

## A STOCHASTIC MODEL FOR WATER-VEGETATION SYSTEMS AND THE EFFECT OF DECREASING PRECIPITATION ON SEMI-ARID ENVIRONMENTS

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(Communicated by Christopher M. Kribs)

**ABSTRACT.** Current climate change trends are affecting the magnitude and recurrence of extreme weather events. In particular, several semi-arid regions around the planet are confronting more intense and prolonged lack of precipitation, slowly transforming part of these regions into deserts in some cases. Although it is documented that a decreasing tendency in precipitation might induce earlier disappearance of vegetation, quantifying the relationship between decrease of precipitation and vegetation endurance remains a challenging task due to the inherent complexities involved in distinct scenarios. In this paper we present a model for precipitation-vegetation dynamics in semi-arid landscapes that can be used to explore numerically the impact of decreasing precipitation trends on appearance of desertification events. The model, a stochastic differential equation approximation derived from a Markov jump process, is used to generate extensive simulations that suggest a relationship between precipitation reduction and the desertification process, which might take several years in some instances.

**1. Introduction.** Relatively large areas in the western region of the United States are classified as arid or semi-arid environments, which are characterized in part by their limited and variable precipitation. Semi-arid regions are expected to receive around 10 to 30 inches of average annual precipitation ( $\approx 254\text{-}762 \text{ mm year}^{-1}$ ) but climate change is predicted to increase the intensity and frequency of droughts globally [24]. For instance, in August of 2016, abnormally dry to moderate drought

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2010 *Mathematics Subject Classification.* Primary: 92B05; Secondary: 60K30.

*Key words and phrases.* Desertification, water-vegetation model, precipitation, semi-arid environments, drylands.

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conditions were observed in several locations of the western region of the United States, with ranges between severe to extreme drought occurring in the northern portions, and severe to exceptional drought extending from California into Nevada [20]. The lack of precipitation puts indigenous species of plants and animals in semi-arid environments under unusual stress and the parallel habitat loss might pose a threat to local biodiversity [16]. Under these circumstances, having educated guesses of the potential vegetation biomass responses in semi-arid landscapes to long term changes in precipitation could serve to put forward the design of adaptation and conservation policies [9]. The estimation of the expected time of transition to a desert state (or bare-soil), as a conceivable measure of those responses, presents difficulties due to the complexities associated with specific water-vegetation systems. For instance, the inherent non-linear characteristics of semi-arid landscapes may trigger desertification in response to slow changes [23]. Results from simulations of simple mathematical models could still offer a hint of the relationship with the parameters that might be driving decline.

Extensive mathematical modeling and analysis of semi-arid water-vegetation systems emerged for at least the past twenty years, especially since the appearance of *deterministic* ecohydrological models with broad developments focused on vegetation pattern formation, see for instance [5, 12, 19, 22, 25, 26, 27]. The effects of noise on dryland ecosystems that are usually described by deterministic models showing bistability have also been analysed, [6, 21], suggesting the possibility of creation or disappearance of vegetated states in the form of noise-induced transitions, [11]. In contrast, the goals in this paper are to present a stochastic differential equation approximation for an idealized water-vegetation (non-spatial) discrete system and the estimation of mean transition times into a desert state. We emphasize that our model is *not* obtained by adding “noise” to a differential equation as previously done [6, 21], but by the construction of a diffusion approximation.

As a first step, we set up a Markov jump process that incorporates the interactions in an idealized water-vegetation system. Similar conceptual models have been used successfully in other biological contexts, see for instance [17] and [18, 15]. The model involves only water and vegetation biomass, in an environment of limited capacity. When this capacity (or “system size”) increases it gives rise to a deterministic system of differential equations for the mean densities. We deduce an intermediate *mesoscale* stochastic model between the jump process and the differential equations obtained for the means. Using estimated parameters for vegetation and precipitation in semi-arid landscapes from the literature, and data for state precipitation anomalies in California as baseline, we estimate the mean times for a system to reach desertification in a range of realistic precipitation anomalies, i.e. departures from long term mean. With these results we finally quantify, for this simple model, the dependence between changes in precipitation anomalies and mean transition times to the desert state.

