GLOBAL STABILITY AND UNIFORM PERSISTENCE OF THE REACTION-CONVECTION-DIFFUSION CHOLERA EPIDEMIC MODEL

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Abstract. We study the global stability issue of the reaction-convection-diffusion cholera epidemic PDE model and show that the basic reproduction number serves as a threshold parameter that predicts whether cholera will persist or become globally extinct. Specifically, when the basic reproduction number is beneath one, we show that the disease-free-equilibrium is globally attractive. On the other hand, when the basic reproduction number exceeds one, if the infectious hosts or the concentration of bacteria in the contaminated water are not initially identically zero, we prove the uniform persistence result and that there exists at least one positive steady state.

1. **Introduction.** Cholera is an ancient intestinal disease for humans. It has a renowned place in epidemiology with John Snow's famous investigations of London cholera in 1850's which established the link between contaminated water and cholera outbreak. Cholera is caused by bacterium *vibrio cholerae*. The disease transmission consists of two routes: indirect environment-to-human (through ingesting the contaminated water) and direct person-to-person transmission routes. Even though cholera has been an object of intense study for over a hundred years, it remains to be a major public health concern in developing world; the disease has resulted in a number of outbreaks including the recent devastating outbreaks in Zimbabwe and Haiti, and renders more than 1.4 million cases of infection and 28,000 deaths worldwide every year [35].

It is well known that the transmission and spread of infectious diseases are complicated by spatial variation that involves distinctions in ecological and geographical environments, population sizes, socio-economic and demographic structures, human

activity levels, contact and mixing patterns, and many other factors. In particular, for cholera, spatial movements of humans and water can play an important role in shaping complex disease dynamics and patterns [6, 18]. There have been many studies published in recent years on cholera modeling and analysis (see, e.g., [1, 3, 4, 11, 16, 17, 20, 25, 26, 29, 30, 31, 32, 37]). However, only a few mathematical models among this large body of cholera models have considered human and water movement so far. Specifically, Bertuzzo et al. incorporated both water and human movement and formulated a simple PDE model [1, 19] and a patch model [2], in which only considered indirect transmission route. Chao et al. [5] proposed a stochastic model to study vaccination strategies and accessed its impact on spatial cholera outbreak in Haiti by using the model and data, for which both direct and indirect transmission were included. Tien, van den Driessche and their collaborators used network ODE models incorporating both water and human movement between geographic regions, and their results establish the connection in disease threshold between network and regions [7, 27]. Wang et al. [31] developed a generalized PDE model to study the spatial spread of cholera dynamics along a theoretical river, employing general incidence functions for direct and indirect transmission and intrinsic bacterial growth and incorporating both human/pathogen diffusion and bacterial convection.

In the present paper, we shall pay our attention to a reaction-diffusion-convection cholera model, which employs a most general formulation incorporating all different factors. This PDE model was first proposed in [31] and received investigations [31, 37]. Let us now describe this model explicitly in the following section.

2. **Statement of main results.** We study the following SIRS-B epidemic PDE model for cholera dynamics with $x \in [0,1], t > 0$:

$$\frac{\partial S}{\partial t} = D_1 \frac{\partial^2 S}{\partial x^2} + b - \beta_1 SI - \beta_2 S \frac{B}{B+K} - dS + \sigma R, \tag{1a}$$

$$\frac{\partial I}{\partial t} = D_2 \frac{\partial^2 I}{\partial x^2} + \beta_1 SI + \beta_2 S \frac{B}{B+K} - I(d+\gamma), \tag{1b}$$

$$\frac{\partial R}{\partial t} = D_3 \frac{\partial^2 R}{\partial x^2} + \gamma I - R(d+\sigma), \tag{1c}$$

$$\frac{\partial B}{\partial t} = D_4 \frac{\partial^2 B}{\partial x^2} - U \frac{\partial B}{\partial x} + \xi I + gB\left(1 - \frac{B}{K_B}\right) - \delta B, \tag{1d}$$

(cf. [31]) subjected to the following initial and Neumann and Robin boundary conditions respectively:

$$S(x,0) = \phi_1(x), \quad I(x,0) = \phi_2(x), \quad R(x,0) = \phi_3(x), \quad B(x,0) = \phi_4(x), \tag{2}$$

where each $\phi_i (i = 1,2,3,4)$ is assumed to be nonnegative and continuous in space x, and

$$\left.\frac{\partial Z}{\partial x}(x,t)\right|_{x=0,1} = 0, \qquad\qquad Z = S, I, R, \tag{3a}$$

$$D_4 \frac{\partial B}{\partial x}(x,t) - UB(x,t)|_{x=0} = \frac{\partial B}{\partial x}(x,t)|_{x=1} = 0. \tag{3b}$$

Here $S = S(x,t), I = I(x,t)$, and $R = R(x,t)$ measure the number of susceptible, infectious, and recovered human hosts at location x and time t, respectively. $B = B(x,t)$ denotes the concentration of the bacteria (vibrios) in the water environment. The definition of model parameters is provided in Table 1.

