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INVASION ENTIRE SOLUTIONS IN A TIME PERIODIC LOTKA-VOLTERRA COMPETITION SYSTEM WITH DIFFUSION

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Abstract. This paper is concerned with invasion entire solutions of a monostable time periodic Lotka-Volterra competition-diffusion system. We first give the asymptotic behaviors of time periodic traveling wave solutions at infinity by a dynamical approach coupled with the two-sided Laplace transform. According to these asymptotic behaviors, we then obtain some key estimates which are crucial for the construction of an appropriate pair of sub-super solutions. Finally, using the sub-super solutions method and comparison principle, we establish the existence of invasion entire solutions which behave as two periodic traveling fronts with different speeds propagating from both sides of x-axis. In other words, we formulate a new invasion way of the superior species to the inferior one in a time periodic environment.

1. Introduction.

In this paper, we consider the following time periodic Lotka-Volterra competition-diffusion system

$$\begin{cases} u_t = u_{xx} + u(r_1(t) - a_1(t)u - b_1(t)v), \\ v_t = dv_{xx} + v(r_2(t) - a_2(t)u - b_2(t)v), \end{cases} \tag{1.1}$$

where $u = u(t,x)$ and $v = v(t,x)$ denote the densities of two competing species at time $t \in \mathbb{R}^+$ and $x \in \mathbb{R}$, $d \in (0,1]$ denotes the relatively diffusive coefficient of the two species, $r_i(t), a_i(t)$ and $b_i(t)$ are T-periodic continuous functions, $a_i(\cdot)$ and $b_i(\cdot)$ are positive in [0,T], and $\overline{r_i} := \frac{1}{T}\int_0^T r_i(t)dt > 0$, where $i = 1, 2$. Systems like (1.1) arise in interactive populations which live in a fluctuating environment, for instance, physical environmental conditions such as temperature and humidity and the availability of food, water and other resources usually vary in time with seasonal or daily variations [45]. Time periodic traveling waves of (1.1) are solutions with the form

$$\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} = \begin{pmatrix} X(t,x-ct) \\ Y(t,x-ct) \end{pmatrix}$$

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satisfying

$$\begin{pmatrix} X(t+T,z) \\ Y(t+T,z) \end{pmatrix} = \begin{pmatrix} X(t,z) \\ Y(t,z) \end{pmatrix}$$

and

$$\begin{pmatrix} X(t,\pm\infty) \\ Y(t,\pm\infty) \end{pmatrix} := \lim_{z\to\pm\infty} \begin{pmatrix} X(t,z) \\ Y(t,z) \end{pmatrix} = \begin{pmatrix} u^{\pm}(t) \\ v^{\pm}(t) \end{pmatrix},$$

where $c \in \mathbb{R}$ is the wave speed, $z = x - ct$ is the co-moving frame coordinate, and $(u^{\pm}(t), v^{\pm}(t))$ are periodic solutions of the corresponding kinetic system

$$\begin{cases} \frac{du}{dt} = u(r_1(t) - a_1(t)u - b_1(t)v), \\ \frac{dv}{dt} = v(r_2(t) - a_2(t)u - b_2(t)v). \end{cases} \tag{1.2}$$

Traveling wave solutions of system (1.1) with autonomous nonlinearities have been extensively studied. In particular, we can refer to Hosono [17] and Kan-on [20] for the monostable case, Conley and Gardner [8], Gardner [11] and Kan [19] for the bistable case, Tang and Fife [36] and Vuuren [37] for the coexistence case. At the same time, during the past decades, there have been many works on the space/time periodic traveling waves of scalar reaction-diffusion equations. For instance, one can see Alikakos et al. [1], Bates and Chen [4] and Shen [33] on time periodic traveling waves of the local, nonlocal and lattice equations, respectively, Berestycki and Hamel [5] and Hamel [14] on space periodic traveling waves, and Nadin [30, 31] and Nolen et al. [32] on the space-time periodic traveling waves.

It is well known that traveling wave solutions are special examples of the so-called *entire solutions* defined for all time and whole space. As we all know, it is of great significance in studying the entire solution since it is essential for a full understanding of the transient dynamics and structures of the global attractor. In addition, entire solutions can be used to describe the dynamics of two solutions that have distinct histories in the configuration, though their asymptotic profiles as $t \to +\infty$ coincide. The study of new types of entire solutions can be traced back to the works of Hamel and Nadirashvili [15, 16] and Yagisita [41], see also [10, 7, 12, 28, 22, 39] for equations with and without delays, and [21, 35] with nonlocal dispersal. Note that all these works mainly concentrate on entire solutions of scalar space-time homogeneous equations. Recently, some researchers paid attention to the study of entire solutions for space/time periodic equations (see [34, 6, 23, 25]). With regard to some systems, Morita and Tachibana [29] first established the existence of entire solutions for a homogeneous Lotka-Volterra competition-diffusion system while Li et al [24] lately considered the corresponding nonlocal dispersal system. The basic idea in establishing such entire solutions is to use traveling fronts propagating from both sides of the x-axis to construct sub-super solutions, and then obtain the existence of entire solutions by comparison principle. A similar result was established by Guo and Wu [13] for the discrete system. One can also see Zhang et al. [42] for a nonlocal dispersal epidemic system.

However, to the best of our knowledge, the issue on constructing new types of entire solutions other than traveling waves for time periodic reaction-diffusion systems is still open, which is the motivation of our present work. More precisely, we deal with the time periodic system (1.1) focusing on the following monostable case

$$\overline{r_1} > \max_{t\in[0,T]} \left(\frac{b_1(t)}{b_2(t)}\right) \overline{r_2} > 0, \quad \min_{t\in[0,T]} \left(\frac{a_2(t)}{a_1(t)}\right) \overline{r_1} \geq \overline{r_2} > 0, \tag{1.3}$$

which implies that (1.2) has only three nonnegative T-periodic solutions $(0,0)$, $(p(t),0)$ and $(0,q(t))$, with $(p(t),0)$ globally stable and $(0,q(t))$ unstable in the positive quadrant $\mathbb{R}_+^2=\{(u,v)|\ u\geq 0,\ v\geq 0\}$, where $p(t)$ and $q(t)$ are given by

$$\begin{cases} p(t)=\frac{p_0 e^{\int_0^t r_1(s)ds}}{1+p_0\int_0^t e^{\int_0^s r_1(\tau)d\tau}a_1(s)ds},\ p_0=\frac{e^{\int_0^T r_1(s)ds}-1}{\int_0^T e^{\int_0^s r_1(\tau)d\tau}a_1(s)ds}, \\ q(t)=\frac{q_0 e^{\int_0^t r_2(s)ds}}{1+q_0\int_0^t e^{\int_0^s r_2(\tau)d\tau}b_2(s)ds},\ q_0=\frac{e^{\int_0^T r_2(s)ds}-1}{\int_0^T e^{\int_0^s r_2(\tau)d\tau}b_2(s)ds}. \end{cases}$$

For system (1.1), the time periodic traveling wave solution $(X(t,z),Y(t,z))$ connecting $(0,q(t))$ and $(p(t),0)$ actually satisfies

$$\begin{cases} X_t=X_{zz}+cX_z+X(r_1(t)-a_1(t)X-b_1(t)Y), \\ Y_t=dY_{zz}+cY_z+Y(r_2(t)-a_2(t)X-b_2(t)Y), \\ (X(t,z),Y(t,z))=(X(t+T,z),Y(t+T,z)), \\ \lim\limits_{z\to-\infty}(X,Y)=(0,q(t)),\ \lim\limits_{z\to+\infty}(X,Y)=(p(t),0). \end{cases} \tag{1.4}$$

In the past few years, there were a few works devoted to the study of this issue. In particular, Zhao and Ruan [43] established the existence, uniqueness and stability of time periodic traveling waves under the monostable assumption (1.3). In 2014, the authors extended the results to a class of more general time-periodic advection-reaction-diffusion systems in [44]. In addition, Bao and Wang [3] obtained the existence and stability of time periodic traveling waves for the bistable case. Very recently, Bao et al. [2] further studied the existence, non-existence and asymptotic stability of bistable time-periodic traveling curved fronts in two-dimensional spatial space.

In our present paper, we shall consider the invasion entire solutions of system (1.1), that is, an entire solution $(u(t,x),v(t,x))$ satisfying (1.1) as well as the following conditions

$$\lim_{t\to-\infty}\{|u(t,x)|+|v(t,x))-q(t)|\}=0 \text{ locally in } x\in\mathbb{R},$$
$$\lim_{t\to+\infty}\{|u(t,x)-p(t)|+|v(t,x)|\}=0 \text{ locally in } x\in\mathbb{R}.$$

For future reference, we denote a vector by $\boldsymbol{u}=(u_1,\ldots,u_n)$, where u_i stands for the ith component of $\boldsymbol{u}$. Let I, $\Gamma\subset\mathbb{R}$ be two (possibly unbounded) intervals and $M\subset\mathbb{R}^n$. Denote by $C(I\times\Gamma,M)$ the space of continuous functions $\boldsymbol{u}:I\times\Gamma\to M$, $C_b(I\times\Gamma,M)$ the space of functions $\boldsymbol{u}\in C(I\times\Gamma,M)$ with $|\boldsymbol{u}|_\infty<\infty$, $C^{k,l}(I\times\Gamma,M)$ the space of functions $\boldsymbol{u}\in C(I\times\Gamma,M)$ such that $\boldsymbol{u}(\cdot,x)$ is k-time continuously differentiable and $\boldsymbol{u}(t,\cdot)$ is l-time continuously differentiable, $C_b^{k,l}(I\times\Gamma,M)$ the space of functions $\boldsymbol{u}\in C^{k,l}(I\times\Gamma,M)$ such that all partial derivatives of $\boldsymbol{u}$ are uniformly bounded. Throughout the paper, we always assume that

(A1): $r_i,\ a_i,\ b_i\in C^\theta(\mathbb{R},\mathbb{R})$ for some $0<\theta<1$, $r_i(t+T)=r_i(t)$, $a_i(t+T)=a_i(t)$, $b_i(t+T)=b_i(t)$, $i=1,2$.

(A2): $\overline{r_i}>0,\ a_i(t)>0,\ b_i(t)>0$ for all $t\in[0,T]$. Moreover, $\overline{r_1}>\max\limits_{t\in[0,T]}(\frac{b_1}{b_2})\overline{r_2}$ and $\overline{r_2}\leq\min\limits_{t\in[0,T]}(\frac{a_2}{a_1})\overline{r_1}$.

(A3): $a_1(t)p(t)-b_1(t)q(t)\geq a_2(t)p(t)-b_2(t)q(t)\geq 0$ for all $t\in[0,T]$.

Now let $u^*(t,x) = \frac{u(t,x)}{p(t)}$ and $v^*(t,x) = \frac{q(t)-v(t,x)}{q(t)}$, then (1.1) becomes (omitting $*$ for simplicity)

$$\begin{cases} u_t = u_{xx} + a_1 pu[1 - N_1(t) - u + N_1(t)v], \\ v_t = dv_{xx} + b_2 q(1-v)[N_2(t)u - v)], \end{cases} \tag{1.5}$$

where $N_1(t) = \frac{b_1(t)q(t)}{a_1(t)p(t)} \le 1$ and $N_2(t) = \frac{a_2(t)p(t)}{b_2(t)q(t)} \ge 1$ for all $t \in \mathbb{R}$. It is easy to see that

$$(P(t,z), Q(t,z)) := \left(\frac{X(t,z)}{p(t)}, \frac{q(t) - Y(t,z)}{q(t)}\right)$$

is a periodic traveling wave solution of (1.5) connecting $(0,0)$ and $(1,1)$, that is

$$\begin{cases} P_t = P_{zz} + cP_z + a_1 pP[1 - N_1(t) - P + N_1(t)Q], \\ Q_t = dQ_{zz} + cQ_z + b_2 q(1-Q)[N_2(t)P - Q)], \\ (P(t,z), Q(t,z)) = (P(t+T,z), Q(t+T,z)), \\ \lim\limits_{z\to-\infty}(P,Q) = (0,0), \ \lim\limits_{z\to+\infty}(P,Q) = (1,1). \end{cases} \tag{1.6}$$

Clearly, if $(P(t,z), Q(t,z)) = (P(t, x-ct), Q(t, x-ct))$ is a periodic traveling wave solution of (1.5) with speed c, then $(\tilde{P}(t,z), \tilde{Q}(t,z)) := (P(t,-x-ct), Q(t,-x-ct))$ is a periodic traveling wave solution of (1.5) as well, with speed $\tilde{c} := -c > 0$ and satisfies

$$\lim_{z\to-\infty}(\tilde{P}, \tilde{Q}) = (1,1) \text{ and } \lim_{z\to+\infty}(\tilde{P}, \tilde{Q}) = (0,0).$$

Under assumptions (A1)–(A3), for any $c \le c^*$, (1.5) admits a time periodic traveling wave solution $(P,Q) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R}, \mathbb{R}^2)$ with $(P_z, Q_z) > (0,0)$ and $(P,Q) \le (1,1)$ for all $(t,z) \in \mathbb{R}\times\mathbb{R}$, where

$$c^* = -2\sqrt{\overline{(a_1 p - b_1 q)}}$$

is the maximal wave speed (see [43]). In particular, the authors in [43] obtained the exact exponential decay rate of solutions of (1.6) as $z \to -\infty$ in establishing the uniqueness of the periodic traveling wave solution. Next we will consider the cooperative system (1.5) to obtain invasion entire solutions of system (1.1). In order to employ the basic idea developed in [29, 13, 27] to establish the existence of such entire solutions, we essentially need some estimates which are concerned with the asymptotic behavior of the periodic traveling wave solution. One of the main difficulties arises in obtaining the exact exponential decay rate of the periodic traveling wave as it tends to its limiting state. In the autonomous case, the asymptotic behavior is usually obtained by investigating the linearized equations at the equilibrium points (see e.g. [38, 18]), which can not be applied to system (1.1) since the presence of time dependent nonlinearities. Inspired by [43], we employ the two-sided Laplace transform method to obtain the exact exponential decay rate, which is essentially based on some a priori exponential decay estimates of the periodic traveling wave tails as $z \to +\infty$. In particular, unlike the a priori exponential estimates as $z \to -\infty$ characterized by the principle eigenvalue of the linear periodic eigenvalue problem associated with the linearized system at the unstable limiting state (see [43, Lemma 3.3]), the exponential estimates as $z \to +\infty$ can only be characterized by a small perturbation of the corresponding principle eigenvalue (see '$\lambda_{c,\epsilon}^{\pm}$' in Lemma 2.2). Fortunately, this small perturbation can be declined small enough such that it imposes no influence on the Laurent development of the resolvent near the isolated principle eigenvalue.

