GLOBAL DYNAMICS OF A VECTOR-HOST EPIDEMIC MODEL WITH AGE OF INFECTION

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Abstract. In this paper, a partial differential equation (PDE) model is proposed to explore the transmission dynamics of vector-borne diseases. The model includes both incubation age of the exposed hosts and infection age of the infectious hosts which describe incubation-age dependent removal rates in the latent period and the variable infectiousness in the infectious period, respectively. The reproductive number $\mathcal{R}_0$ is derived. By using the method of Lyapunov function, the global dynamics of the PDE model is further established, and the results show that the basic reproduction number $\mathcal{R}_0$ determines the transmission dynamics of vector-borne diseases: the disease-free equilibrium is globally asymptotically stable if $\mathcal{R}_0 \leq 1$, and the endemic equilibrium is globally asymptotically stable if $\mathcal{R}_0 > 1$. The results suggest that an effective strategy to contain vector-borne diseases is decreasing the basic reproduction number $\mathcal{R}_0$ below one.

1. **Introduction.** Vector-borne diseases are infectious diseases caused by pathogens and parasites in human populations that are transmitted to people by blood-sucking arthropods, such as mosquitoes, ticks and fleas. They include some of the world's most destructive diseases, for instance, malaria, schistosomiasis, plague, and dengue fever. According to WHO [1], vector-borne diseases account for more than 17% of all infectious diseases, causing more than 1 million deaths annually. In the past two decades, some vector-borne diseases, such as malaria and schistosomiasis, have continued to threaten human health. Furthermore, other vector-borne diseases have reemerged and broken out in different parts of the world, such as the 2014 Guangzhou outbreak of dengue fever and the outbreak of West Nile virus in

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North America since 1999. Any outbreak of the vector-borne diseases causes great harm to public health. As far as the 2014 Guangzhou outbreak of dengue fever is concerned, the total number of dengue fever cases reached $36,889$ as of October 21st, 2014 [2], according to the provincial health and family planning commission. Due to the great harm to the public health caused by the vector diseases, it is imperative to understand the transmission dynamics of the vector-borne diseases firstly, and then discuss strategies to prevent and contain their outbreaks.

Mathematical modeling has contributed significantly to our understanding of the epidemiology of infectious diseases [3, 5]. Over the past two decades, there have been many published mathematical models focused on understanding the transmission dynamics of the vector-borne diseases ([4, 12, 23, 26, 27, 30] and references therein). These models provided useful insights into the transmission dynamics of the vector-borne diseases. Almost all of the above models are described by ordinary differential equations (ODEs); therefore, some of the assumptions implicitly made in the formulation of these models [28] include: (1) infectious individuals are equally infectious during their infectious period; (2) the stage durations of the latent and infectious periods are exponentially distributed. Although in many cases these simplifying assumptions may provide a reasonable approximation to the biological process being modeled, it is important to examine how the model results may be influenced by these assumptions, which calls for an investigation of models that use more realistic assumptions [28].

In this paper, we develop an age-structured model to study how transmission dynamics of the vector-borne diseases are affected by the incubation and infectious ages. The model studied in the paper incorporates both incubation age of the exposed hosts and infection age of the infectious hosts. Incubation age of the exposed hosts describes the different removal rates in the latent period, and infection age of the infectious hosts describes the variable infectiousness in the infectious period. Several recent studies [6, 16, 19, 20, 24, 25, 29] on age structured models have shown that age of infection may play an important role in the transmission dynamics of infectious diseases. Thieme and Castillo-Chavez [29] studied the effect of infection-age-independent infectivity on dynamics of HIV transmission, and showed that undamped oscillations may occur in particular if the variable infectivity is highly concentrated at certain parts of the incubation period. Lloyd [19, 20] studied the epidemic model with the inclusion of non-exponential distributions of infectious periods. The results indicated that the inclusion of more realistic description of the recovery process may cause a significant destabilization of the model, and less dispersed distributions are seen to have two important epidemiological consequences: (1) less stable behavior is seen within the model; (2) disease persistence is diminished.

Epidemic models with age of infection are usually described by first order partial differential equations, whose complexity makes them more difficult to theoretically analyze, particularly, their global behavior. Most existing studies on age-structured models focus only on the existence of non-trivial steady states [17, 11] or give local stability results [32]. The stability analysis of nonlinear dynamical systems has always been a topic of both theoretical and practical importance since global stability is one of the most important issues related to their dynamic behaviors. However, proving the global stability is a very challenging task, especially for nonlinear systems described by PDEs due to the lack of generically applicable tools.

The global stability results for the age-structured epidemic models were first obtained in [7, 8, 9]. The method of Lyapunov functions is the most common tool used to prove the global stability, especially for ODE models [14, 15, 18]. In recent years, Lyapunov function has been also used to study the global stability of epidemic models with age of infection [21, 22, 31].

In this paper, we also use Lyapunov functions to study the global dynamics of a vector-borne disease model with incubation age of the exposed hosts and infection age of the infectious hosts. By using a class of Lyapunov functions we show that the global dynamics of the system is completely determined by the basic reproduction number $\mathcal{R}_0$: if $\mathcal{R}_0 < 1$ the disease-free equilibrium is globally asymptotically stable; if $\mathcal{R}_0 > 1$, a unique endemic equilibrium is globally asymptotically stable.

This paper is organized as follows. In the next section we formulate a vector-borne epidemic model with incubation age of exposed hosts and infection age of infectious hosts. The two infection ages describe the different removal rates in the latent stage and the variable infectiousness in the infectious stage, respectively. We obtain an explicit formula for the basic reproduction number of system. Then we discuss the trivial and non-trivial equilibria and their stabilities. In Section 3, the global stability of the infection-free equilibrium of the system is analyzed by constructing a Lyapunov function. In Section 4, we show uniform strong persistence of the vector-borne disease if $\mathcal{R}_0 > 1$. In Section 5, we again use a Lyapunov function to derive the global stability of the epidemic equilibrium. Finally, a brief discussion is given in Section 6.

2. **The vector-borne disease model with two ages of infection and the local stabilities.** To introduce the model, we divide the host population under consideration into four groups: susceptible hosts at time t, denoted by $S_h(t)$, infected but not infectious individuals $E_h(\tau, t)$, infected and infectious individuals $I_h(a, t)$, and the number of recovered or immune individuals, denoted by $R_h(t)$. The vector population, on the other hand, is divided into three compartmental classes: susceptible vector at time t, denoted by $S_v(t)$, the number of recovered or immune vectors, denoted by $R_v(t)$, and infected/infectious vectors $I_v(t)$.

With the above notation, we study the following infection-age-structured vector-borne epidemic model:

$$
\begin{cases}
S_v'(t) = \Lambda_v - S_v(t) \displaystyle\int_0^\infty \beta_v(a) I_h(a, t) da - \mu_v S_v(t), \\[2mm]
I_v'(t) = S_v(t) \displaystyle\int_0^\infty \beta_v(a) I_h(a, t) da - (\mu_v + \alpha_v) I_v(t), \\[2mm]
R_v'(t) = \alpha_v I_v(t) - \mu_v R_v(t), \\[1mm]
S_h'(t) = \Lambda_h - \beta_h S_h(t) I_v(t) - \mu_h S_h(t), \\[2mm]
\dfrac{\partial E_h(\tau, t)}{\partial \tau} + \dfrac{\partial E_h(\tau, t)}{\partial t} = -(\mu_h + m(\tau)) E_h(\tau, t), \\[2mm]
E_h(0, t) = \beta_h S_h(t) I_v(t), \\[2mm]
\dfrac{\partial I_h(a, t)}{\partial a} + \dfrac{\partial I_h(a, t)}{\partial t} = -(\mu_h + \alpha_h(a) + r_h(a)) I_h(a, t), \\[2mm]
I_h(0, t) = \displaystyle\int_0^\infty m(\tau) E_h(\tau, t) d\tau, \\[2mm]
R_h'(t) = \displaystyle\int_0^\infty r_h(a) I_h(a, t) da - \mu_h R_h(t).
\end{cases}
\tag{1}
$$

In equation (1), Λ_h is the birth /recruitment rate of the host population. Let μ_v, μ_h be the natural death rate of the vectors and the host, respectively. $m(\tau)$ denotes the removal rate of the infected hosts with age of incubation τ from the latent period. $\alpha_h(a)$ gives the additional disease induced death rate due to vector-borne disease at age of infection a. α_v denotes the recovery rate of the infected vectors. $r_h(a)$ denotes the recovery rate of the infected hosts with age of infection a. Furthermore, $\beta_v(a)$ is the transmission coefficient of the infected host individuals at age of infection a, and β_h is the transmission coefficient from infected vectors to healthy host individuals.

To understand the model, notice that susceptible host individuals are recruited at a rate Λ_h. Susceptible host individuals can become infected by a bite of an infected mosquito with disease. Upon infection through biting by infected mosquitoes, the newly infected individuals move to the latent class, then progress into the infectious class with the progression rate $m(\tau)$. The non-infectious and infectious individuals infected by disease with age-since-infection equal to zero move to the boundary condition. The number total recovery rate from the infected class $I_h(a, t)$ is given by the integral over all ages-since-infection. The susceptible vectors are recruited at a rate Λ_v. Susceptible mosquitos can become infected through biting on an infected individual of any age-since-infection at a specific age-infection transmission rate. As a consequence, the force of infection of susceptible vectors is given by the integral over all ages-since-infection. The total recovery rate from the infected vector class $I_v(t)$ is given by $\alpha_v I_v(t)$.

We notice that the equations for the recovered individuals and the recovered vectors are decoupled from the system and the analysis of system (1) is equivalent to the analysis of the system

$$
\begin{cases}
S_v'(t) = \Lambda_v - S_v(t) \int_0^\infty \beta_v(a) I_h(a,t) da - \mu_v S_v(t), \\
I_v'(t) = S_v(t) \int_0^\infty \beta_v(a) I_h(a,t) da - (\mu_v + \alpha_v) I_v(t), \\
S_h'(t) = \Lambda_h - \beta_h S_h(t) I_v(t) - \mu_h S_h(t), \\
\dfrac{\partial E_h(\tau,t)}{\partial \tau} + \dfrac{\partial E_h(\tau,t)}{\partial t} = -(\mu_h + m(\tau)) E_h(\tau,t), \\
E_h(0,t) = \beta_h S_h(t) I_v(t), \\
\dfrac{\partial I_h(a,t)}{\partial a} + \dfrac{\partial I_h(a,t)}{\partial t} = -(\mu_h + \alpha_h(a) + r_h(a)) I_h(a,t), \\
I_h(0,t) = \int_0^\infty m(\tau) E_h(\tau,t) d\tau.
\end{cases}
\tag{2}
$$

Model (2) is equipped with the following initial conditions:

$$ S_v(0) = S_{v_0}, \quad I_v(0) = I_{v_0}, \quad S_h(0) = S_{h_0}, \quad E_h(\tau,0) = \varphi(\tau), \quad I_h(a,0) = \psi(a). $$

All parameters are nonnegative, $\Lambda_v > 0$, $\Lambda_h > 0$, and $\beta_h > 0$, $\mu_v > 0$, $\mu_h > 0$. We make the following assumptions on the parameter-functions.