## 2. Theoretical framework for water-vegetation dynamics.

**2.1. The stochastic and ODE models.** We start by defining a Markov jump process that represents a simplified version of the real interactions between water and vegetation at a small scale. We do this through the discretization of (alive) vegetation biomass and water volume in small units (individuals), for which a specific set of stochastic events can be explicitly characterized. Naturally, as we transition into larger scales discreteness is lost, and the continuous state space takes place.

Event	Transition	Jump	Jump rate
Vegetation biomass loss	$(n, m) \rightarrow (n - 1, m)$	$(-1, 0)$	$d$
Incoming water	$(n, m) \rightarrow (n, m + 1)$	$(0, 1)$	$s$
Water evaporation	$(n, m) \rightarrow (n, m - 1)$	$(0, -1)$	$v$
Increase vegetation by water take up	$(n, m) \rightarrow (n + 1, m - 1)$	$(1, -1)$	$b$

TABLE 1. Possible transition events with their associated jumps if the system is at state  $(n, m)$ , where  $n$  and  $m$  represent units of biomass and water, respectively.

We first consider a patch with finite capacity, say  $N$ , containing three types of individuals: (i) vegetation biomass units; (ii) water volume units; (iii) empty locations. The dynamics of plant biomass and water interactions is driven by events involving a few processes: vegetation biomass loss, incoming water, water evaporation, and increase vegetation yield by taking up water. If we represent the state of the system, i.e. the number of biomass units,  $n$ , and the number of water volume units,  $m$ , by the pair  $(n, m)$  then these events correspond to the transitions detailed in Table 1.

It is then straightforward to find the probability rates of transition from a state  $(n, m)$  to  $(n', m')$ ,  $T(n', m'|n, m)$ :

- (a)  $T(n + 1, m - 1|n, m) = b \frac{n}{N} \frac{m}{N-1}$
- (b)  $T(n - 1, m|n, m) = d \frac{n}{N}$
- (c)  $T(n, m + 1|n, m) = s \frac{N-n-m}{N}$
- (d)  $T(n, m - 1|n, m) = v \frac{m}{N}$

Using these rates we can write the associated Kolmogorov equation (see [7] for instance),

$$\begin{aligned} \frac{dP(n, m, t)}{dt} &= T(n, m|n - 1, m + 1)P(n - 1, m + 1, t) \\ &+ T(n, m|n + 1, m)P(n + 1, m, t) \\ &+ T(n, m|n, m + 1)P(n, m + 1, t) + T(n, m|n, m - 1)P(n, m - 1, t) \\ &- (T(n + 1, m - 1|n, m) + T(n - 1, m|n, m) + T(n, m - 1|n, m) \\ &+ T(n, m + 1|n, m)) P(n, m, t). \end{aligned}$$

where  $P(n, m, t)$  is the probability that the system is at the state  $(n, m)$  at time  $t$ . Imposing zero boundary conditions, multiplying the Kolmogorov equation by  $n$ , and summing over  $m$  and  $n$  gives the expression for the rate of change of the mean biomass,

$$\begin{aligned} \frac{d\langle n \rangle}{dt} &= \sum_{n,m=0}^N [T(n + 1, m - 1|n, m) - T(n - 1, m|n, m)] P(n, m, t) \\ &= b \frac{\langle n \rangle}{N} \frac{\langle m \rangle}{N - 1} - d \frac{\langle n \rangle}{N}, \end{aligned} \tag{1}$$

where the correlations between the random variables are neglected under the assumption of a large  $N$ . Writing the mean density of vegetation as  $\rho_v = \langle n \rangle / N$  finally gives