The rest of this paper is organized as follows. In Section 2, we study the exact exponential decay rate of a periodic traveling wave solution of (1.5) as it approaches its stable limiting state. We then establish some key and useful estimates in Section 3. In Section 4, we establish the existence and qualitative properties of entire solutions by a comparing argument.

2. **Asymptotic behavior of periodic traveling wave fronts.** In this section we shall study the asymptotic behavior of time periodic traveling waves of (1.5).

Denote

$$\kappa=\overline{a_1p-b_1q},\ \phi(t)=e^{\int_0^t (a_1(s)p(s)-b_1(s)q(s))ds-\kappa t},\ \lambda_c^+=\frac{-c-\sqrt{c^2-4\kappa}}{2}$$

if $c\le c^*=-2\sqrt{\kappa}$, and

$$\begin{cases}\phi_d(t)=\phi_d(0)e^{-\int_0^t (b_2(s)q(s)+\rho_0)ds}+\int_0^t e^{-\int_s^t (b_2(\tau)q(\tau)+\rho_0)d\tau}a_2(s)p(s)\phi(s)ds,\\ \phi_d(0)=(1-e^{-\int_0^T (b_2(t)q(t)+\rho_0)dt})^{-1}\int_0^T e^{-\int_s^T (b_2(\tau)q(\tau)+\rho_0)d\tau}a_2(s)p(s)\phi(s)ds,\end{cases}$$

where $\rho_0=\kappa+(1-d)(\lambda_c^+)^2$. For completeness, we first state the following asymptotic behavior of solutions of system (1.6) as it approaches its unstable limiting state (see [43, Theorem 3.8]).

Proposition 1. *Assume (A1)-(A3) hold, and $(P(t,z),Q(t,z))\in C_b^{1,2}(\mathbb{R}\times\mathbb{R},\mathbb{R}^2)$ and c solve (1.6). Then*

$$\lim_{z\to-\infty}\frac{P(t,z)}{k_1|z|^l e^{\lambda_c^+ z}\phi(t)}=1,\quad \lim_{z\to-\infty}\frac{Q(t,z)}{k_1|z|^l e^{\lambda_c^+ z}\phi_d(t)}=1\quad \textit{uniformly in } t\in\mathbb{R},$$

and

$$\lim_{z\to-\infty}\frac{P_z(t,z)}{k_1|z|^l e^{\lambda_c^+ z}\phi(t)}=\lambda_c^+,\quad \lim_{z\to-\infty}\frac{Q_z(t,z)}{k_1|z|^l e^{\lambda_c^+ z}\phi_d(t)}=\lambda_c^+\quad \textit{uniformly in } t\in\mathbb{R},$$

where $k_1>0$ is a constant, $l=0$ if $c<c^*$ and $l=1$ if $c=c^*$.

In order to characterize the asymptotic behavior of time periodic traveling waves as $z\to+\infty$, we now list a useful lemma of the Harnack inequalities for cooperative parabolic system, which was given in [9] (see also [43, 3]).

Lemma 2.1. *Let*

$$L_k:=\sum_{i,j=1}^n a_{i,j}^k(t,\boldsymbol{x})\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^n b_i^k(t,x)\frac{\partial}{\partial x_i}-\frac{\partial}{\partial t}\ (k=1,2,\cdots,l)$$

be uniformly parabolic in an open domain $(\tau,M)\times\Omega$ of $(t,\boldsymbol{x})\subset\mathbb{R}\times\mathbb{R}^n$, that is, there is $\alpha_0>0$ such that $a_{i,j}^k(t,\boldsymbol{x})\xi_i\xi_j\ge\alpha_0\sum_{i=1}^n{\xi_i}^2$ for any n-tuples of real numbers $(\xi_1,\xi_2,\cdots,\xi_n)$, where $-\infty<\tau<M\le+\infty$ and ω is open and bounded. Suppose that $a_{i,j}^k,\ b_i^k\in C((\tau,M)\times\Omega,\mathbb{R})$ and

$$\underset{(\tau,M)\times\Omega}{sup}\left(\left|a_{i,j}^k(t,\boldsymbol{x})\right|+\left|b_i^k(t,\boldsymbol{x})\right|\right)\le\beta_0 \textit{ for some } \beta_0>0.$$

Assume that $\boldsymbol{w}=(w_1,w_2,\cdots,w_l)\in C([\tau,M)\times\overline{\Omega},\mathbb{R}^l)\cap C^{1,2}((\tau,M)\times\Omega,\mathbb{R}^l)$ satisfies:

$$\sum_{s=1}^l c^{k,s}(t,\boldsymbol{x})w_s+L_kw_k\le 0,\quad (t,\boldsymbol{x})\in(\tau,M)\times\Omega,\ k=1,2,\cdots,l,\tag{2.1}$$

where $c^{k,s} \in C((\tau, M) \times \Omega, \mathbb{R})$ *and* $c^{k,s} \geq 0$ *if* $k \neq s$, *and*

$$\sup_{(t,\boldsymbol{x}) \in (\tau, M) \times \Omega} \left| c^{k,s}(t, \boldsymbol{x}) \right| \leq \gamma_0 \ (k, s = 1, 2, \cdots, l) \text{ for some } \gamma_0 > 0.$$

Let D *and* U *be domains in* ω *such that* $D \subset\subset U$, $dist(\overline{D}, \partial U) > \rho$ *and* $|D| > \varepsilon$ *for certain positive constants* ρ *and* $\varepsilon > 0$. *Let* θ *be a positive constant with* $\tau + 4\theta < M$. *Then there exist positive constants* p, ω_1 *and* ω_2, *determined by* α_0, β_0, γ_0, ρ, ϵ, n, θ *and* $diam\Omega$, *such that*

$$\inf_{(\tau+3\theta, \tau+4\theta) \times D} w_k \geq \omega_1 \left\| (w_k)^+ \right\|_{L^p((\tau+\theta, \tau+2\theta) \times D)} - \omega_2 \max_{j=1,\cdots,k} \sup_{\partial_p((\tau, \tau+4\theta) \times U)} (w_j)^-,$$

here $(w_k)^+ = \max\{w_k, 0\}$, $(w_k)^- = \max\{-w_k, 0\}$ *and* $\partial_p((\tau, \tau + 4\theta) \times U) = \{\tau\} \times U \cup [\tau, \tau + 4\theta] \times \partial U$. *Moreover, if all inequalities in (2.1) are replaced by equalities, then the conclusion holds with* $p = +\infty$, *and with* ω_1, ω_2 *independent of* ϵ.

Let

$$(U(t, z), V(t, z)) = \left(\frac{p(t) - X(t, z)}{p(t)}, \frac{Y(t, z)}{q(t)} \right),$$

then we have

$$\begin{cases} U_t = U_{zz} + cU_z + g(t, U, V), \\ V_t = dV_{zz} + cV_z + h(t, U, V), \\ (U(t, z), V(t, z)) = (U(t + T, z), V(t + T, z)), \\ \lim\limits_{z \to -\infty} (U, V) = (1, 1), \ \lim\limits_{z \to +\infty} (U, V) = (0, 0), \end{cases} \tag{2.2}$$

where

$$\begin{cases} g(t, u, v) = -(1 - u)[a_1(t)p(t)u - b_1(t)q(t)v], \\ h(t, u, v) = -v[a_2(t)p(t)(1 - u) - b_2(t)q(t)(1 - v)]. \end{cases}$$

For any $c \leq c^* = -2\sqrt{\kappa}$, denote

$$\kappa_0 = -\overline{h_v(t, 0, 0)} = \overline{a_2 p - b_2 q}, \ \lambda_c^- = \frac{-c - \sqrt{c^2 + 4d\kappa_0}}{2d}, \ \psi(t) = e^{\int_0^t h_v(s,0,0)ds + \kappa_0 t},$$

and

$$\kappa_1 = -\overline{g_u(t, 0, 0)} = \overline{a_1 p}, \ \lambda_c = \frac{-c - \sqrt{c^2 + 4\kappa_1}}{2}, \ \widetilde{\psi}(t) = e^{\int_0^t g_u(s,0,0)ds + \kappa_1 t}.$$

To be specific and convenient, we give an additional assumption on the periodic coefficients.

(A4): $a_1(t)p(t) < 5\, b_1(t)q(t)$ for all $t \in [0, T]$.

Remark 1. It follows from (A4) and (A3) that $b_1(t)q(t) \leq a_1(t)p(t) < 5\, b_1(t)q(t)$ for all $t \in [0, T]$, that is, assumption (A4) is compatible with (A3). It should be emphasized here that (A4) is a technique assumption that ensures $\lambda_c^- > \lambda_c$ for any $c \leq c^*$, which is essential in obtaining the exact exponential decay rate of u in our present work. Indeed, (A3) yields that $\kappa_0 \leq \kappa < \kappa_1$, then a direct computation shows that

$$\begin{aligned}
(A4) \Rightarrow & \frac{\kappa}{\kappa_1} < \frac{4}{5} \\
\Rightarrow & \frac{\kappa_0}{\kappa_1} \le \frac{\kappa}{\kappa_1} < \frac{2}{\sqrt{1+\frac{\kappa_1}{\kappa}}+1} = \frac{2}{\sqrt{1+\frac{4\kappa_1}{(c^*)^2}}+1} = \min_{c\le c^*} \frac{2}{\sqrt{1+\frac{4\kappa_1}{c^2}}+1} \\
\Rightarrow & \frac{\kappa_0}{\kappa_1} < \frac{2}{\sqrt{1+\frac{4\kappa_1}{c^2}}+1} \quad \text{for any } c \le c^*, \\
\Leftrightarrow & \frac{-4\kappa_0}{2c} < \frac{4\kappa_1}{\sqrt{c^2+4\kappa_1}-c} \quad \text{for any } c \le c^*.
\end{aligned}$$

Thus,

$$\begin{aligned}
2(\lambda_c - \lambda_c^-) =& \frac{c+\sqrt{c^2+4d\kappa_0}}{d} - (c+\sqrt{c^2+4\kappa_1}) \\
=& \frac{4\kappa_0}{\sqrt{c^2+4d\kappa_0}-c} - \frac{4\kappa_1}{\sqrt{c^2+4\kappa_1}-c} \\
<& \frac{-4\kappa_0}{2c} - \frac{4\kappa_1}{\sqrt{c^2+4\kappa_1}-c} < 0 \quad \text{for any } c \le c^*.
\end{aligned}$$

Remark 2. We also remark here that when $d = 1$, condition (A4) can be deleted since $\lambda_c^- > \lambda_c$ holds certainly under condition (A3).

Noting that $g(t,0,0) = h(t,0,0) = g(t,1,1) = h(t,1,1) = 0$, system (2.2) can be written as

$$\begin{cases} U_t = U_{zz} + cU_z + U\int_0^1 g_u(t,\tau U,\tau V)d\tau + V\int_0^1 g_v(t,\tau U,\tau V)d\tau, \\ V_t = dV_{zz} + cV_z + U\int_0^1 h_u(t,\tau U,\tau V)d\tau + V\int_0^1 h_v(t,\tau U,\tau V)d\tau. \end{cases} \tag{2.3}$$

Let $D = (z-\frac{1}{4}, z+\frac{1}{4})$, $U = (z-\frac{1}{2}, z+\frac{1}{2})$, $\Omega = (z-1, z+1)$ with $z \in \mathbb{R}$, $\tau = 0$, and $\theta = T$. Since $U(\cdot,z)$ and $V(\cdot,z)$ are periodic of t, and (U,V) are both positive and bounded, Lemma 2.1 then implies that there is some $N > 0$ such that

$$(U(t,z),V(t,z)) \le N(U(t',z),V(t',z)) \quad \text{for all } z\in\mathbb{R},\ t,\ t'\in\mathbb{R}. \tag{2.4}$$

We now state an essential lemma for system (2.2) on the exponential decay estimates of the periodic traveling wave tails as $z \to +\infty$ by using the method similar to [2].

Lemma 2.2. *Assume (A1)-(A4) hold. Let $(U(t,z),V(t,z)) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R},\mathbb{R}^2)$ be a solution of (2.2). Then for any $0<\epsilon<\min\{1,\frac{\kappa_0}{C^+}\}$, there exist some constants $K_i > 0$ $(i = 1,2,3,4)$ and $\sigma < \lambda_c$ such that*

$$K_1 e^{\sigma z} \le U(t,z) \le K_2 e^{\lambda_{c,\epsilon}^+ z} \quad \textit{for any } (t,z)\in\mathbb{R}\times[0,+\infty) \tag{2.5}$$

and

$$K_3 e^{\lambda_{c,\epsilon}^- z} \le V(t,z) \le K_4 e^{\lambda_{c,\epsilon}^+ z} \quad \textit{for any } (t,z)\in\mathbb{R}\times[0,+\infty), \tag{2.6}$$

where $\lambda_{c,\epsilon}^{\pm} = \frac{-c-\sqrt{c^2+4d(\kappa_0\mp C^{\pm}\epsilon)}}{2d}$, $C^+ = \max_{[0,T]} a_2(t)p(t)$ *and* $C^- = \max_{[0,T]} b_2(t)q(t)$.

Proof. According to definitions of $\psi(t)$ and $\widetilde{\psi}(t)$, $\frac{U(t,z)}{\widetilde{\psi}(t)}$ and $\frac{V(t,z)}{\psi(t)}$ are $T-$periodic in t for any $z\in\mathbb{R}$. Let

$$\hat{u}(z) = \int_0^T \frac{U(t,z)}{\widetilde{\psi}(t)}dt, \quad \hat{v}(z) = \int_0^T \frac{V(t,z)}{\psi(t)}dt \quad \text{for any } z\in\mathbb{R},$$

then a direct calculation yields that

$$\begin{cases} \hat{u}_{zz} + c\hat{u}_z - \kappa_1 \hat{u} + \int_0^T \frac{b_1(t)q(t)V(t,z)}{\widetilde{\psi}(t)} dt \\ \qquad\qquad + \int_0^T \frac{a_1(t)p(t)U^2(t,z) - b_1(t)q(t)U(t,z)V(t,z)}{\widetilde{\psi}(t)} dt = 0, \\ d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} + \int_0^T \frac{a_2(t)p(t)U(t,z)V(t,z) - b_2(t)q(t)V^2(t,z)}{\psi(t)} dt = 0. \end{cases} \tag{2.7}$$

Since

$$\lim_{z\to+\infty} (U(t,z), V(t,z)) = (0,0) \text{ uniformly in } t \in \mathbb{R},$$

for any $0 < \epsilon < \min\{1, \frac{\kappa_0}{C^+}\}$, we can choose constant $M >> 1$ such that $(0,0) < (U(t,z), V(t,z)) \le (\epsilon, \epsilon)$ for any $(t,z) \in [0,T] \times [M, +\infty)$.

