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AN AGE-STRUCTURED VECTOR-BORNE DISEASE MODEL WITH HORIZONTAL TRANSMISSION IN THE HOST

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Abstract. We concern with a vector-borne disease model with horizontal transmission and infection age in the host population. With the approach of Lyapunov functionals, we establish a threshold dynamics, which is completely determined by the basic reproduction number. Roughly speaking, if the basic reproduction number is less than one then the infection-free equilibrium is globally asymptotically stable while if the basic reproduction number is larger than one then the infected equilibrium attracts all solutions with initial infection. These theoretical results are illustrated with numerical simulations.

1. **Introduction.** Vector-borne diseases such as malaria, dengue, schistomiasis, Chagas disease, and yellow fever are illnesses that are transmitted by *vectors*, which include mosquitos, ticks, and fleas. They account for over 17% of all infectious diseases and are great threat to the health of human and animal. Every year there are more than 1 billion cases and over 1 million deaths from vector-borne diseases.

Mathematical modeling has been successfully used to better understand the mechanisms underlying vector-borne disease spread and to provide efficient control strategies. The Ross-Macdonald model on vector-borne diseases was described by ordinary differential equations [14, 19, 20]. Macdonald [14] established a threshold condition on the invasion and persistence of infection, which is determined by the basic reproduction number (defined as the average number of secondary cases produced by an index case during its infectious period). Most of the existing vector-borne disease models, especially those on malaria that investigate complications arising from host superinfection, immunity, and other factors, are based on this fundamental model [3, 5, 8, 12, 18, 21, 23, 24]. In particular, Lashari and Zaman [12] considered the following vector-borne disease model with horizontal

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Table 1. Biological meanings of parameters in (1)

Parameter	Meaning
λ_h	Per capita host birth rate
μ_h	Host death rate
β_1	Rate of horizontal transmission of the disease
β_2	Rate of a pathogen carrying mosquito biting susceptible host
α_h	Inverse of host latent period
δ_h	Disease related death rate of host
γ_h	Recovery rate of host
λ_v	Per capita vector birth rate
k	Biting rate of per susceptible vector per host per unit time
μ_v	Vector death rate
α_v	Inverse of vector latent period
δ_v	Disease related death rate of vectors

transmission in the host population,

$$\begin{cases} \frac{dS_h(t)}{dt} &= \lambda_h - \mu_h S_h - \beta_1 S_h I_h - \beta_2 S_h I_v, \\ \frac{dE_h(t)}{dt} &= \beta_1 S_h I_h + \beta_2 S_h I_v - (\alpha_h + \mu_h) E_h, \\ \frac{dI_h(t)}{dt} &= \alpha_h E_h - (\mu_h + \delta_h + \gamma_h) I_h, \\ \frac{dR_h(t)}{dt} &= \gamma_h I_h - \mu_h R_h, \\ \frac{dS_v(t)}{dt} &= \lambda_v - k S_v I_h - \mu_v S_v, \\ \frac{dE_v(t)}{dt} &= k S_v I_h - (\alpha_v + \mu_v) E_v, \\ \frac{dI_v(t)}{dt} &= \alpha_v E_v - (\mu_v + \delta_v) I_v, \end{cases} \tag{1}$$

where S_h, E_h, I_h, and R_h denote the susceptible, exposed, infectious, and recovered epidemiological classes in the host, respectively, while S_v, E_v, and I_v denote the susceptible, exposed, and infectious epidemiological classes in the vector, respectively. There is no recovered class for the vector (mosquitos) because no infected mosquito can recover from the infection. The biological meanings of the parameters in (1) are summarized in Table 1.

It is well known that the infectivity varies during the infectious period and hence the time passed since being infected, called infection age, affects the number of secondary infections. In recent years, epidemic models with infection age have been extensively studied. For works on vector-borne diseases, not much has been done [10, 13, 17, 25], where only the host has infection age. In [10], an SI(host)SI(vector) model is proposed, which incorporated horizontal transmission. Under additional condition besides the basic reproduction ratio $\mathcal{R}_0 < 1$, it is shown that the disease-free steady state is globally asymptotically stable. Moreover, only the local stability of the endemic steady state is discussed. In [13], Lou and Zhao considered a periodic SEIRS(host)SEI(vector) model with standard incidence. It is shown that there exists at least one positive periodic state and that the disease persists when the basic reproduction ratio $\mathcal{R}_0 > 1$ while the disease will die out if $\mathcal{R}_0 < 1$. One of the models in [25] is an SIR(host)SI(vector) model with constant vector population and a threshold dynamics characterized by the basic reproduction number is obtained.

The purpose of this paper is to modify (1) by introducing infection age into the host and study the dynamics of the resulted model. The remaining part of this paper is organized as follows. In the next section, we introduce the model and state some preliminary results on solutions. Then, in Section 3, we study the existence of equilibria and their local stability. Section 4 is the main part of this paper, where we establish a threshold dynamics with the approach of Lyapunov functional. The threshold dynamics is characterized only by the basic reproduction number. Here, to obtain the stability of the infected equilibrium, we need the existence of a global attractor and the uniformly strong persistence. The theoretical results are illustrated with numerical simulations in Section 5. The paper concludes with a brief summary.

2. The model and preliminary results.

Our model is based on model (1). To build it, we further subdivide the infectious host according to the infection age a. Let $i_h(t,a)$ be the density of infectious hosts at time t with infection age a. Then $\int_{a_1}^{a_2} i_h(t,a)da$ is the number of infectious hosts with infection ages between a_1 and a_2 at time t and the total number of infectious hosts at time t is $I_h(t) = \int_0^\infty i_h(t,a)da$. We assume that the infectivity of infectious hosts, the biting rate of an infectious host by a susceptible vector, disease-induced death rate of infectious hosts, and the recovery rate of infectious hosts all depend on the infection age a and denote them by $\beta_1(a)$, $k(a)$, $\delta_h(a)$, and $\gamma(a)$, respectively. Then the rate of horizontal transmission of the disease from infectious hosts to susceptible hosts is $\int_0^\infty \beta_1(a)i_h(t,a)da$ and the force of infection of the host to susceptible vectors is $\int_0^\infty k(a)i_h(t,a)da$. Since the recovered hosts have permanent immunity, there is no need to consider the evolution of R_h in time. Based on our assumptions and model (1), the vector-borne disease model with infection age in host to be studied in this paper is as follows,

$$\begin{cases} \frac{dS_h(t)}{dt} = \lambda_h - S_h(t)\int_0^\infty \beta_1(a)i_h(t,a)da - \beta_2 S_h(t)I_v(t) - \mu_h S_h(t), \\ \frac{dE_h(t)}{dt} = S_h(t)\int_0^\infty \beta_1(a)i_h(t,a)da + \beta_2 S_h(t)I_v(t) - (\alpha_h+\mu_h)E_h(t), \\ \frac{\partial i_h(t,a)}{\partial t} + \frac{\partial i_h(t,a)}{\partial a} = -\delta(a)i_h(t,a), \\ \frac{dS_v(t)}{dt} = \lambda_v - \int_0^\infty k(a)S_v(t)i_h(t,a)da - \mu_v S_v(t), \\ \frac{dE_v(t)}{dt} = \int_0^\infty k(a)S_v(t)i_h(t,a)da - (\alpha_v+\mu_v)E_v(t), \\ \frac{dI_v(t)}{dt} = \alpha_v E_v(t) - \mu_v I_v(t), \\ i_h(t,0) = \alpha_h E_h(t), t>0, \\ S_h(0) = S_{h0}\in R_+, \quad E_h(0) = E_{h0}\in R_+, \quad i_h(0,\cdot) = i_{h0}\in L_+^1(0,\infty), \\ S_v(0) = S_{v0}\in R_+, \quad E_v(0) = E_{v0}\in R_+, \quad I_v(0) = I_{v0}\in R_+, \end{cases} \tag{2}$$

where $\delta(a) = \mu_h + \delta_h(a) + \gamma(a)$, $R_+ = [0,\infty)$, and $L_+^1(0,\infty)$ is the nonnegative cone of $L^1(0,\infty)$.

To continue our discussion, in the sequel, we assume that $k(\cdot)\in L_+^\infty(0,\infty)\setminus\{0\}$ and $\beta_1(\cdot),\ \gamma(\cdot)\in L_+^\infty(0,\infty)$, where $L_+^\infty(0,\infty)$ is the nonnegative cone of $L^\infty(0,\infty)$. Clearly, $\delta(a)\geq\mu_h$ for $a\in R_+$. For (2), there should be an inherent relationship between the initial value and the boundary value for the partial differential equation, that is, $i_h(0,0) = i_{h0}(0)$. Therefore, we always assume that the initial values satisfy $\alpha_h E_{h0} = i_{h0}(0)$.

