IMPACT OF BEHAVIORAL CHANGE ON THE EPIDEMIC CHARACTERISTICS OF AN EPIDEMIC MODEL WITHOUT VITAL DYNAMICS

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ABSTRACT. The epidemic characteristics of an epidemic model with behavioral change in [V. Capasso, G. Serio, A generalizaition of the Kermack-McKendrick deterministic epidemic model, Math. Bios., 42 (1978), 43-61] are investigated, including the epidemic size, peak and turning point. The conditions on the appearance of the peak state and turning point are represented clearly, and the expressions determining the corresponding time for the peak state and turning point are described explicitly. Moreover, the impact of behavioral change on the characteristics is discussed.

1. **Introduction.** Dynamical models for epidemic spread have made a great contribution to understanding transmission mechanism of the infection and controlling the spread. In 1927, Kermack and McKendrick [5] established the following simple SIR epidemic model to investigate the outbreak of the Great Plague lasting from 1665 to 1666 in London

$$
\begin{cases}
S' = -\beta S I, \\
I' = \beta S I - \alpha I, \\
R' = \alpha R,
\end{cases}
\tag{1}
$$

where the population is divided into three classes, susceptible (S), infective (I) and removed (R); $S(t), I(t)$ and $R(t)$ denote their numbers respectively at time t; β is the infection rate coefficient, and α is the removal rate coefficient. A threshold theorem of epidemic spread was found by Kermack and McKendrick [5] for system (1). Since then a lot of epidemic models are established based on model (1)[9, 2] and references therein.

In 1978, after a study of the cholera epidemic spread in Bari in 1973, Capasso and Serio [3] proposed a saturation incidence rate $\beta S I / (1 + \varepsilon I)$ to measure the inhibition effect due to behavioral change (e.g. reduction of contact rate, strengthening of protection measures, etc.) of the susceptible individuals when the number of

infective individuals increases, where ε is referred to as *inhibition parameter* reflecting the intensity of behavioral change of susceptible individuals during the disease spread. The replacement of the bilinear incidence in model (1) with the saturation incidence yields the following model with behavioral change

$$\begin{cases} S' = -\frac{\beta SI}{1+\varepsilon I}, \\ I' = \frac{\beta SI}{1+\varepsilon I} - \alpha I, \\ R' = \alpha R, \end{cases} \tag{2}$$

For model (1), the *epidemic final size* (i.e. the number of individuals who are infected over the course of the epidemic) can be determined easily by dividing the first two equations of the model and then integrating it[1, 6, 8, 10], and the *epidemic peak* (i.e. the largest number of real-time infected individuals in the population (not cumulative cases)) can also be found directly from the first integral and the second equation of model (1) [6].

In [3], Capasso and Serio compared the two models (1) and (2) in a qualitative way, and extended the threshold theorem for model (1) by replacing the threshold line of model (1) with the threshold curve of model (2). But with respect to the epidemic final size and peak state of model (2), there is not an investigation in detail. Especially, the role of the inhibition parameter ε for the epidemic characteristics has not been discussed.

During an epidemic outbreak, for the local public health department to control the spread of the disease, while concerning about the peak state of disease spread and the epidemic final size, the *turning point* and the associated state of population are also the important characteristics that need to be paid attention too. The turning point denotes the time at which the rate of cumulative cases changes from increasing to decreasing or vice versa [4]. Recently, we theoretically investigated the epidemic characteristics including the epidemic final size, the peak and the turning point of some simple epidemic models without vita dynamics including model (1), and analyzed the dependence of the related quantities on the initial conditions [6, 7, 11]. However, in the preceding models considered by us, no behavioral intervention was involved. In this paper, our aim is to investigate the impact of behavioral change on the epidemic characteristics for model (2). Based on some fundamental and elegant mathematical deductions, the dependence of the epidemic characteristics on the initial condition and the inhibition parameter is established.

2. **Formulation of model and preliminary.** Since the variable R does not appear in the first two equations of (2), and the system (called as SI model) consisting of the first two equations of (2) can determine its dynamics and epidemic characteristics, we consider the reduced model

$$\begin{cases} S' = -\frac{\beta SI}{1+\varepsilon I}, \\ I' = \frac{\beta SI}{1+\varepsilon I} - \alpha I \end{cases} \tag{3}$$

with the initial condition $S(0) = S_0 > 0$ and $I(0) = I_0 > 0$, where α may represent the sum of the recovery and disease-induced death rates.