We first show (2.6). Let

$$V^+(z) = \rho e^{\lambda^+_{c,\epsilon} z} \text{ with } \lambda^+_{c,\epsilon} = \frac{-c - \sqrt{c^2 + 4d(\kappa_0 - C^+\epsilon)}}{2d} < 0,$$

then $V^+(z)$ is a solution of the linear equation

$$dv_{zz} + cv_z - \kappa_0 v + C^+ \epsilon v = 0. \tag{2.8}$$

Since $\hat{v}$ is bounded, we can choose $\rho > 0$ large enough such that $\hat{v}(M) \le \rho e^{\lambda^+_{c,\epsilon} M}$. In addition, we obtain from the second equation of (2.7) that

$$\begin{aligned} 0 &= d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} + \int_0^T \frac{a_2(t)p(t)U(t,z)V(t,z) - b_2(t)q(t)V^2(t,z)}{\psi(t)} dt \\ &\le d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} + \int_0^T \frac{a_2(t)p(t)U(t,z)V(t,z)}{\psi(t)} dt \\ &\le d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} + C^+ \epsilon \hat{v} \end{aligned}$$

for any $z \in [M, +\infty)$, which implies that $\hat{v}(z)$ is a subsolution of (2.8) in $[M, +\infty)$. Since $\lim_{z\to+\infty} \hat{v}(z) = \lim_{z\to+\infty} V^+(z) = 0$, the maximum principle then yields that $\hat{v}(z) \le \rho e^{\lambda^+_{c,\epsilon} z}$ for any $z \in [M, +\infty)$. Hence, there exists constant $\rho' \ge \rho$ such that $\hat{v}(z) \le \rho' e^{\lambda^+_{c,\epsilon} z}$ for any $z \in [0, +\infty)$. Furthermore, we have $\inf_{[0,T]} V(t,z) \le \frac{C_1}{T} \hat{v}(z) \le \frac{C_1 \rho'}{T} e^{\lambda^+_{c,\epsilon} z}$ for any $z \in [0, +\infty)$ with some constant $C_1 > 0$, which combining the Harnack inequalities (2.4) shows that there exists $K_4 > 0$ such that $V(t,z) \le K_4 e^{\lambda^+_{c,\epsilon} z}$ for any $(t,z) \in \mathbb{R} \times [0, +\infty)$. Similarly, let $V^-(z) = \eta e^{\lambda^-_{c,\epsilon} z}$, where $\lambda^-_{c,\epsilon} = \frac{-c - \sqrt{c^2 + 4d(\kappa_0 + C^-\epsilon)}}{2d} < 0$ and $\hat{v}(M) \ge \eta e^{\lambda^-_{c,\epsilon} M}$ for some $\eta > 0$. Then $V^-(z)$ satisfies

$$dv_{zz} + cv_z - \kappa_0 v - C^- \epsilon v = 0. \tag{2.9}$$

On the other hand, by the second equation of (2.7), we have

$$\begin{aligned} 0 &= d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} + \int_0^T \frac{a_2(t)p(t)U(t,z)V(t,z) - b_2(t)q(t)V^2(t,z)}{\psi(t)} dt \\ &\ge d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} - \int_0^T \frac{b_2(t)q(t)V^2(t,z)}{\psi(t)} dt \\ &\ge d\hat{v}_{zz} + c\hat{v}_z - \kappa_0 \hat{v} - C^- \epsilon \hat{v} \end{aligned}$$

for any $z \in [M, +\infty)$, thus $\hat{v}(z)$ is a supersolution of (2.9) in $[M, +\infty)$. Since $\lim_{z\to+\infty} \hat{v}(z) = \lim_{z\to+\infty} V^-(z) = 0$, it follows from the maximum principle that $\hat{v}(z) \ge$

$\eta e^{\lambda^-_{c,\epsilon}z}$ for any $z\in[M,+\infty)$. A similar argument yields that there exists $K_3>0$ such that $V(t,z)\geq K_3e^{\lambda^-_{c,\epsilon}z}$ for any $(t,z)\in\mathbb{R}\times[0,+\infty)$.

We now prove (2.5). Note that (A4) implies $\lambda_c<\lambda_c^-<\lambda^+_{c,\epsilon}<0$, it then follows from the definition of λ_c that $L:=-(\lambda^+_{c,\epsilon})^2-c\lambda^+_{c,\epsilon}+\kappa_1>0$. In view of $V(t,z)\leq K_4e^{\lambda^+_{c,\epsilon}z}$ for any $(t,z)\in\mathbb{R}\times[0,+\infty)$, the Harnack inequalities (2.4) imply that there exist positive constants M_1 and M_2 such that

$$\begin{aligned}\int_0^T\frac{b_1(t)q(t)V(t,z)}{\widetilde{\psi}(t)}dt&\leq T\max_{[0,T]}\frac{b_1(t)q(t)}{\widetilde{\psi}(t)}K_4e^{\lambda^+_{c,\epsilon}z}\leq M_1e^{\lambda^+_{c,\epsilon}z},\\ \int_0^T\frac{a_1(t)p(t)U^2(t,z)}{\widetilde{\psi}(t)}dt&\leq\max_{[0,T]}[a_1(t)p(t)\widetilde{\psi}(t)]\int_0^T\frac{U^2(t,z)}{\widetilde{\psi}^2(t)}dt\leq M_2(\hat{u}(z))^2\end{aligned}$$

for any $z\in[0,+\infty)$. Moreover, we know from $\lim\limits_{z\to+\infty}\hat{u}(z)=0$ that there exists a constant $M'>0$ such that $M_2\hat{u}(z)\leq\frac{1}{2}\min\{L,\kappa_1\}$ for any $z\in[M',+\infty)$. The first equation of (2.7) indicates that

$$\begin{aligned}0&=\hat{u}_{zz}+c\hat{u}_z-\kappa_1\hat{u}+\int_0^T\frac{b_1(t)q(t)V(t,z)}{\widetilde{\psi}(t)}dt\\&\quad+\int_0^T\frac{a_1(t)p(t)U^2(t,z)-b_1(t)q(t)U(t,z)V(t,z)}{\widetilde{\psi}(t)}dt\\&\leq\hat{u}_{zz}+c\hat{u}_z-\kappa_1\hat{u}+\int_0^T\frac{b_1(t)q(t)V(t,z)}{\widetilde{\psi}(t)}dt+\int_0^T\frac{a_1(t)p(t)U^2(t,z)}{\widetilde{\psi}(t)}dt\\&\leq\hat{u}_{zz}+c\hat{u}_z-\kappa_1\hat{u}+M_1e^{\lambda^+_{c,\epsilon}z}+M_2(\hat{u}(z))^2\\&\leq\hat{u}_{zz}+c\hat{u}_z-\kappa_1\hat{u}+M_1e^{\lambda^+_{c,\epsilon}z}+\frac{L}{2}\hat{u}\end{aligned}$$

for any $z\in[M',+\infty)$, that is, $\hat{u}$ is a subsolution of equation

$$-u_{zz}-cu_z+\kappa_1u-\frac{L}{2}u-M_1e^{\lambda^+_{c,\epsilon}z}=0,\quad z\in[M',+\infty).\tag{2.10}$$

Let $U^+(z)=\delta e^{\lambda^+_{c,\epsilon}z}$ with $\delta\geq\frac{2M_1}{L}$ large enough such that $\hat{u}(M')\leq\delta e^{\lambda^+_{c,\epsilon}M'}$, then

$$\begin{aligned}&-U^+_{zz}-cU^+_z+\kappa_1U^+-\frac{L}{2}U^+-M_1e^{\lambda^+_{c,\epsilon}z}\\&=[-(\lambda^+_{c,\epsilon})^2-c\lambda^+_{c,\epsilon}+\kappa_1-\frac{L}{2}-\frac{M_1}{\delta}]\delta e^{\lambda^+_{c,\epsilon}z}\\&=\left(\frac{L}{2}-\frac{M_1}{\delta}\right)\delta e^{\lambda^+_{c,\epsilon}z}\geq0\end{aligned}$$

for any $z\in[M',+\infty)$, which shows that $U^+(z)$ is a supersolution of (2.10). Since $\lim\limits_{z\to+\infty}\hat{u}(z)=\lim\limits_{z\to+\infty}U^+(z)=0$, the maximum principle yields that $\hat{u}(z)\leq U^+(z)$ for all $z\in[M',+\infty)$. On the other hand, there exists $N_3>0$ such that

$$\int_0^T\frac{b_1(t)q(t)U(t,z)V(t,z)}{\widetilde{\psi}(t)}dt\leq\max_{[0,T]}(b_1q)K_4e^{\lambda^+_{c,\epsilon}z}\cdot\hat{u}\leq N_3e^{\lambda^+_{c,\epsilon}z}\hat{u}$$

for any $z \in [0, +\infty)$, and we then have

$$
\begin{aligned}
0 &= \hat{u}_{zz} + c\hat{u}_z - \kappa_1 \hat{u} + \int_0^T \frac{b_1(t)q(t)V(t,z)}{\widetilde{\psi}(t)} dt \\
&\quad + \int_0^T \frac{a_1(t)p(t)U^2(t,z) - b_1(t)q(t)U(t,z)V(t,z)}{\widetilde{\psi}(t)} dt \\
&\geq \hat{u}_{zz} + c\hat{u}_z - \kappa_1 \hat{u} - \int_0^T \frac{b_1(t)q(t)U(t,z)V(t,z)}{\widetilde{\psi}(t)} dt \\
&\geq \hat{u}_{zz} + c\hat{u}_z - \kappa_1 \hat{u} - N_3 e^{\lambda^+_{c,\epsilon} z} \hat{u}
\end{aligned}
$$

for any $z \in [0, +\infty)$, that is, $\hat{u}(z)$ is a supersolution of the equation

$$
U_{zz} + cU_z - \kappa_1 U - N_3 e^{\lambda^+_{c,\epsilon} z} U = 0, \quad z \in [0, +\infty). \tag{2.11}
$$

By the definition of λ_c, we have $\sigma^2 + c\sigma - \kappa_1 > 0$ for $\sigma < \lambda_c$. Let $M'' \geq M'$ such that $e^{\lambda^+_{c,\epsilon} M''} \leq \frac{\sigma^2 + c\sigma - \kappa_1}{N_3}$. Taking $U^-(z) = \delta' e^{\sigma z}$ with $\delta' > 0$ large enough satisfying $\hat{u}(M'') \geq \delta' e^{\sigma M''}$, it then follows that

$$
\begin{aligned}
U^-_{zz} + cU^-_z - \kappa_1 U^- - N_3 e^{\lambda^+_{c,\epsilon} z} U^- &= (\sigma^2 + c\sigma - \kappa_1 - N_3 e^{\lambda^+_{c,\epsilon} z}) \delta' e^{\sigma z} \\
&\geq (\sigma^2 + c\sigma - \kappa_1 - N_3 e^{\lambda^+_{c,\epsilon} M''}) \delta e^{\sigma z} \geq 0
\end{aligned}
$$

for any $z \in [M'', +\infty)$, that is, $U^-(z)$ is a subsolution of (2.11). Note that $\lim_{z \to +\infty} \hat{u}(z) = \lim_{z \to +\infty} U^-(z) = 0$, again the maximum principle yields that $\hat{u}(z) \geq U^-(z)$ for any $z \in [M'', +\infty)$. Thus $\delta' e^{\sigma z} \leq \hat{u}(z) \leq \delta e^{\lambda^+_{c,\epsilon} z}$ for all $z \in [M'', +\infty)$, then (2.5) follows from the same argument as (2.6). The proof is complete. $\square$

Remark 3. The definitions of $\lambda^\pm_{c,\epsilon}$ indicate that there exist $\varepsilon^\pm = \varepsilon^\pm(\epsilon) > 0$ such that for any $c \leq c^*$ and $0 < \epsilon < \min\{1, \frac{\kappa_0}{C^+}\}$, there holds $\lambda^\pm_{c,\epsilon} = \lambda^-_c \pm \varepsilon^\pm$ with $\varepsilon^\pm(\epsilon) \to 0^+$ as $\epsilon \to 0^+$. Actually, we know from the proof of Lemma 2.2 that the exponential decay as $z \to +\infty$ can only be estimated by the perturbation $\lambda^\pm_{c,\epsilon}$ rather than λ^-_c, since there is no such exponential type sub-super solutions that equipped with λ^-_c as the exponential decay rate.

Lemma 2.3. *Suppose (A1)-(A4) hold, let $(U(t,z), V(t,z)) \in C^{1,2}_b(\mathbb{R} \times \mathbb{R}, \mathbb{R}^2)$ be a solution of (2.2). Then there exist $C_1 > 0$ and $C_2 > 0$ such that*

$$
|U(t,z)| + |U_z(t,z)| + |U_{zz}(t,z)| \leq C_1 e^{\lambda^+_{c,\epsilon} z} \quad \text{for any } (t,z) \in R \times [0, +\infty),
$$

$$
|V(t,z)| + |V_z(t,z)| + |V_{zz}(t,z)| \leq C_2 e^{\lambda^+_{c,\epsilon} z} \quad \text{for any } (t,z) \in R \times [0, +\infty),
$$

where $\lambda^+_{c,\epsilon}$ are defined as in Lemma 2.2.

Proof. The proof is similar to [43, Proposition 3.4], using the interior parabolic estimates and Lemma 2.1, so we omit the details here. $\square$

We next establish the exact exponential decay rate of the solution of (2.2) as $z \to +\infty$. Specifically, regarding variable z as the evolution variable, we employ the Laplace transform method and spectral theory for this purpose. In the following of the current section, we denote $(u(t,z), v(t,z)) := (U(t,z), V(t,z))$ for convenience of writing.