Note that the partial differential equation in (2) is a linear transport equation with decay. With integration along the characteristic line $t - a =$ const., one can solve

$$\begin{cases} \frac{\partial i_h(t,a)}{\partial t} + \frac{\partial i_h(t,a)}{\partial a} = -\delta(a) i_h(t,a) \\ i_h(t,0) = \alpha_h E_h(t), \qquad t \geq 0 \end{cases}$$

to get

$$i_h(t,a) = \begin{cases} \sigma(a)\alpha_h E_h(t-a) & \text{if } t > a \geq 0, \\ \frac{\sigma(a)}{\sigma(a-t)} i_h(0, a-t) & \text{if } a \geq t > 0, \end{cases}$$

where $\sigma(a) = \exp(-\int_0^a \delta(s)ds)$ represents the probability that an infectious host survives to infection age a. Then we obtain the following equivalent system of integro-differential equations to (2),

$$\begin{cases} \frac{dS_h(t)}{dt} = \lambda_h - S_h(t)\int_0^\infty \beta_1(a) i_h(t,a)da - \beta_2 S_h(t) I_v(t) - \mu_h S_h(t), \\ \frac{dE_h(t)}{dt} = S_h(t)\int_0^\infty \beta_1(a) i_h(t,a)da + \beta_2 S_h(t) I_v(t) - (\alpha_h + \mu_h) E_h(t), \\ i_h(t,a) = \sigma(a)\alpha_h E_h(t-a)\mathbf{1}_{t>a} + \frac{\sigma(a)}{\sigma(a-t)} i_h(0,a-t)\mathbf{1}_{a>t}, \\ \frac{dS_v(t)}{dt} = \lambda_v - \int_0^\infty k(a) S_v(t) i_h(t,a)da - \mu_v S_v(t), \\ \frac{dE_v(t)}{dt} = \int_0^\infty k(a) S_v(t) i_h(t,a)da - (\alpha_v + \mu_v) E_v(t), \\ \frac{dI_v(t)}{dt} = \alpha_v E_v(t) - \mu_v I_v(t), \end{cases} \tag{3}$$

where

$$\mathbf{1}_{t>a} = \begin{cases} 1 & \text{if } t > a \geq 0 \\ 0 & \text{if } a \geq t \geq 0 \end{cases} \quad \text{and} \quad \mathbf{1}_{a>t} = \begin{cases} 0 & \text{if } t > a \geq 0, \\ 1 & \text{if } a \geq t \geq 0. \end{cases}$$

Let

$$X_+ = R_+^2 \times L_+^1(0,\infty) \times R_+^3,$$

which is the nonnegative cone of the Banach space $X = R^2 \times L^1(0,\infty) \times R^3$ equipped with norm $\|\cdot\|$ defined by

$$\|x\| = |x_1| + |x_2| + \|x_3\|_1 + |x_4| + |x_5| + |x_6|$$

for $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in X$. With a reasonable modification of the proofs of Theorem 2.1 and Lemma 2.2 in Browne and Pilyugin [1], we can prove the existence and nonnegativeness of solutions to (3) and hence to (2).

Theorem 2.1. *For any $x \in X_+$, system (2) has a unique solution on R_+, which depends continuously on the initial value and time. Moreover, $(S_h(t), E_h(t), i_h(t,\cdot), S_v(t), E_v(t), I_v(t)) \in X_+$ for $t \in R_+$.*

In fact, every solution is bounded. On the one hand, let

$$N_h(t) = S_h(t) + E_h(t) + \int_0^\infty i_h(t,a)da.$$

Then we have $\frac{dN_h(t)}{dt} \leq \lambda_h - \mu_h N_h(t)$ and hence $\limsup_{t\to\infty} N_h(t) \leq \lambda_h/\mu_h$. On the other hand, let

$$N_v(t) = S_v(t) + E_v(t) + I_v(t).$$

Then $\frac{dN_v(t)}{dt} = \lambda_v - \mu_v N_v(t)$, which implies that $\lim_{t\to\infty} N_v(t) = \lambda_v/\mu_v$. Denote

$$\Omega = \left\{ (S_h, E_h, i_h, S_v, E_v, I_v) \in X_+ \;\middle|\; \begin{array}{l} S_h + E_h + \|i_h\|_1 \le \frac{\lambda_h}{\mu_h}, \\ S_v + E_v + I_v = \frac{\lambda_v}{\mu_v} \end{array} \right\}.$$

Then we have shown that Ω is an attracting set for (2). Moreover, one can easily see that Ω is also a positively invariant set for (2).

3. **The existence of equilibria and their local stability.** In this section, we study the local dynamics of (2). We first consider the existence of equilibria. It turns out that this only depends on the basic reproduction number R_0, which is defined as

$$R_0 = \frac{\lambda_h[\xi\alpha_h(\alpha_v + \mu_v)\mu_v^2 + \beta_2\lambda_v\alpha_v\eta\alpha_h]}{\mu_h\mu_v^2(\alpha_h + \mu_h)(\alpha_v + \mu_v)},$$

where $\eta = \int_0^\infty k(a)\sigma(a)da$ and $\xi = \int_0^\infty \beta_1(a)\sigma(a)da$.

Clearly, (2) always has the infection-free equilibrium $E^0 = (S_h^0, 0, 0, S_v^0, 0, 0) \in \Omega$, where $S_h^0 = \lambda_h/\mu_h, S_v^0 = \lambda_v/\mu_v$. Let $E^* = (S_h^*, E_h^*, i_h^*, S_v^*, E_v^*, I_v^*)$ be an equilibrium. Then we have

$$\begin{cases} \lambda_h - \mu_h S_h^* - \beta_2 S_h^* I_v^* - S_h^* \int_0^\infty \beta_1(a) i_h^*(a)da = 0, \\ S_h^* \int_0^\infty \beta_1(a) i_h^*(a)da + \beta_2 S_h^* I_v^* = (\alpha_h + \mu_h)E_h^*, \\ \frac{di_h^*(a)}{da} = -\delta(a) i_h^*(a), \\ i_h^*(0) = \alpha_h E_h^*, \\ \lambda_v - \int_0^\infty k(a) S_v^* i_h^*(a)da - \mu_v S_v^* = 0, \\ \int_0^\infty k(a) S_v^* i_h^*(a)da = (\alpha_v + \mu_v)E_v^*, \\ \alpha_v E_v^* = \mu_v I_v^*. \end{cases} \tag{4}$$

It is easy to see that an equilibrium other than E^0 must be infected, that is, all components are positive. For an infected equilibrium, it is not difficult to deduce from (4) that

$$\begin{aligned} S_h^* &= \frac{\lambda_h - (\alpha_h + \mu_h)E_h^*}{\mu_h}, \\ i_h^*(a) &= \alpha_h \sigma(a) E_h^*, \\ S_v^* &= \frac{\lambda_v}{\mu_v + \eta\alpha_h E_h^*}, \\ E_v^* &= \frac{\lambda_v \eta \alpha_h E_h^*}{(\alpha_v + \mu_v)(\mu_v + \eta\alpha_h E_h^*)}, \\ I_v^* &= \frac{\lambda_v \alpha_v \eta \alpha_h E_h^*}{\mu_v(\alpha_v + \mu_v)(\mu_v + \eta\alpha_h E_h^*)}, \end{aligned} \tag{5}$$

where E_h^* is a positive zero of H with

$$\begin{aligned} H(x) = {} & \lambda_h[\xi\alpha_h(\alpha_v + \mu_v)(\mu_v + \eta\alpha_h x)\mu_v + \beta_2\alpha_v\lambda_v\eta\alpha_h] \\ & -\mu_v\mu_h(\alpha_h + \mu_h)(\alpha_v + \mu_v)(\mu_v + \eta\alpha_h x) \\ & -\beta_2\alpha_v\lambda_v\eta\alpha_h x(\alpha_h + \mu_h) \\ & -(\alpha_h + \mu_h)(\alpha_v + \mu_v)\mu_v\xi\alpha_h x(\mu_v + \eta\alpha_h x). \end{aligned} \tag{6}$$

Theorem 3.1. *(i) Suppose $R_0 \le 1$. Then (2) only has the infection-free equilibrium E^0.*

(ii) Suppose $R_0 > 1$. Then, besides E^0, (2) also has a unique infected equilibrium $E^* = (S_h^*, E_h^*, i_h^*, S_v^*, E_v^*, I_v^*)$, where E_h^* is the unique positive zero of H defined by (6) and the other components are determined by (5).