Obviously, the initial condition implies that $S(t) > 0$ and $I(t) > 0$ for $t > 0$. Moreover, $S(t)$ is decreasing since $S'(t) < 0$ for $t > 0$. Thus $S(t) \le S_0$ for $t \ge 0$.

From model (3) we have

$$\frac{dI}{dS} = \frac{\alpha(1 + \varepsilon I) - \beta S}{\beta S}. \tag{4}$$

Correspondingly, $I|_{S=S_0} = I_0$. Then equation (4) with condition $I|_{S=S_0} = I_0$ has the following solution

$$I = I_0 \frac{S}{S_0} + \frac{\alpha}{\beta}\left(\frac{S}{S_0} - 1\right) - S \ln \frac{S}{S_0} \quad \text{for} \quad \beta = \alpha\varepsilon, \tag{5}$$

and

$$I = I_0 \left(\frac{S}{S_0}\right)^{\frac{\alpha\varepsilon}{\beta}} - \frac{1}{\varepsilon}\left[1 - \left(\frac{S}{S_0}\right)^{\frac{\alpha\varepsilon}{\beta}}\right] - \frac{\beta S}{\beta - \alpha\varepsilon}\left[1 - \left(\frac{S}{S_0}\right)^{\frac{\alpha\varepsilon}{\beta} - 1}\right] \quad \text{for} \quad \beta \neq \alpha\varepsilon. \tag{6}$$

For simplicity, we denote $x = S/S_0$ and $\sigma = \alpha\varepsilon/\beta$. Here $x = x(t) \in (0, 1]$ for $t > 0$ since $S(t) \leq S_0$ for $t \geq 0$.

Further, for $\sigma = 1$ and $\sigma \neq 1$, (5) and (6) can be rewritten as

$$I = \frac{1}{\varepsilon}\left[(\varepsilon I_0 + 1)x - 1 - \varepsilon S_0 x \ln x\right] =: \frac{1}{\varepsilon}f_1(x), \tag{7}$$

and

$$I = \frac{1}{\varepsilon}\left[\left(\varepsilon I_0 + 1 + \frac{\varepsilon S_0}{1 - \sigma}\right)x^\sigma - 1 - \frac{\varepsilon S_0 x}{1 - \sigma}\right] =: \frac{1}{\varepsilon}f_2(x), \tag{8}$$

respectively. In particular, when $\sigma = 1 + \varepsilon S_0/(1 + \varepsilon I_0)$, function $f_2(x)$ becomes $f_0(x) = (1 + \varepsilon I_0)x - 1$.

Additionally, we state the following lemmas and inequalities, which will be used in our later inferences.

Lemma 2.1. *For function $g(x) \in C^2[a, b]$ with $g''(x) \leq 0$ (≥ 0), there is a unique zero point in (a, b) if $g(a)g(b) < 0$; it is always positive (negative) in $[a, b]$ if $g''(x) \leq 0$ (≥ 0) and both $g(a)$ and $g(b)$ are positive (negative).*

Lemma 2.2. *For function $g(x) \in C^1[a, b]$, if there is at most one local extreme point, then there is a unique zero of function $g(x)$ in (a, b) if and only if $g(a)g(b) < 0$.*

Lemma 2.3. *The inequality*

$$\Phi(u) = u - 1 - \ln u \geq 0$$

for $u > 0$, and the equality holds if and only if $u = 1$.

Lemma 2.4. *For any positive number σ with $\sigma \neq 1$, the inequality*

$$\Psi(u) = u^\sigma + \frac{\sigma u^{\sigma - 1}}{1 - \sigma} - \frac{1}{1 - \sigma} \geq 0$$

for $u > 0$, and the equality holds if and only if $u = 1$.

It is easy to prove the above lemmas and inequalities by applying the fundamental knowledge of differential calculus, so we omit them.

3. **Analysis of epidemic characteristics.** In this section, we analyze the epidemic characteristics of SI model (3) including the final state, the peak state and the turning point by means of Lemmas and the relation between variable S and I obtained in Section 2.

3.1. **Epidemic final state.** Epidemic final size is the number of the cumulative cases. According to the character of SI model (3), there is no reinfection for the model. Then the size can be obtained by subtracting the number of the individuals, who have not been infected when the spread of the disease terminates, from the initial number of susceptible individuals. The termination of infection is indicated by the fact that there is no infected individuals.

