for all $t \in [t_{k-1}, t_k], k \in \mathbb{N}$, whenever $qV(t, \psi(0)) \geq V(t+s, \psi(s))$ for $s \in [-\tau, 0]$, where $q > e^{2\lambda\alpha}$ is a constant;
- (iii) $V(t_k, \psi(0) + I_k(t_k, \psi)) \leq d_k V(t_k^-, \psi(0))$, where $d_k > 0$, $\forall k \in \mathbb{N}$, are constants;
- (iv) $\tau \leq t_k - t_{k-1} \leq \alpha$ and $\ln(d_k) + \lambda\alpha < -\lambda(t_{k+1} - t_k)$.

Then the trivial solution of (1) is globally exponentially stable and the convergence rate is $\frac{\lambda}{p}$.

For all the above studies [63, 105, 103], authors have investigated for the uniform asymptotical stability and global exponential stability of IDSs under the assumption that $\tau < t_k - t_{k-1}$, where τ is the finite delay or some positive constant. From this point of view, author in [36] studied the globally exponential stabilization of impulsive functional differential equations with infinite delays or finite delays by using Lyapunov functions and improved Razumikhin technique, where there is no any restriction on the lower bound of the impulse interval. Hence, the obtained results in [36] have wider applications.

Consider the following impulsive functional differential equations:

$$\begin{cases} \dot{x}(t) = F(t, x(\cdot)), & t > t_0, t \neq t_k, \\ \Delta x = I_k(t, x(t^-)), & t = t_k, k \in \mathbb{N}, \\ x_{t_0} = \phi(s), & \alpha \leq s \leq 0. \end{cases} \quad (8)$$

Some detailed information can be found in [36]; here we omit it.

Theorem 4.3. [36]. Assume that there exist a function $V \in v_0$ and constants $p > 0$, $q > 1$, $c_1 > 0$, $c_2 > 0$, $m > 0$ and $\gamma > 0$ such that

- (i) $c_1 \|x\|^m \leq V(t, x) \leq c_2 \|x\|^m$, for any $(t, x) \in [0, +\infty) \times \mathbb{R}^n$;
- (ii) For any $t_0 \in \mathbb{R}_+$ and $\psi \in PC_\tau$, if $e^{\gamma\theta} V(t+\theta, \psi(\theta)) \leq qV(t, \psi(0))$, $\theta \in [\alpha, 0]$, $t \neq t_k$, then $D^+V(t, \psi(0)) \leq pV(t, \psi(0))$;
- (iii) For all $(t_k, \psi) \in \mathbb{R}_+ \times PC_\tau$, $V(t_k, \psi(0) + I_k(t_k, \psi)) \leq 1/qV(t_k^-, \psi(0))$;
- (iv) $\sup_{k \in \mathbb{Z}_+} t_k - t_{k-1} < \ln q/p$.

Then the trivial solution of (8) is globally exponentially stable.

It should be noted that in [36], for $t \geq t_0 \geq 0 > \alpha \geq -\infty$, $f(t, x_t)$ in IDSs (1) was replaced by $f(t, x(s))$, where $s \in [t + \alpha, t]$ or $f(t, x(\cdot))$ be a Volterra type functional. In [36], let $\alpha = -\infty$. Moreover, it follows from [36] that Theorem 4.3 can be applied to systems with finite or/and infinite delays since $\alpha \in [-\infty, 0]$. Based on this point, [49] considered the existence, uniqueness, and global stability of periodic solutions for a class of recurrent neural networks with discrete and continuously distributed delays. By using contraction mapping theorem, some new sufficient conditions ensuring the existence, uniqueness, and global stability of periodic solutions were obtained. As we know, since the periodic oscillations can be presented by systems model with periodic coefficients, the properties of periodic solution are very important to study the dynamical behaviors of systems. With the development of impulsive control theory, it is known that an equilibrium point can be viewed as a special periodic solution of impulsive control systems with arbitrary period. Hence, the analysis of periodic solutions of impulsive control systems is more general than that of an equilibrium point.

Hence, investigation of periodic solution for system is indispensable for practical design and engineering applications of models. The periodic solution problem of IDSs has found many applications such as associative memories, pattern recognition, machine learning, robot motion control, and so on [76, 97, 82, 27]. For example, in [49], a class of recurrent neural networks with discrete and continuously distributed delays was considered. Sufficient conditions for the existence, uniqueness, and global exponential stability of a periodic solution were obtained by using contraction mapping theorem and stability theory on impulsive functional differential equations. [38] dealt with the periodic solutions problem for impulsive differential equations. By using Lyapunov's second method and the contraction mapping principle, some conditions ensuring the existence and global attractiveness of unique periodic solutions were derived, and so on [37, 84, 101, 26].