Proof. (i) Since $R_0 \le 1$, we have $\lambda_h \xi \alpha_h \le \mu_h(\alpha_h + \mu_h)$. Note that H is a quadratic function with negative coefficient for x^2. Moreover, the coefficient of x in $H(x)$ is

$$\begin{aligned} & \lambda_h \xi \alpha_h (\alpha_v + \mu_v) \eta \alpha_h \mu_v - \mu_v \mu_h (\alpha_h + \mu_h)(\alpha_v + \mu_v) \eta \alpha_h \\ & -\beta_2 \alpha_v \lambda_v \eta \alpha_h (\alpha_h + \mu_h) - (\alpha_h + \mu_h)(\alpha_v + \mu_v) \mu_v \xi \alpha_h \mu_v \\ < \ & \mu_h (\alpha_h + \mu_h)(\alpha_v + \mu_v) \eta \alpha_h \mu_v - \mu_v \mu_h (\alpha_h + \mu_h)(\alpha_v + \mu_v) \eta \alpha_h \\ = \ & 0 \end{aligned}$$

and $H(0) = \mu_h \mu_v^2 (\alpha_h + \mu_h)(\alpha_v + \mu_v)(R_0 - 1) \le 0$. It follows that $H(x)$ has no positive zeros and hence there is no infected equilibrium.

(ii) Now, since $R_0 > 1$, we have $H(0) = \mu_h \mu_v^2 (\alpha_h + \mu_h)(\alpha_v + \mu_v)(R_0 - 1) > 0$. Then $H(x)$ has a unique positive zero since $H(x)$ is a quadratic polynomial with negative coefficient for x^2. Therefore, there is a unique infected equilibrium as described in the statement. This completes the proof. □

Now, we study the stability of the equilibria by linearization. For more detail, see Iannelli [9].

Theorem 3.2. *(i) The infection-free equilibrium E^0 of (2) is locally asymptotically stable if $R_0 < 1$ and it is unstable if $R_0 > 1$.*

(ii) If $R_0 > 1$, then the infected equilibrium E^* of (2) is locally asymptotically stable.

Proof. (i) The characteristic equation at E^0 is

$$\begin{aligned} 0 &= F(\tau) \\ &\triangleq (\tau + \alpha_h + \mu_h)(\tau + \mu_v)(\tau + \alpha_v + \mu_v) \\ &\quad -\alpha_h S_h^0 \Bigg[(\tau + \mu_v)(\tau + \alpha_v + \mu_v) \int_0^\infty \beta_1(a) \sigma(a) e^{-\tau a} da \\ &\quad + \beta_2 \alpha_v S_v^0 \int_0^\infty k(a) \sigma(a) e^{-\tau a} da \Bigg]. \end{aligned}$$

First, assume $R_0 > 1$. Then $F(0) = \mu_v(\alpha_h + \mu_h)(\alpha_v + \mu_v)(1 - R_0) < 0$ and $\lim_{\tau \to \infty} F(\tau) = \infty$. By the Intermediate Value Theorem, F has a positive zero and hence E^0 is unstable if $R_0 > 1$.

Next, assume $R_0 < 1$. It suffices to show that all zeros of F have negative real parts. If this is not true, then F has a zero τ_0 with $\mathrm{Re}(\tau_0) \ge 0$. It follows that

$$\begin{aligned} 1 &= \frac{\left|\alpha_h S_h^0 \left[(\tau_0 + \mu_v)(\tau_0 + \alpha_v + \mu_v) \int_0^\infty \beta_1(a) \sigma(a) e^{-\tau_0 a} da + \beta_2 \alpha_v S_v^0 \int_0^\infty k(a) \sigma(a) e^{-\tau_0 a} da \right]\right|}{|(\tau_0 + \alpha_h + \mu_h)(\tau_0 + \mu_v)(\tau_0 + \alpha_v + \mu_v)|} \\ &\le \frac{\alpha_h S_h^0 \xi}{\alpha_h + \mu_h} + \frac{\beta_2 S_h^0 \alpha_h \alpha_v S_v^0 \eta}{(\alpha_h + \mu_h) \mu_v (\alpha_v + \mu_v)} \\ &= R_0, \end{aligned}$$

which contradicts with $R_0 < 1$. Therefore, E^0 is locally asymptotically stable if $R_0 < 1$.

(ii) For the infected equilibrium E^*, the associated characteristic equation is,

$$
\begin{aligned}
&(\tau + A_1)(\tau + \alpha_h + \mu_h)(\tau + \mu_v)(\tau + A_2)(\tau + \alpha_v + \mu_v)\\
&-(\tau + \mu_h)\alpha_h S_h^* \left[(\tau + \mu_v)(\tau + \alpha_v + \mu_v) \int_0^\infty \beta_1(a)\sigma(a)e^{-\tau a}da(\tau + A_2)\right.\\
&\left.+ \beta_2\alpha_v S_v^*(\tau + \mu_v) \int_0^\infty k(a)\sigma(a)e^{-\tau a}da\right] = 0,
\end{aligned}
\tag{7}
$$

where $A_1 = \mu_h + \beta_2 I_v^* + \int_0^\infty \beta_1(a)i_h^*(a)da$ and $A_2 = \mu_v + \int_0^\infty k(a)i_h^*(a)da$. We claim that (7) has no root with a nonnegative real part. If the claim is not true, then (7) has a root $\hat{\tau}$ with $\mathrm{Re}(\hat{\tau}) \geq 0$. On the one hand,

$$
\begin{aligned}
1 =&\frac{|(\hat{\tau}+\mu_h)(\hat{\tau}+\mu_v)\alpha_h S_h^*\left[(\hat{\tau}+\alpha_v+\mu_v)\int_0^\infty \beta_1(a)\sigma(a)e^{-\hat{\tau}a}da(\hat{\tau}+A_2)+\beta_2\alpha_v S_v^*\int_0^\infty k(a)\sigma(a)e^{-\hat{\tau}a}da\right]|}{|(\hat{\tau}+A_1)(\hat{\tau}+\alpha_h+\mu_h)(\hat{\tau}+\mu_v)(\hat{\tau}+A_2)(\hat{\tau}+\alpha_v+\mu_v)|}\\
\leq&\frac{|(\hat{\tau}+\mu_h)\alpha_h S_h^*\int_0^\infty \beta_1(a)\sigma(a)da|}{|(\hat{\tau}+A_1)(\hat{\tau}+\alpha_h+\mu_h)|} + \frac{|(\hat{\tau}+\mu_h)(\hat{\tau}+\mu_v)\alpha_h S_h^*\beta_2\alpha_v S_v^*\int_0^\infty k(a)\sigma(a)da|}{|(\hat{\tau}+A_1)(\hat{\tau}+\alpha_h+\mu_h)(\hat{\tau}+\mu_v)(\hat{\tau}+A_2)(\hat{\tau}+\alpha_v+\mu_v)|}\\
<&\frac{|\alpha_h S_h^*\int_0^\infty \beta_1(a)\sigma(a)da|}{|(\hat{\tau}+\alpha_h+\mu_h)|} + \frac{|\alpha_h S_h^*\beta_2\alpha_v S_v^*\int_0^\infty k(a)\sigma(a)da|}{|(\hat{\tau}+\alpha_h+\mu_h)(\hat{\tau}+\mu_v)(\hat{\tau}+\alpha_v+\mu_v)|}\\
\leq&\frac{\alpha_h S_h^*\xi}{\alpha_h+\mu_h} + \frac{\alpha_h S_h^*\beta_2\alpha_v S_v^*\eta}{(\alpha_h+\mu_h)\mu_v(\alpha_h+\mu_v)}.
\end{aligned}
\tag{8}
$$

On the other hand, it follows from (4) that

$$
I_v^* = \frac{\alpha_v}{\mu_v}E_v^* = \frac{\alpha_v}{\mu_v}\frac{\eta S_v^* i_h^*(0)}{\alpha_v+\mu_v}
$$

and

$$
\begin{aligned}
i_h^*(0) &= \alpha_h E_h^*\\
&= \alpha_h \cdot \frac{\xi S_h^* i_h^*(0)}{\alpha_h+\mu_h} + \alpha_h \cdot \frac{\beta_2 S_h^* I_v^*}{\alpha_h+\mu_h}\\
&= \alpha_h \cdot \frac{\xi S_h^* i_h^*(0)}{\alpha_h+\mu_h} + \alpha_h \cdot \frac{\beta_2 S_h^*}{\alpha_h+\mu_h} \cdot \frac{\alpha_v}{\mu_v}\frac{\eta S_v^* i_h^*(0)}{\alpha_v+\mu_v}.
\end{aligned}
$$

This implies that $\frac{\alpha_h S_h^*\xi}{\alpha_h+\mu_h} + \frac{\alpha_h S_h^*\beta_2\alpha_v S_v^*\eta}{(\alpha_h+\mu_h)\mu_v(\alpha_h+\mu_v)} = 1$, a contradiction with (8). Therefore, the infected equilibrium E^* of (2) is locally asymptotically stable when $R_0 > 1$. □

4. **Global stability.**

We first study the global stability of the infection-free equilibrium E^0.