In what follows, we will determine the final size by analyzing the zeros of $f_i(x)$ $(i = 0, 1, 2)$ on $(0, 1]$. It is easy to know that

$$\lim_{x \to 0^+} f_1(x) = f_2(0) = -1,$$

and

$$f_1(1) = f_2(1) = \varepsilon I_0.$$

Moreover, the direct calculation gives

$$f_1''(x) = -\frac{\varepsilon S_0}{x}, \qquad f_2''(x) = \sigma(\sigma - 1)\left(\varepsilon I_0 + 1 + \frac{\varepsilon S_0}{1 - \sigma}\right) x^{\sigma - 2}.$$

So the signs of $f_1''(x)$ and $f_2''(x)$ are unchanged for $\sigma \neq 1 + \varepsilon S_0/(1 + \varepsilon I_0)$. Then, according to Lemma 2.1, when $\sigma \neq 1 + \varepsilon S_0/(1 + \varepsilon I_0)$, both $f_1(x)$ and $f_2(x)$ have a unique zero in the interval $(0, 1)$, denoted by x_∞. Further, we know that $f_i(x) < 0$ for $0 < x < x_\infty$ and $f_i(x) > 0$ for $x_\infty < x < 1$ $(i = 1, 2)$.

For $\sigma = 1 + \varepsilon S_0/(1 + \varepsilon I_0)$, it is obvious that $f_0(x)$ has a unique positive zero, $1/(1 + \varepsilon I_0)$, denoted by $x_{0\infty}$.

The above inference implies that x_∞ or $x_{0\infty}$ represents the fraction of the susceptible individuals, who have not been infected when the spread of the disease terminates, and that the feasible region of the variable x is $(x_\infty, 1]$ for $\sigma \neq 1 + \varepsilon S_0/(1 + \varepsilon I_0)$, and $(x_{0\infty}, 1]$ for $\sigma = 1 + \varepsilon S_0/(1 + \varepsilon I_0)$. Therefore, when the infection terminates, the number of susceptible individuals is $S_0 x_\infty$ for $\sigma \neq 1 + \varepsilon S_0/(1 + \varepsilon I_0)$, and $S_0/(1 + \varepsilon I_0)$ for $\sigma = 1 + \varepsilon S_0/(1 + \varepsilon I_0)$. Correspondingly, the epidemic final size is $S_0(1 - x_\infty)$ for $\sigma \neq 1 + \varepsilon S_0/(1 + \varepsilon I_0)$, and $\varepsilon S_0 I_0/(1 + \varepsilon I_0)$ for $\sigma = 1 + \varepsilon S_0/(1 + \varepsilon I_0)$.

Remark 1. Although we have introduced how to determine the epidemic state, the related expressions are not formulated explicitly since the equations determining them could be a transcendental one. Then it is necessary to turn to mathematical softwares for finding them.

Additionally, with respect to x_∞, we have the following statement which will be used later.

Proposition 1. *When $\sigma = 1$, $x_\infty < 1/(\varepsilon S_0)$. When $\sigma \neq 1$ and $\sigma \neq 1 + \varepsilon S_0/(1 + \varepsilon I_0)$, $x_\infty < \sigma/(\varepsilon S_0)$.*

Proof. First, substituting $x = 1/(\varepsilon S_0)$ into function $f_1(x)$ yields

$$f_1\left(\frac{1}{\varepsilon S_0}\right) = \frac{I_0}{S_0} + \Phi\left(\frac{1}{\varepsilon S_0}\right).$$

From Lemma 2.3 it follows that $f_1(1/(\varepsilon S_0)) > 0$. By the property of function $f_1(x)$ we know that $x_\infty < 1/(\varepsilon S_0)$.

Next the substitution of $x = \sigma/(\varepsilon S_0)$ into function $f_2(x)$ yields

$$f_2\left(\frac{\sigma}{\varepsilon S_0}\right) = \varepsilon I_0 \left(\frac{\sigma}{\varepsilon S_0}\right)^\sigma + \Psi\left(\frac{\sigma}{\varepsilon S_0}\right).$$

By Lemma 2.4 we know that $f_2\left(\sigma/(\varepsilon S_0)\right) > 0$. Further, the property of function $f_2(x)$ implies that $x_\infty < \sigma/(\varepsilon S_0)$.