Assumption 2.1 *The parameter-functions satisfy the following.*

1. *The functions $\beta_v(a)$ is bounded and uniformly continuous. When $\beta_v(a)$ is of compact support,   the support has non-zero Lebesgue measure;*
2. *The functions $m(\tau)$, $\alpha_h(a)$, $r_h(a)$ belong to $L^\infty(0, \infty)$;*

3. The functions $\varphi(\tau)$, $\psi(a)$ are integrable.

Define the space of functions

$$X = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (L^1(0, \infty)) \times (L^1(0, \infty)).$$

It can be verified that solutions of (2) with nonnegative initial conditions belong to the positive cone for $t \geq 0$. Furthermore, adding the first and the second equations we have

$$\frac{d}{dt}\left(S_v(t) + I_v(t)\right) \leq \Lambda_v - \mu_v\left(S_v(t) + I_v(t)\right).$$

Hence,

$$\limsup_t \left(S_v(t) + I_v(t)\right) \leq \frac{\Lambda_v}{\mu_v}.$$

The number of the hosts can be bounded as follows:

$$\frac{d}{dt}\left(S_h(t) + \int_0^\infty E_h(\tau, t)d\tau + \int_0^\infty I_h(a, t)da\right)$$

$$\leq \Lambda_h - \mu_h\left(S_h(t) + \int_0^\infty E_h(\tau, t)d\tau + \int_0^\infty I_h(a, t)da\right).$$

Hence,

$$\limsup_t \left(S_h(t) + \int_0^\infty E_h(\tau, t)d\tau + \int_0^\infty I_h(a, t)da\right) \leq \frac{\Lambda_h}{\mu_h}.$$

Therefore, the following set is positively invariant for system

$$\Omega = \left\{(S_v, I_v, S_h, E_h, I_h) \in X_+ \,\middle|\, \left(S_v(t) + I_v(t)\right) \leq \frac{\Lambda_v}{\mu_v},\right.$$

$$\left.\left(S_h(t) + \int_0^\infty E_h(\tau, t)d\tau + \int_0^\infty I_h(a, t)da\right) \leq \frac{\Lambda_h}{\mu_h}\right\}.$$

Finally, since the exit rate of exposed host individuals from the incubation compartment is given by $\mu_h + m(\tau)$, then the probability of still being latent after τ time units is given by

$$\pi_1(\tau) = e^{-\mu_h \tau}e^{-\int_0^\tau m(\sigma)d\sigma}. \tag{3}$$

The exit rate of infected individuals from the infective compartment is given by $\mu_h + \alpha_h(a) + r_h(a)$, thus the probability of still being infectious after a time units is given by

$$\pi_2(a) = e^{-\mu_h a}e^{-\int_0^a (\alpha_h(\sigma) + r_h(\sigma))d\sigma}. \tag{4}$$

The reproduction number of disease in system (2) is given by the following expression

$$\mathcal{R}_0 = \frac{\beta_h \Lambda_v \Lambda_h}{\mu_v \mu_h (\mu_v + \alpha_v)} \int_0^\infty m(\tau)\pi_1(\tau)d\tau \int_0^\infty \beta_v(a)\pi_2(a)da. \tag{5}$$

The reproduction number of disease gives the number of secondary infections produced in an entirely susceptible population by a typical infected individual during its entire infectious period. $\mathcal{R}_0$ gives the strength of vector-borne disease to invade when rare and alone. In particular, we notice that the reproduction number for vector-borne diseases is a product of the reproduction numbers of the two transmission processes: human-to-vector $\mathcal{R}_h$ and vector-to-human $\mathcal{R}_v$,

$$\mathcal{R}_h = \frac{\Lambda_v}{\mu_v} \int_0^\infty \beta_v(a)\pi_2(a)da, \qquad \mathcal{R}_v = \frac{\beta_h \Lambda_h}{\mu_h(\mu_v + \alpha_v)} \int_0^\infty m(\tau)\pi_1(\tau)d\tau,$$

that is $\mathcal{R}_0 = \mathcal{R}_v \mathcal{R}_h$. In the next section we compute explicit expressions for the equilibria and establish their local stability.

System (2) always has a unique disease-free equilibrium $\mathcal{E}_0$, which is given by

$$\mathcal{E}_0 = \left(S^*_{v_0}, \; 0, \; S^*_{h_0}, \; 0, \; 0 \right),$$

where

$$S^*_{v_0} = \frac{\Lambda_v}{\mu_v}, \qquad S^*_{h_0} = \frac{\Lambda_h}{\mu_h}.$$

In addition, for Dengue virus there is a corresponding endemic equilibrium $\mathcal{E}_1$ given by

$$\mathcal{E}_1 = (S^*_v, \; I^*_v, \; S^*_h, \; E^*_h(\tau), \; I^*_h(a)).$$

We denote by

$$\Lambda = \frac{\beta_h \Lambda_h \Lambda_v}{\mu_h \mu_v (\mu_v + \alpha_v)},$$

$$b = \int_0^\infty m(\tau)\pi_1(\tau)d\tau \int_0^\infty \beta_v(a)\pi_2(a)da, \tag{6}$$

$$b(\lambda) = \int_0^\infty m(\tau)e^{-\lambda\tau}\pi_1(\tau)d\tau \int_0^\infty \beta_v(a)e^{-\lambda a}\pi_2(a)da.$$

The non-zero components of the equilibrium $\mathcal{E}_1$ are given by

$$I^*_v = \frac{\mu_v \mu_h (\mathcal{R}_0 - 1)}{\beta_h (\Lambda_h b + \mu_v)}, \quad S^*_v = \frac{\Lambda_v - (\mu_v + \alpha_v) I^*_v}{\mu_v}, \quad S^*_h = \frac{\Lambda_h}{\beta_h I^*_v + \mu_h},$$

$$E^*_h(\tau) = E^*_h(0)\pi_1(\tau), \qquad E^*_h(0) = \beta_h S^*_h I^*_v, \tag{7}$$

$$I^*_h(a) = I^*_h(0)\pi_2(a), \qquad I^*_h(0) = E^*_h(0) \int_0^\infty m(\tau)\pi_1(\tau)d\tau.$$

Next, we turn to the linearized equations for the disease-free equilibrium. To introduce the linearization at the disease-free equilibrium $\mathcal{E}_0$, we let $S_v(t) = S^*_{v_0} + x_v(t)$, $I_v(t) = y_v(t)$, $S_h(t) = S^*_{h_0} + x_h(t)$, $E_h(\tau, t) = z_h(\tau, t)$, $I_h(a, t) = y_h(a, t)$. The linearized system becomes

$$\begin{cases} \dfrac{dx_v(t)}{dt} = -S^*_{v_0} \displaystyle\int_0^\infty \beta_v(a)y_h(a,t)da - \mu_v x_v(t), \\[2mm] \dfrac{dy_v(t)}{dt} = S^*_{v_0} \displaystyle\int_0^\infty \beta_v(a)y_h(a,t)da - (\mu_v + \alpha_v)y_v(t), \\[2mm] \dfrac{dx_h(t)}{dt} = -\beta_h S^*_{h_0} y_v(t) - \mu_h x_h(t), \\[2mm] \dfrac{\partial z_h(\tau,t)}{\partial \tau} + \dfrac{\partial z_h(\tau,t)}{\partial t} = -(\mu_h + m(\tau))z_h(\tau,t), \\[2mm] z_h(0,t) = \beta_h S^*_{h_0} y_v(t), \\[2mm] \dfrac{\partial y_h(a,t)}{\partial a} + \dfrac{\partial y_h(a,t)}{\partial t} = -(\mu_h + \alpha_h(a) + r_h(a))y_h(a,t), \\[2mm] y_h(0,t) = \displaystyle\int_0^\infty m(\tau)z_h(\tau,t)d\tau. \end{cases} \tag{8}$$

To study system (2), we look for solutions of the form $x_v(t) = \bar{x}_v e^{\lambda t}$, $y_v(t) = \bar{y}_v e^{\lambda t}$, $x_h(t) = \bar{x}_h e^{\lambda t}$, $z_h(\tau, t) = \bar{z}_h(\tau) e^{\lambda t}$ and $y_h(a, t) = \bar{y}_h(a) e^{\lambda t}$. We obtain the following eigenvalue problem

$$
\begin{cases}
\lambda \bar{x}_v = -S_{v_0}^* \int_0^\infty \beta_v(a) \bar{y}_h(a) da - \mu_v \bar{x}_v, \\[2mm]
\lambda \bar{y}_v = S_{v_0}^* \int_0^\infty \beta_v(a) \bar{y}_h(a) da - (\mu_v + \alpha_v) \bar{y}_v, \\[2mm]
\lambda \bar{x}_h = -\beta_h S_{h_0}^* \bar{y}_v - \mu_h \bar{x}_h, \\[2mm]
\dfrac{d \bar{z}_h(\tau)}{d\tau} = -(\lambda + \mu_h + m(\tau)) \bar{z}_h(\tau), \\[2mm]
\bar{z}_h(0) = \beta_h S_{h_0}^* \bar{y}_v, \\[2mm]
\dfrac{d \bar{y}_h(a)}{da} = -(\lambda + \mu_h + \alpha_h(a) + r_h(a)) \bar{y}_h(a), \\[2mm]
\bar{y}_h(0) = \int_0^\infty m(\tau) \bar{z}_h(\tau) d\tau.
\end{cases}
\tag{9}
$$

We notice that the two equations for $\bar{x}_v$ and $\bar{x}_h$ are decoupled from the equation for $\bar{y}_v$, $\bar{z}_h$, $\bar{y}_h$. Hence, the equations for $\bar{x}_v$ and $\bar{x}_h$ are independent from the equations for $\bar{y}_v$, $\bar{z}_h$, $\bar{y}_h$. Solving the differential equations for $\bar{z}_h$, $\bar{y}_h$, we have

$$
\begin{aligned}
\bar{z}_h(\tau) &= \bar{z}_h(0) \ e^{-\lambda \tau} \pi_1(\tau) = \beta_h S_{h_0}^* \bar{y}_v \ e^{-\lambda \tau} \pi_1(\tau), \\
\bar{y}_h(a) &= \bar{y}_h(0) \ e^{-\lambda a} \pi_2(a) = \beta_h S_{h_0}^* \bar{y}_v \ e^{-\lambda a} \pi_2(a) \int_0^\infty m(\tau) \ e^{-\lambda \tau} \pi_1(\tau) d\tau.
\end{aligned}
\tag{10}
$$

Substituting for $\bar{y}_h(a)$ in the second equation of (9), we can obtain the following equation

$$
\lambda + \mu_v + \alpha_v = \beta_h S_{v_0}^* S_{h_0}^* \int_0^\infty m(\tau) e^{-\lambda \tau} \pi_1(\tau) d\tau \int_0^\infty \beta_v(a) e^{-\lambda a} \pi_2(a) da.
\tag{11}
$$

Now we are ready to establish the following result.

Proposition 1. *If*

$$
\mathcal{R}_0 < 1,
$$

then the disease-free equilibrium is locally asymptotically stable. If $\mathcal{R}_0 > 1$, it is unstable.

Proof. Assume

$$
\mathcal{R}_0 < 1.
$$

We set

$$
\begin{aligned}
LHS &\overset{def}{=} \lambda + \mu_v + \alpha_v, \\
RHS &\overset{def}{=} \mathcal{G}_1(\lambda) = \beta_h S_{v_0}^* S_{h_0}^* \int_0^\infty m(\tau) e^{-\lambda \tau} \pi_1(\tau) d\tau \int_0^\infty \beta_v(a) e^{-\lambda a} \pi_2(a) da.
\end{aligned}
\tag{12}
$$

Consider λ with $\Re\lambda \geq 0$. For such λ, following from (12), we have that

$$
\begin{aligned}
|LHS| \quad &\geq \mu_v + \alpha_v, \\
|RHS| \quad &\leq \mathcal{G}_1(\Re\lambda) \leq \mathcal{G}_1(0) = \beta_h S_{v_0}^* S_{h_0}^* \int_0^\infty m(\tau)\pi_1(\tau)d\tau \int_0^\infty \beta_v(a)\pi_2(a)da \\
&= \frac{\beta_h \Lambda_v \Lambda_h}{\mu_v \mu_h} \int_0^\infty m(\tau)\pi_1(\tau)d\tau \int_0^\infty \beta_v(a)\pi_2(a)da \\
&= \mathcal{R}_0(\mu_v + \alpha_v) < |LHS|.
\end{aligned}
$$

This gives a contradiction. Hence, we have shown that equation (11) cannot have any roots with non-negative real parts. Therefore, the disease-free equilibrium $\mathcal{E}_0$ depends on the eigenvalues of the equations for x_v and x_h. It is evident that $\lambda = -\mu_v$ and $\lambda = -\mu_h$, so the disease-free equilibrium $\mathcal{E}_0$ is locally asymptotically stable if $\mathcal{R}_0 < 1$.