Theorem 4.1. *If $R_0 < 1$, then the infection-free equilibrium E^0 of (2) is globally asymptotically stable.*

Proof. Define

$$
\begin{aligned}
\rho_1(a) &= \int_a^\infty k(\theta)e^{-\int_a^\theta \delta(s)ds}d\theta,\\
\rho_2(a) &= \int_a^\infty \beta_1(\theta)e^{-\int_a^\theta \delta(s)ds}d\theta.
\end{aligned}
\tag{9}
$$

Obviously, $\rho_1(0) = \eta$ and $\rho_2(0) = \xi$. Moreover, $\rho_1(a)$ and $\rho_2(a)$ are bounded and satisfy

$$
\rho_1'(a) = \rho_1(a)\delta(a) - k(a) \qquad \text{and} \qquad \rho_2'(a) = \rho_2(a)\delta(a) - \beta_1(a)
$$

for $a \in R_+$, respectively. Define the Lyapunov functional

$$
L = L(S_h, E_h, i_h, S_v, E_v, I_v) = L_1 + L_2 + L_3,
$$

where

$$\begin{aligned} L_1 &= \frac{1}{S_h^0}\Big(S_h - S_h^0 - S_h^0 \ln \frac{S_h}{S_h^0}\Big) + \frac{1}{S_h^0}E_h, \\ L_2 &= \frac{\beta_2\alpha_v S_v^0}{(\alpha_v+\mu_v)\mu_v}\int_0^\infty \rho_1(a)i_h(t,a)da + \int_0^\infty \rho_2(a)i_h(t,a)da, \\ L_3 &= \frac{\beta_2\alpha_v}{(\alpha_v+\mu_v)\mu_v}\Big(S_v - S_v^0 - S_v^0 \ln\frac{S_v}{S_v^0}\Big) + \frac{\beta_2\alpha_v E_v}{(\alpha_v+\mu_v)\mu_v} + \frac{\beta_2}{\mu_v}I_v. \end{aligned}$$

Clearly, $L(\cdot)$ is non-negative and $L(x)=0$ if and only if $x=E^0$.

Now, we calculate the time derivatives of L_1, L_2, and L_3 along solutions of (2) one by one. First,

$$\begin{aligned} & \frac{dL_1}{dt} \\ = \ & \frac{1}{S_h^0}\Big(1-\frac{S_h^0}{S_h}\Big)\Big(\lambda_h - S_h\int_0^\infty \beta_1(a)i_h(t,a)da - \beta_2 S_h I_v - \mu_h S_h\Big) \\ & +\frac{1}{S_h^0}\Big(S_h\int_0^\infty \beta_1(a)i_h(t,a)da + \beta_2 S_h I_v - (\alpha_h+\mu_h)E_h\Big) \\ = \ & \mu_h\Big(1-\frac{S_h^0}{S_h}\Big)\Big(1-\frac{S_h}{S_h^0}\Big) - \Big(1-\frac{S_h^0}{S_h}\Big)\frac{S_h}{S_h^0}\int_0^\infty \beta_1(a)i_h(t,a)da \\ & -\beta_2 I_v\frac{S_h}{S_h^0}\Big(1-\frac{S_h^0}{S_h}\Big) + \frac{S_h}{S_h^0}\int_0^\infty \beta_1(a)i_h(t,a)da \\ & +\beta_2 I_v\frac{S_h}{S_h^0} - \frac{\alpha_h+\mu_h}{S_h^0}E_h \\ = \ & \mu_h\Big(2-\frac{S_h^0}{S_h}-\frac{S_h}{S_h^0}\Big) + \int_0^\infty \beta_1(a)i_h(t,a)da + \beta_2 I_v - \frac{\alpha_h+\mu_h}{S_h^0}E_h. \end{aligned}$$

Next, applying integration by parts gives

$$\begin{aligned} \frac{dL_2}{dt} &= \frac{\beta_2\alpha_v S_v^0}{(\alpha_v+\mu_v)\mu_v}\int_0^\infty \rho_1(a)\Big(-\frac{\partial i_h(t,a)}{\partial a} - \delta(a)i_h(t,a)\Big)da \\ & \quad -\int_0^\infty \rho_2(a)\Big(\frac{\partial i_h(t,a)}{\partial a} + \delta(a)i_h(t,a)\Big)da \\ &= \frac{\beta_2\alpha_v S_v^0}{(\alpha_v+\mu_v)\mu_v}\int_0^\infty (\rho_1'(a)-\rho_1(a)\delta(a))i_h(t,a)da \\ & \quad +\frac{\beta_2\alpha_v S_v^0}{(\alpha_v+\mu_v)\mu_v}\rho_1(0)i_h(t,0) \\ & \quad +\int_0^\infty (\rho_2'(a)-\rho_2(a)\delta(a))i_h(t,a)da + \rho_2(0)i_h(t,0) \\ &= -\frac{\beta_2\alpha_v S_v^0}{(\alpha_v+\mu_v)\mu_v}\int_0^\infty k(a)i_h(t,a)da - \int_0^\infty \beta_1(a)i_h(t,a)da \\ & \quad +\frac{\beta_2\alpha_v S_v^0}{(\alpha_v+\mu_v)\mu_v}\eta\alpha_h E_h + \xi\alpha_h E_h. \end{aligned}$$

Finally,

$$\frac{dL_3}{dt} = \frac{\beta_2\alpha_v}{(\alpha_v+\mu_v)\mu_v}\Big(1-\frac{S_v^0}{S_v}\Big)\Big(\lambda_v - \int_0^\infty k(a)S_v i_h(t,a)da - \mu_v S_v\Big)$$

$$
\begin{aligned}
&+\frac{\beta_2\alpha_v}{(\alpha_v+\mu_v)\mu_v}\Big(\int_0^\infty k(a)S_v i_h(t,a)da-(\alpha_v+\mu_v)E_v\Big)\\
&+\frac{\beta_2}{\mu_v}(\alpha_v E_v-\mu_v I_v)\\
=\;&\frac{\beta_2\alpha_v S_v^0}{\alpha_v+\mu_v}\Big(2-\frac{S_v^0}{S_v}-\frac{S_v}{S_v^0}\Big)\\
&+\frac{\beta_2\alpha_v S_v^0}{(\alpha_v+\mu_v)\mu_v}\int_0^\infty k(a)i_h(t,a)da-\beta_2 I_v.
\end{aligned}
$$

Here we have used $\lambda_v=\mu_v S_v^0$.

In summary, we have shown that

$$
\begin{aligned}
\frac{dL}{dt} &= \frac{dL_1}{dt}+\frac{dL_2}{dt}+\frac{dL_3}{dt}\\
&= \mu_h\Big(2-\frac{S_h^0}{S_h}-\frac{S_h}{S_h^0}\Big)+\frac{\beta_2\alpha_v S_v^0}{\alpha_v+\mu_v}\Big(2-\frac{S_v^0}{S_v}-\frac{S_v}{S_v^0}\Big)\\
&\quad+\Big(\frac{\beta_2\alpha_v S_v^0}{(\alpha_v+\mu_v)\mu_v}\eta\alpha_h+\xi\alpha_h-\frac{\alpha_h+\mu_h}{S_h^0}\Big)E_h\\
&= \mu_h\Big(2-\frac{S_h^0}{S_h}-\frac{S_h}{S_h^0}\Big)+\frac{\beta_2\alpha_v S_v^0}{\alpha_v+\mu_v}\Big(2-\frac{S_v^0}{S_v}-\frac{S_v}{S_v^0}\Big)\\
&\quad+\frac{(\alpha_h+\mu_h)\mu_h}{\lambda_h}(R_0-1)E_h.
\end{aligned}
$$

It follows that $\frac{dL}{dt}\le 0$ if $R_0<1$. Furthermore, the equality $\frac{dL}{dt}=0$ holds if and only if $S_h(t)=S_h^0$, $S_v(t)=S_v^0$, and $E_h(t)=0$ for $t\in R_+$. It is easy to see that $\{E^0\}$ is the largest invariant set in $\{\frac{dL}{dt}=0\}$. By the LaSalle invariance principle [11], E^0 is globally attractive. This, combined with Theorem 3.2, implies that E^0 is globally asymptotically stable. □

In order to study the global stability of the infected equilibrium E^*, we need the following preparation.