This completes the proof of Proposition 1. □

3.2. **Peak state.** The peak state corresponds to time at which the number of infected individuals attains the maximum. It can be found by determining the state at which $I'(t) = 0$. That is, the peak state satisfies the equation

$$\frac{\beta S}{1 + \varepsilon I} = \alpha. \tag{9}$$

Note that $\sigma = \alpha\varepsilon/\beta$. Then (9) can become

$$\varepsilon I + 1 - \frac{\varepsilon}{\sigma} S = 0. \tag{10}$$

In Section 2, we have expressed I by a function of x. Then, for the cases $\sigma = 1 + \varepsilon S_0/(1 + \varepsilon I_0)$, $\sigma = 1$ and $\sigma \neq 1, 1 + \varepsilon S_0/(1 + \varepsilon I_0)$, substituting $S = S_0 x$ and $I = f_i(x)$ $(i = 0, 1, 2)$ into the left hand side of (10) gives $g_i(x) = 0$, where

$$g_0(x) = -\frac{\varepsilon S_0 x}{\sigma(1-\sigma)},$$

$$g_1(x) = x(1 + \varepsilon I_0 - \varepsilon S_0 - \varepsilon S_0 \ln x) =: x\bar{g}_1(x),$$

$$g_2(x) = x\left[\left(\varepsilon I_0 + 1 + \frac{\varepsilon S_0}{1-\sigma}\right)x^{\sigma-1} - \frac{\varepsilon S_0}{\sigma(1-\sigma)}\right] =: x\bar{g}_2(x).$$

Thus, the zero of functions $g_i(x)$ $(i = 0, 1, 2)$ corresponds to the peak state of the associated cases.

Obviously, $g_0(x) > 0$ for $x \in (x_{0\infty}, 1)$. Then there is no peak state as $\sigma = 1 + \varepsilon S_0/(1 + \varepsilon I_0)$.

For $\sigma = 1$, from $f_1(x_\infty) = 0$, i.e. $1 + \varepsilon I_0 - \varepsilon S_0 \ln x_\infty = 1/x_\infty$, it follows that $\bar{g}_1(x_\infty) = (1 - \varepsilon S_0 x_\infty)/x_\infty$. From Proposition 1 we have $\bar{g}_1(x_\infty) > 0$. Then, according to the monotonicity of $\bar{g}_1(x)$, there is a unique zero of $\bar{g}_1(x)$ (i.e. $g_1(x)$), x_{1p}, in the interval $(x_\infty, 1)$ if and only if $g_1(1) = 1 + \varepsilon I_0 - \varepsilon S_0 < 0$, i.e. $\varepsilon S_0/(1 + \varepsilon I_0) > 1$, where

$$x_{1p} = e^{(1+\varepsilon I_0 - \varepsilon S_0)/(\varepsilon S_0)}.$$

For function $\bar{g}_2(x)$ with $\sigma \neq 1, 1 + \varepsilon S_0/(1 + \varepsilon I_0)$, applying $f_2(x_\infty) = 0$ gives $\bar{g}_2(x_\infty) = 1/x_\infty - \varepsilon S_0/\sigma$. From Proposition 1 we know that $\bar{g}_2(x_\infty) > 0$. It is evident that $\bar{g}_2(x)$ is monotone, then $\bar{g}_2(x)$ (i.e. $g_2(x)$) has a unique zero x_{2p} in the interval $(x_\infty, 1)$ if and only if $\bar{g}_2(1) = 1 + \varepsilon I_0 - \varepsilon S_0/\sigma < 0$, i.e., $\sigma < \varepsilon S_0/(1 + \varepsilon I_0)$, where

$$x_{2p} = \left\{\frac{\varepsilon S_0}{\sigma[(1-\sigma)(\varepsilon I_0 + 1) + \varepsilon S_0]}\right\}^{1/(\sigma-1)}.$$

Further, substituting $x = x_{ip}$ $(i = 1, 2)$ into $S = S_0 x$ and $I = f_i(x)/\varepsilon$ $(i = 1, 2)$, we obtain the peak state (S_{1p}, I_{1p}) for $\sigma = 1$, and (S_{2p}, I_{2p}) for $\sigma \neq 1, 1 + \varepsilon S_0/(1 + \varepsilon I_0)$, respectively, where

$$S_{1p} = S_0 e^{\frac{1+\varepsilon I_0 - \varepsilon S_0}{\varepsilon S_0}}, I_{1p} = S_0 e^{\frac{1+\varepsilon I_0 - \varepsilon S_0}{\varepsilon S_0}} - \frac{1}{\varepsilon};$$

$$S_{2p} = S_0 \left\{\frac{\varepsilon S_0}{\sigma[(1-\sigma)(\varepsilon I_0 + 1) + \varepsilon S_0]}\right\}^{\frac{1}{\sigma-1}}, I_{2p} = \frac{S_0}{\sigma}\left\{\frac{\varepsilon S_0}{\sigma[(1-\sigma)(\varepsilon I_0 + 1) + \varepsilon S_0]}\right\}^{\frac{1}{\sigma-1}} - \frac{1}{\varepsilon}.$$