Now assume

$$
\mathcal{R}_0 > 1.
$$

We rewrite the characteristic equation (11) in the form

$$
(\lambda + \mu_v + \alpha_v) - \beta_h S_{v_0}^* S_{h_0}^* \int_0^\infty m(\tau)e^{-\lambda\tau}\pi_1(\tau)d\tau \int_0^\infty \beta_v(a)e^{-\lambda a}\pi_2(a)da = 0.
$$

(13)

We denote

$$
\mathcal{G}_2(\lambda) = (\lambda + \mu_v + \alpha_v) - \beta_h S_{v_0}^* S_{h_0}^* \int_0^\infty m(\tau)e^{-\lambda\tau}\pi_1(\tau)d\tau \int_0^\infty \beta_v(a)e^{-\lambda a}\pi_2(a)da.
$$

(14)

Thus equation (13) has turned into the following characteristic equation

$$
\mathcal{G}_2(\lambda) = 0. \tag{15}
$$

For λ real we have

$$
\begin{aligned}
\mathcal{G}_2(0) \quad &= (\mu_v + \alpha_v) - \beta_h S_{v_0}^* S_{h_0}^* \int_0^\infty m(\tau)\pi_1(\tau)d\tau \int_0^\infty \beta_v(a)\pi_2(a)da \\
&= (\mu_v + \alpha_v)(1 - \mathcal{R}_0) < 0.
\end{aligned}
$$

Furthermore, $\lim_{\lambda \to \infty} \mathcal{G}_2(\lambda) = +\infty$. Hence, the characteristic equation (15) has a real positive root. Therefore, the endemic equilibrium $\mathcal{E}_0$ is unstable. This concludes the proof. $\qquad\square$

Now we turn to the local stability of the endemic equilibrium $\mathcal{E}_1$ if $\mathcal{R}_0 > 1$. The result on local stability of the equilibrium $\mathcal{E}_1$ is summarized below

Proposition 2. *Assume $\mathcal{R}_0 > 1$, then the endemic equilibrium $\mathcal{E}_1$ is locally asymptotically stable.*

Proof. We study the linearized equation around the endemic equilibrium $\mathcal{E}_1$. We introduce the following notation for the perturbations $S_v(t) = S_v^* + x_v(t)$, $I_v(t) = I_v^* + y_v(t)$, $S_h(t) = S_h^* + x_h(t)$, $E_h(\tau, t) = E_h^*(\tau) + z_h(\tau, t)$, $I_h(a, t) = I_h^*(a) + y_h(a, t)$. The system for the perturbations becomes (16)

$$\begin{cases} \dfrac{dx_v(t)}{dt} = -S_v^* \displaystyle\int_0^\infty \beta_v(a)y_h(a,t)da - x_v(t)\int_0^\infty \beta_v(a)I_h^*(a)da - \mu_v x_v(t), \\[2mm] \dfrac{dy_v(t)}{dt} = S_v^* \displaystyle\int_0^\infty \beta_v(a)y_h(a,t)da + x_v(t)\int_0^\infty \beta_v(a)I_h^*(a)da - (\mu_v + \alpha_v)y_v(t), \\[2mm] \dfrac{dx_h(t)}{dt} = -\beta_h S_h^* y_v(t) - \beta_h x_h(t)I_v^* - \mu_h x_h(t), \\[2mm] \dfrac{dz_h(\tau)}{d\tau} = -(\lambda + \mu_h + m(\tau))z_h(\tau,t), \\[2mm] z_h(0,t) = \beta_h S_h^* y_v(t) + \beta_h x_h(t)I_v^*, \\[2mm] \dfrac{dy_h(a)}{da} = -(\lambda + \mu_h + \alpha_h(a) + r_h(a))y_h(a,t), \\[2mm] y_h(0,t) = \displaystyle\int_0^\infty m(\tau)z_h(\tau,t)d\tau. \end{cases} \tag{16}$$

An approach similar to [8] (see Appendix B in [8]) can show that the linear stability of the system is in fact determined by the eigenvalues of the linearized system (16). To investigate the point spectrum, we look for exponential solutions (see the case of the disease-free equilibrium) and obtain a linear eigenvalue problem.

$$\begin{cases} \lambda x_v = -S_v^* \displaystyle\int_0^\infty \beta_v(a)y_h(a)da - x_v \int_0^\infty \beta_v(a)I_h^*(a)da - \mu_v x_v, \\[2mm] \lambda y_v = S_v^* \displaystyle\int_0^\infty \beta_v(a)y_h(a)da + x_v \int_0^\infty \beta_v(a)I_h^*(a)da - (\mu_v + \alpha_v)y_v, \\[2mm] \lambda x_h = -z_h(0) - \mu_h x_h, \\[2mm] \dfrac{dz_h(\tau)}{d\tau} = -(\lambda + \mu_h + m(\tau))z_h(\tau), \\[2mm] z_h(0) = \beta_h S_h^* y_v + \beta_h I_v^* x_h, \\[2mm] \dfrac{dy_h(a)}{da} = -(\lambda + \mu_h + \alpha_h(a) + r_h(a))y_h(a), \\[2mm] y_h(0) = \displaystyle\int_0^\infty m(\tau)z_h(\tau)d\tau. \end{cases} \tag{17}$$

Solving the differential equation, we have

$$z_h(\tau) = z_h(0)\, e^{-\lambda\tau}\pi_1(\tau),$$

$$y_h(a) = y_h(0)\, e^{-\lambda a}\pi_2(a) = z_h(0)\, e^{-\lambda a}\pi_2(a)\int_0^\infty m(\tau)\, e^{-\lambda\tau}\pi_1(\tau)d\tau.$$

Substituting for y_h in the second equation of (17), we can obtain the following equation

$$\begin{cases} (\lambda + \mu_v + \displaystyle\int_0^\infty \beta_v(a)I_h^*(a)da)x_v + S_v^* b(\lambda)z_h(0) = 0, \\[2mm] -x_v \displaystyle\int_0^\infty \beta_v(a)I_h^*(a)da + (\lambda + \mu_v + \alpha_v)y_v - S_v^* b(\lambda)z_h(0) = 0, \\[2mm] (\lambda + \mu_h)x_h + z_h(0) = 0, \\[2mm] -\beta_h I_v^* x_h - \beta_h S_h^* y_v + z_h(0) = 0. \end{cases} \tag{18}$$

By direct calculation, we obtain the following characteristic equation:

$$(\lambda + \mu_v + \int_0^\infty \beta_v(a)I_h^*(a)da)(\lambda + \mu_v + \alpha_v)(\lambda + \mu_h + \beta_h I_v^*)$$
$$= \beta_h S_h^* S_v^* b(\lambda)(\lambda + \mu_v)(\lambda + \mu_h). \tag{19}$$

We divide both sides by $(\lambda + \mu_v)(\lambda + \mu_h)$, then we introduce the following notation.

$$\mathcal{G}_3(\lambda) = \frac{(\lambda + \mu_v + \int_0^\infty \beta_v(a)I_h^*(a)da)(\lambda + \mu_v + \alpha_v)(\lambda + \mu_h + \beta_h I_v^*)}{(\lambda + \mu_v)(\lambda + \mu_h)},$$

$$\mathcal{G}_4(\lambda) = \beta_h S_h^* S_v^* b(\lambda) = \beta_h S_h^* S_v^* \int_0^\infty m(\tau)e^{-\lambda\tau}\pi_1(\tau)d\tau \int_0^\infty \beta_v(a)e^{-\lambda a}\pi_2(a)da. \tag{20}$$

Thus (19) can be expressed as the the equation

$$\mathcal{G}_3(\lambda) = \mathcal{G}_4(\lambda). \tag{21}$$

If λ is a root with $\Re\lambda \geq 0$, it follows from equation (20) that

$$|\mathcal{G}_3(\lambda)| > |\lambda + \mu_v + \alpha_v| \geq \mu_v + \alpha_v. \tag{22}$$

From system (2), we have

$$\beta_h S_v^* S_h^* \int_0^\infty m(\tau)\pi_1(\tau)d\tau \int_0^\infty \beta_v(a)\pi_2(a)da = \mu_v + \alpha_v.$$

Hence,

$$|\mathcal{G}_4(\lambda)| \leq |\mathcal{G}_4(\Re\lambda)| \leq \mathcal{G}_4(0) \quad = \beta_h S_v^* S_h^* \int_0^\infty m(\tau)\pi_1(\tau)d\tau \int_0^\infty \beta_v(a)\pi_2(a)da$$
$$= \mu_v + \alpha_v < |\mathcal{G}_3(\lambda)|. \tag{23}$$

This leads to contradiction. Hence, for $\Re\lambda \geq 0$, (21) has no solutions. Thus, the characteristic equation (19) has only solutions with negative real parts. Therefore, the endemic equilibrium $\mathcal{E}_1$ is locally asymptotically stable if $\mathcal{R}_0 > 1$. This concludes the proof. □

3. **Global stability of the disease-free equilibrium.** In the previous section, we have established that equilibria are locally stable, that is, given the conditions on the parameters, if the initial conditions are close enough to the equilibrium, the solution will converge to that equilibrium. In this section our objective is to extend these results to global results. That is, given the conditions on the parameters, convergence to the equilibrium occurs independently of the initial conditions.

As a first step, we establish the global stability of the disease-free equilibrium. We will use a Lyapunov function to approach the problem. We need to integrate the differential equation along the characteristic lines. Denote the initial condition by $B_E(t)$, $B_I(t)$:

$$B_E(t) = E_h(0,t), \quad B_I(t) = I_h(0,t).$$

Integrating along the characteristic lines, we obtain

$$E_h(\tau, t) = \begin{cases} B_E(t - \tau)\pi_1(\tau), & t > \tau, \\ \varphi(\tau - t)\dfrac{\pi_1(\tau)}{\pi_1(\tau - t)}, & t < \tau, \end{cases}$$

$$I_h(a, t) = \begin{cases} B_I(t - a)\pi_2(a), & t > a, \\ \psi(a - t)\dfrac{\pi_2(a)}{\pi_2(a - t)}, & t < a. \end{cases} \qquad (24)$$

Theorem 3.1. *Assume*

$$\mathcal{R}_0 \leq 1.$$

Then the disease-free equilibrium $\mathcal{E}_0$ is globally asymptotically stable.

Proof. We will use a Lyapunov function. We adopt the Volterra-type function used in [7, 10, 13]. Define

$$f(x) = x - 1 - \ln x.$$

We note that $f(x) \geq 0$ for all $x > 0$. $f(x)$ achieves its global minimum at one, with $f(1) = 0$. Let

$$q(a) = \int_a^\infty \beta_v(s)e^{-\int_a^s (\mu_h + \alpha_h(\sigma) + r_h(\sigma))d\sigma} ds,$$

$$p(\tau) = \frac{\beta_h \Lambda_h \Lambda_v}{\mu_h \mu_v (\mu_v + \alpha_v)} q(0) \int_\tau^\infty m(s)e^{-\int_\tau^s (\mu_h + m(\sigma))d\sigma} ds. \qquad (25)$$

We notice that

$$p(0) = \mathcal{R}_0.$$

Differentiating (25) first, we obtain

$$q'(a) = -\beta_v(a) + (\mu_h + \alpha_h(a) + r_h(a))q(a),$$

$$p'(\tau) = -\frac{\beta_h \Lambda_h \Lambda_v}{\mu_h \mu_v (\mu_v + \alpha_v)} q(0)m(\tau) + (\mu_h + m(\tau))p(\tau). \qquad (26)$$

According to (26), we have $\Lambda = \frac{\beta_h \Lambda_h \Lambda_v}{\mu_h \mu_v (\mu_v + \alpha_v)}$. So we define the following Lyapunov function:

$$U_1(t) = U_{1_1}(t) + U_{1_2}(t) + U_{1_3}(t) + U_{1_4}(t) + U_{1_5}(t), \qquad (27)$$

where

$$U_{1_1}(t) = \Lambda f(\frac{S_v}{S_{v_0}^*}), \quad U_{1_2}(t) = \frac{\Lambda}{S_{v_0}^*}I_v(t), \quad U_{1_3}(t) = S_{h_0}^* f(\frac{S_h}{S_{h_0}^*}),$$
$$U_{1_4}(t) = \int_0^\infty p(\tau)E_h(\tau, t)d\tau, \quad U_{1_5}(t) = \Lambda \int_0^\infty q(a)I_h(a, t)da.$$