According to Theorem 2.1, there is a continuous solution semiflow of (2), denoted by $\Phi: R_+\times X_+\to X_+$, where

$$
\Phi(t,x)=(S_h(t),E_h(t),i_h(t,\cdot),S_v(t),E_v(t),I_v(t))\quad\text{for } (t,x)\in R_+\times X_+
$$

with $(S_h(t),E_h(t),i_h(t,\cdot),S_v(t),E_v(t),I_v(t))$ being the solution of (2) with the initial value $(S_{h0},E_{h0},i_{h0},S_{v0},E_{v0},I_{v0})=x$. The semiflow Φ is also written as $\{\Phi(t)\}_{t\in R_+}$.

Define $\rho: X_+\to R_+$ by

$$
\rho(S_h,E_h,i_h,S_v,E_v,I_v)=S_h\int_0^\infty \beta_1(a)i_h(a)da+\beta_2 S_h I_v
$$

for $(S_h,E_h,i_h,S_v,E_v,I_v)\in X_+$. Let

$$
X_+^0=\{x\in X_+ \mid \text{there exists } t_0\in R_+ \text{ such that } \rho(\Phi(t_0,x))>0\}.
$$

Clearly, if $x\in X_+\setminus X_+^0$ then $\Phi(t,x)\to E^0$ as $t\to\infty$. With the help of Lemma 3.2 of Hale [7] and Theorem 2.3 of Thieme [22], one can obtain the following results with standard arguments (see, for example, Chen et al. [2]).

Theorem 4.2. *Suppose $R_0>1$. Then the following statements are true.*

(i) There exists a global attractor $\mathcal{A}$ for the solution semiflow Φ of (2) in X_+^0.

(ii) System (2) is uniformly strongly ρ-persistent, that is, there exists an $\varepsilon_0 > 0$ (independent of initial values) such that

$$\liminf_{t\to\infty} \rho(\Phi(t,x)) > \varepsilon_0 \qquad \text{for } x \in X_+^0.$$

Note that the global attractor $\mathcal{A}$ only can contain points with total trajectories through them since it must be invariant. A *total trajectory* of Φ is a function $X : R \to X_+$ such that $\Phi(s, X(t)) = X(t+s)$ for all $t \in R$ and all $s \in R_+$. For a total trajectory,

$$i_h(t,a) = i_h(t-a)\sigma(a) \qquad \text{for all } t \in R \text{ and } a \in R_+.$$

The alpha limit of a total trajectory $X(t)$ passing through $X(0) = X_0$ is

$$\alpha(X_0) = \cap_{t\le 0}\overline{\cup_{s\le t}\{X(s)\}} \subseteq \mathcal{A} \cap X_+^0.$$

Corollary 1. *Suppose $R_0 > 1$. Then there exists an $\varepsilon_0 > 0$ such that $S_h(t)$, $E_h(t)$, $i_h(t,0)$, $S_v(t)$, $E_v(t)$, $I_v(t) \ge \varepsilon_0$ for all $t \in R$, where $(S_h(t), E_h(t), i_h(t,\cdot), S_v(t), E_v(t), I_v(t))$ is any total trajectory in $\mathcal{A}$.*

Proof. First, since Ω is attracting and invariant, there exists $T \in R_+$ such that, for $t \ge T$,

$$S_h(t), E_h(t), \int_0^\infty i_h(t,a)da \le \frac{3\lambda_h}{2\mu_h}$$

and

$$S_v(t), E_v(t), I_v(t) \le \frac{3\lambda_v}{2\mu_v}.$$

Then, for $t \ge T$, it follows from the first equation of (2) that

$$\frac{dS_h(t)}{dt} \ge \lambda_h - \left(\mu_h + \frac{3\lambda_h\|\beta_1\|_\infty}{2\mu_h} + \frac{3\lambda_v\beta_2}{2\mu_v}\right) S_h(t),$$

which implies $\liminf_{t\to\infty} S_h(t) \ge \frac{\lambda_h}{\mu_h + \frac{3\lambda_h\|\beta_1\|_\infty}{2\mu_h} + \frac{3\lambda_v\beta_2}{2\mu_v}} \triangleq \varepsilon_1$. By invariance, $S_h(t) \ge \varepsilon_1$ for $t \in R$. Similarly, $S_v(t) \ge \frac{\lambda_v}{\mu_v + \frac{3\lambda_h\|k\|_\infty}{2\mu_h}} \triangleq \varepsilon_2$ for $t \in R$.

Next, by Theorem 4.2 and invariance, there exists $\varepsilon_3 > 0$ such that $S_h(t)\int_0^\infty \beta_1(a) i_h(t,a)da + \beta_2 S_h(t)I_v(t) \ge \varepsilon_3$ for $t \in R$. This, combined with the second equation of (2), gives

$$\frac{dE_h(t)}{dt} \ge \varepsilon_3 - (\alpha_h + \mu_h)E_h(t) \qquad \text{for } t \in R.$$

It follows that $\liminf_{t\to\infty} E_h(t) \ge \frac{\varepsilon_3}{\alpha_h+\mu_h} \triangleq \varepsilon_4$ and hence $E_h(t) \ge \varepsilon_4$ for $t \in R$ by invariance again. Therefore, $i_h(t,0) = \alpha_h E_h(t) \ge \alpha_h\varepsilon_4 \triangleq \varepsilon_5$ for $t \in R$. Then, for $t \in R$,

$$\begin{aligned}
\frac{dE_v(t)}{dt} &\ge \varepsilon_2 \int_0^\infty k(a) i_h(t-a,0)\sigma(a)da - (\alpha_v + \mu_v)E_v(t) \\
&\ge \varepsilon_2\varepsilon_5 \int_0^\infty k(a)\sigma(a)da - (\alpha_v + \mu_v)E_v(t) \\
&= \varepsilon_2\varepsilon_5\eta - (\alpha_v + \mu_v)E_v(t),
\end{aligned}$$

which implies that $E_v(t) \ge \frac{\varepsilon_2\varepsilon_5\eta}{\alpha_v+\mu_v} \triangleq \varepsilon_6$ for $t \in R$. Finally, from $\frac{dI_v(t)}{dt} \ge \alpha_v\varepsilon_6 - \mu_v I_v(t)$ for $t \in R$, we can similarly get $I_v(t) \ge \frac{\alpha_v\varepsilon_6}{\mu_v} \triangleq \varepsilon_7$ for $t \in R$.

Letting $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7\}$ finishes the proof. $\square$

Now, we are ready to establish the global stability of the infected equilibrium E^* with the approach of Lyapunov functionals.

Theorem 4.3. *If $R_0 > 1$, then the infected equilibrium E^* of (2) is globally asymptotically stable in X_+^0.*

Proof. By Theorem 3.2, it suffices to show that $\mathcal{A} = \{E^*\}$. To build a Lyapunov functional, we need the function $g : (0,\infty) \ni z \to z - 1 - \ln z \in R$. Note that $g(z) \geq 0$ for $z \in (0,\infty)$ and $g(z) = 0$ if and only if $z = 1$.

Let $X(t) = (S_h(t), E_h(t), i_h(t,\cdot), S_v(t), E_v(t), I_v(t))$ be a total trajectory in $\mathcal{A}$. Note that all $S_h(t)$, $E_h(t)$, $i_h(t,0)$, $S_v(t)$, $E_v(t)$, and $I_v(t)$ are bounded above. Moreover, by Corollary 1, they are also bounded away from 0. Therefore, there exists an $\varepsilon_0 > 0$ such that $0 \leq g(z) < \varepsilon_0$ for $z = \frac{S_h(t)}{S_h^*}$, $\frac{E_h(t)}{E_h^*}$, $\frac{i_h(t,0)}{i_h^*(0)}$, $\frac{S_v(t)}{S_v^*}$, $\frac{E_v(t)}{E_v^*}$, and $\frac{I_v(t)}{I_v^*}$ for all $t \in R$. Noting $\frac{i_h(t,a)}{i_h^*(a)} = \frac{i_h(t-a,0)}{i_h^*(0)}$, we have $0 \leq g(\frac{i_h(t,a)}{i_h^*(a)}) < \varepsilon_0$ for all $t \in R$ and $a \in R_+$.