$$\frac{d\rho_v}{dt} = \tilde{b}\rho_v\rho_w - \tilde{d}\rho_v, \tag{2}$$

where  $\tilde{b} = b/(N - 1)$  and  $\tilde{d} = d/N$ . Similarly,

$$\begin{aligned} \frac{d\langle m \rangle}{dt} &= \sum_{n,m=0}^N [T(n, m + 1|n, m) - T(n, m - 1|n, m)] P(n, m, t) \\ &= s - s \frac{\langle n + m \rangle}{N} - b \frac{\langle n \rangle}{N} \frac{\langle m \rangle}{N - 1} - v \frac{\langle m \rangle}{N} \end{aligned} \quad (3)$$

Similarly, writing the mean density of water by  $\rho_w = \langle m \rangle / N$  then

$$\frac{d\rho_w}{dt} = \tilde{s}(1 - (\rho_v + \rho_w)) - \tilde{b}\rho_v\rho_w - \tilde{v}\rho_w, \quad (4)$$

where  $\tilde{s} = s/N$  and  $\tilde{v} = v/N$ . We remark at this point that water infiltration in the soil is generally improved by the presence of vegetation. As a consequence, the process of taking up water by plants gets more efficient. This fact can be incorporated into the model by letting the rate for taking up water be density dependent, that is, to the jump (1,-1) we associate a new rate  $bn/N$ . This leads to having  $T(n + 1, m - 1|n, m) = b \frac{n^2}{N^2} \frac{m}{N-1}$ , which replaced into the equation for  $d\langle n \rangle / dt$  produces

$$\frac{d\rho_v}{dt} = \tilde{b}\rho_v^2\rho_w - \tilde{d}\rho_v \quad (5)$$

instead of equation (2). A similar change happens in equation (4), which is now

$$\frac{d\rho_w}{dt} = \tilde{s}(1 - (\rho_v + \rho_w)) - \tilde{b}\rho_v^2\rho_w - \tilde{v}\rho_w. \quad (6)$$

Equations (5) and (6) constitute a system of differential equations that serves as approximation to the dynamics of the mean densities for large values of  $N$ , also known as *mean field* equations in the Physics literature. We remark that for  $N$  relatively large the factor  $1 - (\rho_v + \rho_w)$  is close to one, making our equations an approximation to the *reaction* part of Klausmeier's reaction-diffusion-advection equations for water-vegetation systems, [12].

**2.2. The diffusion approximation.** The diffusion approximation to our model (the *mesoscale* model) describes the system as an intermediate approximation that emerges between the Markov jump process model (microscale model) and the differential equations for the mean densities (macroscale model). For this approximation the state variables are continuous but include random fluctuations. We expect the new model to incorporate the differential equations and terms that express random fluctuations around the mean densities.

To obtain a representation of the Markov jump process as a diffusion process one can follow either Kurtz's method [14], or find the same equations via the Fokker-Planck equation [18, 7]; see [4] for a nice introduction. In Kurtz's approach, which is the one we use here, the jump process is represented by

$$X(t) = X(0) + \sum_{\mathbf{r}} \mathbf{r} N^{(\mathbf{r})} \left( \int_0^t N \Phi \left( \frac{1}{N} X(s); \mathbf{r} \right) ds \right), \quad (7)$$

where  $\mathbf{r}$  is a jump (see the third column in Table 1),  $N^{(\mathbf{r})}(t)$  is a collection of independent rate 1 scalar Poisson processes and  $N \Phi \left( \frac{1}{N} X(s); \mathbf{r} \right)$  is the rate at which a transition with jump  $\mathbf{r}$  occurs. Kurtz showed that the process  $X(t)/N$  can be approximated by a process  $Y(t)$  that satisfies the stochastic differential equation

$$dY(t) = \sum_{\mathbf{r}} \mathbf{r} \Phi(Y(t); \mathbf{r}) dt + \frac{1}{\sqrt{N}} \sum_{\mathbf{r}} \mathbf{r} \sqrt{\Phi(Y(t); \mathbf{r})} dW^{(\mathbf{r})}(t), \quad (8)$$