Moreover, the time of the peak state can also be found by substituting $S = S_0 x$ and $I = f_i(x)/\varepsilon$ $(i = 1, 2)$ into the first equation of (3) and then integrating it. In

detail, we first substitute $S = S_0 x$ and $I = f_i(x)/\varepsilon$ $(i = 1, 2)$ into the first equation of (3) yields

$$\frac{dx}{dt} = -\frac{\beta x f_i(x)}{\varepsilon[1 + f_i(x)]}$$

with $x_\infty < x < 1$, that is,

$$dt = -\frac{\varepsilon[1 + f_i(x)]}{\beta x f_i(x)} dx. \tag{11}$$

Note that $x(0) = 1$ since $S(0) = S_0$. Denote the time of the peak state by t_p. Then t_p can be determined by integrating the right hand side of (11) from 1 to x_{ip}, that is,

$$t_p = -\int_1^{x_{ip}} \frac{\varepsilon[1 + f_i(x)]}{\beta x f_i(x)} dx = \frac{\varepsilon}{\beta} \left[\int_{x_{ip}}^1 \frac{dx}{x f_i(x)} - \ln x_{ip} \right].$$

3.3. **Turning point.** Let $C(t)$ represent the number of the cumulative cases at time t. Then the turning point of the infection spread corresponds to the inflection point of function $C(t)$, and $C'(t) = \beta SI/(1 + \varepsilon I)$ for model (3).

The expressions (5) and (6) show that the variable I can be expressed by the variable S. Then

$$C''(t) = \frac{d}{dS}\left(\frac{\beta SI}{1 + \varepsilon I} \right) \cdot \frac{dS}{dt} = \frac{\beta}{(1 + \varepsilon I)^2} \cdot \left[I(1 + \varepsilon I) + S\frac{dI}{dS} \right] \cdot \frac{dS}{dt}.$$

Since $\frac{dS}{dt} < 0$, the inflection point of function $C(t)$ is determined by equation

$$I(1 + \varepsilon I) + S\frac{dI}{dS} = 0. \tag{12}$$

Substituting $S = S_0 x$ and $I = f_i(x)/\varepsilon$ $(i = 0, 1, 2)$ into the left hand side of (12) gives

$$\begin{aligned}
h_0(x) =&\ \frac{(\varepsilon I_0 + 1)^2 x^2}{\varepsilon}, \\
h_1(x) =&\ \frac{1}{\varepsilon}\left[f_1(x)(1 + f_1(x)) + x f_1'(x) \right] \\
=&\ \frac{x^2}{\varepsilon}\left\{ [(\varepsilon I_0 + 1) - \varepsilon S_0 \ln x]^2 - \frac{\varepsilon S_0}{x} \right\} =: \frac{x^2}{\varepsilon} h_1^+(x) h_1^-(x), \\
h_2(x) =&\ \frac{1}{\varepsilon}\left[f_2(x)(1 + f_2(x)) + x f_2'(x) \right] =: \frac{x^2}{\varepsilon} \bar{h}_2(x),
\end{aligned}$$

respectively, where

$$\begin{aligned}
h_1^+(x) &= (\varepsilon I_0 + 1) - \varepsilon S_0 \ln x + \sqrt{\tfrac{\varepsilon S_0}{x}}, \\
h_1^-(x) &= (\varepsilon I_0 + 1) - \varepsilon S_0 \ln x - \sqrt{\tfrac{\varepsilon S_0}{x}}, \\
\bar{h}_2(x) &= \left[(\varepsilon I_0 + 1)x^{\sigma-1} - \tfrac{\varepsilon S_0(1 - x^{\sigma-1})}{1-\sigma} \right]^2 - \left[(1-\sigma)(\varepsilon I_0 + 1) + \varepsilon S_0 \right] x^{\sigma-2}.
\end{aligned}$$

Thus, the zeros of functions $h_i(x)$ $(i = 0, 1, 2)$ in $(x_\infty, 1)$ correspond to the inflection points of $C(t)$ (i.e. the turning point of the infection) under the cases $\sigma = 1 + \varepsilon S_0/(1 + \varepsilon I_0)$, $\sigma = 1$ and $\sigma \neq 1, 1 + \varepsilon S_0/(1 + \varepsilon I_0)$, respectively.

(1) Note that $h_0(x) > 0$. Then there is no inflection point of function $C(t)$ for $\sigma = 1 + \varepsilon S_0/(1 + \varepsilon I_0)$.