Define a Lyapunov functional

$$W = W(S_h, E_h, i_h, S_v, E_v, I_v) = W_1 + W_2 + W_3,$$

where

$$\begin{aligned}
W_1 &= g\left(\frac{S_h}{S_h^*}\right) + \frac{E_h^*}{S_h^*} g\Big(\frac{E_h}{E_h^*}\Big),\\
W_2 &= \frac{\beta_2 I_v^*}{\eta\alpha_h E_h^*} \int_0^\infty \rho_1(a) i_h^*(a) g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big) da\\
&\quad + \int_0^\infty \rho_2(a) i_h^*(a) g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big) da,\\
W_3 &= \frac{\beta_2\alpha_v S_v^*}{(\alpha_v+\mu_v)\mu_v} g\Big(\frac{S_v}{S_v^*}\Big) + \frac{\beta_2\alpha_v E_v^*}{(\alpha_v+\mu_v)\mu_v} g\Big(\frac{E_v}{E_v^*}\Big)\\
&\quad + \frac{\beta_2 I_v^*}{\mu_v} g\Big(\frac{I_v}{I_v^*}\Big).
\end{aligned}$$

Here ρ_1 and ρ_2 are those functions defined by (9). Then W is well-defined and is bounded on $X(t)$. In the following, we calculate the time derivative of the components of W along solutions of (2) one by one.

Firstly,

$$\begin{aligned}
&\frac{dW_1}{dt}\\
={}& \frac{1}{S_h^*}\Big(1 - \frac{S_h^*}{S_h}\Big)\Big(\lambda_h - S_h \int_0^\infty \beta_1(a) i_h(t,a) da - \beta_2 S_h I_v - \mu_h S_h\Big)\\
&+ \frac{1}{S_h^*}\Big(1 - \frac{E_h^*}{E_h}\Big)\Big(S_h \int_0^\infty \beta_1(a) i_h(t,a) da + \beta_2 S_h I_v - (\alpha_h+\mu_h) E_h\Big).
\end{aligned}$$

This, combined with

$$\begin{aligned}
\lambda_h &= S_h^*(\xi\alpha_h E_h^* + \mu_h + \beta_2 I_v^*),\\
\frac{(\alpha_h+\mu_h)E_h^*}{S_h^*} &= \xi\alpha_h E_h^* + \beta_2 I_v^*,\\
\frac{(\alpha_h+\mu_h)E_h}{S_h^*} &= \xi\alpha_h E_h + \beta_2 I_v^* \frac{E_h}{E_h^*},
\end{aligned}$$

gives

$$\begin{aligned}
\frac{dW_1}{dt} &= \frac{1}{S_h^*}\Big(1-\frac{S_h^*}{S_h}\Big)\Big(\mu_h S_h^* - \mu_h S_h + S_h^*(\xi\alpha_h E_h^* + \beta_2 I_v^*)\Big) \\
&\quad -\frac{1}{S_h^*}\Big(1-\frac{S_h^*}{S_h}\Big)S_h\int_0^\infty \beta_1(a) i_h(t,a)da \\
&\quad -\frac{1}{S_h^*}\Big(1-\frac{S_h^*}{S_h}\Big)\beta_2 S_h I_v + \frac{S_h}{S_h^*}\int_0^\infty \beta_1(a) i_h(t,a)da \\
&\quad -\frac{E_h^* S_h}{E_h S_h^*}\int_0^\infty \beta_1(a) i_h(t,a)da + \beta_2 I_v \frac{S_h}{S_h^*} \\
&\quad -\beta_2 I_v^* \frac{I_v S_h E_h^*}{I_v^* S_h^* E_h} - \frac{\alpha_h+\mu_h}{S_h^*}E_h + \frac{\alpha_h+\mu_h}{S_h^*}E_h^* \\
&= \mu_h\Big(2-\frac{S_h^*}{S_h}-\frac{S_h}{S_h^*}\Big) + (\xi\alpha_h E_h^* + \beta_2 I_v^*)\Big(1-\frac{S_h^*}{S_h}\Big) \\
&\quad +\int_0^\infty \beta_1(a) i_h(t,a)da + \beta_2 I_v \\
&\quad -\int_0^\infty \beta_1(a) i_h(t,a)\frac{E_h^* S_h}{E_h S_h^*}da - \beta_2 I_v^* \frac{I_v S_h E_h^*}{I_v^* S_h^* E_h} \\
&\quad -\xi\alpha_h E_h - \beta_2 I_v^* \frac{E_h}{E_h^*} + \xi\alpha_h E_h^* + \beta_2 I_v^*.
\end{aligned}$$

Secondly,

$$\begin{aligned}
&\frac{dW_2}{dt} \\
= \;&\frac{\beta_2 I_v^*}{\eta\alpha_h E_h^*}\int_0^\infty \rho_1(a)\Big(1-\frac{i_h^*(a)}{i_h(t,a)}\Big)\Big(-\frac{\partial i_h(t,a)}{\partial a} - \delta(a) i_h(t,a)\Big)da \\
&+\int_0^\infty \rho_2(a)\Big(1-\frac{i_h^*(a)}{i_h(t,a)}\Big)\Big(-\frac{\partial i_h(t,a)}{\partial a} - \delta(a) i_h(t,a)\Big)da.
\end{aligned}$$

Note that $\rho_1(0)=\eta$ and

$$i_h^*(a)\frac{\partial}{\partial a}\Big(g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)\Big) = \Big(1-\frac{i_h^*(a)}{i_h(t,a)}\Big)\Big(\frac{\partial i_h(t,a)}{\partial a} + \delta(a) i_h(t,a)\Big).$$

Then, with integration by parts, we obtain

$$\begin{aligned}
&\int_0^\infty \rho_1(a)\Big(1-\frac{i_h^*(a)}{i_h(t,a)}\Big)\Big(\frac{\partial i_h(t,a)}{\partial a} + \delta(a) i_h(t,a)\Big)da \\
= \;&\int_0^\infty \rho_1(a) i_h^*(a)\frac{\partial}{\partial a}\Big(g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)\Big)da \\
= \;&\rho_1(a) i_h^*(a) g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)\Big|_{a=0}^{a=\infty} \\
&-\int_0^\infty g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)(\rho_1'(a) i_h^*(a) + \rho_1(a) i_h^{*\prime}(a))da \\
= \;&\rho_1(a) i_h^*(a) g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)\Big|_{a=\infty} - \rho_1(0) i_h^*(0) g\Big(\frac{i_h(t,0)}{i_h^*(0)}\Big) \\
&+\int_0^\infty k(a) i_h^*(a) g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)da.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^\infty \rho_2(a)\Big(1-\frac{i_h^*(a)}{i_h(t,a)}\Big)\Big(\frac{\partial i_h(t,a)}{\partial a}+\delta(a)i_h(t,a)\Big)da\\
=\;& \rho_2(a)i_h^*(a)g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)\Big|_{a=\infty}-\rho_2(0)i_h^*(0)g\Big(\frac{i_h(t,0)}{i_h^*(0)}\Big)\\
&+\int_0^\infty \beta_1(a)i_h^*(a)g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)da.
\end{aligned}$$

Therefore, with the help of $i_h(t,0)=\alpha_h E_h$ and $i_h^*(0)=\alpha_h E_h^*$, we get

$$\begin{aligned}
&\frac{dW_2}{dt}\\
=\;& \beta_2 I_v^* g\Big(\frac{E_h}{E_h^*}\Big)-\frac{\beta_2 I_v^*}{\eta\alpha_h E_h^*}\rho_1(a)i_h^*(a)g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)\Big|_{a=\infty}\\
&-\frac{\beta_2 I_v^*}{\eta\alpha_h E_h^*}\int_0^\infty k(a)i_h^*(a)g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)da+\xi\alpha_h E_h^* g\Big(\frac{E_h}{E_h^*}\Big)\\
&-\rho_2(a)i_h^*(a)g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)\Big|_{a=\infty}-\int_0^\infty \beta_1(a)i_h^*(a)g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)da.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\frac{dW_3}{dt}\\
=\;& \frac{\beta_2 I_v^*}{(\alpha_v+\mu_v)E_v^*}\Big(1-\frac{S_v^*}{S_v}\Big)\Big(\lambda_v-\int_0^\infty k(a)S_v i_h(t,a)da-\mu_v S_v\Big)\\
&+\frac{\beta_2 I_v^*}{(\alpha_v+\mu_v)E_v^*}\Big(1-\frac{E_v^*}{E_v}\Big)\Big(\int_0^\infty k(a)S_v i_h(t,a)da-(\alpha_v+\mu_v)E_v\Big)\\
&+\frac{\beta_2}{\mu_v}\Big(1-\frac{I_v^*}{I_v}\Big)(\alpha_v E_v-\mu_v I_v).
\end{aligned}$$