In addition, the stability analysis is much more complicated because of the existence of impulsive effects and stochastic effects at the same time. In [12], based on the Razumikhin techniques and Lyapunov functions, several criteria on the global exponential stability and instability of impulsive stochastic functional differential systems were obtained. The results show that impulses make contribution to the exponential stability of stochastic differential systems with any time delay even they are originally unstable. In [77], authors investigated the p th moment and almost sure exponential stability of impulsive stochastic functional differential equations with finite delay by using Lyapunov method. The obtained results do not need the strong condition of impulsive gain $|d_k| < 1$, which is more general than those given in [12]. By using Lyapunov functions method [77] and stochastic analysis approaches, stability theorems were derived in terms of linear matrix inequality (LMI), which can overcome the effect of time-delay and impulses. Those results can guarantee

the neural networks to be robustly exponentially stable in the mean square. [78] studied robust stability for a class of uncertain stochastic neural networks, where the ‘uncertainties’ mean the uncertain parameters, which take values in some intervals. [6] was devoted to prove some sufficient conditions ensuring practical p th moment exponential and almost sure exponential stability of solutions to impulsive stochastic functional differential equations.

In what follows, an example is given to illustrate the existence and attractiveness of the periodic solution for impulsive control system.

Example 4.4. [49]. Consider the following neural networks with discrete delays:

$$\left\{ \begin{array}{l} \dot{x}_1(t) = -\left[0.15 + 0.05 \sin \frac{2\pi}{\omega}t\right]x_1(t) + \sum_{j=1}^2 \left[0.3 - 0.01 \cos \frac{2\pi}{\omega}(t+j)\right] \\ \quad \times f_j(x_j(t) + \sum_{j=1}^2 \left[0.1 + 0.04 \sin \frac{2\pi}{\omega}(t+j)\right]f_j(x_j(t-1)) \\ \quad + \cos \frac{2\pi}{\omega}t, \quad t \neq t_k, \\ \dot{x}_2(t) = -\left[0.2 + 0.1 \sin \frac{2\pi}{\omega}t\right]x_2(t) + \sum_{j=1}^2 \left[0.17 + 0.02 \sin \frac{2\pi}{\omega}(t+j)\right] \\ \quad \times f_j(x_j(t) + \sum_{j=1}^2 \left[0.1 + 0.01 \sin \frac{2\pi}{\omega}(t+j)\right]f_j(x_j(t-1)) \\ \quad + \sin \frac{2\pi}{\omega}t, \quad t \neq t_k, \end{array} \right. \tag{9}$$

subject to impulses:

$$x_i(t_k) = \frac{1}{\rho}x_i(t_k^-), k \in \mathbb{Z}_+, i = 1, 2,$$

where $f_1 = f_2 = \tanh(x)$, $\omega > 0$ and $\rho > 1$ are some real constants.

In the simulations, one may observe that system (9) with $\omega = 2$ or 4 has no periodic solution which is globally exponentially stable when there is no impulsive effect, which can be seen from Fig.4. 2.(a-d). However, via the impulsive control strategies that we established, system (9) may admit a unique periodic solution which is globally exponentially stable. For instance, when $\omega = 2$ or 4 , one may choose $\rho = 1.5$ and $\mu = 0.5$ such that the conditions in [49] hold and so the periodic solution can be guaranteed. The corresponding numerical simulations are shown in Fig.4. 3.(a-d).

5. Impulsive perturbation on IDSs. In this case where a given equation without impulses is stable, and it can remain the stability behavior under certain impulsive interference, it is regarded as impulsive perturbation problem. Considering the IDSs (1), authors in [63] established a criteria on uniform asymptotic stability, where there exists impulsive perturbation.