Because of the complexity of the expressions, we take the derivative of each component of the Lyapunov function separately

$$U_{1_1}'(t) = \frac{\Lambda}{S_{v_0}^*}\left(1 - \frac{S_{v_0}^*}{S_v}\right)\left(\Lambda_v - S_v \int_0^\infty \beta_v(a)I_h(a, t)da - \mu_v S_v\right)$$

$$= \frac{\Lambda}{S_{v_0}^*}\left(1 - \frac{S_{v_0}^*}{S_v}\right)\left(\mu_v S_{v_0}^* - \mu_v S_v - S_v \int_0^\infty \beta_v(a)I_h(a, t)da\right)$$

$$= -\frac{\Lambda \mu_v (S_v - S_{v_0}^*)^2}{S_v S_{v_0}^*} - \frac{\Lambda}{S_{v_0}^*}S_v \int_0^\infty \beta_v(a)I_h(a, t)da + \Lambda \int_0^\infty \beta_v(a)I_h(a, t)da.$$

$$(28)$$

$$
\begin{aligned}
U'_{1_2}(t) &= \frac{\Lambda}{S_{v_0}^*}\left[S_v \int_0^\infty \beta_v(a)I_h(a,t)da - (\mu_v + \alpha_v)I_v\right] \\
&= \frac{\Lambda}{S_{v_0}^*}S_v \int_0^\infty \beta_v(a)I_h(a,t)da - \beta_h S_{h_0}^* I_v.
\end{aligned}
\tag{29}
$$

Noting that $E_h(0,t) = \beta_h S_h I_v$, we have

$$
\begin{aligned}
U'_{1_3}(t) &= \left(1 - \frac{S_{h_0}^*}{S_h}\right)\left(\Lambda_h - \beta_h S_h I_v - \mu_h S_h\right) \\
&= \left(1 - \frac{S_{h_0}^*}{S_h}\right)\left(\mu_h S_{h_0}^* - \mu_h S_h - \beta_h S_h I_v\right) \\
&= -\frac{\mu_h(S_h - S_{h_0}^*)^2}{S_h} - E_h(0,t) + \beta_h S_{h_0}^* I_v.
\end{aligned}
\tag{30}
$$

$$
\begin{aligned}
U'_{1_4}(t) &= \int_0^\infty p(\tau)\frac{\partial E_h(\tau,t)}{\partial t}d\tau \\
&= -\int_0^\infty p(\tau)\left[\frac{\partial E_h(\tau,t)}{\partial \tau} + (\mu_h + m(\tau))E_h(\tau,t)\right]d\tau \\
&= -\left[\int_0^\infty p(\tau)dE_h(\tau,t) + \int_0^\infty (\mu_h + m(\tau))p(\tau)E_h(\tau,t)d\tau\right] \\
&= -\left[p(\tau)E_h(\tau,t)|_0^\infty - \int_0^\infty E_h(\tau,t)dp(\tau) + \int_0^\infty (\mu_h + m(\tau))p(\tau)E_h(\tau,t)d\tau\right] \\
&= p(0)E_h(0,t) - \Lambda q(0)\int_0^\infty m(\tau)E_h(\tau,t)d\tau \\
&= \mathcal{R}_0 E_h(0,t) - \Lambda q(0)I_h(0,t).
\end{aligned}
\tag{31}
$$

Similarly to (31), we obtain

$$
\begin{aligned}
U'_{1_5}(t) &= -\Lambda \int_0^\infty q(a)\left[\frac{\partial I_h(a,t)}{\partial a} + (\mu_h + \alpha_h(a) + r_h(a))I_h(a,t)\right]da \\
&= \Lambda q(0)I_h(0,t) - \Lambda \int_0^\infty \beta_v(a)I_h(a,t)da.
\end{aligned}
\tag{32}
$$

Now differentiating (27) we have

$$
\begin{aligned}
U'_1(t) &= -\frac{\Lambda \mu_v(S_v - S_{v_0}^*)^2}{S_v S_{v_0}^*} - \frac{\Lambda}{S_{v_0}^*}S_v \int_0^\infty \beta_v(a)I_h(a,t)da + \Lambda \int_0^\infty \beta_v(a)I_h(a,t)da \\
&\quad + \frac{\Lambda}{S_{v_0}^*}S_v \int_0^\infty \beta_v(a)I_h(a,t)da - \beta_h S_{h_0}^* I_v \\
&\quad - \frac{\mu_h(S_h - S_{h_0}^*)^2}{S_h} - E_h(0,t) + \beta_h S_{h_0}^* I_v + \mathcal{R}_0 E_h(0,t) - \Lambda q(0)I_h(0,t) \\
&\quad + \Lambda q(0)I_h(0,t) - \Lambda \int_0^\infty \beta_v(a)I_h(a,t)da.
\end{aligned}
\tag{33}
$$

Canceling all terms that cancel, we simplify the above expression:

$$
U'_1(t) = -\frac{\Lambda \mu_v(S_v - S_{v_0}^*)^2}{S_v S_{v_0}^*} - \frac{\mu_h(S_h - S_{h_0}^*)^2}{S_h} + (\mathcal{R}_0 - 1)E_h(0,t).
\tag{34}
$$

The last inequality follows from the fact that $\mathcal{R}_0 \leq 1$. Notice that U_1' equals zero implies that $S_v = S_{v_0}^*$, $S_h = S_{h_0}^*$, $E_h(0,t) = 0$. We define a set

$$\Theta_1 = \left\{ (S_v, I_v, S_h, E_h, I_h) \in \Omega \middle| U_1'(t) = 0 \right\}.$$

LaSalle's Invariance Principle [9] implies that the bounded solutions of (2) converge to the largest compact invariant set of Θ_1. We will show that this largest compact invariant set is the singleton given by the disease-free equilibrium. First, we notice that equality in (34) occurs if and only if $S_v = S_{v_0}^*$, $S_h = S_{h_0}^*$, $E_h(0,t) = 0$. Thus, from the solution for the equation along the characteristic line (24), we have that $E_h(\tau, t) = E_h(0, t - \tau)\pi_1(\tau) = 0$ for all $t > \tau$. Hence, $\lim_{t\to\infty} E_h(\tau, t) = 0$ for $t > \tau$. Noting that

$$I_h(0,t) = \int_0^\infty m(\tau)E(\tau,t)d\tau.$$

So we have $\lim_{t\to\infty} I_h(0,t) = 0$. Thus, we have

$$\lim_{t\to\infty} I_h(a,t) = 0, \ t > a.$$

Therefore, we conclude that the disease-free equilibrium is globally stable. This completes the proof. $\qquad\square$

Our next step is to show the global asymptotic stability of the epidemic equilibrium in system (2)

4. **The uniform strong persistence of the vector-borne disease.** In the previous section, we saw that if the reproduction number is less or equal to one, The vector-borne disease dies out. In this section, we assume that for $\mathcal{R}_0 > 1$, we will show that the vector-borne disease persists.

From Proposition 2 we know that under the specified conditions the equilibrium $\mathcal{E}_1$ is locally asymptotically stable. It remains to be established that $\mathcal{E}_1$ is globally stable. We expect to show this result using a Lyapunov function, similar to the one used in [7, 10, 13]. With $f(x) = x - 1 - \ln x$, we define the following Lyapunov function

$$U_2(t) = U_{2_1}(t) + U_{2_2}(t) + U_{2_3}(t) + U_{2_4}(t) + U_{2_5}(t) + U_{2_6}(t) + U_{2_7}(t) + U_{2_8}(t),$$
(35)

where

$$\begin{cases} U_{2_1}(t) = \dfrac{1}{q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau} f\left(\dfrac{S_v}{S_v^*}\right), \\[3mm] U_{2_2}(t) = \dfrac{1}{S_v^* q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau} I_v^* f\left(\dfrac{I_v}{I_v^*}\right), \\[3mm] U_{2_3}(t) = S_h^* f\left(\dfrac{S_h}{S_h^*}\right), \\[3mm] U_{2_4}(t) = \dfrac{1}{\mathcal{R}_0} \int_0^\infty p(\tau)E_h^*(\tau) f\left(\dfrac{E_h(\tau,t)}{E_h^*(\tau)}\right)d\tau, \\[3mm] U_{2_5}(t) = \dfrac{1}{q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau} \int_0^\infty q(a)I_h^*(a) f\left(\dfrac{I_h(a,t)}{I_h^*(a)}\right)da, \\[3mm] U_{2_6}(t) = \int_t^\infty \dfrac{S_h^*}{S_h(s)} E_h(0,s)ds, \\[3mm] U_{2_7}(t) = \int_t^\infty \dfrac{S_h(s)}{S_h^*} \dfrac{(E_h^*(0))^2}{E_h(0,s)}ds, \\[3mm] U_{2_8}(t) = 2E_h^*(0)t. \end{cases}$$
(36)

One difficulty with the Lyapunov function U_2 above is that the component U_{2_1} is not defined if $S_v = 0$, the component U_{2_2} is not defined if $I_v = 0$, the component U_{2_3}, U_{2_6} is not defined if $S_h = 0$, the component U_{2_4} is not defined if $E_h(\tau, t) = 0$, and the component U_{2_5} is not defined if $I_h(a, t) = 0$. To show that the Lyapunov function above is valid, we need to show that the vector borne disease persists both in the hosts and in the vectors. For this to be the case, we need to guarantee that the initial conditions we start from are non-trivial, that is, the initial conditions are such that they lead to new infections of individuals and vectors either initially or at some future point. Mathematically speaking this means that the support of the initial density of latent individuals $\varphi(\tau)$ intersects the support of $m(\tau)$ or the support of the initial density of infectious individuals $\psi(a)$ intersects the support of $\beta_v(a)$ either initially or at some future point. Thus, we define the following set

$$\hat{\Omega}_1 = \left\{ \varphi \in L^1_+(0, \infty) \middle| \exists s \geq 0 : \int_0^\infty m(\tau + s)\varphi(\tau)d\tau > 0 \right\},$$

$$\hat{\Omega}_2 = \left\{ \psi \in L^1_+(0, \infty) \middle| \exists s \geq 0 : \int_0^\infty \beta_v(a + s)\psi(a)da > 0 \right\}.$$

Define

$$\Omega_0 = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \hat{\Omega}_1 \times \hat{\Omega}_2.$$

Finally, define $X_0 = \Omega \cap \Omega_0$. We notice that X_0 is forward invariant. It is not hard to see that Ω is a forward invariant. To see that $\hat{\Omega}_2$ is forward invariant, let us assume that the inequality holds for the initial condition. The inequality says that the condition is such that if the support of $\beta_v(a)$ is transferred s units to the right, it will intersect the support of the initial condition. But if that happens for the initial time, it will happen for any other time since the support of the initial condition only moves to the right. Similarly, $\hat{\Omega}_1$ is also forward invariant.

We want to formulate the persistence result for the vector-borne disease which on one side will justify the use of the Lyapunov functional $U_2(t)$, and on the other, will show that when $\mathcal{R}_0 > 1$ the disease persists in the form of the endemic equilibrium. Consequently, we identify conditions which lead to the prevalence in individuals and vectors being bounded away from zero. There are many different types of persistence [21]. We identify here the two that we will be working with.

Definition 4.1. We call the vector-borne disease uniformly weakly persistent if there exists some $\gamma > 0$ independent of the initial conditions such that

$$\limsup_{t\to\infty} \int_0^\infty E_h(\tau, t)d\tau > \gamma \quad \text{whenever} \quad \int_0^\infty \varphi(\tau)d\tau > 0,$$

$$\limsup_{t\to\infty} \int_0^\infty I_h(a, t)da > \gamma \quad \text{whenever} \quad \int_0^\infty \psi(a)da > 0,$$

and

$$\limsup_{t\to\infty} I_v(t) > \gamma \quad \text{whenever} \quad I_{v_0} > 0.$$

for all solutions of model (2).

One of the important implications of uniform weak persistence of the disease is that the disease-free equilibrium is unstable.

Definition 4.2. We call the vector borne diease uniformly strongly persistent if there exists some $\gamma > 0$ independent of the initial conditions such that

$$\liminf_{t \to \infty} \int_0^\infty E_h(\tau, t)d\tau > \gamma \quad \text{whenever} \quad \int_0^\infty \varphi(\tau)d\tau > 0,$$

$$\liminf_{t \to \infty} \int_0^\infty I_h(a, t)da > \gamma \quad \text{whenever} \quad \int_0^\infty \psi(a)da > 0,$$

and

$$\liminf_{t \to \infty} I_v(t) > \gamma \quad \text{whenever} \quad I_{v_0} > 0.$$

for all solutions of model (2).

It is evident from the definitions that, if the disease is uniformly strongly persistent, it is also uniformly weakly persistent. To show uniform strong persistence for the vector-borne disease, we need to show two components.

1. We have to show that the vector-borne disease is uniformly weakly persistent.
2. We need to show that the solution semiflow of system (2.2) has a global compact attractor $\mathfrak{T}$.