(2) When $\sigma = 1$, from $f_1(x) > 0$ for $x_\infty < x < 1$ it follows that $(\varepsilon I_0 + 1) - \varepsilon S_0 \ln x > 1/x > 0$ for $x_\infty < x < 1$. Then $h_1^+(x) > 0$ for $x_\infty \leq x < 1$.

From $f_1(x_\infty) = 0$ it follows that $h_1^-(x_\infty) = (1 - \sqrt{\varepsilon S_0 x_\infty})/x_\infty$. Further we know that $h_1^-(x_\infty) > 0$ by Proposition 1. Note that

$$h_1^{-\prime}(x) = \sqrt{\varepsilon S_0}\left(\frac{1}{2\sqrt{x}} - \sqrt{\varepsilon S_0}\right)\frac{1}{x}$$

implies that function $\bar{h}_1(x)$ has at most one local extreme point. Then from Lemma 2.2 it follows that there is one zero of $h_1^-(x)$ in $(x_\infty, 1)$ if and only if $h_1^-(1) = (\varepsilon I_0 + 1) - \sqrt{\varepsilon S_0} < 0$, i.e. $\varepsilon I_0 + 1 < \sqrt{\varepsilon S_0}$.

Therefore, when $\sigma = 1$, there is a unique inflection point of $C(t)$ for $x \in (x_\infty, 1)$ if and only if $\varepsilon I_0 + 1 < \sqrt{\varepsilon S_0}$.

(3) It is easy to see that function $\bar{h}_2(x) > 0$ (i.e. $h_2(x) > 0$) for $x_\infty < x < 1$ when $\sigma > 1 + \varepsilon S_0/(1 + \varepsilon I_0)$.

When $\sigma < 1 + \varepsilon S_0/(1 + \varepsilon I_0)$ and $\sigma \neq 1$, function $\bar{h}_2(x)$ can be expressed as $\bar{h}_2(x) = h_2^+(x)h_2^-(x)$, where

$$h_2^+(x) = \left[(\varepsilon I_0 + 1)x^{\sigma-1} - \frac{\varepsilon S_0(1-x^{\sigma-1})}{1-\sigma}\right] + \sqrt{[(1-\sigma)(\varepsilon I_0 + 1) + \varepsilon S_0]\,x^{\sigma-2}},$$

$$h_2^-(x) = \left[(\varepsilon I_0 + 1)x^{\sigma-1} - \frac{\varepsilon S_0(1-x^{\sigma-1})}{1-\sigma}\right] - \sqrt{[(1-\sigma)(\varepsilon I_0 + 1) + \varepsilon S_0]\,x^{\sigma-2}}.$$

From $f_2(x) > 0$ for $x_\infty < x < 1$ we know that

$$(\varepsilon I_0 + 1)x^{\sigma-1} - \frac{\varepsilon S_0(1 - x^{\sigma-1})}{1 - \sigma} > \frac{1}{x} > 0$$

for $x_\infty < x < 1$. Then $h_2^+(x) > 0$ for $x_\infty \leq x < 1$.

For function $h_2^-(x)$,

$$h_2^{-\prime}(x) = -\sqrt{(1-\sigma)(\varepsilon I_0 + 1) + \varepsilon S_0}\, x^{\frac{\sigma}{2}-2}$$
$$\times\left\{\sqrt{(1-\sigma)(\varepsilon I_0 + 1) + \varepsilon S_0}\, x^{\frac{\sigma}{2}} + \frac{\sigma-2}{2}\right\}.$$

Then function $h_2^-(x)$ has at most one local extreme point in $(x_\infty, 1)$.

On the other hand, applying $f_2(x_\infty) = 0$ gives $h_2(x_\infty) = (\sigma - \varepsilon S_0 x_\infty)/\varepsilon > 0$ by Proposition 1. Further we have $h_2^-(x_\infty) > 0$ since $h_2^+(x) > 0$ for $x_\infty \leq x < 1$.