Theorem 5.1. [63]. Assume that there exist functions $a, b, c \in \mathbb{K}, p \in PC(\mathbb{R}_+, \mathbb{R}_+)$, $g, \hat{g} \in \mathbb{K}$, and $V \in v_0$, such that the following conditions hold:

- (i) $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$, for all $(t, x) \in [t_0 - \tau, +\infty) \times \mathbb{R}^n$;
- (ii) $D^+V(t, \psi(0)) \leq -p(t)c(V(t, \psi(0)))$, for all $t \neq t_k$ in \mathbb{R}_+ and $\psi \in PC_\tau$ whenever

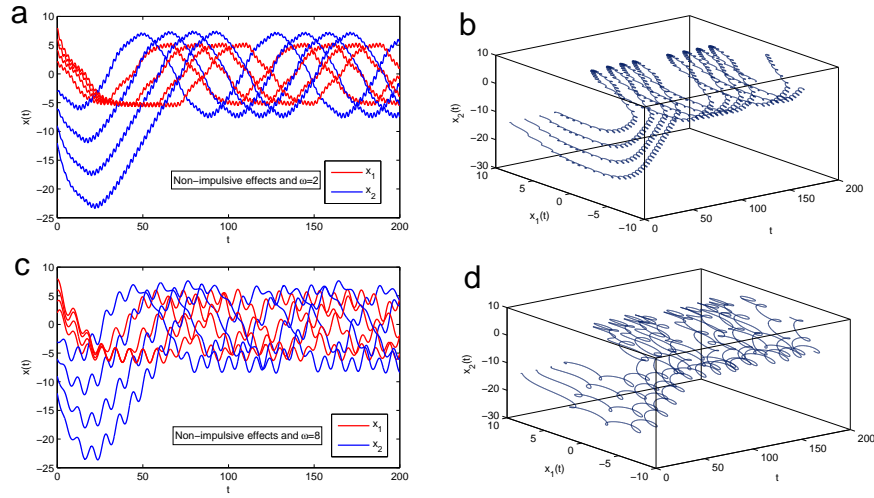


Fig.4.2. (a) State trajectories of system (9) with $\omega=2$ without impulsive control. (b) Phase portraits of system (9) with $\omega=2$ without impulsive control. (c) State trajectories of system (9) with $\omega=8$ without impulsive control. (d) Phase portraits of system (9) with $\omega=8$ without impulsive control.

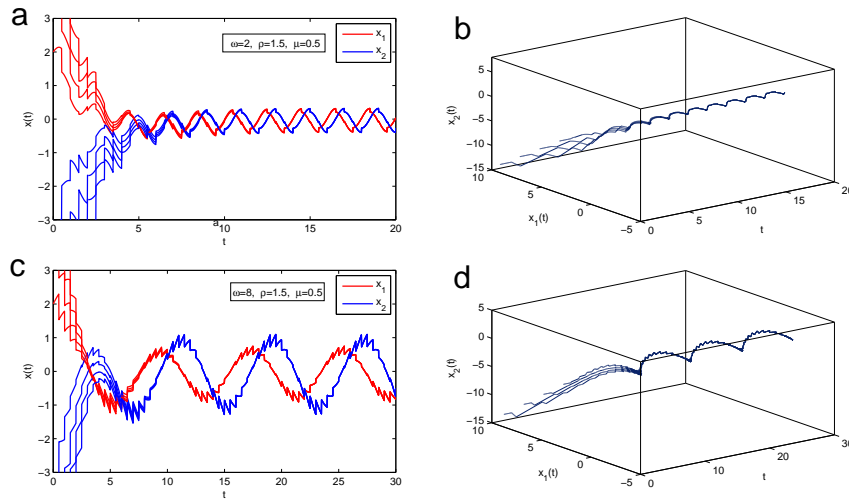


Fig.4.3. (a) State trajectory of system (9) with $\omega = 2, \rho = 1.5, \mu = 0.5$. (b) Phase portraits of system (9) with $\omega = 2, \rho = 1.5, \mu = 0.5$. (c) State trajectory of system (9) with $\omega = 8, \rho = 1.5, \mu = 0.5$. (d) Phase portraits of system (9) with $\omega = 8, \rho = 1.5, \mu = 0.5$.

$g(V(t, \psi(0))) \geq V(t + s, \psi(s))$ for $s \in [-\tau, 0]$;

(iii) $V(t_k, \psi(0) + I_k(t_k, \psi)) \leq \hat{g}(V(t_k^-, \psi(0)))$, for all $(t_k, \psi) \in \mathbb{R}_+ \times PC_\tau$ for which $\psi(0^-) = \psi(0)$;

(iv) $\mu = \inf_{k \in \mathbb{Z}} \{t_k - t_{k-1}\} > 0$, $M_2 = \sup_{q > 0} \int_q^{g(q)} (ds)/(c(s))$, $M_1 = \inf_{t \geq 0} \int_t^{t+\mu} p(s)ds > M_2$,

where $s < \hat{g}(s) < g(s)$. Then the trivial solution of (1) is uniformly asymptotically stable.