First, we show uniform weak persistence of the vector-borne disease. The following proposition states that result.

Proposition 3. *Assume $\mathcal{R}_0 > 1$. Then, for all initial conditions that belong to X_0, the vector-borne disease is uniformly weakly persistent, that is, there exists $\gamma > 0$ such that*

$$\limsup_{t} \beta_h I_v(t) \geq \gamma, \quad \limsup_{t} \int_0^\infty m(\tau)E_h(\tau, t)d\tau \geq \gamma,$$

$$\limsup_{t} \int_0^\infty \beta_v(a)I_h(a, t)da \geq \gamma.$$

Proof. We argue by contradiction. Assume that the vector-borne disease dies out. In particular, assume that for every $\varepsilon > 0$ and an initial condition in X_0 we have

$$\limsup_{t} \beta_h I_v(t) < \varepsilon, \limsup_{t} \int_0^\infty m(\tau)E_h(\tau, t)d\tau < \varepsilon, \limsup_{t} \int_0^\infty \beta_v(a)I_h(a, t)da < \varepsilon.$$

Hence, there exist $T > 0$ such that for all $t > T$, we have

$$\beta_h I_v(t) < \varepsilon, \quad \int_0^\infty m(\tau)E_h(\tau, t)d\tau < \varepsilon, \quad \int_0^\infty \beta_v(a)I_h(a, t)da < \varepsilon.$$

By shifting the dynamical system we may assume that the above inequality holds for all $t \geq 0$. From the first equation in (2), and taking into account the above inequality, we have

$$S_v'(t) \geq \Lambda_v - \varepsilon S_v - \mu_v S_v, \quad S_h'(t) \geq \Lambda_h - \varepsilon S_h - \mu_h S_h.$$

Therefore,

$$\limsup_{t \to \infty} S_v(t) \geq \liminf_{t \to \infty} S_v(t) \geq \frac{\Lambda_v}{\varepsilon + \mu_v}, \quad \limsup_{t \to \infty} S_h(t) \geq \liminf_{t \to \infty} S_h(t) \geq \frac{\Lambda_h}{\varepsilon + \mu_h}.$$

Recall that we are using the following notation $B_E(t) = E_h(0, t)$, $B_I(t) = I_h(0, t)$. Using the inequality above we obtain

$$\begin{cases} B_E(t) = E_h(0, t) = \beta_h S_h I_v \geq \beta_h \dfrac{\Lambda_h}{\varepsilon + \mu_h} I_v, \\ \dfrac{dI_v(t)}{dt} \geq \dfrac{\Lambda_v}{\varepsilon + \mu_v} \displaystyle\int_0^\infty \beta_v(a) I_h(a, t) da - (\mu_v + \alpha_v) I_v. \end{cases} \tag{37}$$

Now, we apply expression (24) to obtain the following system of inequalities in $B_E(t)$, $B_I(t)$ and $I_v(t)$:

$$\begin{cases} B_E(t) \geq \beta_h \dfrac{\Lambda_h}{\varepsilon + \mu_h} I_v, \\ B_I(t) = \displaystyle\int_0^\infty m(\tau) E_h(\tau, t) d\tau \geq \int_0^t m(\tau) B_E(t - \tau) \pi_1(\tau) d\tau, \\ \dfrac{dI_v(t)}{dt} \geq \dfrac{\Lambda_v}{\varepsilon + \mu_v} \displaystyle\int_0^t \beta_v(a) B_I(t - a) \pi_2(a) da - (\mu_v + \alpha_v) I_v. \end{cases} \tag{38}$$

We will take the Laplace transform of both sides of inequalities (38). Since all functions above are bounded, their Laplace transform exists for $\lambda > 0$. We denote by $\hat{B}_E(\lambda)$ the Laplace transform of $B_E(t)$, by $\hat{B}_I(\lambda)$ the Laplace transform of $B_I(t)$, and by $\hat{I}_v(\lambda)$ the Laplace transform of $I_v(t)$. Furthermore,

$$\hat{K}_1(\lambda) = \int_0^\infty m(\tau) \pi_1(\tau) e^{-\lambda\tau} d\tau, \quad \hat{K}_2(\lambda) = \int_0^\infty \beta_v(a) \pi_2(a) e^{-\lambda a} da. \tag{39}$$

Taking the Laplace transform of inequalities (38) and using the convolution property of the Laplace transform, we obtain the following system of inequalities for $\hat{B}_E(\lambda)$, $\hat{B}_I(\lambda)$ and $\hat{I}_v(\lambda)$.

$$\begin{cases} \hat{B}_E(\lambda) \geq \beta_h \dfrac{\Lambda_h}{\varepsilon + \mu_h} \hat{I}_v(\lambda), \\ \hat{B}_I(\lambda) \geq \hat{K}_1(\lambda) \hat{B}_E(\lambda), \\ \lambda \hat{I}_v(\lambda) - I_v(0) \geq \dfrac{\Lambda_v}{\varepsilon + \mu_v} \hat{K}_2(\lambda) \hat{B}_I(\lambda) - (\mu_v + \alpha_v) \hat{I}_v(\lambda). \end{cases} \tag{40}$$

Eliminating $\hat{B}_I(\lambda)$ and $\hat{I}_v(\lambda)$ from the system above, we obtain

$$\hat{B}_E(\lambda) \geq \frac{\beta_h \Lambda_v \Lambda_h \hat{K}_1(\lambda) \hat{K}_2(\lambda)}{(\varepsilon + \mu_v)(\varepsilon + \mu_h)(\lambda + \mu_v + \alpha_v)} \hat{B}_E(\lambda) + \frac{\beta_h \Lambda_h}{(\varepsilon + \mu_h)(\lambda + \mu_v + \alpha_v)} I_v(0).$$

This last inequality should hold for the given $\varepsilon \approx 0$ and for any $\lambda > 0$. But this is impossible since for $\varepsilon \approx 0$ and $\lambda \approx 0$, the coefficient in front $\hat{B}_E(\lambda)$ on the right hand side is approximately $\mathcal{R}_0 > 1$, that is,

$$\frac{\beta_h \Lambda_v \Lambda_h \hat{K}_1(\lambda) \hat{K}_2(\lambda)}{(\varepsilon + \mu_v)(\varepsilon + \mu_h)(\lambda + \mu_v + \alpha_v)} \approx \mathcal{R}_0 > 1.$$

In addition, there is another positive term on the right side of this equality. This is a contradiction with our assumption that

$$\limsup_t \beta_h I_v(t) < \varepsilon, \quad \limsup_t \int_0^\infty m(\tau) E_h(\tau, t) d\tau < \varepsilon,$$

$$\limsup_t \int_0^\infty \beta_v(a) I_h(a, t) da < \varepsilon. \tag{41}$$

Therefore, there exists at least one limit supremum which is bounded below by γ for any initial condition in X_0 and some $\gamma > 0$.

Note that

$$
\begin{aligned}
E_h(0,t) &= S_h \beta_h I_v(t) \leq \frac{\Lambda_h}{\mu_h} \beta_h I_v(t) \\
I_h(0,t) &= \int_0^\infty m(\tau) E_h(\tau, t) d\tau \\
&= \int_0^t m(\tau) E_h(0, t-\tau) \pi_1(\tau) d\tau + \int_t^\infty m(\tau) \varphi(\tau - t) \frac{\pi_1(\tau)}{\pi_1(\tau - t)} d\tau \quad (42) \\
\frac{dI_v(t)}{dt} &= \int_0^\infty \beta_v(a) I_h(a,t) da - (\mu_v + \alpha_v) I_v(t) \\
&= \int_0^t \beta_v(a) I_h(0, t-a) \pi_2(a) da + \int_t^\infty \beta_v(a) \psi(a - t) \frac{\pi_2(a)}{\pi_2(a - t)} da \\
&\quad - (\mu_v + \alpha_v) I_v(t).
\end{aligned}
$$

Following (42), we get

$$
\begin{aligned}
\limsup_t E_h(0,t) &\leq \frac{\Lambda_h}{\mu_h} \limsup_t \beta_h I_v(t) \\
\limsup_t I_h(0,t) &\leq \int_0^\infty m(\tau) \pi_1(\tau) d\tau \limsup_t E_h(0,t) \\
&\leq \bar{m} \int_0^\infty e^{-\mu_h \tau} d\tau \limsup_t E_h(0,t) = \frac{\bar{m}}{\mu_h} \limsup_t E_h(0,t) \\
\limsup_t \frac{dI_v(t)}{dt} &\leq \int_0^\infty \beta_v(a) \pi_2(a) da \limsup_t I_h(0,t) - (\mu_v + \alpha_v) \limsup_t I_v(t) \\
&\leq m_0 \int_0^\infty e^{-\mu_h a} da \limsup_t I_h(0,t) - (\mu_v + \alpha_v) \limsup_t I_v(t) \\
&= \frac{m_0}{\mu_h} \limsup_t I_h(0,t) - (\mu_v + \alpha_v) \limsup_t I_v(t),
\end{aligned}
$$
$$(43)$$

where $\bar{m} = \sup_\tau \{m(\tau)\}$, $m_0 = \sup_a \{\beta_v(a)\}$. The last inequality means that

$$
\limsup_t I_v(t) \leq \frac{m_0}{\mu_h(\mu_v + \alpha_v)} \limsup_t I_h(0,t).
$$

Thus we obtain that if any inequality in (41) holds, all the three inequalities are less than *a constant*$\times \varepsilon$. There is another contradiction with the above result that there exists at least one limit supremum which is bounded below by γ. As a result, there exists $\gamma > 0$ such that for any initial condition in X_0, we have

$$
\limsup_t \beta_h I_v(t) \geq \gamma, \limsup_t \int_0^\infty m(\tau) E_h(\tau, t) d\tau \geq \gamma, \limsup_t \int_0^\infty \beta_v(a) I_h(a,t) da \geq \gamma.
$$

In addition, the differential equation for I_v can be rewritten in the form

$$
\frac{dI_v}{dt} \geq \frac{\Lambda_v \gamma}{\gamma + \mu_v} - (\mu_v + \alpha_v) I_v,
$$

which in turn, implies a lower bound for I_v. This concludes the proof. □

Our next goal is to prove that system (2) has a global compact attractor $\mathfrak{T}$. As a first step, we define the semiflow Ψ of the solutions of system (2)

$$
\Psi\left(t : S_{v_0}, I_{v_0}, S_{h_0}, \varphi(\cdot), \psi(\cdot)\right) = \left(S_v(t), I_v(t), S_h(t), E_h(\tau, t), I_h(a, t)\right).
$$

Definition 4.3. The semiflow is a mapping $\Psi : [0, \infty) \times X_0 \to X_0$. A set $\mathfrak{T}$ in X_0 is called a global compact attractor for Ψ, if $\mathfrak{T}$ is a maximal compact invariant set and if for all open sets $\mathfrak{U}$ containing $\mathfrak{T}$ and all bounded sets $\mathcal{B}$ of X_0 there exists some $T > 0$ such that $\Psi(t, \mathcal{B}) \subseteq \mathfrak{U}$, for all $t > T$.

The following proposition establishes the presence of a global compact attractor.