TABLE 1. Definition of parameters in model (1)

Parameter	Definition
b	Recruitment rate of susceptible hosts
d	Natural death rate of human hosts
γ	Recovery rate of infectious hosts
σ	Rate of host immunity loss
δ	Natural death rate of bacteria
ξ	Shedding rate of bacteria by infectious hosts
β_1	Direct transmission parameter
β_2	Indirect transmission parameter
K	Half saturation rate of bacteria
U	Bacterial convection coefficient
K_B	Maximal carrying capacity of bacteria in the environment

We assume all of these parameters to be positive. Hereafter let us write $\partial_t, \partial_x, \partial_{xx}^2$ for $\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}$, respectively.

To state our results clearly, let us denote the solution

$$u = (u_1, u_2, u_3, u_4) \triangleq (S, I, R, B) \in \mathbb{R}^4, \qquad \phi \triangleq (\phi_1, \phi_2, \phi_3, \phi_4). \tag{4}$$

We also denote the Lebesgue spaces L^p with their norms by $\|\cdot\|_{L^p}$, $p \in [1, \infty]$. Finally, we denote

$$X \triangleq C([0,1], \mathbb{R}^4) = \prod_{i=1}^{4} X_i, \qquad X_i \triangleq C([0,1], \mathbb{R}), \tag{5}$$

the space of $\mathbb{R}^4$-valued functions continuous in $x \in [0,1]$ with the usual sup norm

$$\|u\|_{C([0,1])} \triangleq \|S\|_{C([0,1])} + \|I\|_{C([0,1])} + \|R\|_{C([0,1])} + \|B\|_{C([0,1])}. \tag{6}$$

We define analogously

$$X^+ \triangleq C([0,1], \mathbb{R}_+^4) = \prod_{i=1}^{4} X_i^+, \quad X_i^+ \triangleq \{f \in C([0,1], \mathbb{R}) : f \geq 0\}.$$

Understanding the global dynamical behavior of cholera modeling problems is crucial in order to suggest effective measures to control the growth of the disease. To the best of our knowledge, the existing literature has only studied local dynamics of solutions of this general PDE model. The focus of the present work is global disease threshold dynamics, which will be established in terms of the basic reproduction number R_0 [12, 24, 33]. To that end, we conduct a rigorous investigation on the disease using the model, and analyze both model parameters and the system dynamics for a better understanding of disease mechanisms. Particularly, we perform a careful analysis on the global threshold dynamics of the disease.

In review of previous results, firstly the authors in [32] defined $\mathcal{R}_0^{ODE}$ for the SIRS-B ODE model, which can be extended to the SIRS-B PDE model as follows: denoting

$$\Theta_1 \triangleq \begin{pmatrix} m^*\beta_1 & m^*\frac{\beta_2}{K} \\ \xi & g \end{pmatrix}, \qquad \Theta_2 \triangleq \begin{pmatrix} D_2\partial_{xx}^2 - (d+\gamma) & 0 \\ 0 & D_4\partial_{xx}^2 - U\partial_x - \delta \end{pmatrix}, \tag{7}$$

where $m^* \triangleq \frac{b}{d}$, we have $\mathcal{R}_0^{PDE} \triangleq r(-\Theta_1 \Theta_2^{-1})$, the spectral radius of $-\Theta_1 \Theta_2^{-1}$, for which $\mathcal{R}_0^{ODE}$ is same except that the operators Θ_1, Θ_2 in (7) would have no diffusive operators ∂_{xx}^2. Moreover, the authors in [32] proved that when $\mathcal{R}_0^{ODE} \leq 1$, the model has the disease-free-equilibrium (DFE) $(S, I, R, B) = (m^*, 0, 0, 0)$ which is globally asymptotically stable (see Theorem 2.1 of [32]). On the other hand, when $\mathcal{R}_0^{ODE} > 1$, it was proven that this ODE model has two equilibriums, namely the DFE which is unstable and endemic equilibrium which is globally asymptotically stable (see Theorem 2.1 [32]). For the SIRS-B PDE model with diffusion, the authors in [37] used spectral analysis tools from [24] to show that when $\mathcal{R}_0^{PDE} < 1$, the DFE is locally asymptotically stable while if $\mathcal{R}_0^{PDE} > 1$, then there exists $\eta > 0$ such that any positive solution of (1) linearized at the DFE satisfies