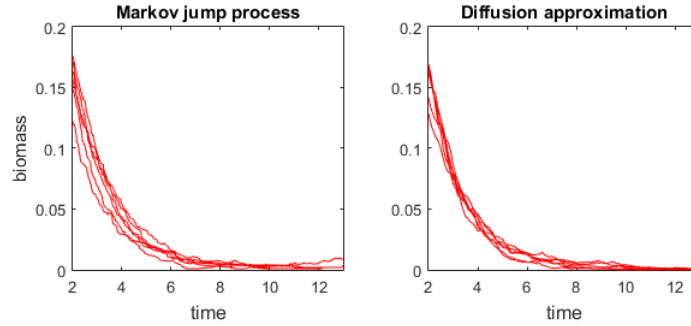


FIGURE 1. Simulations corresponding to the Markov jump process (left) and the diffusion approximation (right). For comparison purposes the paths in both panels were generated using the same parameters and the same scaled time.

where the  $W^{(r)}$  are the Brownian motions associated with the jump types of the Markov jump process. The error introduced on bounded intervals of time by replacing  $X(t)/N$  with  $Y(t)$  is  $O(\log N/N)$  as  $N \rightarrow \infty$ . The first term in the sum of the right hand side of (8) corresponds to the vector field of mean densities, that is,

$$\mathbf{A} = \begin{bmatrix} A_v \\ A_w \end{bmatrix} = \begin{bmatrix} \tilde{b}\rho_v^2\rho_w - \tilde{d}\rho_v, \\ \tilde{s}(1 - \rho_v - \rho_w) - \tilde{b}\rho_v^2\rho_w - \tilde{v}\rho_w, \end{bmatrix}.$$

If we denote by  $\mathbf{B}$  the covariance matrix

$$\mathbf{B} = \begin{bmatrix} B_{vv} & B_{vw} \\ B_{wv} & B_{ww} \end{bmatrix},$$

where

$$\begin{aligned} B_{vv} &= \tilde{b}\rho_v^2\rho_w + \tilde{d}\rho_v, \\ B_{vw} &= B_{wv} = -\tilde{b}\rho_v^2\rho_w, \\ B_{ww} &= \tilde{b}\rho_v^2\rho_w + \tilde{s}(1 - \rho_v - \rho_w) + \tilde{v}\rho_w, \end{aligned}$$

and factor it as  $\mathbf{B} = \mathbf{g}\mathbf{g}^T$  for some  $\mathbf{g}$ , then the stochastic differential equations system (8) can be written as

$$dY(t) = \mathbf{A}dt + \frac{1}{\sqrt{N}}\mathbf{g}dW, \tag{9}$$

where  $dW$  is a two dimensional Brownian motion. Figure 1 compares several paths that correspond to the Markov jump process and the diffusion approximation, generated with the same set of (arbitrarily chosen) parameters. It is apparent that the stochastic differential equation provides a good approximation to the jump process.

**3. Simulations.** We use the stochastic differential equation (9) to simulate the water-vegetation system and obtain averages of the expected time to desertification (see [10] or [13] for a quick or an extensive introduction respectively to the numerical solution of stochastic differential equations).

By identifying the parameters of the nondimensional deterministic (non-spatial) model in [12] with the mean field system obtained above ( $A_v = A_v(\rho)$  and  $A_w = A_w(\rho)$ , where  $\rho = (\rho_v, \rho_w)$ ) we obtain  $S = AR^{1/2}J/L^{3/2}$  and  $\tilde{d} = M/L$ , with the meaning and realistic values for these parameters listed in Table 3. Thus, for