Since

$$\begin{aligned}
\lambda_v &= \mu_v S_v^*+\eta\alpha_h E_h^* S_v^*,\\
\eta\alpha_h E_h^* S_v^* &= (\alpha_v+\mu_v)E_v^*,\\
\frac{\beta_2 I_v^*}{(\alpha_v+\mu_v)E_v^*} &= \frac{\beta_2 I_v^*}{\eta\alpha_h E_h^* S_v^*},
\end{aligned}$$

we have

$$\begin{aligned}
\frac{dW_3}{dt} =\;& \frac{\beta_2 I_v^*}{\eta\alpha_h E_h^* S_v^*}\Big(1-\frac{S_v^*}{S_v}\Big)(\mu_v S_v^*-\mu_v S_v)\\
&+\frac{\beta_2 I_v^*}{\eta\alpha_h E_h^* S_v^*}\Big(1-\frac{S_v^*}{S_v}\Big)\eta\alpha_h E_h^* S_v^*\\
&-\frac{\beta_2 I_v^*}{\eta\alpha_h E_h^* S_v^*}\Big(1-\frac{S_v^*}{S_v}\Big)S_v\int_0^\infty k(a)i_h(t,a)da\\
&+\frac{\beta_2 I_v^*}{\eta\alpha_h E_h^* S_v^*}\Big(1-\frac{E_v^*}{E_v}\Big)S_v\int_0^\infty k(a)i_h(t,a)da\\
&-\beta_2 I_v^*\Big(1-\frac{E_v^*}{E_v}\Big)\frac{E_v}{E_v^*}+\beta_2 I_v^*\frac{E_v}{E_v^*}-\beta_2 I_v-\beta_2 I_v^*\frac{I_v^* E_v}{I_v E_v^*}+\beta_2 I_v^*
\end{aligned}$$

$$
\begin{aligned}
=\ & \frac{\beta_2 I_v^* \mu_v}{\eta\alpha_h E_h^*}\Big(2-\frac{S_v^*}{S_v}-\frac{S_v}{S_v^*}\Big)-\beta_2 I_v^*\Big(\frac{S_v^*}{S_v}-1\Big) \\
& +\beta_2 I_v^* \int_0^\infty k(a) i_h(t,a)da-\frac{\beta_2 I_v^*}{\eta\alpha_h E_h^*}\frac{E_v^* S_v}{E_v S_v^*}\int_0^\infty k(a)i_h(t,a)da \\
& +\beta_2 I_v^*-\beta_2 I_v-\beta_2 I_v^*\frac{I_v^* E_v}{I_v E_v^*}+\beta_2 I_v^*.
\end{aligned}
$$

To summarize, we have obtained

$$
\begin{aligned}
\frac{dW}{dt} =\ & \mu_h\Big(2-\frac{S_h^*}{S_h}-\frac{S_h}{S_h^*}\Big)-(\xi\alpha_h E_h^*+\beta_2 I_v^*)\Big(\frac{S_h^*}{S_h}-1-\ln\frac{S_h^*}{S_h}\Big) \\
& -\int_0^\infty \beta_1(a) i_h^*(a)\Big(\frac{i_h(t,a)E_h^* S_h}{i_h^*(a)E_h S_h^*}-1-\ln\frac{i_h(t,a)E_h^* S_h}{i_h^*(a)E_h S_h^*}\Big)da \\
& -\beta_2 I_v^*\frac{I_v S_h E_h^*}{I_v^* S_h^* E_h}-(\xi\alpha_h E_h^*+\beta_2 I_v^*)\Big(\frac{E_h}{E_h^*}-1-\ln\frac{E_h}{E_h^*}\Big) \\
& +\beta_2 I_v^* g\Big(\frac{E_h}{E_h^*}\Big)+\xi\alpha_h E_h^* g\Big(\frac{E_h}{E_h^*}\Big) \\
& -\Big(\frac{\beta_2 I_v^*}{\eta\alpha_h E_h^*}\rho_1(a)+\rho_2(a)\Big)i_h^*(a)g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)\Big|_{a=\infty} \\
& -\frac{\beta_2 I_v^*}{\eta\alpha_h E_h^*}\int_0^* k(a)i_h^*(a)\Big(\frac{i_h(t,a)E_v^* S_v}{i_h^*(a)E_v S_v^*}-1-\ln\frac{i_h(t,a)E_v^* S_v}{i_h^*(a)E_v S_v^*}\Big)da \\
& +\frac{\beta_2 I_v^* \mu_v}{\eta\alpha_h E_h^*}\Big(2-\frac{S_v^*}{S_v}-\frac{S_v}{S_v^*}\Big)-\beta_2 I_v^*\Big(\frac{S_v^*}{S_v}-1-\ln\frac{S_v^*}{S_v}\Big) \\
& -\beta_2 I_v^*\Big(\frac{I_v^* E_v}{I_v E_v^*}-1-\ln\frac{I_v^* E_v}{I_v E_v^*}\Big)+\beta_2 I_v^*\Big(1+\ln\frac{I_v S_h E_h^*}{I_v^* S_h^* E_h}\Big) \\
=\ & \mu_h\Big(2-\frac{S_h^*}{S_h}-\frac{S_h}{S_h^*}\Big)-(\xi\alpha_h E_h^*+\beta_2 I_v^*)g\Big(\frac{S_h^*}{S_h}\Big) \\
& -\int_0^\infty \beta_1(a)i_h^*(a)g\Big(\frac{i_h(t,a)E_h^* S_h}{i_h^*(a)E_h S_h^*}\Big)da-\beta_2 I_v^* g\Big(\frac{I_v S_h E_h^*}{I_v^* S_h^* E_h}\Big) \\
& -\Big(\frac{\beta_2 I_v^*}{\eta\alpha_h E_h^*}\rho_1(a)+\rho_2(a)\Big)i_h^*(a)g\Big(\frac{i_h(t,a)}{i_h^*(a)}\Big)\Big|_{a=\infty} \\
& -\frac{\beta_2 I_v^*}{\eta\alpha_h E_h^*}\int_0^\infty k(a)i_h^*(a)g\Big(\frac{i_h(t,a)E_v^* S_v}{i_h^*(a)E_v S_v^*}\Big)da \\
& +\frac{\beta_2 I_v^* \mu_v}{\eta\alpha_h E_h^*}\Big(2-\frac{S_v^*}{S_v}-\frac{S_v}{S_v^*}\Big)-\beta_2 I_v^* g\Big(\frac{S_v^*}{S_v}\Big)-\beta_2 I_v^* g\Big(\frac{I_v^* E_v}{I_v E_v^*}\Big) \\
\leq\ & 0.
\end{aligned}
$$

Therefore, W is nonincreasing. Since W is bounded on $X(\cdot)$, the alpha limit set of $X(\cdot)$ must be contained in the largest invariant subset of $\{\frac{dW}{dt}=0\}$, which is easily identified to be $\{E^*\}$. It follows that $W(X(t)) \leq W(E^*)$ for all $t \in R$. This gives $X(t) \equiv E^*$ and hence $\mathcal{A}=\{E^*\}$, which completes the proof. □

5. **Numerical simulations.** In this section, we illustrate the theoretical results obtained in Section 4 with numerical simulations. For this purpose, we take $k(a)=k$. For the form of $\beta_1(a)$, we give some explanation. In general, when the infection age a is relatively small, the age-dependent horizontal transmission rate $\beta_1(a)$ of the disease from the infectious hosts to susceptible hosts is relatively small. With

the increase of the infection age, the infection rate also increases and then tends to a constant. When the infection age is very large, the infection rate is reduced to 0 due to the loss of infectivity. Similar explanation can be given for the form of the age-dependent recovery rate $\gamma(a)$. For some more details, we refer readers to [4, 6]. Therefore, we take the following forms for β_1 and γ in the simulations.

$$\beta_1(a) = \begin{cases} c_1, & 0 \le a < 10, \\ c_1 + c_2(a-10)e^{-0.009(a-25)^2}, & 10 \le a < 25, \\ c_2, & 25 \le a < 50, \\ 0, & a \ge 50, \end{cases}$$

and

$$\gamma(a) = \begin{cases} 0, & 0 \le a < 10, \\ \frac{c_3}{15}(\arctan 50)(a-10), & 10 \le a < 25, \\ c_3 \arctan(75-a), & 25 \le a < 50, \\ c_3 \arctan(25), & a \ge 50. \end{cases}$$

We first set parameters $\lambda_h = 10$, $\lambda_v = 500$, $\mu_h = 0.008$, $\beta_2 = 1.81 \times 10^{-7}$, $\mu_v = 0.05$, $\alpha_h = 0.833$, $\alpha_v = 0.05$, $k = 4.1665 \times 10^{-5}$, $c_1 = 0.00003$, $c_2 = 0.00005$, and $c_3 = 5.6 \times 10^{-6}$, which are chosen from some recent studies [4, 6]. With these parameters, the basic reproductive number is $R_0 = 0.8286 < 1$. Thus, the infection-free equilibrium E^0 is globally asymptotically stable by Theorem 4.1. Fig. 1 shows the time evolution of the solution with the initial value $(1000, 100, 5(a+3)e^{-0.2(a+3)}, 10000, 100, 1000)$.