Let $Y = L^2_T \times L^2_T$, where $L^2_T := \{\int_0^T |h(t+s)|^2 ds < \infty,\ h(t+T) = h(t)\}$ is equipped with the norm $\|h\|_{L^2_T} = (\int_0^T |h(s)|^2 ds)^{\frac{1}{2}}$, and $H^1_T = \{h \in L^2_T,\ \sup_{t\in\mathbb{R}} \int_0^T |h'(t+s)|^2 ds < \infty\}$. Define $\mathcal{A} : D(\mathcal{A}) \subset Y \to Y$ as

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \frac{1}{d}(\partial_t - h_v(t,0,0)) & -\frac{c}{d} \end{pmatrix}. \tag{2.12}$$

It is easy to see that $\mathcal{A}$ is closed and densely defined in $D(\mathcal{A}) = H^1_T \times L^2_T$. Now let $w = v_z$, then the v–equation of (2.2) can be written as a first order system

$$\frac{d}{dz}\begin{pmatrix} v \\ w \end{pmatrix} = \mathcal{A}\begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{d}[h_v(t,0,0)v - h(t,u,v)] \end{pmatrix}.$$

Similar to [43, Lemma 3.5], we have that $\lambda_c^- \in \sigma(\mathcal{A})$ (the spectrum of $\mathcal{A}$) and

$$ker(\lambda_c^- I - \mathcal{A})^n = ker(\lambda_c^- I - \mathcal{A}) = span\left\{\begin{pmatrix} \psi(t) \\ \lambda_c^-\psi(t) \end{pmatrix}\right\} \text{ for } n = 2, 3, \cdots,$$

which implies that λ_c^- is a simple pole of $(\lambda I - \mathcal{A})^{-1}$ (see [26, Remark A.2.4]). Moreover, a similar argument as [43, Proposition 3.6] shows that there exists $\varepsilon' > 0$ such that $\Theta_{\varepsilon'} \cap \sigma(\mathcal{A}) = \{\lambda_c^-\}$, where $\Theta_{\varepsilon'} = \{\lambda \in \mathbb{C} | \lambda_c^- - \varepsilon' \le \mathrm{Re}\,\lambda \le \lambda_c^- + \varepsilon'\}$ is the vertical strip containing the vertical line $Re\lambda = \lambda_c^-$. Thus, λ_c^- is the only singular point of $\lambda I - \mathcal{A}$ in $\Theta_{\varepsilon'}$. Then by [26], we have

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_c^-)^n S^{n+1} + \frac{P}{(\lambda - \lambda_c^-)} + \sum_{n=1}^{\infty} (\lambda - \lambda_c^-)^{n+1} D^n, \tag{2.13}$$

where $P = \frac{1}{2\pi i}\int_\Gamma (\lambda I - \mathcal{A})^{-1} d\lambda$ is the spectral projection with $\Gamma : |\lambda - \lambda_c^-| < \varepsilon'$ for some small $\varepsilon' > 0$ and

$$S = \frac{1}{2\pi i}\int_\Gamma \frac{(\lambda I - \mathcal{A})^{-1}}{\lambda - \lambda_c^-} d\lambda = \lim_{\lambda\to\lambda_c^-} (I - P)(\lambda I - \mathcal{A})^{-1},$$

$D = (\mathcal{A} - \lambda_c^- I)P$. Since λ_c^- is a simple pole of $(\lambda I - \mathcal{A})^{-1}$, [26, Proposition A.2.2] then implies that $R(P) = \ker(\lambda_c^- I - \mathcal{A})$ for any $c \le c^*$, hence $D^n = 0$ for all $n \in N^+$ and (2.13) becomes

$$(\lambda I - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_c^-)^n S^{n+1} + \frac{P}{(\lambda - \lambda_c^-)}. \tag{2.14}$$

The formula (2.14) is therefore the Laurent series of $(\lambda I - \mathcal{A})^{-1}$ near $\lambda = \lambda_c^-$, and the projection P is the residue of $(\lambda I - \mathcal{A})^{-1}$ at $\lambda = \lambda_c^-$. On the other hand, if we let $\lambda = \mu + i\eta \in \rho(\mathcal{A})$ with $\mu,\ \eta \in \mathbb{R}$, denote by

$$S = \left\{\begin{pmatrix} 0 \\ j \end{pmatrix}\middle| j \in L^2_T\right\} \subset Y$$

and $(\lambda I - \mathcal{A})^{-1}_S$ the restriction of $(\lambda I - \mathcal{A})^{-1}$ to S, similar to [43, Remark 3.7], there exist positive constants C and ϱ such that for any $\mu \in [\lambda_c^- - \varepsilon', \lambda_c^- + \varepsilon']$, there holds

$$\left\|(\lambda I - \mathcal{A})^{-1}_S\right\| \le \frac{C}{|\eta|} \text{ for } |\eta| \ge \varrho. \tag{2.15}$$

Now we state the main results of this section as follows.

Theorem 2.4. *Assume (A1)-(A4) hold. Let $(u(t,z),v(t,z)) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R},\mathbb{R}^2)$ be a solution of (2.2). Then for any $c \le c^*$, we have*

$$\lim_{z\to+\infty}\frac{u(t,z)}{k_2e^{\lambda_c^- z}\widetilde{\phi}(t)}=1,\ \lim_{z\to+\infty}\frac{v(t,z)}{k_2e^{\lambda_c^- z}\psi(t)}=1,\ \text{uniformly in}\ \ t\in\mathbb{R} \tag{2.16}$$

and

$$\lim_{z\to+\infty}\frac{u_z(t,z)}{k_2e^{\lambda_c^- z}\widetilde{\phi}(t)}=\lambda_c^-,\ \lim_{z\to+\infty}\frac{v_z(t,z)}{k_2e^{\lambda_c^- z}\psi(t)}=\lambda_c^-,\ \text{uniformly in}\ \ t\in\mathbb{R}, \tag{2.17}$$

where $k_2>0$ is some constant and

$$\begin{cases}\widetilde{\phi}(t)=\widetilde{\phi}(0)e^{\int_0^t(\rho+g_u(s,0,0))ds}+\int_0^t e^{\int_s^t(\rho+g_u(\tau,0,0))d\tau}g_v(s,0,0)\psi(s)ds,\\ \widetilde{\phi}(0)=\left(1-e^{\int_0^T(\rho+g_u(s,0,0))ds}\right)^{-1}\int_0^T e^{\int_s^T(\rho+g_u(\tau,0,0))d\tau}g_v(s,0,0)\psi(s)ds\end{cases} \tag{2.18}$$

with $\rho=(\lambda_c^-)^2+c\lambda_c^-$.

Proof. The proof is divided into two steps.

Step I. We prove that there exists $k_2>0$ such that

$$\lim_{z\to+\infty}\frac{v(t,z)}{k_2e^{\lambda_c^- z}\psi(t)}=1,\ \lim_{z\to+\infty}\frac{v_z(t,z)}{k_2e^{\lambda_c^- z}\psi(t)}=\lambda_c^-,\ \text{uniformly in}\ \ t\in\mathbb{R}.$$

Now we introduce an auxiliary function

$$\left\{\chi\in C_b^3(R,R)\left|\begin{array}{l}(i).\ \chi(z)\equiv1,\ z\ge0;\\(ii).\ \chi(z)\equiv0,\ z<-1;\\(iii).\ |\chi'|+|\chi''|+|\chi'''|<\infty\end{array}\right.\quad\text{for all } z\in\mathbb{R}\right\}$$

and set $w=v_z$, $\breve{v}=\chi v$, $\breve{w}=(\chi v)_z$. A direct calculation yields that

$$\breve{w}_z+\frac{c}{d}\breve{w}-\frac{1}{d}\breve{v}_t=-\frac{1}{d}h_v(t,0,0)\breve{v}+\frac{1}{d}\chi[h_v(t,0,0)v-h(t,u,v)]+\chi''v+2\chi'v_z+\frac{c}{d}\chi'v. \tag{2.19}$$

Let

$$\widetilde{g}(t,z)=\frac{1}{d}\chi[h_v(t,0,0)v-h(t,u,v)]+\chi''v+2\chi'v_z+\frac{c}{d}\chi'v,$$

then we can rewrite (2.19) as a first order system

$$\frac{d}{dz}\begin{pmatrix}\breve{v}\\ \breve{w}\end{pmatrix}=\mathcal{A}\begin{pmatrix}\breve{v}\\ \breve{w}\end{pmatrix}+\begin{pmatrix}0\\ \widetilde{g}(t,z)\end{pmatrix}. \tag{2.20}$$

Taking $0<\varepsilon'<\min\{-\frac{\lambda_c^-}{4},\lambda_c^--\lambda_c\}$ sufficiently small such that $\Theta_{\varepsilon'}\cap\sigma(\mathcal{A})=\{\lambda_c^-\}$. For this fixed $\varepsilon'>0$, it follows from Remark 3 that there exists $0<\epsilon<\min\{1,\frac{\kappa_0}{C^+}\}$ small enough such that $\pm(\lambda_{c,\epsilon}^\pm-\lambda_c^-)=\varepsilon^\pm=\varepsilon^\pm(\epsilon)<\frac{1}{2}\varepsilon'$. Due to Lemma 2.3, we have $\sup_{t\in R}(|\breve{v}|+|\breve{w}|+|\breve{v}_z|+|\breve{w}_z|)=O(e^{\lambda_{c,\epsilon}^+ z})$ as $z\to+\infty$. Thus for any $Re\lambda\in(\lambda_{c,\epsilon}^+,\lambda_c^-+\varepsilon']$, there holds $(e^{-\lambda z}\breve{v},e^{-\lambda z}\breve{w})\in W^{1,1}(\mathbb{R},Y)\cap W^{1,\infty}(\mathbb{R},Y)$. We now take the two-sided Laplace transform of (2.20) with respect to z and obtain that

$$\begin{pmatrix}\int_{\mathbb{R}}e^{-\lambda s}\breve{v}(\cdot,s)ds\\ \int_{\mathbb{R}}e^{-\lambda s}\breve{w}(\cdot,s)ds\end{pmatrix}=\mathcal{F}(\lambda):=(\lambda I-\mathcal{A})^{-1}\begin{pmatrix}0\\ \int_{\mathbb{R}}e^{-\lambda s}\widetilde{g}(\cdot,s)ds\end{pmatrix}, \tag{2.21}$$

where $\lambda_{c,\epsilon}^+<Re\lambda\le\lambda_c^-+\varepsilon'$. It follows from the expression of $\widetilde{g}$ and Lemma 2.3 that $\sup_{t\in\mathbb{R}}(|\widetilde{g}|+|\widetilde{g}_z|)=O(e^{2\lambda_{c,\epsilon}^+ z})$ as $z\to+\infty$. Hence $\int_{\mathbb{R}}e^{-\lambda s}\widetilde{g}(\cdot,s)ds$ and $\int_{\mathbb{R}}e^{-\lambda s}\widetilde{g}_z(\cdot,s)ds$

are analytic for λ with $Re\lambda \in (\lambda_{c,\epsilon}^{+} - 3\varepsilon', 0)$. Let $\lambda = \mu + i\eta$, then $\int_{\mathbb{R}} e^{-\lambda s}\widetilde{g}(\cdot, s)ds = \int_{\mathbb{R}} e^{-i\eta s} \cdot e^{-\mu s}\widetilde{g}(\cdot, s)ds = \hat{f}_{\mu}(\eta)$, where $\hat{f}_{\mu}$ is the Fourier transform of $f_{\mu}(s) := e^{-\mu s}\widetilde{g}(\cdot, s)$. It is easy to see that $f_{\mu}(s) \in W^{1,1}(\mathbb{R}, L_T^2) \cap W^{1,\infty}(\mathbb{R}, L_T^2)$ for any fixed $\mu \in [\lambda_{c,\varepsilon}^{+} - \frac{5}{2}\varepsilon', -\frac{1}{2}\varepsilon']$. Particularly, $\|e^{-\mu s}\widetilde{g}\|_{W^{1,1}(\mathbb{R}, L_T^2)}$ is uniformly bounded in $\mu \in [\lambda_{c,\varepsilon}^{+} - \frac{5}{2}\varepsilon', -\frac{1}{2}\varepsilon']$, hence there exist positive constants C_1 and ϱ_0 such that $\left\|\hat{f}(\eta)\right\|_{L_T^2} = \left\|\int_{\mathbb{R}} e^{-\lambda s}\widetilde{g}(\cdot, s)ds\right\|_{L_T^2} \leq \frac{C_1}{|\eta|}$ for any $|\eta| \geq \varrho_0$ whenever $\mu \in [\lambda_{c,\varepsilon}^{+} - \frac{5}{2}\varepsilon', -\frac{1}{2}\varepsilon']$. Inequality (2.15) then yields that there exist $C_2 > 0$ and $\varrho > 0$ such that

$$\left\|(\lambda I - \mathcal{A})^{-1} G(\lambda)\right\|_Y \leq \frac{C_2}{|\eta|^2} \quad \text{for any } |\eta| \geq \varrho, \tag{2.22}$$

whenever $\mu \in [\lambda_c^- - \varepsilon', \lambda_c^- + \frac{1}{2}\varepsilon'] \backslash \{\lambda_c^-\}$, where $G(\lambda) = \begin{pmatrix} 0 \\ \int_{\mathbb{R}} e^{-\lambda s}\widetilde{g}(\cdot, s)ds \end{pmatrix}$. Thus we have that $\mathcal{F}(\lambda) = \mathcal{F}(\mu + i\eta) \in L^1(\mathbb{R}, Y) \cap L^{\infty}(\mathbb{R}, Y)$ for any fixed $\mu \in [\lambda_c^- - \varepsilon', \lambda_c^- + \frac{1}{2}\varepsilon'] \backslash \{\lambda_c^-\}$.