Proposition 4. *Assume* $\mathcal{R}_0 > 1$. *Then, there exists* $\mathfrak{T}$, *a compact subset of* X_0, *which is a global attractor for the solution semiflow* Ψ *of* (2) *in* X_0. *Moreover,* $\mathfrak{T}$ *is invariant under the solution semiflow, that is*

$$\Psi(t, x^0) \subseteq \mathfrak{T} \quad for\ every \quad x^0 \in \mathfrak{T},\ \forall t \geq 0.$$

Proof. To establish this result, we will apply Lemma 3.1.3 and Theorem 3.4.6 in [22]. To show the assumptions of Lemma 3.1.3 and Theorem 3.4.6 in [22], we split the solution semiflow into two components. For an initial condition $x^0 \in X_0$ we have $\Psi(t, x^0) = \hat{\Psi}(t, x^0) + \tilde{\Psi}(t, x^0)$. The splitting is done in such a way that $\hat{\Psi}(t, x^0) \to 0$ as $t \to \infty$ for every $x^0 \in X_0$, and for a fixed t and any bounded set $\mathcal{B}$ in X_0, the set $\{\tilde{\Psi}(t, x^0) : x^0 \in \mathcal{B}\}$ is precompact. The two components of the semiflow are defined as follows:

$$\hat{\Psi}\left(t : S_{v_0}, I_{v_0}, S_{h_0}, \varphi(\cdot), \psi(\cdot)\right) = \left(0, 0, 0, \hat{E}_h(\cdot, t), \hat{I}_h(\cdot, t)\right)$$

$$\tilde{\Psi}\left(t : S_{v_0}, I_{v_0}, S_{h_0}, \varphi(\cdot), \psi(\cdot)\right) = \left(S_v(t), I_v(t), S_h(t), \tilde{E}_h(\cdot, t), \tilde{I}_h(\cdot, t)\right),$$

(44)

where $E_h(\tau, t) = \hat{E}_h(\tau, t) + \tilde{E}_h(\tau, t)$, $I_h(a, t) = \hat{I}_h(a, t) + \tilde{I}_h(a, t)$ and $\hat{E}_h(\tau, t), \hat{I}_h(a, t)$, $\tilde{E}_h(\tau, t), \tilde{I}_h(a, t)$ are the solutions of the following equations (the remaining equations are as in system (2)

$$\begin{cases} \dfrac{\partial \hat{E}_h}{\partial t} + \dfrac{\partial \hat{E}_h}{\partial \tau} = -(\mu_h + m(\tau))\hat{E}_h(\tau, t), \\[2mm] \hat{E}_h(0, t) = 0, \\[2mm] \hat{E}_h(\tau, 0) = \varphi(\tau), \end{cases}$$

(45)

$$\begin{cases} \dfrac{\partial \hat{I}_h}{\partial t} + \dfrac{\partial \hat{I}_h}{\partial a} = -(\mu_h + \alpha_h(a) + r_h(a))\hat{I}_h(\tau, t), \\[2mm] \hat{I}_h(0, t) = 0, \\[2mm] \hat{I}_h(a, 0) = \psi(a), \end{cases}$$

(46)

and

$$\begin{cases} \dfrac{\partial \tilde{E}_h}{\partial t} + \dfrac{\partial \tilde{E}_h}{\partial \tau} = -(\mu + m(\tau))\tilde{E}_h(\tau, t), \\[2mm] \tilde{E}_h(0, t) = \beta_h S_h I_v, \\[2mm] \tilde{E}_h(\tau, 0) = 0, \end{cases}$$

(47)

$$\begin{cases} \dfrac{\partial \tilde{I}_h}{\partial t} + \dfrac{\partial \tilde{I}_h}{\partial a} = -(\mu_h + \alpha_h(a) + r_h(a))\tilde{I}_h(\tau, t), \\[2mm] \tilde{I}_h(0, t) = \displaystyle\int_0^\infty m(\tau)\tilde{E}_h(\tau, t)d\tau, \\[2mm] \tilde{I}_h(\tau, 0) = 0. \end{cases}$$

(48)

System (45) is decoupled from the remaining equations. Using the formula (24) to integrate along the characteristic lines, we obtain

$$\hat{E}_h(\tau, t) = \begin{cases} 0, & t > \tau, \\ \varphi(\tau - t)\dfrac{\pi_1(\tau)}{\pi_1(\tau - t)}, & t < \tau, \end{cases} \tag{49}$$

$$\hat{I}_h(a, t) = \begin{cases} 0, & t > a, \\ \psi(a - t)\dfrac{\pi_2(a)}{\pi_2(a - t)}, & t < a. \end{cases} \tag{50}$$

Integrating $\hat{E}_h$ with respect to τ, we obtain:

$$\int_t^\infty \varphi(\tau - t)\frac{\pi_1(\tau)}{\pi_1(\tau - t)}d\tau = \int_0^\infty \varphi(\tau)\frac{\pi_1(t + \tau)}{\pi_1(\tau)}d\tau \le e^{-\mu_h t}\int_0^\infty \varphi(\tau)d\tau \to 0,$$

as $t \to +\infty$. Integrating $\hat{I}_h$ with respect to a, we obtain:

$$\int_t^\infty \psi(a - t)\frac{\pi_2(a)}{\pi_2(a - t)}da = \int_0^\infty \psi(a)\frac{\pi_2(t + a)}{\pi_2(a)}da \le e^{-\mu_h t}\int_0^\infty \psi(a)da \to 0,$$

as $t \to +\infty$. This shows the first claim, that is, it shows that $\hat{\Psi}(t, x^0) \to 0$ as $t \to +\infty$ uniformly for every $x^0 \in \mathcal{B} \subseteq X_0$, where $\mathcal{B}$ is a ball of a given radius.

To show the second claim, we need to show compactness. We fix t and let $x^0 \in X_0$. Note that X_0 is bounded. We have to show that for that fixed t the family of functions defined by

$$\tilde{\Psi}(t, x^0) = \left(S_v(t), I_v(t), S_h(t), \tilde{E}_h(\tau, t), \tilde{I}_h(a, t) \right),$$

obtained by taking different initial conditions in X_0 is a compact family of functions. The family

$$\{\tilde{\Psi}(t, x^0) | x^0 \in X_0, t - \text{fixed}\} \subseteq X_0,$$

and, therefore, it is bounded. Thus, we have established the boundedness of the set. To show compactness we first see that the remaining conditions of the Frechet-Kolmogorov Theorem [19]. The third condition in the Frechet-Kolmogorov Theorem for compactness in L^1 is trivially satisfied since $\tilde{E}_h(\tau, t) = 0$ for $\tau > t$ and $\tilde{I}_h(a, t) = 0$ for $a > t$. To see the second condition of that Theorem, we have to bound by two constants the L^1-norms of $\partial E_h/\partial \tau$ and $\partial I_h/\partial a$. To derive that bound, first notice that

$$\tilde{E}_h(\tau, t) = \begin{cases} \tilde{B}_E(t - \tau)\pi_1(\tau), & t > \tau, \\ 0, & t < \tau, \end{cases}$$

$$\tilde{I}_h(a, t) = \begin{cases} \tilde{B}_I(t - a)\pi_2(a), & t > a, \\ 0, & t < a, \end{cases} \tag{51}$$

where

$$\tilde{B}_E(t) = \beta_h S_h(t)I_v(t),$$

$$\tilde{B}_I(t) = \int_0^\infty m(\tau)\tilde{E}_h(\tau, t)d\tau = \int_0^t m(\tau)\tilde{B}_E(t - \tau)\pi_1(\tau)d\tau. \tag{52}$$

First, we notice that for $x^0 \in X_0$, $\tilde{B}_E(t)$ is bounded. We can see that by recalling that S_h and I_v are bounded. Hence, the $\tilde{B}_E(t)$ satisfies

$$\tilde{B}_E(t) \le k_1.$$

Then, we obtain

$$\tilde{B}_I(t) = \int_0^t m(\tau)\tilde{B}_E(t-\tau)\pi_1(\tau)d\tau \le k_2 \int_0^t \tilde{B}_E(t-\tau)d\tau = k_2 \int_0^t \tilde{B}_E(\tau)d\tau \le k_1 k_2 t.$$

Next, we differentiate (51) with respect to τ and a:

$$\left|\frac{\partial \tilde{E}_h(\tau,t)}{\partial \tau}\right| \le \begin{cases} |\tilde{B}'_E(t-\tau)|\pi_1(\tau) + \tilde{B}_E(t-\tau)|\pi'_1(\tau)|, & t > \tau, \\ 0, & t < \tau, \end{cases}$$

$$\left|\frac{\partial \tilde{I}_h(a,t)}{\partial a}\right| \le \begin{cases} |\tilde{B}'_I(t-a)|\pi_2(a) + \tilde{B}_I(t-a)|\pi'_2(a)|, & t > a, \\ 0, & t < a. \end{cases}$$

We have to see that $|\tilde{B}'_E(t-\tau)|$, $|\tilde{B}'_I(t-a)|$ are bounded. Differentiating (52), we obtain

$$\tilde{B}'_E(t) = \beta_h\left(S'_h(t)I_v(t) + S_h(t)I'_v(t)\right),$$
$$\tilde{B}'_I(t) = m(t)\tilde{B}_E(0)\pi_1(t) + \int_0^t m(\tau)\tilde{B}'_E(t-\tau)\pi_1(\tau)d\tau. \tag{53}$$

Taking an absolute value and bounding all terms, we can rewrite the above equality as the following inequality:

$$|\tilde{B}'_E(t)| \le k_3, \quad |\tilde{B}'_I(t)| \le k_4.$$

Putting all these bounds together, we have

$$\| \partial_\tau \tilde{E}_h \| \le k_3 \int_0^\infty \pi_1(\tau)d\tau + k_1(\mu_h + \bar{m})\int_0^\infty \pi_1(\tau)d\tau < \mathfrak{b}_1,$$

$$\| \partial_a \tilde{I}_h \| \le k_4 \int_0^\infty \pi_2(a)da + k_1 k_2(\mu_h + \bar{\alpha}_h + \bar{r}_h)t \int_0^\infty \pi_2(a)da < \mathfrak{b}_2,$$

where $\bar{m} = \sup_\tau\{m(\tau)\}$, $\bar{\alpha}_h = \sup_a\{\alpha_h(a)\}$, $\bar{r}_h = \sup_a\{r_h(a)\}$. To complete the proof, we notice that

$$\int_0^\infty |\tilde{E}_h(\tau+h,t) - \tilde{E}_h(\tau,t)|d\tau \le \| \partial_\tau \tilde{E}_h \| |h| \le \mathfrak{b}_1|h|,$$

$$\int_0^\infty |\tilde{I}_h(a+h,t) - \tilde{I}_h(a,t)|d\tau \le \| \partial_a \tilde{I}_h \| |h| \le \mathfrak{b}_2|h|.$$

Thus, the integral can be made arbitrary small uniformly in the family of functions. That establishes the second requirement of the Frechét-Kolmogorov Theorem. We conclude that the family is compact. □

Now we have all components to establish the uniform strong persistence. The next proposition states the uniform strong persistence of I_v, E_h and I_h.

Proposition 5. *Assume $\mathcal{R}_0 > 1$. Then, for all initial conditions that belong to X_0, The vector-borne disease persists, that is, there exists $\gamma > 0$ such that*

$$\liminf_t \beta_h I_v(t) \ge \gamma, \liminf_t \int_0^\infty m(\tau)E_h(\tau,t)d\tau \ge \gamma, \liminf_t \int_0^\infty \beta_v(a)I_h(a,t)da \ge \gamma.$$

Proof. We apply Theorem 2.6 in [29]. We consider the solution semiflow Ψ on X_0. We define three functionals $\rho_j : X_0 \to R_+$, $j = 1, 2, 3$ as follows:

$$\left\{ \begin{array}{l} \rho_1(\Psi(t, x^0)) = \beta_h I_v(t), \\[2mm] \rho_2(\Psi(t, x^0)) = \displaystyle\int_0^\infty m(\tau)\tilde{E}_h(\tau, t)d\tau, \\[2mm] \rho_3(\Psi(t, x^0)) = \displaystyle\int_0^\infty \beta_v(a)\tilde{I}_h(a, t)da. \end{array} \right.$$

Proposition 3 implies that the semiflow is uniformly weakly ρ-persistent. Proposition 4 shows that the solution semiflow has a global compact attractor $\mathfrak{T}$. Total orbits are solutions to the system (2) defined for all times $t \in \mathbb{R}$. Since the solution semiflow is nonnegative, we have that for any s and any $t > s$

$$\beta_h I_v(t) \geq \beta_h I_v(s)e^{-(\mu_v + \alpha_v)(t-s)},$$

$$\begin{aligned} \int_0^\infty m(\tau)\tilde{E}_h(\tau, t)d\tau &= \tilde{B}_I(t) = \int_0^t m(\tau)\tilde{B}_E(t - \tau)\pi_1(\tau)d\tau \geq k^1 \int_0^t \tilde{B}_E(t - \tau)d\tau \\ &= k^1 \int_0^t \tilde{B}_E(\tau)d\tau = k^1 \int_0^t \beta_h S_h(\tau)I_v(\tau)d\tau \\ &\geq k^2 \int_0^t I_v(\tau)d\tau = k^2 \int_0^t I_v(s)e^{-(\mu_v + \alpha_v)(\tau - s)}d\tau \\ &= \frac{k^2 I_v(s)}{\mu_v + \alpha_v}e^{(\mu_v + \alpha_v)s}(1 - e^{-(\mu_v + \alpha_v)t}), \end{aligned}$$

$$\begin{aligned} \int_0^\infty \beta_v(a)\tilde{I}_h(a, t)da &= \int_0^t \beta_v(a)\tilde{B}_I(t - a)\pi_2(a)da \geq k^3 \int_0^t \tilde{B}_I(t - a)da \\ &= k^3 \int_0^t \tilde{B}_I(a)da \\ &\geq \frac{k^2 k^3 I_v(s)}{\mu_v + \alpha_v}e^{(\mu_v + \alpha_v)s} \int_0^t (1 - e^{-(\mu_v + \alpha_v)a})da. \end{aligned}$$