Theorem 5.1 is in some ways the opposite of Theorem 4.1. Here the derivative of V is always non-positive, which implies that it is nonincreasing along solutions between impulses. In the absence of impulses, the trivial solution of system (1) is uniformly asymptotically stable. Theorem 5.1 allows for significant increases in V , and correspondingly the solutions themselves, at impulse times but only as long as these are balanced sufficiently by the decrease of V between impulses. These techniques are based in part on earlier work in the study of boundedness properties of solutions of impulsive differential equations without delay (see [4, 5]). Years later, Liu et al. employed the method of Lyapunov functionals for the study of exponential stability of impulsive systems with time delay in [66]. Several exponential stability criteria were established as follows:

Theorem 5.2. [66]. Assume that hypotheses $(H_1) - (H_3)$ in [66] are satisfied and there exist $V_1 \in v_0$, $V_2 \in v_0^*(\cdot)$, $p_1, p_2 > 0$ with $p_1 \leq p_2$ and constants $\alpha, l, c, c_1, c_2, c_3 > 0, d_k \geq 0, k \in \mathbb{N}$, such that

(i) $c_1 \|x\|^{p_1} \leq V_1(t, x) \leq c_2 \|x\|^{p_1}, 0 \leq V_2(t, \psi) \leq c_3 \|\psi\|_{\tau}^{p_2}$, for any $t \in \mathbb{R}_+, x \in \mathbb{R}^n, \psi \in PC_{\tau}$;

(ii) $V_1(t_k, x + I_k(t_k, x)) \leq d_k V(t_k^-, x)$, where $x \in \mathbb{R}^n, k \in \mathbb{N}$;

(iii) for $V(t, \psi) = V_1(t, \psi(0)) + V_2(t, \psi)$, $D^+V(t, \psi) \leq cV(t, \psi)$, for $t \in [t_{k-1}, t_k), \psi \in PC_{\tau}, k \in \mathbb{N}$,

(iv) For any $k \in \mathbb{N}, \tau \leq t_k - t_{k-1} \leq l$, and $\ln(d_k + \frac{c_3}{c_1} e^{(\frac{p_2}{p_1} - 1)ckl}) \leq -(\alpha + c)l$.

Then the trivial solution of (1) is exponentially stable.

Based on the idea in [66], ref. [60] investigated input-to-state stability (ISS) and integral input-to-state stability (iISS) of impulsive and switching hybrid systems with time-delay, using the method of multiple Lyapunov-Krasovskii functionals. It is also shown that the results in the present paper can be applied to systems with arbitrarily large delays and, therefore, improve the results in [66]. In addition, [68] presented a new concept AII, which can be used to describe impulsive signals with a wider range of impulsive interval. Based on this concept, a unified synchronization criterion was obtained for impulsive directed dynamical networks with desynchronizing impulses or synchronizing impulses. However, there is no time delay in it. [114] addressed the stability problem of a class of delayed neural networks with time-varying impulses. Different from the results in Lu et al. [68], the impulses considered here are time-varying, which can cover the results in Lu et al. [68]. From the obtained results in [114], we can see that even if destabilizing impulsive effects occur frequently, the delayed neural networks can also be stable if stabilizing impulses can prevail over the influence of destabilizing impulsive effects.

To data, impulsive perturbation problem has been widely investigated [89, 91, 13, 113, 92, 40, 65, 58, 38, 43, 73, 67, 102, 90, 69, 70, 120]. For example, in [113], the global exponential stability of complex-valued impulsive systems was addressed. Some new sufficient conditions were obtained to guarantee the global exponential stability by the Lyapunov-Razumikhin theory. Authors in [92] studied the problem of impulsive effects on global exponential stability for a class of impulsive n -dimensional neural networks with unbounded delays and supremums. The robust exponential stability of nonlinear impulsive switched delayed systems was investigated in [67]. In [48], the stability problem of impulsive functional differential

equations (IFDEs) was considered. Several criteria ensuring the uniform stability of IFDEs with finite or infinite delay were derived by establishing some new Razumikhin conditions. authors investigated the stability problem of time-delay systems with persistent impulses and focused on the discussion of systems with unbounded time-varying delays in [40]. The above results have been discussed the stability for impulsive perturbation systems, where the impulsive perturbation exists in system but not destroy the stability property.