Choose $\mu \in (\lambda_{c,\varepsilon}^{+},\ \lambda_c^- + \frac{1}{2}\varepsilon']$. By the inverse Laplace transform we get that

$$\begin{pmatrix} \breve{v}(\cdot, z) \\ \breve{w}(\cdot, z) \end{pmatrix} = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} e^{\lambda z} (\lambda I - \mathcal{A})^{-1} G(\lambda) d\lambda.$$

Since $(\breve{v}(\cdot, z), \breve{w}(\cdot, z)) = (v(\cdot, z), w(\cdot, z))$ for $z \geq 0$, it follows that

$$\begin{pmatrix} v(\cdot, z) \\ w(\cdot, z) \end{pmatrix} = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} e^{\lambda z} (\lambda I - \mathcal{A})^{-1} G(\lambda) d\lambda \quad \text{for any } z \geq 0. \tag{2.23}$$

Let $\widetilde{\mu} = \lambda_c^- - \varepsilon'$, then λ_c^- is the only pole of $\mathcal{F}(\lambda)$ in $Re\lambda \in (\widetilde{\mu}, \lambda_{c,\varepsilon}^{+}]$. In view of (2.22), we have

$$\lim_{|\eta| \to \infty} \int_{\lambda_c^- - \varepsilon'}^{\mu} \left\| e^{(\tau + i\eta)z} ((\tau + i\eta)I - \mathcal{A})^{-1} G(\tau + i\eta) \right\|_Y d\tau = 0 \quad \text{for any } z \geq 0.$$

Therefore, the path of integral in (2.23) can be shifted to $Re\lambda = \widetilde{\mu}$ such that

$$\begin{pmatrix} v(\cdot, z) \\ w(\cdot, z) \end{pmatrix} = \frac{1}{2\pi i} \int_{\lambda_c^- - \varepsilon' - i\infty}^{\lambda_c^- - \varepsilon' + i\infty} e^{\lambda z} (\lambda I - \mathcal{A})^{-1} G(\lambda) d\lambda + Res(e^{\lambda z}\mathcal{F}(\lambda), \lambda_c^-),\ z \geq 0, \tag{2.24}$$

where $Res(g, \lambda_0) := \frac{1}{2\pi i} \int_{\Gamma: |\lambda - \lambda_0| < \varepsilon'} g(\lambda) d\lambda$ denotes the residue of g at λ_0 with $\varepsilon' > 0$ sufficiently small. Furthermore, with the aid of

$$(\lambda I - \mathcal{A})^{-1} G(\lambda) = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_c^-)^n S^{n+1} G(\lambda) + \frac{PG(\lambda_c^-)}{(\lambda - \lambda_c^-)} - \frac{P[G(\lambda_c^-) - G(\lambda)]}{\lambda - \lambda_c^-}$$

for $|\lambda - \lambda_c^-| < \varepsilon'$,

$$PG \subset \ker(\lambda_c^- I - \mathcal{A}) = span \left\{ \begin{pmatrix} \psi(t) \\ \lambda_c^- \psi(t) \end{pmatrix} \right\}$$

and $G(\lambda)$ is analytic in $Re\lambda \in (\lambda_{c,\varepsilon}^{+} - 3\varepsilon', 0)$, using the residue theorem, we obtain that

$$\begin{aligned}\begin{pmatrix} v(t,z) \\ w(t,z) \end{pmatrix} =& \frac{e^{(\lambda_c^- - \varepsilon')z}}{2\pi} \int_{-\infty}^{+\infty} e^{i\eta z}((\lambda_c^- - \varepsilon' + i\eta)I - \mathcal{A})^{-1} G(\lambda_c^- - \varepsilon' + i\eta) d\eta \\ &+ k_2 e^{\lambda_c^- z} \begin{pmatrix} \psi(t) \\ \lambda_c^- \psi(t) \end{pmatrix}, \ z \geq 0,\end{aligned} \tag{2.25}$$

where $k_2 \geq 0$ is a constant. Let $\zeta(t,z) = v(t,z) - k_2 e^{\lambda_c^- z}\psi(t)$ for all $(t,z) \in \mathbb{R} \times \mathbb{R}^+$. Note that $\zeta(t,z)$ is $T-$periodic in t, then (2.25) and (2.22) imply that there exists $C_3 > 0$ such that $\left(\int_{z-1}^{z+1}\int_0^{2T} |\zeta(\tau,s)|^2 d\tau ds\right)^{\frac{1}{2}} \leq C_3 e^{(\lambda_c^- - \varepsilon')z}$ for any $z \geq 0$. Since $\zeta(t,z)$ satisfies

$$[h(t,u,v) - h_v(t,0,0)v] + h_v(t,0,0)\zeta + d\zeta_{zz} + c\zeta_z - \zeta_t = 0 \quad \text{for all } (t,z) \in R \times R^+$$

and Lemma 2.3 yields that $[h(t,u,v) - h_v(t,0,0)v] = O(e^{2\lambda_{c,\varepsilon}^{+} z})$ as $z \to +\infty$, by the interior parabolic estimates, we infer that there exists $C_4 > 0$ independent of z such that

$$\left(\int_{z-\frac{1}{2}}^{z+\frac{1}{2}} \int_T^{2T} \left(|\zeta_{zz}(\tau,s)|^2 + |\zeta_z(\tau,s)|^2 + |\zeta_t(\tau,s)|^2\right) d\tau ds\right)^{\frac{1}{2}} \leq C_4 e^{(\lambda_c^- - \varepsilon')z}$$

for any $z \geq 0$. Sobolev embedding theorem then implies that $\sup_{t\in[0,T]} |\zeta(t,z)| \leq C_5 e^{(\lambda_c^- - \varepsilon')z}$ for all $z \geq 0$, where $C_5 > 0$ is constant. Noting that $\lambda_{c,\varepsilon}^- = \lambda_c^- - \varepsilon^- > \lambda_c^- - \varepsilon'$, then (2.6) yields that $k_2 > 0$, and thus

$$\lim_{z\to+\infty} \frac{v(t,z)}{k_2 e^{\lambda_c^- z}\psi(t)} = 1 \quad \text{uniformly in } t \in \mathbb{R}.$$

Now set $\widetilde{\zeta}(t,z) = v_z(t,z) - k_2\lambda_c^- e^{\lambda_c^- z}\psi(t)$ for any $(t,z) \in \mathbb{R} \times \mathbb{R}^+$. We know from (2.22) that there exits $C_6 > 0$ such that $\left(\int_{z-1}^{z+1}\int_0^{2T} \left|\widetilde{\zeta}(\tau,s)\right|^2 d\tau ds\right)^{\frac{1}{2}} \leq C_6 e^{(\lambda_c^- - \varepsilon')z}$ for any $z \geq 0$. Noting that for any $(t,z) \in \mathbb{R} \times \mathbb{R}^+$, $\widetilde{\zeta}$ satisfies

$$[h_u(t,u,v)u_z + h_v(t,u,v)v_z - h_v(t,0,0)v_z] + h_v(t,0,0)\widetilde{\zeta} + d\widetilde{\zeta}_{zz} + c\widetilde{\zeta}_z - \widetilde{\zeta}_t = 0$$

and $[h_u(t,u,v)u_z + h_v(t,u,v)v_z - h_v(t,0,0)v_z] = O(e^{2\lambda_{c,\varepsilon}^{+} z})$ as $z \to +\infty$ by Lemma 2.3. Through the same argument as above, we know that there exists $C_7 > 0$ such that $\sup_{t\in[0,T]} \left|\widetilde{\zeta}(t,z)\right| \leq C_7 e^{(\lambda_c^- - \varepsilon')z}$ for any $z \geq 0$, and hence

$$\lim_{z\to+\infty} \frac{v_z(t,z)}{k_2 e^{\lambda_c^- z}\psi(t)} = \lambda_c^- \quad \text{uniformly in } t \in \mathbb{R}.$$

Step II. We study the asymptotic behavior of u. Let $\widetilde{\rho} = \rho + \overline{g_u(t,0,0)}$, where $\rho = (\lambda_c^-)^2 + c\lambda_c^-$, then $\widetilde{\rho} = \rho - \kappa_1 < 0$. Hence the equation

$$g_v(t,0,0)\psi(t) + [(\lambda_c^-)^2 + c\lambda_c^- + g_u(t,0,0)]w - w_t = 0$$

has a unique positive periodic solution $\widetilde{\phi}(t)$ given by (2.18). A direct calculation shows that $\omega(t,z) := k_2 e^{\lambda_c^- z}\widetilde{\phi}(t)$ satisfies

$$g_u(t,0,0)\omega + g_v(t,0,0)k_2 e^{\lambda_c^- z}\psi(t) + \omega_{zz} + c\omega_z - \omega_t = 0.$$

Now let

$$\xi(t,z)=\frac{u(t,z)-k_2e^{\lambda_c^- z}\widetilde{\phi}(t)}{\widetilde{\psi}(t)},\quad \eta(t,z)=\frac{v(t,z)-k_2e^{\lambda_c^- z}\psi(t)}{\widetilde{\psi}(t)},\ (t,z)\in\mathbb{R}\times\mathbb{R}^+.$$

Then $\xi(t,z)$ satisfies $R(t,z)-\kappa_1\xi+\xi_{zz}+c\xi_z-\xi_t=0$ for any $z\geq 0$, where

$$R(t,z)=[g(t,u,v)-g_v(t,0,0)v-g_u(t,0,0)u]\widetilde{\psi}^{-1}+g_v(t,0,0)\eta.$$

We know from Step I that $\sup_{t\in\mathbb{R}}|\eta(t,z)|=O(e^{(\lambda_c^- -\varepsilon')z})$ as $z\to+\infty$. In addition, we know from Lemma 2.3 that $\sup_{t\in R}[g(t,u,v)-g_v(t,0,0)v-g_u(t,0,0)u]=O(e^{2\lambda_{c,\varepsilon}^- z})$ as $z\to+\infty$. Thus there exist positive constants M and K_M such that

$$|R(t,z)|\leq\left|[g(t,u,v)-g_v(t,0,0)v-g_u(t,0,0)u]\widetilde{\psi}^{-1}\right|+|g_v(t,0,0)\eta|\leq K_Me^{(\lambda_c^- -\varepsilon')z}$$

for all $(t,z)\in\mathbb{R}\times[M,+\infty)$. Next we show that $\sup_{t\in\mathbb{R}}|\xi(t,z)|=o(e^{\lambda_c^- z})$ as $z\to+\infty$. In view of Lemma 2.2, we have $\sup_{t\in\mathbb{R}}|\xi(t,z)|=O(e^{\lambda_{c,\varepsilon}^+ z})$ as $z\to+\infty$. Notice that $\lambda_c^- -\varepsilon'>\lambda_c$, then $Q:=(\lambda_c^- -\varepsilon')^2+c(\lambda_c^- -\varepsilon')-\kappa_1<0$. It is easy to verify that $\pm Ke^{(\lambda_c^- -\varepsilon')z}$ satisfy respectively

$$R(t,z)-\kappa_1\omega+\omega_{zz}+c\omega_z-\omega_t\leq(\geq)\ 0\quad\text{for all } z\geq M,$$

whenever $K\geq\frac{K_M}{|Q|}$. Since $|\xi(t,z)|$ is bounded in $(t,z)\in\mathbb{R}\times\mathbb{R}^+$, then there exists $K_Q\geq\frac{K_M}{|Q|}$ such that $|\xi(t,M)|\leq K_Qe^{(\lambda_c^- -\varepsilon')M}$ for all $t\in\mathbb{R}$, hence

$$-K_Qe^{(\lambda_c^- -\varepsilon')z}\leq\xi(t,z)\leq K_Qe^{(\lambda_c^- -\varepsilon')z}\quad\text{for all } (t,z)\in\mathbb{R}\times[M,+\infty).\tag{2.26}$$

Indeed, set $\omega^{\pm}(t,z)=\pm K_Qe^{(\lambda_c^- -\varepsilon')z}-\xi(t,z)$ for all $(t,z)\in\mathbb{R}\times[M,+\infty)$, then we have

$$\omega_{zz}^+ +c\omega_z^+ -\omega_t^+ -\kappa_1\omega^+\leq 0,\ \omega_{zz}^- +c\omega_z^- -\omega_t^- -\kappa_1\omega^-\geq 0.\tag{2.27}$$

Since $\omega^{\pm}(t,z)$ is $T-$ periodic in t, it is sufficient to show that $\omega^+(t,z)\geq 0$ for $(t,z)\in(0,2T)\times[M,+\infty)$, while the similar argument holds for $\omega^-(t,z)\leq 0$. Assume to the contrary that $\inf_{(t,z)\in(0,2T)\times[M,+\infty)}\omega^+(t,z)<0$, since $\lim_{z\to+\infty}\sup_{t\in[0,2T]}\omega^+(t,z)=0$, it follows that there exists $(t^*,z^*)\in(0,2T)\times(M,+\infty)$ such that $\omega^+(t^*,z^*)=\inf_{(t,z)\in(0,2T)\times[M,+\infty)}\omega^+(t,z)<0$, and hence $\left[\omega_{zz}^+ +c\omega_z^+ -\omega_t^+ -\kappa_1\omega^+\right]_{(t^*,z^*)}>0$, which contradicts to (2.27). Hence (2.26) implies that $\sup_{t\in\mathbb{R}}|\xi(t,z)|=o(e^{\lambda_c^- z})$ as $z\to+\infty$. Thus, we know from the definition of $\xi(t,z)$ that

$$\lim_{z\to+\infty}\frac{u(t,z)}{k_2e^{\lambda_c^- z}\widetilde{\phi}(t)}=1\quad\text{uniformly in } t\in\mathbb{R}.$$

The argument for $u_z(t,z)$ is similar and we only give a brief sketch here. Let

$$\widetilde{\xi}(t,z)=\frac{u_z(t,z)-k_2\lambda_c^- e^{\lambda_c^- z}\widetilde{\phi}(t)}{\widetilde{\psi}(t)},\ \widetilde{\eta}(t,z)=\frac{v_z(t,z)-k_2\lambda_c^- e^{\lambda_c^- z}\psi(t)}{\widetilde{\psi}(t)}$$

for $(t,z)\in\mathbb{R}\times\mathbb{R}^+$, then

$$\widetilde{R}(t,z)-\kappa_1\widetilde{\xi}+\widetilde{\xi}_{zz}+c\widetilde{\xi}_z-\widetilde{\xi}_t=0\quad\text{for all } z\geq 0$$

with

$$\widetilde{R}(t,z)=[(g_u(t,u,v)-g_u(t,0,0))u_z+(g_v(t,u,v)-g_v(t,0,0))v_z]\widetilde{\psi}^{-1}+g_v(t,0,0)\widetilde{\eta}.$$

The same argument as above implies that $\sup_{t\in\mathbb{R}}\left|\widetilde{\xi}(t,z)\right|=o(e^{\lambda_c^- z})$ as $z\to+\infty$, and then

$$\lim_{z\to+\infty}\frac{u_z(t,z)}{k_2e^{\lambda_c^- z}\widetilde{\phi}(t)}=\lambda_c^- \quad \text{uniformly in } t\in\mathbb{R}.$$

Now we complete all the proof. □

The following is a direct result of Theorem 2.4.

Corollary 1. *Assume (A1)-(A4) hold. Let $(P(t,z),Q(t,z))\in C_b^{1,2}(\mathbb{R}\times\mathbb{R},\mathbb{R}^2)$ be a solution of (1.6). Then*

$$\lim_{z\to+\infty}\frac{1-P(t,z)}{k_2e^{\lambda_c^- z}\widetilde{\phi}(t)}=1,\quad \lim_{z\to+\infty}\frac{1-Q(t,z)}{k_2e^{\lambda_c^- z}\psi(t)}=1 \text{ uniformly in } t\in\mathbb{R},\ c\le c^*,$$

and

$$\lim_{z\to+\infty}\frac{P_z(t,z)}{k_2e^{\lambda_c^- z}\widetilde{\phi}(t)}=-\lambda_c^-,\quad \lim_{z\to+\infty}\frac{Q_z(t,z)}{k_2e^{\lambda_c^- z}\psi(t)}=-\lambda_c^- \text{ uniformly in } t\in\mathbb{R},\ c\le c^*,$$

for some constant $k_2>0$.