Parameter	Definition	Estimated values	Units
$R$	uptake rate of water	1.5(trees) - 100(grass)	mm year <sup>-1</sup> (kg dry mass) <sup>-2</sup>
$J$	yield of plant biomass	0.002(trees) - 0.003(grass)	kg dry mass (mm) <sup>-1</sup>
$M$	mortality rate	0.18(trees) - 1.8(grass)	year <sup>-1</sup>
$A$	precipitation	250 - 750	mm year <sup>-1</sup>
$L$	evaporation rate	4	year <sup>-1</sup>

TABLE 2. Parameters for semi-arid landscapes, taken from [12].

instance, Klausmeier's parameter ranges for  $S$  are  $[0.077, 0.23]$  and  $\tilde{d} = 0.045$  for trees, and  $[0.94, 2.81]$  and  $\tilde{d} = 0.45$  for grass. Also, the corresponding value for  $\tilde{b}$  and  $\tilde{v}$  in both cases is one. Regarding the average evaporation rate, we follow Klausmeier assuming that the equilibrium of water (in his deterministic model) is at  $w^* = 75$  mm, and then computing the associated evaporation rate given the averaged annual precipitation, [12]. For example, with  $A = 300$  mm year<sup>-1</sup> the evaporation rate is  $A/w^* = 4$  year<sup>-1</sup>.

Precipitation anomalies records, i.e. records of the deviations from a long term precipitation mean, have a negative trend in specific geographic drought events. In the state of California for instance, which experienced unusually long drought conditions, the (state) average of the precipitation anomalies for the past 16 years is  $\approx -2.07$  (inches year<sup>-1</sup>) (i.e.  $-52.58$  mm year<sup>-1</sup>), see Figure 2. Although there is a lot of variability across the state, we use this value for illustration purposes, and plot the data as if this negative deviation from the long term precipitation average were steady in time.

The results of the simulations are shown in Figure 3, portraying a roughly linear relationship between the time to desertification and the anomalies in precipitation in the range selected. The simulations were run using parameters for trees (top panel) and grass (low panel), with system capacity  $N = 500$ . If we denote by  $T$  the average time to desertification and by  $P$  the average annual precipitation we can define the (dimensionless) sensitivity index  $S_0$  as the ratio  $S_0 = (\Delta T/T)/(\Delta P/P)$ , where  $\Delta T$  and  $\Delta P$  represent the absolute change in the variables  $T$  and  $P$ , i.e.  $S_0$  measures the relative change in  $T$  with respect to the relative change in  $P$ , see for instance [2]. Direct computation from the averaged results gives  $S_0 \approx 2$ . Similar results were obtained with the larger system capacity  $N = 10^4$ .

**4. Conclusions and discussion.** A traditional approach for modeling interacting populations at the macroscopic level assumes that the terms in the equations that drive the dynamics represent the average effects of individual interactions in a general, all-inclusive way. Subsequent developments use those models as departing points for building theoretical extensions by incorporating further complexity to the equations, like the inclusion of spatial dependence by adding diffusion, or the introduction of "noise" terms. A different modeling approach is to start at the individual level, with explicit rules describing the interactions between individuals and their environment, [17]. This alternative implies the definition of a Markov jump process that constitutes the foundation for developing definitive model approximations that relate macro and microscopic dynamical levels. In this paper we have taken the latter approach for constructing a stochastic differential equation (continuous state space) that approximates the dynamics of an idealized water-vegetation system, initially conceived as a Markov jump process (with discrete states as a proxy for small scale). Thus, our work complements the existing literature on modeling noise in drylands, [21].