Therefore, $\beta_h I_v(t) > 0$, $\int_0^\infty m(\tau)\tilde{E}_h(\tau, t)d\tau > 0$, $\int_0^\infty \beta_v(a)\tilde{I}_h(a, t)da > 0$ for all $t > s$, provided $I_v(s) > 0$. Theorem 2.6 in [29] now implies that the semiflow is uniformly strongly ρ-persistent. Hence, there exists γ such that

$$\liminf_t \beta_h I_v(t) \geq \gamma, \liminf_t \int_0^\infty m(\tau)E_h(\tau, t)d\tau \geq \gamma, \liminf_t \int_0^\infty \beta_v(a)I_h(a, t)da \geq \gamma.$$

$\square$

Corollary 1. *Assume $\mathcal{R}_0 > 1$. There exists constants $\vartheta > 0$ and $M > 0$ such that for each orbit $(S_v(t), I_v, S_h(t), E_h(\tau, t), I_h(a, t))$ of Ψ in $\mathfrak{T}$, we have*

$$\vartheta \leq S_v(t) \leq M, \quad \vartheta \leq S_h(t) \leq M, \quad \forall t \in \mathbb{R},$$

and

$$\vartheta \leq \beta_h I_v(t) \leq M, \vartheta \leq \int_0^\infty m(\tau)E_h(\tau, t)d\tau \leq M, \vartheta \leq \int_0^\infty \beta_v(a)I_h(a, t)da \leq M, \forall t \in \mathbb{R}.$$

In the next section we show that the endemic equilibrium $\mathcal{E}_1$ is globally stable.

5. **Global stability of the endemic equilibrium.** Now we are ready to establish the global stability of the equilibrium $\mathcal{E}_1$. To demonstrate that with the Lyapunov function defined in (35) we have to establish that $U_2'(t) \leq 0$ along the solution curves of system (2). The following Theorem summarizes the result.

Theorem 5.1. *Assume $\mathcal{R}_0 > 1$. Then, equilibrium $\mathcal{E}_1$ is globally asymptotically stable, that is, for any initial condition $x^0 \in X_0$ the solution semiflow converges to $\mathcal{E}_1$.*

Proof. Since $\mathcal{R}_0 > 1$, for any initial condition $x^0 \in X_0$ we can find a complete orbit $(S_v(t), I_v(t), S_h(t), E_h(\tau, t), I_h(a, t))$ of Ψ in $\mathfrak{T}$ (similarly to the proof of Proposition 4) for which the inequalities in Corollary 1 hold and, consequently, there exist $\varepsilon_1 > 0$ and $M_1 > 0$ such that

$$\varepsilon_1 \leq \frac{I_v}{I_v^*} \leq M_1, \quad \varepsilon_1 \leq \frac{E_h(\tau, t)}{E_h^*(\tau)} \leq M_1, \quad \varepsilon_1 \leq \frac{I_h(a, t)}{I_h^*(a)} \leq M_1.$$

This makes the Lyapunov function defined in (35) well defined.

Because of the complexity of the expressions, we make the derivative of each component of the Lyapunov function separately (see (35)).

$$U_{2_1}'(t)$$

$$= \frac{(1 - \frac{S_v^*}{S_v})(\Lambda_v - S_v \int_0^\infty \beta_v(a) I_h(a, t) da - \mu_v S_v)}{S_v^* q(0) \int_0^\infty m(\tau) \pi_1(\tau) d\tau}$$

$$= \frac{(1 - \frac{S_v^*}{S_v})[S_v^* \int_0^\infty \beta_v(a) I_h^*(a) da + \mu_v S_v^* - S_v \int_0^\infty \beta_v(a) I_h(a, t) da - \mu_v S_v]}{S_v^* q(0) \int_0^\infty m(\tau) \pi_1(\tau) d\tau}$$

$$= -\frac{\mu_v (S_v - S_v^*)^2}{S_v^* S_v q(0) \int_0^\infty m(\tau) \pi_1(\tau) d\tau} + \frac{\int_0^\infty \beta_v(a) I_h^*(a)(1 - \frac{S_v^*}{S_v} - \frac{S_v I_h(a,t)}{S_v^* I_h^*(a)} + \frac{I_h(a,t)}{I_h^*(a)}) da}{q(0) \int_0^\infty m(\tau) \pi_1(\tau) d\tau}.$$

$$\tag{54}$$

Next, we need to take the time derivative of U_{2_2}.

$$\begin{aligned} U_{2_2}'(t) &= \frac{(1 - \frac{I_v^*}{I_v})[S_v \int_0^\infty \beta_v(a) I_h(a, t) da - (\mu_v + \alpha_v) I_v]}{S_v^* q(0) \int_0^\infty m(\tau) \pi_1(\tau) d\tau} \\[2mm] &= \frac{(1 - \frac{I_v^*}{I_v})(S_v \int_0^\infty \beta_v(a) I_h(a, t) da - \frac{S_v^* \int_0^\infty \beta_v(a) I_h^*(a) da}{I_v^*} I_v)}{S_v^* q(0) \int_0^\infty m(\tau) \pi_1(\tau) d\tau} \\[2mm] &= \frac{(1 - \frac{I_v^*}{I_v}) S_v^* \int_0^\infty \beta_v(a) I_h^*(a)(\frac{S_v I_h(a,t)}{S_v^* I_h^*(a)} - \frac{I_v}{I_v^*}) da}{S_v^* q(0) \int_0^\infty m(\tau) \pi_1(\tau) d\tau} \\[2mm] &= \frac{\int_0^\infty \beta_v(a) I_h^*(a)(\frac{S_v I_h(a,t)}{S_v^* I_h^*(a)} - \frac{I_v}{I_v^*} - \frac{S_v I_h(a,t) I_v^*}{S_v^* I_h^*(a) I_v} + 1) da}{q(0) \int_0^\infty m(\tau) \pi_1(\tau) d\tau}, \end{aligned}$$

$$\tag{55}$$

and

$$
\begin{aligned}
U'_{2_3}(t) \quad &= \left(1 - \frac{S_h^*}{S_h}\right)\left(\Lambda_h - \beta_h S_h I_v - \mu_h S_h\right) \\
&= \left(1 - \frac{S_h^*}{S_h}\right)\left(E_h^*(0) + \mu_h S_h^* - E_h(0,t) - \mu_h S_h\right) \\
&= -\frac{\mu_h(S_h - S_h^*)^2}{S_h} + \left(E_h^*(0) - E_h(0,t) - \frac{S_h^*}{S_h}E_h^*(0) + \frac{S_h^*}{S_h}E_h(0,t)\right).
\end{aligned}
\tag{56}
$$

Differentiating $U_{2_4}(t)$, we have

$$
\begin{aligned}
&U'_{2_4}(t) \\
&= \frac{1}{\mathcal{R}_0}\int_0^\infty p(\tau)E_h^*(\tau)f'\left(\frac{E_h(\tau,t)}{E_h^*(\tau)}\right)\frac{1}{E_h^*(\tau)}\frac{\partial E_h(\tau,t)}{\partial t}d\tau \\
&= -\frac{1}{\mathcal{R}_0}\int_0^\infty \frac{p(\tau)E_h^*(\tau)}{E_h^*(\tau)}f'\left(\frac{E_h(\tau,t)}{E_h^*(\tau)}\right)\left(\frac{\partial E_h(\tau,t)}{\partial \tau} + (\mu_h + m(\tau))E_h(\tau,t)\right)d\tau \\
&= -\frac{1}{\mathcal{R}_0}\int_0^\infty p(\tau)E_h^*(\tau)df\left(\frac{E_h(\tau,t)}{E_h^*(\tau)}\right) \\
&= -\frac{1}{\mathcal{R}_0}\left[p(\tau)E_h^*(\tau)f\left(\frac{E_h(\tau,t)}{E_h^*(\tau)}\right)\Big|_0^\infty - \int_0^\infty f\left(\frac{E_h(\tau,t)}{E_h^*(\tau)}\right)d\Big(p(\tau)E_h^*(\tau)\Big)\right] \\
&= \frac{1}{\mathcal{R}_0}\left[p(0)E_h^*(0)f\left(\frac{E_h(0,t)}{E_h^*(0)}\right) - \Lambda q(0)\int_0^\infty m(\tau)E_h^*(\tau)f\left(\frac{E_h(\tau,t)}{E_h^*(\tau)}\right)d\tau\right] \\
&= E_h^*(0)f\left(\frac{E_h(0,t)}{E_h^*(0)}\right) - \frac{\int_0^\infty m(\tau)E_h^*(\tau)f(\frac{E_h(\tau,t)}{E_h^*(\tau)})d\tau}{\int_0^\infty m(\tau)\pi_1(\tau)d\tau} \\
&= E_h(0,t) - E_h^*(0) - E_h^*(0)\ln\frac{E_h(0,t)}{E_h^*(0)} - \frac{\int_0^\infty m(\tau)E_h^*(\tau)f(\frac{E_h(\tau,t)}{E_h^*(\tau)})d\tau}{\int_0^\infty m(\tau)\pi_1(\tau)d\tau}.
\end{aligned}
\tag{57}
$$

The above equality follows from (35) and the fact

$$
p'(\tau)E_h^*(\tau) + p(\tau)E_h'^*(\tau)
$$

$$
= \left[-\Lambda q(0)m(\tau) + (\mu_h + m(\tau))p(\tau)\right]E_h^*(\tau) - p(\tau)(\mu_h + m(\tau))E_h^*(\tau)
$$

$$
= -\Lambda q(0)m(\tau)E_h^*(\tau).
$$

We also have

$$
q'(\tau)I_h^*(a) + q(a)I_h'^*(a)
$$

$$
= \left[-\beta_v(a) + (\mu_h + \alpha_h(a) + r_h(a))q(a)\right]I_h^*(a) - q(a)(\mu_h + \alpha_h(a) + r_h(a))I_h^*(a)
$$

$$
= -\beta_v(a)I_h^*(a).
$$

Similar to the differentiation of $U_{2_4}(t)$, we have

$$
\begin{aligned}
U'_{2_5}(t) \quad &= \frac{1}{q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau} \int_0^\infty q(a)I_h^*(a)f'\left(\frac{I_h(a,t)}{I_h^*(a)}\right)\frac{1}{I_h^*(a)}\frac{\partial I_h(a,t)}{\partial t}da \\
&= -\frac{1}{q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau}\int_0^\infty q(a)I_h^*(a)df\left(\frac{I_h(a,t)}{I_h^*(a)}\right) \\
&= \frac{q(0)I_h^*(0)f(\frac{I_h(0,t)}{I_h^*(0)}) - \int_0^\infty \beta_v(a)I_h^*(a)f(\frac{I_h(a,t)}{I_h^*(a)})da}{q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau} \\
&= \frac{\int_0^\infty m(\tau)E_h^*(\tau)(\frac{I_h(0,t)}{I_h^*(0)} - 1 - \ln\frac{I_h(0,t)}{I_h^*(0)})d\tau}{\int_0^\infty m(\tau)\pi_1(\tau)d\tau} - \frac{\int_0^\infty \beta_v(a)I_h^*(a)f(\frac{I_h(a,t)}{I_h^*(a)})da}{q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau}.
\end{aligned}
\tag{58}
$$

Finally, we differentiate $U_{2_6}(t)$, $U_{2_7}(t)$ with respect to t, and we have

$$
\begin{aligned}
U'_{2_6}(t) &= -\frac{S_h^*}{S_h}E_h(0,t), \\
U'_{2_7}(t) &= -\frac{S_h}{S_h^*}\frac{(E_h^*(0))^2}{E_h(0,t)}.
\end{aligned}
\tag{59}
$$