$$\limsup_{t \to \infty} \|(S(\cdot, t), I(\cdot, t), R(\cdot, t), B(\cdot, t)) - (m^*, 0, 0, 0)\|_{C([0,1])} \geq \eta. \tag{8}$$

We emphasize here that both these stability and persistence results were local; specifically the results were obtained via analysis on the (S, I, R, B) that solves the system (1) linearized at the DFE $(m^*, 0, 0, 0)$, not necessary the actual system (1). The major difficulty was that because by definition $\mathcal{R}_0^{PDE}$ gives information only on the linearized system (see the definition $\mathcal{R}_0^{PDE} = r(-\Theta_1 \Theta_2^{-1})$, (7), (13), (14)), it seemed difficult to utilize the hypothesis that $\mathcal{R}_0^{PDE} > 1$ or $\mathcal{R}_0^{PDE} < 1$ to deduce any information on the actual system (1) (see e.g. Theorem 4.3 (ii) of [34]).

In this paper, we overcome this major obstacle and extend these stability results to global; moreover, we obtain the uniform persistence result. We also extend Lemma 1 of [13], which have proven to be useful in various other papers (e.g. Lemma 3.2, [28]) to the case with convection, which we believe will be useful in many future work. For simplicity, let us hereafter denote $\mathcal{R}_0 \triangleq \mathcal{R}_0^{PDE}$, and by $u(x, t, \phi)$ the solution at $(x, t) \in [0, 1] \times [0, \infty)$ that initiated from ϕ:

Theorem 2.1. *Suppose $D = D_1 = D_2 = D_3, \phi \in X^+$. Then the system (1) subjected to (2), (3) admits a unique global nonnegative solution $u(x, t, \phi)$ such that $u(x, 0, \phi) = \phi(x)$. Moreover, if $\mathcal{R}_0 < 1$, then the DFE $(m^*, 0, 0, 0)$ is globally attractive.*

Theorem 2.2. *Suppose $D = D_1 = D_2 = D_3, \phi \in X^+$ and $g < \delta$. Let $u(x, t, \phi)$ be the unique global nonnegative solution of the system (1) subjected to (2), (3) such that $u(x, 0, \phi) = \phi(x)$ and $\Phi_t(\phi) = u(t, \phi)$ be its solution semiflow. If $\mathcal{R}_0 > 1$ and $\phi_2(\cdot) \not\equiv 0$ or $\phi_4(\cdot) \not\equiv 0$, then the system (1) admits at least one positive steady state a_0 and there exists $\eta > 0$ such that*

$$\liminf_{t \to \infty} u_i(x, t) \geq \eta, \quad \forall i = 1, 2, 4, \tag{9}$$

uniformly $\forall x \in [0, 1]$.

Remark 1.   1. We remark that typically the persistence results in the case $\mathcal{R}_0 > 1$ requires a hypothesis that the solution is positive (see e.g. Theorem 4.3 (ii) of [34] and also Theorem 2.3 (2) of [37]). In the statement of Theorem 2.2, we only require that $\phi_2(\cdot) \not\equiv 0$ or $\phi_4(\cdot) \not\equiv 0$. Due to the Proposition 2, we are able to relax these conditions. Moreover, we note that sup in (8) is replaced by inf in (9).

2. The proof was inspired by the work of [13, 28, 33].

3. We remark that it remains unknown what happens when $\mathcal{R}_0 = 1$; for this matter, not global but even in the local case, it remains an open problem (see Theorem 2.3 [37]).

4. In the system (1), we chose a particular case of

$$f_1(I) = \beta_1 I, \quad f_2(B) = \beta_2 \frac{B}{B+K}, \quad h(B) = gB\left(1 - \frac{B}{K_B}\right)$$

where f_1, f_2, h represent the direct, indirect transmission rates, intrinsic growth rate of bacteria respectively (see [32, 31]). We remark for the purpose of our subsequent proof that defining this way, f_1, f_2, h are all Lipschitz. It is clear from the proof that some generalization is possible.