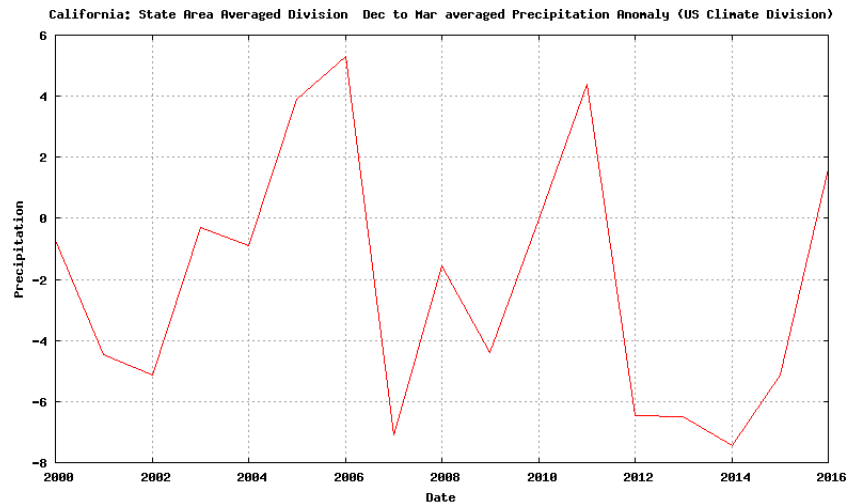


FIGURE 2. State averages of precipitation anomalies for 2000-2016 in California (inches year<sup>-1</sup>). The averaged anomaly (difference from long term average) during that period is  $\approx -2.07$  (inches year<sup>-1</sup>) ( $-52.58$  mm year<sup>-1</sup>). The precipitation increase expected from El Niño for the winter 2015-2016 was scarcely above the long term state average. Data/image provided by the NOAA/OAR/ESRL PSD, Boulder, Colorado, USA, from their Web site at <http://www.esrl.noaa.gov/psd/>

The diffusion approximation obtained, together with parameter values for vegetation and precipitation for semi-arid landscapes extracted from the literature, and data on decreasing precipitation trends in California, were used to estimate average times for desertification. For a fixed system capacity ( $N = 500$ ) the simulations for trees and grass suggests that the sensitivity of the time to desertification from the annual precipitation is roughly similar, and approximately equal to 2, i.e. the relative change in the transition time into a desert state is equal to twice the relative change in precipitation. Repeating the simulations, for different initial conditions and for a larger capacity ( $N = 10^4$ ), provided the same approximate numerical relation. The simulations (see Figure 3) suggest, for instance, that a decrease of roughly 0.4 inches of precipitation ( $10$  mm year<sup>-1</sup>) might reduce times to desertification in some cases by more than 25 years for the case of trees, and around 5 years in the case of grass. Put another way, current trends of desertification could be significantly boosted if the patterns of increasing precipitation anomalies are maintained. However, looking at the basic transition mechanisms considered in formulating the Markov jump process, it is clear that the model should be used with care to draw any conclusions on specific vegetation-water systems.

We remark that extended droughts may resemble desertification, but the return of seasonal precipitation events may recover the vegetation (see for instance [1] where desertification was limited to spatially localized areas). This suggests that the inclusion of patterns of precipitation anomalies restricted to relatively small areas would provide more reliable results. For systems with relatively small capacity we notice that the times to desertification may be reduced dramatically (see Figure 4).

Further work should also include long term variations of other climate related parameters. For instance, it has been documented that higher temperatures increase evapotranspiration rates [3], which have been observed over most of the United States, [8]. Another aspect that deserves attention is the inclusion of changes in the vegetation dynamics during dry periods, where vegetation mortality could be exacerbated. As is clearly pointed out in [28], neglecting the effects of intermittent precipitation on vegetation dynamics may influence the results considerably. Finally, the understanding of desertification will demand treatment with insightful stochastic space-time models.

**Acknowledgments.** The authors are grateful to C. Kribs and three anonymous reviewers for comments and suggestions that led to significant improvement of the paper.