Adding all five components of the Lyapunov function, we have

$$
\begin{aligned}
&U'_2(t) \\
=\quad &-\frac{\mu_v(S_v - S_v^*)^2}{S_v^*S_v q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau} \\
&+\frac{1}{q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau}\int_0^\infty \beta_v(a)I_h^*(a)\left(1 - \frac{S_v^*}{S_v} - \frac{S_v I_h(a,t)}{S_v^* I_h^*(a)} + \frac{I_h(a,t)}{I_h^*(a)}\right)da \\
&+\frac{\int_0^\infty \beta_v(a)I_h^*(a)\left(\frac{S_v I_h(a,t)}{S_v^* I_h^*(a)} - \frac{I_v}{I_v^*} - \frac{S_v I_h(a,t)I_v^*}{S_v^* I_h^*(a)I_v} + 1\right)da}{q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau} \\
&-\frac{\mu_h(S_h - S_h^*)^2}{S_h} + \left(E_h^*(0) - E_h(0,t) - \frac{S_h^*}{S_h}E_h^*(0) + \frac{S_h^*}{S_h}E_h(0,t)\right) \\
&+E_h(0,t) - E_h^*(0) - E_h^*(0)\ln\frac{E_h(0,t)}{E_h^*(0)} - \frac{\int_0^\infty m(\tau)E_h^*(\tau)f(\frac{E_h(\tau,t)}{E_h^*(\tau)})d\tau}{\int_0^\infty m(\tau)\pi_1(\tau)d\tau} \\
&+\frac{\int_0^\infty m(\tau)E_h^*(\tau)(\frac{I_h(0,t)}{I_h^*(0)} - 1 - \ln\frac{I_h(0,t)}{I_h^*(0)})d\tau}{\int_0^\infty m(\tau)\pi_1(\tau)d\tau} - \frac{\int_0^\infty \beta_v(a)I_h^*(a)f(\frac{I_h(a,t)}{I_h^*(a)})da}{q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau} \\
&-\frac{S_h^*}{S_h}E_h(0,t) - \frac{S_h}{S_h^*}\frac{(E_h^*(0))^2}{E_h(0,t)} + 2E_h^*(0).
\end{aligned}
\tag{60}
$$

Canceling all terms that cancel, we simplify (60):

$$
\begin{aligned}
U'_2(t) \quad &= -\frac{\mu_v(S_v - S_v^*)^2}{S_v^*S_v q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau} - \frac{\mu_h(S_h - S_h^*)^2}{S_h} \\
&+\frac{\int_0^\infty \beta_v(a)I_h^*(a)(3 - \frac{S_v^*}{S_v} - \frac{I_v}{I_v^*} - \frac{S_v I_h(a,t)I_v^*}{S_v^* I_h^*(a)I_v} + \ln\frac{I_h(a,t)}{I_h^*(a)})da}{q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau} \\
&-\frac{S_h^*}{S_h}E_h^*(0) - \frac{S_h}{S_h^*}\frac{(E_h^*(0))^2}{E_h(0,t)} - E_h^*(0)\ln\frac{E_h(0,t)}{E_h^*(0)} + 2E_h^*(0) \\
&+\frac{\int_0^\infty m(\tau)E_h^*(\tau)(\frac{I_h(0,t)}{I_h^*(0)} - \frac{E_h(\tau,t)}{E_h^*(\tau)} + \ln\frac{E_h(\tau,t)}{E_h^*(\tau)}\frac{I_h^*(0)}{I_h(0,t)})d\tau}{\int_0^\infty m(\tau)\pi_1(\tau)d\tau}.
\end{aligned}
\tag{61}
$$

Noting that

$$\int_0^\infty m(\tau)E_h^*(\tau)\left(\frac{I_h(0,t)}{I_h^*(0)} - \frac{E_h(\tau,t)}{E_h^*(\tau)}\right)d\tau = 0,$$
$$\int_0^\infty m(\tau)E_h^*(\tau)\left(\frac{E_h(\tau,t)}{E_h^*(\tau)}\frac{I_h^*(0)}{I_h(0,t)} - 1\right) = 0. \tag{62}$$

Indeed,

$$\int_0^\infty m(\tau)E_h^*(\tau)\left(\frac{I_h(0,t)}{I_h^*(0)} - \frac{E_h(\tau,t)}{E_h^*(\tau)}\right)d\tau$$
$$= \frac{I_h(0,t)}{I_h^*(0)}\int_0^\infty m(\tau)E_h^*(\tau)d\tau - \int_0^\infty m(\tau)E_h(\tau,t)d\tau,$$
$$= \frac{I_h(0,t)}{I_h^*(0)}I_h^*(0) - I_h(0,t) = 0,$$
$$\int_0^\infty m(\tau)E_h^*(\tau)\left(\frac{E_h(\tau,t)}{E_h^*(\tau)}\frac{I_h^*(0)}{I_h(0,t)} - 1\right) \tag{63}$$
$$= \frac{I_h^*(0)}{I_h(0,t)}\int_0^\infty m(\tau)E_h(\tau,t)d\tau - \int_0^\infty m(\tau)E_h^*(\tau)d\tau$$
$$= \frac{I_h^*(0)}{I_h(0,t)}I_h(0,t) - I_h^*(0) = 0.$$

Using (62) to simplify (61) we obtain

$$U_2'(t) = -\frac{\mu_v(S_v - S_v^*)^2}{S_v^*S_v q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau} - \frac{\mu_h(S_h - S_h^*)^2}{S_h}$$
$$-\frac{\int_0^\infty \beta_v(a)I_h^*(a)[f(\frac{S_v^*}{S_v}) + f(\frac{I_v}{I_v^*}) + f(\frac{S_v I_h(a,t)I_v^*}{S_v^* I_h^*(a)I_v})]da}{q(0)\int_0^\infty m(\tau)\pi_1(\tau)d\tau} \tag{64}$$
$$-E_h^*(0)\left[f\left(\frac{S_h^*}{S_h}\right) + f\left(\frac{S_h}{S_h^*}\frac{E_h^*(0)}{E_h(0,t)}\right)\right]$$
$$-\frac{1}{\int_0^\infty m(\tau)\pi_1(\tau)d\tau}\int_0^\infty m(\tau)E_h^*(\tau)f\left(\frac{E_h(\tau,t)I_h^*(0)}{E_h^*(\tau)I_h(0,t)}\right)d\tau.$$

Hence, $U_2'(t) \le 0$. Define,

$$\Theta_2 = \left\{(S_v, I_v, S_h, E_h, I_h) \in X_0 \middle| U_2'(t) = 0\right\}.$$

We want to show that the largest invariant set in Θ_2 is the singleton $\mathcal{E}_1$. First, we notice that equality in (64) occurs if and only if $S_v(t) = S_v^*$, $S_h(t) = S_h^*$, $I_v(t) = I_v^*$, and

$$\frac{I_h(a,t)}{I_h^*(a)} = 1, \quad \frac{E_h^*(0)}{E_h(0,t)} = 1, \quad \frac{E_h(\tau,t)I_h^*(0)}{E_h^*(\tau)I_h(0,t)} = 1. \tag{65}$$

Thus, we obtain

$$I_h(a,t) = I_h^*(a), \quad E_h(0,t) = E_h^*(0).$$

According to (35),

$$E_h(\tau,t) = B_E(t-\tau)\pi_1(\tau) = E_h(0,t-\tau)\pi_1(\tau) = E_h^*(0)\pi_1(\tau) = E_h^*(\tau), \ t > \tau.$$

Furthermore, we obtain $E_h(\tau,t) = E_h^*(\tau)$. We conclude that the largest invariant set in Θ_2 is the singleton $\mathcal{E}_1$. Reasoning similarly to [7] can show that the compact global attractor $\mathfrak{T} = \{\mathcal{E}_1\}$. □

6. **Discussion.** In this paper, we formulate a partial differential equation (PDE) model describing the transmission dynamics of a vector-borne disease that incorporates both incubation age of the exposed hosts and infection age of the infectious hosts. An explicit formula for the basic reproduction number $\mathcal{R}_0$ is obtained for the infection-age structured vector-host epidemic model. We show that if $\mathcal{R}_0$ of system (2) is less or equal to one, the disease-free equilibrium is locally and globally asymptotically stable. That means the disease dies out while the endemic equilibrium is not feasible. On the other hand, we show that if $\mathcal{R}_0$ is greater than one, system (2) is permanent and the endemic equilibrium is globally asymptotically stable. Therefore the disease becomes endemic. As a result, the global stability of the equilibria of system (2) is completely determined by its basic reproductive number $\mathcal{R}_0$. Hence, to control the disease, a strategy should be devised to reduce the reproduction number to below one.

Examining the reproduction number more closely reveals that the relative impact of the recruitment rate of susceptible vectors Λ_v, the transmission rate β_h and the specific age-since-infection transmission coefficient $\beta_v(a)$ of the infected host individuals increases $\mathcal{R}_0$. It is easy to see that $\mathcal{R}_0$ is an decreasing function of the death rate of the vector individuals μ_v and the recovery rate of the infected vector individuals α_v. It is also evident that $\mathcal{R}_0$ decreases with the rates $r_h(a)$ and $\alpha_h(a)$ that give recovery and disease-induced mortality of infected hosts.

Furthermore, to see the link between $\mathcal{R}_0$ and the removal rate of the exposed host individuals with the incubation age τ, we first need to transform the reproduction number $\mathcal{R}_0$. We will use the representation of $\mathcal{R}_0$ given in (3) and (5).

$$\mathcal{R}_0$$

$$= \frac{\beta_h \Lambda_v \Lambda_h}{\mu_v \mu_h (\mu_v + \alpha_v)} \int_0^\infty m(\tau) \pi_1(\tau) d\tau \int_0^\infty \beta_v(a) \pi_2(a) da$$

$$= \frac{\beta_h \Lambda_v \Lambda_h}{\mu_v \mu_h (\mu_v + \alpha_v)} \int_0^\infty m(\tau) e^{-\mu_h \tau} e^{-\int_0^\tau m(\sigma) d\sigma} d\tau \int_0^\infty \beta_v(a) \pi_2(a) da$$

$$= \frac{\beta_h \Lambda_v \Lambda_h}{\mu_v \mu_h (\mu_v + \alpha_v)} \left[-\int_0^\infty (\mu_h - \mu_h - m(\tau)) e^{-\mu_h \tau} e^{-\int_0^\tau m(\sigma) d\sigma} d\tau \right] \int_0^\infty \beta_v(a) \pi_2(a) da$$

$$= \frac{\beta_h \Lambda_v \Lambda_h}{\mu_v \mu_h (\mu_v + \alpha_v)} \left[1 - \mu_h \int_0^\infty e^{-\mu_h \tau} e^{-\int_0^\tau m(\sigma) d\sigma} d\tau \right] \int_0^\infty \beta_v(a) \pi_2(a) da.$$

Denoting by

$$\rho = \int_0^\infty e^{-\mu_h \tau} e^{-\int_0^\tau m(\sigma) d\sigma} d\tau.$$

We obtain

$$\mathcal{R}_0 = \frac{\beta_h \Lambda_v \Lambda_h}{\mu_v \mu_h (\mu_v + \alpha_v)} \int_0^\infty \beta_v(a) \pi_2(a) da \left(1 - \mu_h \rho \right).$$

Taking the ρ derivatives of $\mathcal{R}_0$

$$\frac{d\mathcal{R}_0}{d\rho} = -\mu_h \frac{\beta_h \Lambda_v \Lambda_h}{\mu_v \mu_h (\mu_v + \alpha_v)} \int_0^\infty \beta_v(a) \pi_2(a) da < 0.$$

We have that ρ decrease with the increase of $m(\tau)$ and $\mathcal{R}_0$ decreases with increase of ρ. Thus we have that increasing $m(\tau)$ increases the reproduction number $\mathcal{R}_0$.

In conclusion, our model and its analysis suggest that a better strategy of beginning mosquito control is to remove possible breeding grounds, because the larvae and pupae cycle of the mosquito is aquatic. Mosquitoes lay eggs in stagnant water, that is to say, larvae need standing water to prosper, so we must remove items that retain standing water or construct ways to keep the water moving. Furthermore, we can look for shaded rest areas used by adult mosquitoes and eliminate them. When we are outside during the day and evening hours, we can wear long sleeves and pants to prevent the bites of mosquitoes and the transmission of disease. If the infected host individuals who are in the latent period take an active drug therapy in time, the total number of the infected hosts with the virus may become small. At last it is interesting that the disease prevalence will decrease with the increase of the disease induced death rate $\alpha_h(a)$ at the age of infection a.

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