Remark 4. For the autonomous system

$$\begin{cases} u_t=u_{xx}+u(1-u-k_1v), & (t,x)\in\mathbb{R}\times\mathbb{R},\\ v_t=dv_{xx}+av(1-k_2u-v), & (t,x)\in\mathbb{R}\times\mathbb{R}, \end{cases} \tag{2.28}$$

where d,a,k_1,k_2 are positive constants. If we further assume that $1-k_1\ge a(k_2-1)>0$ and $1/5<k_1<1$, then the nonlinearity is monostable and (A3) and (A4) hold for (2.28). The traveling wave solution $(\phi(z),\psi(z))$ $(z=x-ct)$ of (2.28) connecting $(0,1)$ and $(1,0)$ satisfies

$$\begin{cases} 0=\phi''+c\phi'+(1-\phi-k_1\psi),\\ 0=d\psi''+c\psi'+a(1-k_2\phi-\psi),\\ \lim\limits_{z\to-\infty}(\phi,\psi)=(0,1),\ \lim\limits_{z\to+\infty}(\phi,\psi)=(1,0). \end{cases}$$

Then Proposition 1 yields that

$$\phi(z)=\alpha_1|z|^l e^{\lambda_c^+ z}+h.o.t,\quad 1-\psi(z)=\beta_1|z|^l e^{\lambda_c^+ z}+h.o.t \quad \text{as } z\to-\infty,$$

where $l=0$ if $c<c^*$ and $l=1$ if $c=c^*$, and by Corollary 1, we have

$$1-\phi(z)=\alpha_2e^{\lambda_c^- z}+h.o.t,\quad \psi(z)=\beta_2e^{\lambda_c^- z}+h.o.t \quad \text{as } z\to+\infty,\ \text{for all } c\le c^*,$$

where $c^*=-2\sqrt{1-k_1}$, $h.o.t$ denotes the higher-order terms, α_i,β_i $(i=1,2)$ are positive constants, $\lambda_c^+=\frac{-c-\sqrt{c^2-4(1-k_1)}}{2}>0$ and $\lambda_c^-=\frac{-c-\sqrt{c^2+4da(k_2-1)}}{2d}<0$ are roots of linear eigenvalue equations $\lambda^2+c\lambda+(1-k_1)=0$ and $d\lambda^2+c\lambda-a(k_2-1)=0$, respectively. These results are consistent with those in Morita and Tachibana [29].

3. **Key estimates.** In this section, we give some crucial estimates which are helpful for the construction of sub-super solutions. Throughout this section, we always assume that (A1)-(A4) hold. In view of Proposition 1 and Corollary 1, the following lemma holds obviously.

Lemma 3.1. *Let $(P(t,z),Q(t,z)) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R},\mathbb{R}^2)$ be solution of (1.6) with $c \le c^*$. Then there exist positive constants $M(c)$, $N(c)$, $m(c)$, $n(c)$, $\delta_j(c)$, $\gamma_j(c)$ $(j=1,2)$ such that*

$$Q(t,z) \le M(c)P(t,z), \quad t\in\mathbb{R},\ z\le 0, \tag{3.1}$$

$$\delta_1(c)P(t,z) \le P_z(t,z) \le \delta_2(c)P(t,z), \quad t\in\mathbb{R},\ z\le 0, \tag{3.2}$$

$$\gamma_1(c)Q(t,z) \le Q_z(t,z) \le \gamma_2(c)Q(t,z), \quad t\in\mathbb{R},\ z\le 0. \tag{3.3}$$

$$1-Q(t,z) \le N(c)(1-P(t,z)), \quad t\in\mathbb{R},\ z\ge 0, \tag{3.4}$$

$$\delta_1(c)m(c)e^{\lambda_c^- z} \le \delta_1(c)(1-P(t,z)) \le P_z(t,z), \quad t\in\mathbb{R},\ z\ge 0, \tag{3.5}$$

$$\gamma_1(c)n(c)e^{\lambda_c^- z} \le \gamma_1(c)(1-Q(t,z)) \le Q_z(t,z), \quad t\in\mathbb{R},\ z\ge 0. \tag{3.6}$$

In particular, for any $0<\varepsilon<\lambda_c^+$, there exist $K_\varepsilon(c)>0$ such that

$$P(t,z) \le K_\varepsilon(c)e^{(\lambda_c^+-\varepsilon)z}, \quad t\in\mathbb{R},\ z\le 0, \tag{3.7}$$

$$Q(t,z) \le K_\varepsilon(c)e^{(\lambda_c^+-\varepsilon)z}, \quad t\in\mathbb{R},\ z\le 0. \tag{3.8}$$

We now give some key estimates in the following two lemmas.

Lemma 3.2. *Let $(P_i(t,z),Q_i(t,z)) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R},\mathbb{R}^2)$ $(i=1,2)$ be solutions of (1.6). Assume that $c_i \le c^*$ and $p_2 \le p_1 \le 0$. Denote*

$$P_1 = P_1(t,x+p_1),\ P_2 = P_2(t,-x+p_2),\ Q_1 = Q_1(t,x+p_1),\ Q_2 = Q_2(t,-x+p_2)$$

and

$$H_1(t,x) = -2a_1pP_1P_2 + a_1pN_1[P_1Q_2(1-Q_1)+P_2Q_1(1-Q_2)].$$

Then there exist positive constants α_1 and K_1 such that

$$\frac{H_1(t,x)}{P_{1,z}(t,x+p_1)+P_{2,z}(t,-x+p_2)} \le K_1e^{\alpha_1p_1} \quad \text{for any } (t,x)\in\mathbb{R}\times\mathbb{R}. \tag{3.9}$$

Proof. We divide $x\in\mathbb{R}$ into four intervals.

Case A. $p_2 \le x \le 0$. Then $x+p_1 \le 0$ and $-x+p_2 \le 0$. By (3.1), (3.2), (3.7) and (3.8), we have

$$\begin{aligned}
&\frac{H_1(t,x)}{P_{1,z}(t,x+p_1)+P_{2,z}(t,-x+p_2)}\\
&\le \frac{a_1pN_1P_1Q_2}{P_{2,z}(t,-x+p_2)} + \frac{a_1pN_1P_2Q_1}{P_{2,z}(t,-x+p_2)}\\
&\le \max_{t\in[0,T]}(b_1q)\left[\frac{K_\varepsilon(c_1)e^{(\lambda_{c_1}^+-\varepsilon)(x+p_1)}\cdot M(c_2)P_2}{\delta_1(c_2)P_2} + \frac{K_\varepsilon(c_1)e^{(\lambda_{c_1}^+-\varepsilon)(x+p_1)}P_2}{\delta_1(c_2)P_2}\right]\\
&\le \max_{t\in[0,T]}(b_1q)\frac{K_\varepsilon(c_1)(M(c_2)+1)}{\delta_1(c_2)}e^{(\lambda_{c_1}^+-\varepsilon)p_1},\ t\in\mathbb{R}.
\end{aligned}$$

Case B. $0 \le x \le -p_1$. Then $x+p_1 \le 0$ and $-x+p_2 \le 0$. Similar to case A,

$$\begin{aligned}
&\frac{H_1(t,x)}{P_{1,z}(t,x+p_1)+P_{2,z}(t,-x+p_2)}\\
&\le \max_{t\in[0,T]}(b_1q)\frac{K_\varepsilon(c_2)(M(c_1)+1)}{\delta_1(c_1)}e^{(\lambda_{c_2}^+-\varepsilon)p_1},\ t\in\mathbb{R}.
\end{aligned}$$

Case C. $x \geq -p_1$. Then $x+p_1 \geq 0$ and $-x+p_2 \leq 0$. Note that $N_1 \leq 1$ and $P_i, Q_i \leq 1$ $(i=1,2)$, then

$$\begin{aligned} H_1(t,x) &\leq -a_1pN_1P_1P_2 + a_1pN_1[P_1Q_2(1-Q_1)+P_2Q_1] \\ &\leq a_1pN_1Q_2(1-Q_1) + a_1pN_1P_2(1-P_1). \end{aligned}$$

By (3.4), (3.5), (3.7) and (3.8), we have

$$\begin{aligned} &\frac{H_1(t,x)}{P_{1,z}(t,x+p_1)+P_{2,z}(t,-x+p_2)} \leq \frac{a_1pN_1Q_2(1-Q_1)}{P_{1,z}(t,x+p_1)} + \frac{a_1pN_1P_2(1-P_1)}{P_{1,z}(t,x+p_1)} \\ &\leq \max_{t\in[0,T]}(b_1q) \\ &\quad \times \left[\frac{K_\varepsilon(c_2)e^{(\lambda_{c_2}^+-\varepsilon)(-x+p_2)}N(c_1)(1-P_1)}{\delta_1(c_1)(1-P_1)} + \frac{K_\varepsilon(c_2)e^{(\lambda_{c_2}^+-\varepsilon)(-x+p_2)}(1-P_1)}{\delta_1(c_1)(1-P_1)}\right] \\ &\leq \max_{t\in[0,T]}(b_1q)\frac{K_\varepsilon(c_2)(N(c_1)+1)}{\delta_1(c_1)}e^{(\lambda_{c_2}^+-\varepsilon)p_1},\ t\in\mathbb{R}. \end{aligned}$$

Case D. $x \leq p_2$. Then $x+p_1 \leq 0$ and $-x+p_2 \geq 0$. Note that $N_1 \leq 1$, then

$$\begin{aligned} H_1(t,x) &\leq -N_1a_1pP_1P_2 + a_1pN_1[P_1Q_2 + P_2Q_1(1-Q_2)] \\ &\leq a_1pN_1P_1(1-P_2) + a_1pN_1Q_1(1-Q_2), \end{aligned}$$

Similar to case C, we can prove that

$$\begin{aligned} &\frac{H_1(t,x)}{P_{1,z}(t,x+p_1)+P_{2,z}(t,-x+p_2)} \\ &\leq \max_{t\in[0,T]}(b_1q)\frac{K_\varepsilon(c_1)(N(c_2)+1)}{\delta_1(c_2)}e^{(\lambda_{c_1}^+-\varepsilon)p_1},\ t\in\mathbb{R}. \end{aligned}$$

For any fixed $0<\varepsilon<\min\{\lambda_{c_1}^+, \lambda_{c_2}^+\}$, now let $\alpha_1 = \min\{\lambda_{c_1}^+-\varepsilon, \lambda_{c_2}^+-\varepsilon\}$ and $K_1 = \max_{t\in[0,T]}(b_1q)\max_{\substack{i,j=1,2\\ i\neq j}}\left\{\frac{K_\varepsilon(c_i)(M(c_j)+1)}{\delta_1(c_j)}, \frac{K_\varepsilon(c_i)(N(c_j)+1)}{\delta_1(c_j)}\right\}$, then (3.9) holds. □

Lemma 3.3. *Let* $(P_i(t,z), Q_i(t,z)) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R},\mathbb{R}^2)$ $(i=1,2)$ *be solutions of (1.6). Assume that* $c_i \leq c^*$ *and* $p_2 \leq p_1 \leq 0$. *Denote*

$$Q_1 = Q_1(t,x+p_1),\ Q_2 = Q_2(t,-x+p_2)$$

and

$$\tilde{H}_2(t,x) = 2dQ_{1,z}Q_{2,z} + b_2qQ_1Q_2(1-Q_1)(1-Q_2).$$

Then there exist positive constants α_2 *and* K_2 *such that*

$$\frac{\tilde{H}_2(t,x)}{(1-Q_2)Q_{1,z}+(1-Q_1)Q_{2,z}} \leq K_2e^{\alpha_2p_1},\ (t,x)\in\mathbb{R}\times\mathbb{R}. \tag{3.10}$$

Proof. We divide $x\in\mathbb{R}$ into four intervals.

Case A. $p_2 \leq x \leq 0$. Then $x+p_1 \leq 0$ and $-x+p_2 \leq 0$. By (3.3) and (3.8), we have

$$\begin{aligned}
&\frac{\tilde{H}_2(t,x)}{(1-Q_2)Q_{1,z}+(1-Q_1)Q_{2,z}}\\
&\le \frac{2dQ_{1,z}Q_{2,z}+b_2qQ_1Q_2(1-Q_1)(1-Q_2)}{(1-Q_1)Q_{2,z}}\\
&\le \frac{2dQ_{1,z}}{1-Q_1}+\frac{b_2qQ_1Q_2}{Q_{2,z}}\\
&\le \frac{2d\gamma_2(c_1)K_\varepsilon(c_1)e^{(\lambda_{c_1}^+-\varepsilon)(x+p_1)}}{1-Q_1(t,0)}+\frac{b_2qK_\varepsilon(c_1)e^{(\lambda_{c_1}^+-\varepsilon)(x+p_1)}Q_2}{\gamma_1(c_2)Q_2}\\
&\le \left(\frac{2d\gamma_2(c_1)K_\varepsilon(c_1)}{1-Q_1(t,0)}+\max_{t\in[0,T]}(b_2q)\frac{K_\varepsilon(c_1)}{\gamma_1(c_2)}\right)e^{(\lambda_{c_1}^+-\varepsilon)p_1},\ t\in\mathbb{R}.
\end{aligned}$$

Case B. $0\le x\le -p_1$. Then $x+p_1\le 0$ and $-x+p_2\le 0$. Similar to case A,

$$\begin{aligned}
&\frac{\tilde{H}_2(t,x)}{(1-Q_2)Q_{1,z}+(1-Q_1)Q_{2,z}}\\
&\le \left(\frac{2d\gamma_2(c_2)K_\varepsilon(c_2)}{1-Q_2(t,0)}+\max_{t\in[0,T]}(b_2q)\frac{K_\varepsilon(c_2)}{\gamma_1(c_1)}\right)e^{(\lambda_{c_2}^+-\varepsilon)p_1},\ t\in\mathbb{R}.
\end{aligned}$$