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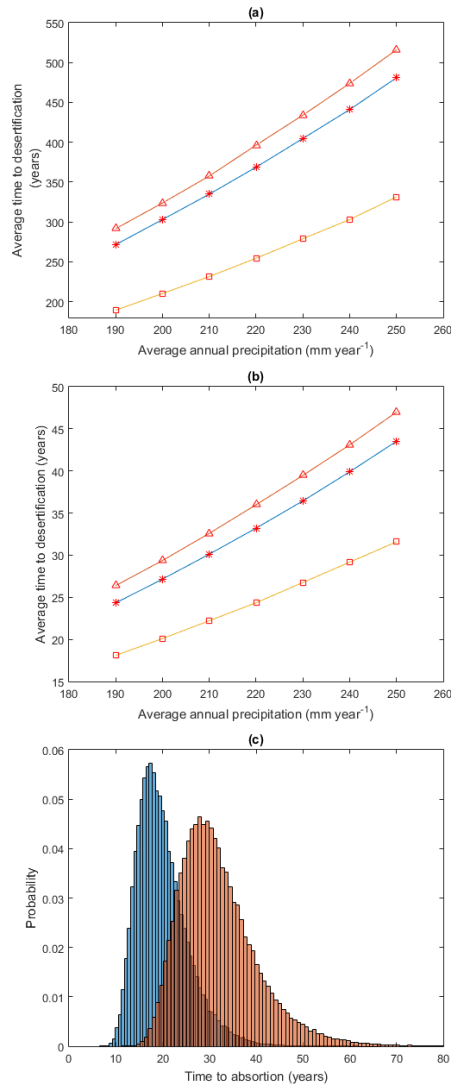


FIGURE 3. Examples of how average time to desertification might be affected by a reduction in average annual precipitation. Parameters for trees were used in panel (a) and for grass in panel (b). The average of negative anomalies similar to that observed for the last years in California is around  $\approx 50 \text{ mm year}^{-1}$ . The model suggests that the sensitivity index  $S_0 \approx 2$ , i.e. relative changes in the mean time to desertification are roughly twice the relative changes in average annual precipitation. For the simulations,  $N = 500$  and the initial conditions were  $\rho(0) = (0.1, 0.1)$  (squares),  $\rho(0) = (0.5, 0.5)$  (stars) and  $\rho(0) = (0.9, 0.1)$  (triangles). Each time average was obtained from 50000 simulations. Panel (c) shows the histograms corresponding to the simulated times to desertification with an average annual precipitation of 200 and 250  $\text{mm year}^{-1}$  (for grass) on the left and right, respectively. The simulations used the same initial conditions  $\rho(0) = (0.1, 0.1)$ .

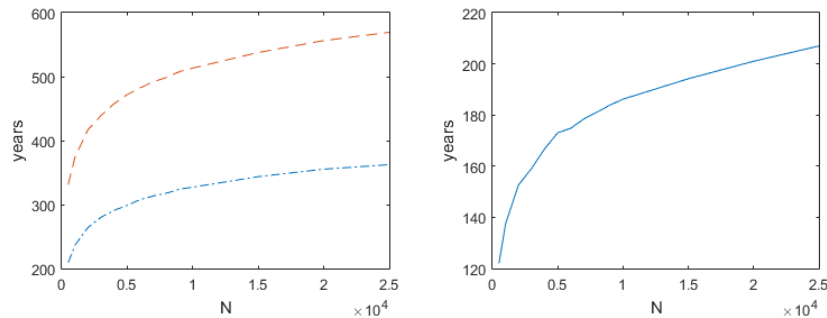


FIGURE 4. Left: time to desertification for  $A = 250$  (dashes) and  $A = 200$  (dot-dashes) as function of the system capacity  $N$ . The sensitivity of the time to desertification from the annual precipitation was computed for  $N = 10000$  showing to be the same as when  $N = 500$ , i.e.  $\approx 2$ . As  $N$  increases both times to desertification also increase, but they get reduced dramatically as  $N$  gets smaller. Right: Difference between the curves in the contiguous plot. Although the difference increases, the sensitivity of the time to desertification from the annual precipitation is apparently similar in relatively large systems.

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Received June 16, 2017; Accepted March 16, 2018.

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