Sometimes, the impulsive perturbation can change the dynamics of a time-delay system. For example, a stable equilibrium may becomes a periodic attractor under impulsive perturbation. To show this observation, we illustrate the following example:

Example 5.3. Consider the 2D impulsive delayed neural networks

$$\begin{cases} \dot{x}(t) = -Ax(t) + Bf(x(t - \tau)), & t \geq t_0, \\ x(k) = Kx(t_k^-), & k \in \mathbb{N}, \end{cases} \quad (10)$$

where

$$A = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.9 \end{pmatrix}, \quad B = \begin{pmatrix} 0.5 & -1 \\ 1 & 0.4 \end{pmatrix}, \quad f(x(t - \tau)) = \tanh(x(t - \tau)), \quad \tau = 1.$$

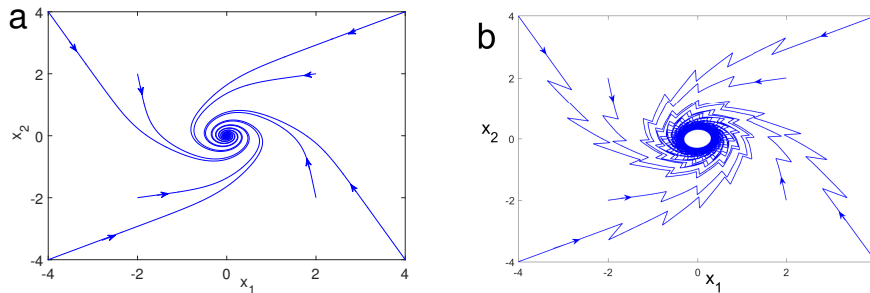


Fig.5.3 (a) State trajectories of (10) without impulse. (b) State trajectories of system (10) with impulsive perturbation.

One may observe that when there is no impulsive perturbation on the system (10), i.e., $K = I$, the system (10) admits a stable equilibrium, see 5.3 (a). When there is impulsive perturbation such as $K = 1.2I$ and $t_k = 0.8$, $k \in \mathbb{N}$, then the system (10) admits a periodic attractor, see Fig. 5.3 (b). Hence, such example shows that impulse can change the dynamics of time-delay systems and leads to complex dynamics.

6. Delayed impulse. Over the past few decades, many stability criteria for IDSs have been proposed. However, most existing results on IDSs do not consider the effect of delayed impulses. Of current interest is the delayed impulses of dynamical systems arising in such applications as automatic control, secure communication and population dynamics, [64, 29, 10, 8, 30]. Delayed impulse describes a phenomenon where impulsive transients depend on not only their current but also historical states of the system, see [16, 107, 11, 52, 51]. For instance, in communication security systems based on impulsive synchronization [8, 30], there exist transmission

and sampling delays during the information transmission process, where the sampling delay created from sampling the impulses at some discrete instances causes the impulsive transients depend on their historical states. Another example, in population dynamics such as fishing industry [52, 51, 99, 50], effective impulsive control such as harvesting and re-leasing can keep the balance of fishing, and the quantities of every impulsive harvesting or releasing are not only measured by the current numbers of fish but also depend on the numbers in recent history due to the fact that the immature fish need some time to grow.

In the previous literature concerning impulsive systems, the impulses are usually assumed to take the form:

$$\Delta x(t_k) = x(t_k) - x(t_k^-) = B_k x(t_k^-),$$

which indicates the state ‘jump’ at the impulse times t_k only depends on the current state. When considering the delay effect in impulse, the impulses are updated as follows:

$$\Delta x(t_k) = x(t_k) - x(t_k^-) = B_k x(t_k - \tau),$$

where τ is the possible time delay in impulses. It indicates that the state ‘jump’ at the impulse times t_k depends on not only the current state but also the state in history. Such impulses are regarded as a better way to model many practical problem. In fact, many practical systems can be modeled as differential systems with delayed impulses [64, 29, 10, 8, 30, 16, 107, 11, 52, 51, 99, 50, 53]. For example, in [64, 29], the stability problem of impulsive systems with stabilizing delayed impulses was studied. More exactly, the asymptotic stability in [29] was investigated for a class of delay-free autonomous systems with linear delayed impulses of the form