Case C. $x\ge -p_1$. Then $x+p_1\ge 0$ and $-x+p_2\le 0$. By (3.3), (3.6)and (3.8), we have

$$\begin{aligned}
&\frac{\tilde{H}_2(t,x)}{(1-Q_2)Q_{1,z}+(1-Q_1)Q_{2,z}}\\
&\le \frac{2dQ_{1,z}Q_{2,z}+b_2qQ_1Q_2(1-Q_1)(1-Q_2)}{(1-Q_2)Q_{1,z}}\\
&\le \frac{2dQ_{2,z}}{1-Q_2}+\frac{b_2qQ_2(1-Q_1)}{Q_{1,z}}\\
&\le \frac{2d\gamma_2(c_2)K_\varepsilon(c_2)e^{(\lambda_{c_2}^+-\varepsilon)(-x+p_2)}}{1-Q_2(t,0)}+\frac{b_2qK_\varepsilon(c_2)e^{(\lambda_{c_2}^+-\varepsilon)(-x+p_2)}(1-Q_1)}{\gamma_1(c_1)(1-Q_1)}\\
&\le \left(\frac{2d\gamma_2(c_2)K_\varepsilon(c_2)}{1-Q_2(t,0)}+\max_{t\in[0,T]}(b_2q)\frac{K_\varepsilon(c_2)}{\gamma_1(c_1)}\right)e^{(\lambda_{c_2}^+-\varepsilon)p_1},\ t\in\mathbb{R}.
\end{aligned}$$

Case D. $x\le p_2$. Then $x+p_1\le 0$ and $-x+p_2\ge 0$. Similar to case C, we have

$$\begin{aligned}
&\frac{\tilde{H}_2(t,x)}{(1-Q_2)Q_{1,z}+(1-Q_1)Q_{2,z}}\\
&\le \left(\frac{2d\gamma_2(c_1)K_\varepsilon(c_1)}{1-Q_1(t,0)}+\max_{t\in[0,T]}(b_2q)\frac{K_\varepsilon(c_1)}{\gamma_1(c_2)}\right)e^{(\lambda_{c_1}^+-\varepsilon)p_1},\ t\in\mathbb{R}.
\end{aligned}$$

For any fixed $0<\varepsilon<\min\{\lambda_{c_1}^+,\lambda_{c_2}^+\}$, now let $\alpha_2=\min\{\lambda_{c_1}^+-\varepsilon,\lambda_{c_2}^+-\varepsilon\}$ and $K_2=\max\limits_{\substack{i,j=1,2,i\neq j\\ t\in[0,T]}}\left\{\frac{2d\gamma_2(c_i)K_\varepsilon(c_i)}{1-Q_i(t,0)}+\max\limits_{t\in[0,T]}(b_2q)\frac{K_\varepsilon(c_i)}{\gamma_1(c_j)}\right\}$, then (3.10) holds. The proof is complete. $\square$

4. **Entire solutions.** In this section, we establish the existence and some qualitative properties of invasion entire solutions by constructing appropriate sub-super

solutions and using the comparison principle. Let

$$\begin{cases} \mathcal{F}_1(t,u,v) = u_t - u_{xx} - f_1(t,u,v), \\ \mathcal{F}_2(t,u,v) = v_t - dv_{xx} - f_2(t,u,v), \end{cases}$$

where $f_1(t,u,v) = a_1 pu(1-N_1(t)-u+N_1(t)v)$ and $f_2(t,u,v) = b_2 q(1-v)(N_2(t)u-v)$. Then (1.5) can be written as

$$\begin{cases} \mathcal{F}_1(t,u,v) = 0, \\ \mathcal{F}_2(t,u,v) = 0. \end{cases}$$

Definition 4.1. Suppose $s < T \le \infty$, a pair $(\overline{U}(t,x), \overline{V}(t,x)) \in C^{1,2}((s,T) \times \mathbb{R}, [0,1]^2)$ is said to be a supersolution of (1.5) in $(t,x) \in (s,T) \times \mathbb{R}$, if there holds

$$\begin{cases} \mathcal{F}_1(t,\overline{U},\overline{V}) \ge 0, \\ \mathcal{F}_2(t,\overline{U},\overline{V}) \ge 0. \end{cases}$$

If for any $s < T$, $(\overline{U}(t,x), \overline{V}(t,x))$ is a supersolution of (1.5) in $(t,x) \in (s,T) \times \mathbb{R}$, then we call that $(\overline{U}(t,x), \overline{V}(t,x))$ is a supersolution of (1.5) in $(t,x) \in (-\infty,T) \times \mathbb{R}$. The subsolution $(\underline{u}(t,x), \underline{v}(t,x))$ can be defined in a similar way by reversing the inequality.

Lemma 4.2. *(i) For any $(0,0) \le (u_0,v_0) \le (1,1)$, system (1.5) admits a unique classical solution $(u(t,x;u_0), v(t,x;v_0))$ with $(u(s,x;u_0), v(s,x;v_0)) = (u_0,v_0)$ which satisfies $(0,0) \le (u,v) \le (1,1)$ for all $(t,x) \in [s,+\infty) \times \mathbb{R}$.*
(ii) Let $(\overline{U},\overline{V})$ and $(\underline{u},\underline{v})$ be supersolution and subsolution of (1.5) in $(t,x) \in (s,T) \times \mathbb{R}$, respectively. If $(\underline{u}(s,\cdot),\underline{v}(s,\cdot)) \le (\overline{U}(s,\cdot),\overline{V}(s,\cdot))$, then $(\underline{u}(t,\cdot),\underline{v}(t,\cdot)) \le (\overline{U}(t,\cdot),\overline{V}(t,\cdot))$ for all $s \le t \le T$.

Proof. The proof is similar to that of [14, Lemma 3.1] and we omit the details here. □

To construct a supersolution of (1.5), we first introduce an auxiliary coupled system of ordinary differential equations

$$\begin{cases} p_1'(t) = -c_1 + Ke^{\alpha p_1(t)}, \ t < 0, \\ p_2'(t) = -c_2 + Ke^{\alpha p_1(t)}, \ t < 0, \\ p_2(0) \le p_1(0) \le 0, \end{cases} \tag{4.1}$$

where $c_2 \le c_1 \le c^*$, α and K are positive constants. Solving the equations explicitly, we obtain

$$\begin{cases} p_1(t) = p_1(0) - c_1 t - \frac{1}{\alpha}\ln\left(1 - \frac{K}{c_1}e^{\alpha p_1(0)}(1-e^{-c_1\alpha t})\right) \le 0 \ (t \le 0), \\ p_2(t) = p_2(0) - c_2 t - \frac{1}{\alpha}\ln\left(1 - \frac{K}{c_1}e^{\alpha p_1(0)}(1-e^{-c_1\alpha t})\right) \le 0 \ (t \le 0). \end{cases}$$

Then $p_i(t)$ is monotone increasing, and by virtue of $p_2'(t) - p_1'(t) = c_1 - c_2 \ge 0$, we have $p_2(t) \le p_1(t) \le 0$ for all $t \le 0$. Let

$$\omega_1 = p_1(0) - \frac{1}{\alpha}\ln\left(1 - \frac{K}{c_1}e^{\alpha p_1(0)}\right), \ \omega_2 = p_2(0) - \frac{1}{\alpha}\ln\left(1 - \frac{K}{c_1}e^{\alpha p_1(0)}\right). \tag{4.2}$$

Then

$$p_i(t) - (-c_i t + \omega_i) = -\frac{1}{\alpha}\ln\left(1 - \frac{\varsigma}{1+\varsigma}e^{-c_1\alpha t}\right) \text{ with } \varsigma = -\frac{K}{c_1}e^{\alpha p_1(0)},$$

and there is a constant $C_0 > 0$ such that

$$0 < p_1(t) - (-c_1 t + \omega_1) = p_2(t) - (-c_2 t + \omega_2) \le C_0 e^{-c_1 \alpha t} \quad \text{for all } t \le 0.$$

Now we can construct a supersolution of (1.5) as follows.

Lemma 4.3. *Let* $(P_i(t,z), Q_i(t,z)) \in C_b^{1,2}(\mathbb{R}\times\mathbb{R},\mathbb{R}^2)$ $(i = 1,2)$ *be the periodic traveling wave of (1.5) with* $c_2 \le c_1 \le c^*$. *Choose* $\alpha = \min\{\alpha_1,\alpha_2\}$ *and* $K = \max\{K_1,K_2\}$ *in (4.1), where* (α_1,K_1) *and* (α_2,K_2) *are defined as in Lemmas 3.2 and 3.3, respectively. Then*

$$\begin{cases} \overline{U}(t,x) := \min\{1, P_1(t,x+p_1(t)) + P_2(t,-x+p_2(t))\}, \\ \overline{V}(t,x) := Q_1(t,x+p_1(t)) + Q_2(t,-x+p_2(t)) \\ \qquad\qquad -Q_1(t,x+p_1(t))Q_2(t,-x+p_2(t)) \end{cases} \tag{4.3}$$

is a supersolution of (1.5) defined in $(t,x) \in (-\infty,0]\times\mathbb{R}$.

Proof. Firstly, we prove $\mathcal{F}_1(t,\overline{U},\overline{V}) \ge 0$. Denote

$$S_1 = \{(t,x) | P_1(t,x+p_1(t)) + P_2(t,-x+p_2(t)) > 1\},$$
$$S_2 = \{(t,x) | P_1(t,x+p_1(t)) + P_2(t,-x+p_2(t)) \le 1\},$$

If $(t,x) \in S_1$, then $\overline{U} \equiv 1$ and thus $\mathcal{F}_1(t,\overline{U},\overline{V}) = b_1 q(1-\overline{V}) \ge 0$. If $(t,x) \in S_2$, then $\overline{U} = P_1(t,x+p_1(t)) + P_2(t,-x+p_2(t))$. Moreover, we have

$$\begin{aligned} &\mathcal{F}_1(t,\overline{U},\overline{V}) \\ &= P_{1,t} - c_1 P_{1,z} - P_{1,zz} + P_{2,t} - c_2 P_{2,z} - P_{2,zz} + Ke^{\alpha p_1}(P_{1,z} + P_{2,z}) \\ &\quad - a_1 p(P_1+P_2)[1 - N_1 - (P_1+P_2) + N_1(Q_1+Q_2-Q_1Q_2)] \\ &= Ke^{\alpha p_1}(P_{1,z}+P_{2,z}) + a_1 p P_1(1 - N_1 - P_1 + N_1 Q_1) \\ &\quad + a_1 p P_2(1 - N_1 - P_2 + N_1 Q_2) \\ &\quad - a_1 p(P_1+P_2)[1 - N_1 - (P_1+P_2) + N_1(Q_1+Q_2-Q_1Q_2)] \\ &= Ke^{\alpha p_1}(P_{1,z}+P_{2,z}) - H_1(t,x), \end{aligned}$$

where $H_1(t,x) = -2a_1 p P_1 P_2 + a_1 p N_1[P_1 Q_2(1-Q_1) + P_2 Q_1(1-Q_2)]$. By Lemma 3.2, there hold

$$\frac{H_1(t,x)}{P_{1,z}(t,x+p_1) + P_{2,z}(t,-x+p_2)} \le K_1 e^{\alpha_1 p_1} \le K e^{\alpha p_1} \text{ for any } (t,x) \in (-\infty,0]\times\mathbb{R},$$

and hence

$$\mathcal{F}_1(t,\overline{U},\overline{V}) = Ke^{\alpha p_1}(P_{1,z}+P_{2,z}) - H_1(t,x) \ge 0 \quad \text{for any } (t,x) \in (-\infty,0]\times\mathbb{R}.$$

Then we prove that $\mathcal{F}_2(t,\overline{U},\overline{V}) \ge 0$. Noting that

$$\begin{aligned} &\mathcal{F}_2(t,\overline{U},\overline{V}) \\ &= (Q_{1,t} - c_1 Q_{1,z} - dQ_{1,zz})(1-Q_2) + (Q_{2,t} - c_2 Q_{2,z} - dQ_{2,zz})(1-Q_1) \\ &\quad - 2dQ_{1,z}Q_{2,z} - b_2 q[1 - (Q_1+Q_2-Q_1Q_2)][N_2\overline{U} - (Q_1+Q_2-Q_1Q_2)] \\ &\quad + Ke^{\alpha p_1}[(1-Q_2)Q_{1,z} + (1-Q_1)Q_{2,z}] \\ &= b_2 q(1-Q_1)(N_2 P_1 - Q_1)(1-Q_2) + b_2 q(1-Q_2)(N_2 P_2 - Q_2)(1-Q_1) \\ &\quad - 2dQ_{1,z}Q_{2,z} - b_2 q[1 - (Q_1+Q_2-Q_1Q_2)][N_2\overline{U} - (Q_1+Q_2-Q_1Q_2)] \\ &\quad + Ke^{\alpha p_1}[(1-Q_2)Q_{1,z} + (1-Q_1)Q_{2,z}] \\ &= Ke^{\alpha p_1}[(1-Q_2)Q_{1,z} + (1-Q_1)Q_{2,z}] - H_2(t,x), \end{aligned}$$

where $H_2(t,x) = 2dQ_{1,z}Q_{2,z} + b_2q(1-Q_1)(1-Q_2)[N_2(\overline{U}-P_1-P_2)+Q_1Q_2]$. It is easy to see that $H_2(t,x) \le 2dQ_{1,z}Q_{2,z} + b_2qQ_1Q_2(1-Q_1)(1-Q_2) = \tilde{H}_2(t,x)$. Then it follows from Lemma 3.3 that

$$\frac{\tilde{H}_2(t,x)}{(1-Q_2)Q_{1,z}+(1-Q_1)Q_{2,z}} \le K_2 e^{\alpha_2 p_1} \le K e^{\alpha p_1} \quad \text{for any } (t,x) \in (-\infty,0]\times\mathbb{R}.$$

Hence

$$\mathcal{F}_2(t,\overline{U},\overline{V}) = Ke^{\alpha p_1}[(1-Q_2)Q_{1,z}+(1-Q_1)Q_{2,z}] - H_2(t,x) \ge 0$$

for any $(t,x) \in (-\infty,0]\times\mathbb{R}$. The proof is complete. □

We now state our main result as follows.