Next, we take another set of parameter values, $\lambda_h = 10$, $\lambda_v = 500$, $\mu_h = 0.008$, $\beta_2 = 1.81 \times 10^{-6}$, $\mu_v = 0.01$, $\alpha_h = 0.08333$, $\alpha_v = 0.05$, $k = 4.1665 \times 10^{-5}$, $c_1 = 0.00003$, $c_2 = 0.00005$, and $c_3 = 5.6 \times 10^{-6}$. In this case, $R_0 = 9.2655 > 1$. Then Theorem 4.3 tells us that the infected equilibrium E^* is globally asymptotically stable. Fig. 2 supports this with the time evolution of the solution with the initial value $(1000, 100, 0.5(a+3)e^{-0.2(a+3)}, 10000, 100, 1000)$.

6. **Conclusion.** Infection age is a very important factor in the transmission of infectious diseases such as malaria, TB, and HIV. In this paper, we incorporated infection age into a vector-host epidemic model with direct transmission. In the model, we also took into account the exposed individuals in both human and vector populations. We assumed that the level of contagiousness and the rate of removal (recovery) of infected hosts depend on the infection age. Therefore, our model is described by a system of ordinary differential equations coupled with a partial differential equation, which is very challenging to study because it is an infinitely dimensional system. With the approach of Lyapunov functionals and some recently developed techniques on global analysis in [15, 16], we have established a threshold dynamics completely determined by the basic reproduction number. That is, the infection-free equilibrium is globally asymptotically stable if the basic reproduction number is less than one while the infected equilibrium is globally asymptotically stable if the basic reproduction number is greater than one. Numerical simulations are conducted to illustrate the stability results.

Our result supports the claim that infection age can affect the number of average secondary infections, that is, the effect of infection age is embodied in the

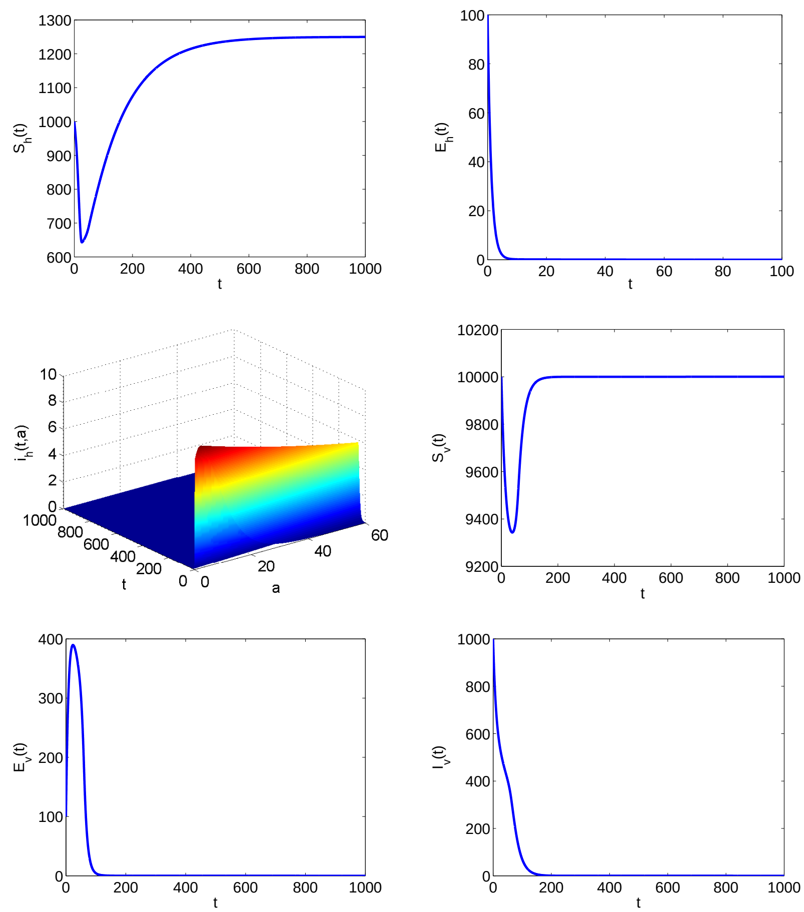

FIGURE 1. When $R_0 < 1$, the infection-free equilibrium E^0 of (2) is globally asymptotically stable. Here since $E_h(t)$ converges to 0 very fast, we use the time interval $[0, 100]$ different from the interval $[0, 1000]$ for other components.

expression of the basic reproduction number R_0. By appropriate control measures, one can decrease the survival probability to infection age a, $\sigma(a)$, the horizontal transmission rate $\beta_1(a)$, and the biting rate $k(a)$. This will decrease the value of R_0 and possibly will eliminate the disease. Even if we cannot eliminate the disease, from the expression of the infected equilibrium one can easily show that this will decrease the levels of E_h^*, $i_h^*(0)$, E_v^*, and I_v^*. Because of the globally asymptotical stability of E^*, we can keep the infection at a tolerance level.

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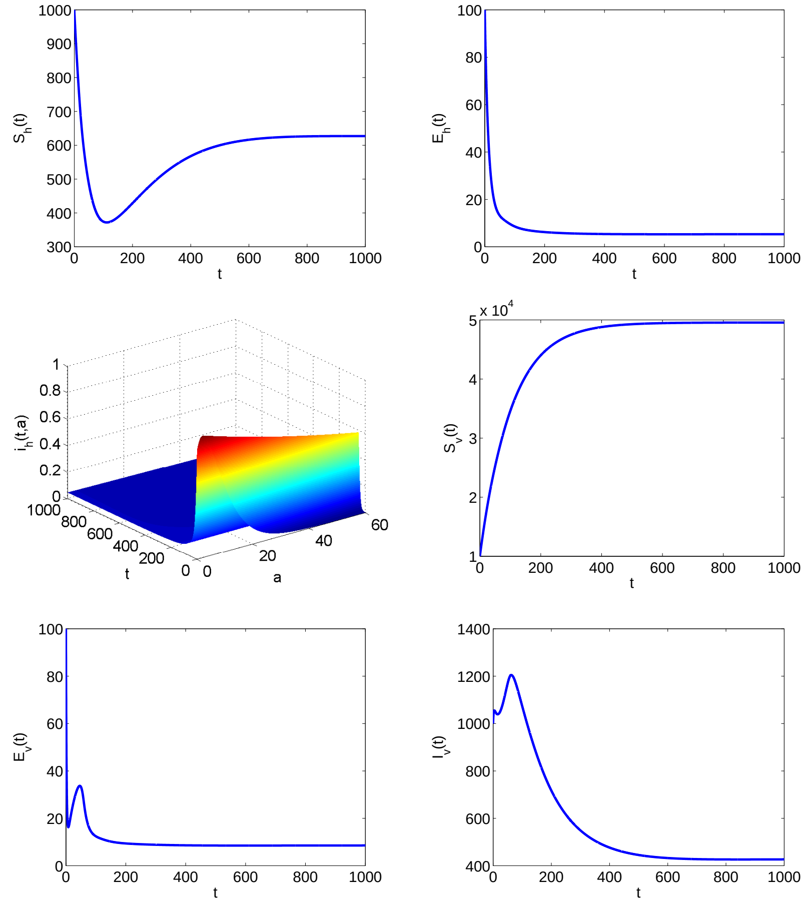

FIGURE 2. When $R_0 > 1$, the infected equilibrium E^* of (2) is globally asymptotically stable.

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