$$x(t) = x(t^-) + B_k x((t - d_k)^-), \quad t = t_k, \quad k \in \mathbb{N},$$

where $\{t_k\}$ is an impulsive time sequence and d_k are impulse input delays. A sufficient condition for asymptotic stability involving the sizes of impulse input delays was derived. [10] studied the problem of exponential stability of nonlinear time-delay systems with more general delayed impulses, which includes the linear delayed impulses of the form

$$x(t) = C_{0k} x(t^-) + C_{1k} x((t - d_k)^-), \quad t = t_k, \quad k \in \mathbb{N},$$

as a special case. The results in [10] dealt with both destabilizing delayed impulses and stabilizing delayed impulses, and derived the corresponding Lyapunov-type sufficient conditions for exponential stability. In the application of networked control systems, due to the finite speed of computation, a type of delayed impulses which are called sensor-to-controller delay and controller-to-actuator delay do exist in a working network [107, 11]. Authors in [52] studied the delayed impulsive control of nonlinear differential systems, where the impulsive control involves the delayed state of the system for which the delay is state-dependent. [51] focused on stability problem of nonlinear differential systems with impulses involving state-dependent delay based on Lyapunov methods. Some general and applicable results for uniform stability, uniform asymptotic stability and exponential stability of the systems were derived in [51] by using the impulsive control theory and some comparison arguments. It shows how restrictions on the change rates of states and impulses should be imposed to achieve systems stability, in comparison with general impulsive delay differential systems with state-dependent delay in the nonlinearity, or the differential systems with constant delays.

With the development of impulsive control theory, some recent works have focused on input-to-state stability (ISS) property of time-delay control system under the delayed impulsive control. For example, [116] addressed the ISS and integral input-to-state stability (iISS) of nonlinear systems with distributed delayed impulses. [53] studied the ISS property of nonlinear systems with delayed impulses and external input affecting both the continuous dynamics and the state impulse map. However, it seems that there have been few results that consider the effect of delayed impulses on ISS property for nonlinear systems, which still remains as an important direction in research fields.

7. Conclusion and future work. IDS is a very important research area with wide applications. Stability analysis is one of the fundamental problems for IDSs. This paper has overviewed the research area of IDSs with emphasis on the following topics:

(i) We described the general IDSs and presented the existence and uniqueness of solutions for IDSs. Moreover, we introduced the effects of impulses on stability for IDSs, which includes impulsive stabilization and impulsive perturbation. Examples were given to illustrate the effects of impulses. In addition, the concept of AII was introduced to characterize how often or how seldom impulses occur.

(ii) We presented sufficient conditions for IDSs, where the impulses contribute to system dynamics. In this sense, an example was illustrated to show the existence of periodic attractor under impulsive control, where there is originally no periodic attractor.

(iii) We presented sufficient conditions for IDSs, where the impulsive effects are harmful. In this sense, we illustrated an example to show the effects of impulsive perturbation for periodic solutions. It indicates that the dynamics of time-delay systems can be changed under impulsive perturbations.

(iv) We introduced the delayed impulses. Some interesting results on stability or ISS properties involving delayed impulse have been presented.

Although IDSs and their control theory have been developed for many years, there are still some shortcomings and problems to be solved:

(1) Many results on delayed impulses have been derived. However, most of them only considered the negative effect of time delay which exists in impulses. How to study the positive effect for such time delay is still a difficult problem.

(2) In recent years, the design and optimization of impulsive controller has been a hot and frontier problem for impulsive control theory, but there has been not much research progress in impulsive systems involving time delays. In particular, how to design an optimal impulsive controller under the constraints of engineering background is a key scientific problem that needs to be solved.

(3) Since impulsive control strategy usually has simple structure in which only discrete control are needed to achieve the desired performance, event-triggered impulsive control deserves increasing attention and some related control strategies have been proposed [93]. However, all the previous works have focused on the design of event-triggered impulsive control strategies for some specific systems, and there is no unified research method for general nonlinear systems, which leads to that the derived results have limitations in applications.

Acknowledgments. This work was supported by National Natural Science Foundation of China (11301308, 61673247, 11601269), and the Research Fund for Distinguished Young Scholars and Excellent Young Scholars of Shandong Province (JQ201719, ZR2016JL024). The paper has not been presented at any conference.

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Received August 03, 2018; revised August 19, 2018.

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