Therefore, from Lemma 2.2 function $h_2^-(x)$ has one zero in $(x_\infty, 1)$ if and only if

$$h_2^-(1) = \frac{(\varepsilon I_0 + 1)(\varepsilon I_0 + \sigma) - \varepsilon S_0}{(\varepsilon I_0 + 1) + \sqrt{(1-\sigma)(\varepsilon I_0 + 1) + \varepsilon S_0}} < 0,$$

i.e., $\sigma < \varepsilon S_0/(1 + \varepsilon I_0) - \varepsilon I_0$. Note that $\varepsilon S_0/(1 + \varepsilon I_0) - \varepsilon I_0 < 1 + \varepsilon S_0/(1 + \varepsilon I_0)$. Hence, when $\sigma \neq 1$, $1 + \varepsilon S_0/(1 + \varepsilon I_0)$, function $h_2^-(x)$(i.e. $h_2(x)$) has one zero in $(x_\infty, 1)$ if and only if $\sigma < \varepsilon S_0/(1 + \varepsilon I_0) - \varepsilon I_0$. That is, when $\sigma \neq 1$, $1 + \varepsilon S_0/(1 + \varepsilon I_0)$, there is a unique inflection point of $C(t)$ for $x \in (x_\infty, 1)$ if and only if $(\varepsilon I_0 + 1)(\varepsilon I_0 + \sigma) < \varepsilon S_0$.

Summarizing the above inference, the conditions on the existence of the turning point can be unified as the expression $(\varepsilon I_0 + \sigma)(\varepsilon I_0 + 1) < \varepsilon S_0$.

Since functions $h_i(x)$ $(i = 1, 2)$ are transcendental functions, the value of x corresponding to the inflection point of $C(t)$ could not expressed explicitly. If it exists, denoted by x_t, then, similar to the determination of time corresponding to the peak state, the time of the turning point of epidemic spread can be found by the following expression

$$t_t = \frac{\varepsilon}{\beta}\left[\int_{x_t}^1 \frac{dx}{xf_i(x)} - \ln x_t\right],$$

and the corresponding state is (S_t, I_t), where $S_t = S_0 x_t$ and $I_t = f_i(x_t)/\varepsilon$ $(i = 1, 2.)$

4. **Conclusion and discussion.** In Section 3, we have discussed the epidemic characteristics of model (3) with behavioral change, obtained the condition on the existence of the peak state, $\sigma(1 + \varepsilon I_0) < \varepsilon S_0$, and the condition on the appearance of the turning point of epidemic spread, $(\varepsilon I_0 + \sigma)(1 + \varepsilon I_0) < \varepsilon S_0$, and provided the methods or expressions determining the associated quantities.

In order to make the dependence of the related conditions on the parameters and the initial conditions more clear, we replace the parameter σ with $\alpha \varepsilon / \beta$ in the associated expressions. Correspondingly, the condition on the existence of the peak state is that $\varepsilon < (\beta S_0/\alpha - 1)/I_0$, and the condition on the appearance of the turning point is that $\varepsilon < [\beta S_0/(\alpha + \beta I_0) - 1]/I_0$. Conversely, when $\varepsilon \geq (\beta S_0/\alpha - 1)/I_0$, the peak can not appear; when $\varepsilon \geq [\beta S_0/(\alpha + \beta I_0) - 1]/I_0$, there is no turning point. Note that $\beta S_0/(\alpha + \beta I_0) < \beta S_0/\alpha$. Then, when $\varepsilon < [\beta S_0/(\alpha + \beta I_0) - 1]/I_0$, both the peak and the turning point can appear; when $[\beta S_0/(\alpha + \beta I_0) - 1]/I_0 \leq \varepsilon < (\beta S_0/\alpha - 1)/I_0$, there is a peak but no turning point; if $\varepsilon \geq (\beta S_0/\alpha - 1)/I_0$, both the two characteristics could not exist. These statements show the impact of the behavioral change on the existence of the two characteristics. From another point of view, when the turning point can appear, there must be the peak state. But there may not be the turning point if the peak state exists. The inequalities above also provide the threshold condition on the appearance of the peak and/or the turning point. The results obtained here would be useful to make the effective control strategy for disease spread.

In order to visually show the impact of behavioral change on the peak state and turning point of disease spread, we plot a set of graphes (Figures 1, 2 and 3) of the curves of $I = I(t)$ and $C = C(t)$, denoting the real-time numbers of infected individuals and cumulative cases, respectively. Here, the chosen values of parameters and the initial conditions do not represent any real data. Corresponding to all the three graphes, parameters $\alpha = 2$ and $\beta = 0.2$, initial values $S(0) = 200$ and $I(0) = 2$, then

$$\frac{1}{I_0} \left(\frac{\beta S_0}{\alpha + \beta I_0} - 1 \right) = 7.8333, \qquad \frac{1}{I_0} \left(\frac{\beta S_0}{\alpha} - 1 \right) = 9.5000.$$

According to the obtained results, there are both the peak and the turning point if $\varepsilon < 7.8333$, there is the peak and no turning point if $7.8333 \leq \varepsilon < 9.8$, and there is neither peak and no turning point for $\varepsilon \geq 9.5$. These theoretic results are verified by Figures 1, 2 and 3. In Figure 1 for $\varepsilon = 0.08$, the peak $I = 75.53400$ achieves at $t = 0.57065$, and the turning point is $(0.16853, 37.07000)$. In Figure 2 for $\varepsilon = 8$, the peak is $I = 2.29549$ at $t = 1.36570$ and there is no turning point. For Figure 3 for $\varepsilon = 9.8$, $I = I(t)$ is decreasing, and $C = C(t)$ is convex upwards. That is, both the peak and the turning point do not appear.