Theorem 4.4. *Assume (A1)-(A4) hold. Let $(P_i(t,z),Q_i(t,z))$ be the periodic traveling wave solution of system (1.5) with $c_2 \le c_1 \le -2\sqrt{\kappa}$. Then for any given constants $\theta_1,\ \theta_2 \in \mathbb{R}$, there exists an entire solution $(U_{\theta_1,\theta_2}(t,x),V_{\theta_1,\theta_2}(t,x))$ of (1.5) such that $(0,0) < (U_{\theta_1,\theta_2},V_{\theta_1,\theta_2}) < (1,1)$, and satisfying*

$$\begin{aligned}\lim_{t\to-\infty}\Big\{&\sup_{x\ge0}|U_{\theta_1,\theta_2}(t,x)-P_1(t,x-c_1t+\theta_1)|\\&+\sup_{x\le0}|U_{\theta_1,\theta_2}(t,x)-P_2(t,-x-c_2t+\theta_2)|\Big\}=0\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}\lim_{t\to-\infty}\Big\{&\sup_{x\ge0}|V_{\theta_1,\theta_2}(t,x)-Q_1(t,x-c_1t+\theta_1)|\\&+\sup_{x\le0}|V_{\theta_1,\theta_2}(t,x)-Q_2(t,-x-c_2t+\theta_2)|\Big\}=0.\end{aligned}\tag{4.5}$$

Furthermore, we have

(i) *$(U_{\theta_1,\theta_2}(t+T,x),V_{\theta_1,\theta_2}(t+T,x)) = (U_{\theta_1,\theta_2}(t,x),V_{\theta_1,\theta_2}(t,x))$ for any $(t,x)\in\mathbb{R}\times\mathbb{R}$ or $(U_{\theta_1,\theta_2},V_{\theta_1,\theta_2})(t+T,x) > (U_{\theta_1,\theta_2},V_{\theta_1,\theta_2})(t,x)$ for any $(t,x)\in\mathbb{R}\times\mathbb{R}$;*
(ii) *$\lim\limits_{t\to+\infty} \sup\limits_{x\in\mathbb{R}}\{|U_{\theta_1,\theta_2}(t,x)-1|+|V_{\theta_1,\theta_2}(t,x)-1|\}=0$;*
(iii) *$\lim\limits_{t\to-\infty} \sup\limits_{x\in(x_1,x_2)}\{|U_{\theta_1,\theta_2}(t,x)|+|V_{\theta_1,\theta_2}(t,x)|\}=0$ for any $x_1<x_2$;*
(iv) *$\lim\limits_{|x|\to+\infty} \sup\limits_{t\ge t_0}\{|U_{\theta_1,\theta_2}(t,x)-1|+|V_{\theta_1,\theta_2}(t,x)-1|\}=0$ for any $t_0\in\mathbb{R}$;*
(v) *$(U_{\theta_1,\theta_2}(t,x),V_{\theta_1,\theta_2}(t,x))$ is monotone increasing with respect to θ_1 and θ_2 for any $(t,x)\in\mathbb{R}^2$;*
(vi) *$(U_{\theta_1,\theta_2}(t,x),V_{\theta_1,\theta_2}(t,x))$ converges to $(1,1)$ locally in $(t,x)\in\mathbb{R}^2$ as $\theta_i\to+\infty$.*

Proof. Let ω_1 and ω_2 be as in (4.2) and define

$$\begin{cases}\underline{u}(t,x)=\max\{P_1(t,x-c_1t+\omega_1),P_2(t,-x-c_2t+\omega_2)\},\\ \underline{v}(t,x)=\max\{Q_1(t,x-c_1t+\omega_1),Q_2(t,-x-c_2t+\omega_2)\},\end{cases}\tag{4.6}$$

then $(\underline{u},\underline{v})$ is a subsolution of (1.5) in $(t,x)\in\mathbb{R}\times\mathbb{R}$, satisfying $(\underline{u},\underline{v}) \le (\overline{U},\overline{V})$ for any $(t,x)\in(-\infty,0]\times\mathbb{R}$, where $(\overline{U},\overline{V})$ is defined in (4.3). Now consider the following initial value problem

$$\begin{cases} u^n_t = u^n_{xx} + f_1(t, u^n, v^n), & (t,x) \in (-n, +\infty) \times \mathbb{R}, \\ v^n_t = dv^n_{xx} + f_2(t, u^n, v^n), & (t,x) \in (-n, +\infty) \times \mathbb{R}, \\ u^n(-n, x) := u^n_0(x) = \underline{u}(-n, x), & x \in \mathbb{R}, \\ v^n(-n, x) := v^n_0(x) = \underline{v}(-n, x), & x \in \mathbb{R}. \end{cases} \tag{4.7}$$

We know from [26] that the problem (4.7) is well posed and the (strong) maximum principle holds since all the coefficients are periodic with respect to t. By virtue of Lemmas 4.2 and 4.3, for $x \in \mathbb{R}$, we have

$$\begin{cases} (\underline{u}(t,x), \underline{v}(t,x)) \le (u^n(t,x), v^n(t,x)) \le (u^{n+1}(t,x), v^{n+1}(t,x)) \le (1,1), t \ge -n, \\ (\underline{u}(t,x), \underline{v}(t,x)) \le (u^n(t,x), v^n(t,x)) \le (\overline{U}(t,x), \overline{V}(t,x)), t \in (-n, 0]. \end{cases}$$

Using the standard parabolic estimates and the diagonal extraction process, there exists a subsequence $\{(u^{n_k}(t,x), v^{n_k}(t,x))\}_{k \in N}$ such that $\{(u^{n_k}(t,x), v^{n_k}(t,x))\}$ converges to a function $(u(t,x), v(t,x))$ locally in $(t,x) \in \mathbb{R} \times \mathbb{R}$ as $k \to +\infty$ ($n_k \to +\infty$). In view of $(u^n(t,x), v^n(t,x)) \le (u^{n+1}(t,x), v^{n+1}(t,x))$ for any $t > -n$, $(u^n(t,x), v^n(t,x))$ converges to $(u(t,x), v(t,x))$ in $(t,x) \in \mathbb{R}^2$ as $n \to +\infty$. Clearly, $(u(t,x), v(t,x))$ is an entire solution of (1.5) and satisfies

$$\begin{cases} (\underline{u}(t,x), \underline{v}(t,x)) \le (u(t,x), v(t,x) \le (1,1), \; x \in \mathbb{R}, \; t \in \mathbb{R}. \\ (\underline{u}(t,x), \underline{v}(t,x)) \le (u(t,x), v(t,x) \le (\overline{U}(t,x), \overline{V}(t,x)), x \in \mathbb{R}, t \in (-\infty, 0]. \end{cases} \tag{4.8}$$

Particularly, the (strong) maximum principle implies that for any $(t,x) \in \mathbb{R} \times \mathbb{R}$, $(0,0) < (u(t,x), v(t,x)) < (1,1)$.

We now prove (4.4) and (4.5). Firstly, we prove

$$\lim_{t \to -\infty} \left\{ \sup_{x \ge 0} |u(t,x) - P_1(t, x - c_1 t + \omega_1)| + \sup_{x \le 0} |u(t,x) - P_2(t, -x - c_2 t + \omega_2)| \right\} = 0. \tag{4.9}$$

For $x \ge 0$, there exists $L_1 > 0$ such that

$$\begin{aligned} 0 &\le u(t,x) - \underline{u}(t,x) \\ &\le u(t,x) - P_1(t, x - c_1 t + \omega_1) \\ &\le \overline{U}(t,x) - P_1(t, x - c_1 t + \omega_1) \\ &\le P_1(t, x + p_1(t)) + P_2(t, -x + p_2(t)) - P_1(t, x - c_1 t + \omega_1) \\ &\le K_\varepsilon(c_2) e^{(\lambda^+_{c_2} - \varepsilon)(-x + p_2)} + \sup_{(t,z) \in [0,T] \times \mathbb{R}} |P_{1,z}(t,z)| \cdot |p_1(t) - (-c_1 t + \omega_1)| \\ &\le K_\varepsilon(c_2) e^{\alpha p_1} + L_1 e^{-c_1 \alpha t} \to 0 \text{ as } t \to -\infty. \end{aligned}$$

For $x \le 0$, there exists $L_2 > 0$ such that

$$\begin{aligned} 0 &\le u(t,x) - \underline{u}(t,x) \\ &\le u(t,x) - P_2(t, -x - c_2 t + \omega_2) \\ &\le \overline{U}(t,x) - P_2(t, -x - c_2 t + \omega_2) \\ &\le P_1(t, x + p_1(t)) + P_2(t, -x + p_2(t)) - P_2(t, -x - c_2 t + \omega_2) \\ &\le K_\varepsilon(c_1) e^{(\lambda^+_{c_1} - \varepsilon)(x + p_1)} + \sup_{(t,z) \in [0,T] \times \mathbb{R}} |P_{2,z}(t,z)| \cdot |p_2(t) - (-c_2 t + \omega_2)| \\ &\le K_\varepsilon(c_1) e^{\alpha p_1} + L_2 e^{-c_1 \alpha t} \to 0 \text{ as } t \to -\infty. \end{aligned}$$

(4.9) then follows.

Note from (4.2) that $\omega_1 = \omega_1(p_1(0))$ and $\omega_2 = \omega_2(p_1(0), p_2(0))$ are defined for any $p_2(0) \leq p_1(0) \leq 0$. Then for any $\theta_1, \theta_2 \in \mathbb{R}$, there exist $p_2(0) \leq p_1(0) \leq 0$ such that $\omega_1 = \omega_1(p_1(0))$ and $\omega_2 = \omega_2(p_1(0), p_2(0))$ satisfy $n^* := \frac{\omega_1-\theta_1+\omega_2-\theta_2}{(c_1+c_2)T} \in \mathbb{Z}$. Define

$$(U_{\theta_1,\theta_2}(t,x), V_{\theta_1,\theta_2}(t,x)) = (u(t+n^*T, x+x_0), v(t+n^*T, x+x_0))$$

with $x_0 = \frac{c_2(\theta_1-\omega_1)-c_1(\theta_2-\omega_2)}{c_1+c_2}$, then $(U_{\theta_1,\theta_2}(t,x), V_{\theta_1,\theta_2}(t,x))$ is an entire solution of (1.5). In view of (4.9), we can easily see (4.4) holds. A similar argument yields that (4.5) holds.

The assertions (ii)-(vi) in Theorem 4.4 are straightforward consequences of (4.8). Therefore, we only prove the assertion (i).

(i) For any $(0,0) \leq (u_0, v_0) \leq (1,1)$, let $(u(t,x;u_0), v(t,x;v_0))$ be the unique classical solution of (1.5) with initial value $(u(0,x;u_0), v(0,x;v_0)) = (u_0, v_0)$, then it is easy to see that $(u^n(t,x), v^n(t,x)) = (u(t+n, x; \underline{u}(-n,\cdot)), v(t+n, x; \underline{v}(-n,\cdot)))$. Note that for any $(t,x) \in \mathbb{R}\times\mathbb{R}$, there is

$$\begin{aligned}
\underline{u}(t+T,x) &= \max\{P_1(t+T, x-c_1(t+T)+\omega_1), P_2(t+T, -x-c_2(t+T)+\omega_2)\} \\
&= \max\{P_1(t, x-c_1(t+T)+\omega_1), P_2(t, -x-c_2(t+T)+\omega_2))\} \\
&> \max\{P_1(t, x-c_1t+\omega_1), P_2(t, -x-c_2t+\omega_2)\} \\
&= \underline{u}(t,x),
\end{aligned}$$

and similarly $\underline{v}(t+T,x) > \underline{v}(t,x)$. it follows from the uniqueness of solutions and the comparison principle that for any $(t,x) \in [-n, +\infty)\times\mathbb{R}$, there hold

$$\begin{aligned}
u^n(t+T,x) &= u(t+T+n, x; \underline{u}(-n,\cdot)) = u(t+n, x; u(T, x; \underline{u}(-n,\cdot))) \\
&\geq u(t+n, x; \underline{u}(T-n,\cdot)) \geq u(t+n, x; \underline{u}(-n,\cdot)) = u^n(t,x),
\end{aligned}$$

and similarly $v^n(t+T,x) > v^n(t,x)$. Then there holds $(u(t+T,x), v(t+T,x)) \geq (u(t,x), v(t,x))$ for any $(t,x) \in \mathbb{R}\times\mathbb{R}$. Therefore, the (strong) maximum principle further implies that $(u(t+T,x), v(t+T,x)) = (u(t,x), v(t,x))$ or $(u(t+T,x), v(t+T,x)) > (u(t,x), v(t,x))$ for any $(t,x) \in \mathbb{R}\times\mathbb{R}$. (i) then follows. This completes the proof. □

Remark 5. For the autonomous Lotka-Volterra competition system with random (local) and nonlocal dispersal, Morita and Tachibana [29] and Li et al. [24] established the existence of invasion entire solutions, respectively. Notice that in their papers, the following condition is needed, which may be technical:

(C): There exists a positive number η_0 such that $\frac{\phi(z)}{1-\varphi(z)} \geq \eta_0$ for $z \leq 0$, where $(\phi(z), \psi(z))$ is the invasion traveling wave solution.

In fact, according to Remark 4, when the time periodic system (1.1) degenerates into the homogeneous case, the condition (C) holds obviously under our assumptions (A1)–(A3). We point out that the following supersolution

$$\begin{cases}
\overline{U}(t,x) := P_1(t, x+p_1(t)) + P_2(t, -x+p_2(t)) - P_1(t, x+p_1(t))P_2(t, -x+p_2(t)), \\
\overline{V}(t,x) := Q_1(t, x+p_1(t)) + Q_2(t, -x+p_2(t)) - Q_1(t, x+p_1(t))Q_2(t, -x+p_2(t)),
\end{cases}$$

which has been used in [29, 24], is also applicable to our problem. In this sense, we generalize the result about entire solutions from autonomous case to periodic case.

Remark 6. By the relation between systems (1.5) and (1.1), we get that (1.1) admits an entire solution $(u(t,x),v(t,x)) := (p(t)U_{\theta_1,\theta_2}(t,x), q(t)(1-V_{\theta_1,\theta_2}(t,x)))$. According to Theorem 4.4 (ii) and (iii), we have

$$\lim_{t\to-\infty}\{|u(t,x)|+|v(t,x))-q(t)|\}=0 \text{ locally in } x\in\mathbb{R},$$
$$\lim_{t\to+\infty}\{|u(t,x)-p(t)|+|v(t,x)|\}=0 \text{ uniformly in } x\in\mathbb{R},$$

which indicates that the entire solution (u,v) exhibits the extinction of the inferior species v by the superior one u invading from both sides of x-axis. In fact, this kind of entire solution describes a different type of biological invasion from one presented by traveling waves in a time periodic environment. On the other hand, we point out in particular that the continuous dependence of such an entire solution on parameters such as wave speeds and the shifted variables is important but still open. For some related works on this issue, one can see Hamel and Nadirashvili [15] for a local dispersal KPP equation, Wang et al. [40] for a delayed lattice differential equation, and Li et al. [23] for a nonlocal dispersal periodic monostable equation. We will leave such problems about our system (1.1) for a future study.

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