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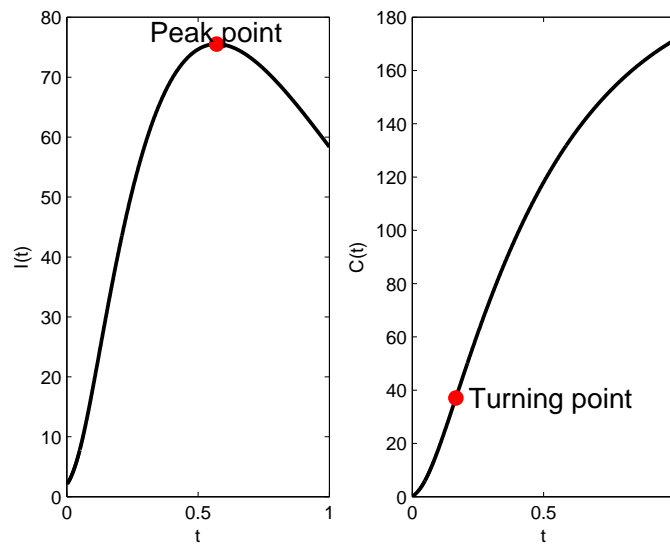
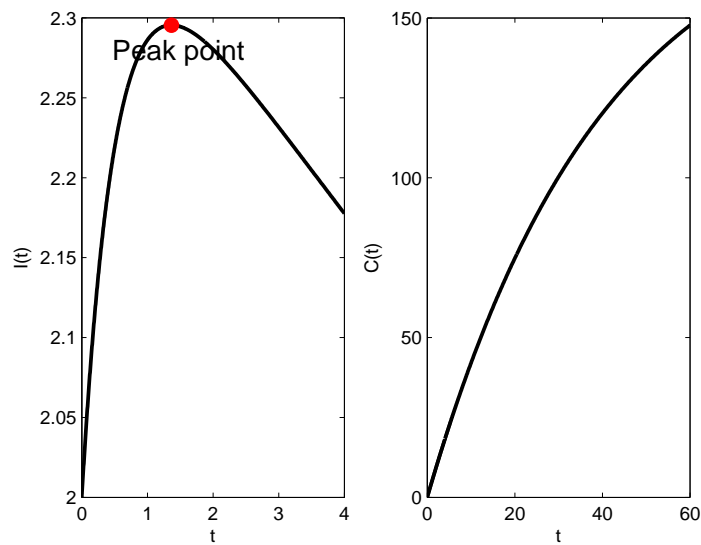FIGURE 1.   The case that both the peak and the turning point can appear.

FIGURE 2.   The case that there is the peak and no turning point.

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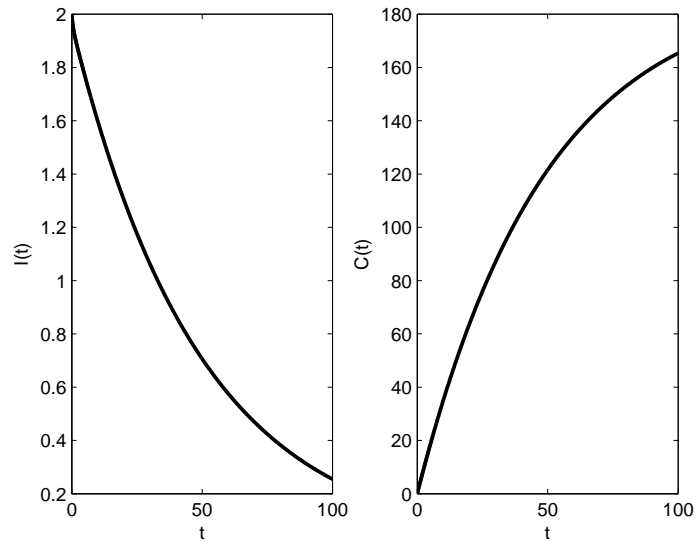
FIGURE 3. The case that both the peak and the turning point can not appear.

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