The rest of the article is organized as follows. The next section presents preliminary results of this study. Section 4 verifies a key proposition as an extension of Lemma 1 of [13], which has proved to be useful in various context. Our main results are established in Sections 5-6. By employing the theory of monotone dynamical systems [38], we prove that (1) the disease free equilibrium (DFE) is globally asymptotically stable if the basic reproduction number R_0 is less than unity; (2) there exists at least one positive steady state and the disease is uniformly persistent in both the human and bacterial populations if $R_0 > 1$. Additionally, we identify a precise condition on model parameters for which the system admits a unique non-negative solution, and study the global attractivity of this solution. In the end, a brief discussion is given in Section 7, followed by Appendix.

3. **Preliminaries.** When there exists a constant $c = c(a,b) \geq 0$ such that $A \leq cB, A = cB$, we write $A \lesssim_{a,b} B, A \approx_{a,b} B$.

Following [21, 37], we let $A_i^0, i = 1, 2, 3$ denote the differentiation operator

$$A_i^0 u_i \triangleq D\partial_{xx}^2 u_i, \quad A_4^0 \triangleq D_4\partial_{xx}^2 u_4 - U\partial_x u_4,$$

defined on their domains

$$D(A_i^0) \triangleq \{\psi \in C^2((0,1)) \cap C^1([0,1]) : A_i^0\psi \in C([0,1]), \partial_x\psi|_{x=0,1} = 0\}, \ i = 1, 2, 3,$$

$$D(A_4^0) \triangleq \{\psi \in C^2((0,1)) \cap C^1([0,1]) :$$

$$A_4^0\psi \in C([0,1]), D_4\partial_x\psi - U\psi|_{x=0} = \partial_x\psi|_{x=1} = 0\},$$

respectively. We can then define $A_i, (i = 1, 2, 3, 4)$ to be the closure of A_i^0 so that A_i on X_i generates an analytic semigroup of bounded linear operator $T_i(t), t \geq 0$ such that $u_i(x,t) = (T_i(t)\phi_i)(x)$ satisfies

$$\partial_t u_i(t) = A_i u_i(t), \quad u_i(0) = \phi_i \in D(A_i)$$

where

$$D(A_i) = \left\{\psi \in X_i : \lim_{t\to 0^+} \frac{(T_i(t) - I)\psi}{t} = A_i\psi \text{ exists}\right\};$$

that is, for $i = 1, 2, 3$,

$$\partial_t u_i(x,t) = D_i\partial_{xx}^2 u_i(x,t), t > 0, x \in (0,1), \quad \partial_x u_i|_{x=0,1} = 0, \quad u_i(x,0) = \phi_i(x),$$

and

$$\begin{cases} \partial_t u_4(x,t) = D_4\partial_{xx}^2 u_4(x,t) - U\partial_x u_4(x,t), & t > 0, x \in (0,1), \\ D_4\partial_x u_4 - Uu_4|_{x=0} = \partial_x u_4|_{x=1} = 0, & u_4(x,0) = \phi_4(x). \end{cases}$$

It follows that each T_i is compact (see e.g. pg. 121 [21]). Moreover, by Corollary 7.2.3, pg. 124 [21], because $X_i^+ = C([0,1], \mathbb{R}_+)$, each $T_i(t)$ is strongly positive (see Definition 3.2).

We now let

$$F_1 \triangleq b - \beta_1 SI - \beta_2 S \left(\frac{B}{B+K} \right) - dS + \sigma R, \tag{10a}$$

$$F_2 \triangleq \beta_1 SI + \beta_2 S \left(\frac{B}{B+K} \right) - I(d + \gamma), \tag{10b}$$

$$F_3 \triangleq \gamma I - R(d + \sigma), \tag{10c}$$

$$F_4 \triangleq \xi I + gB \left(1 - \frac{B}{K_B} \right) - \delta B, \tag{10d}$$

and $F \triangleq (F_1, F_2, F_3, F_4)$. Let $T(t) : X \mapsto X$ be defined by $T(t) \triangleq \prod_{i=1}^{4} T_i(t)$ so that it is a semigroup of operator on X generated by $A \triangleq \prod_{i=1}^{4} A_i$ with domain $D(A) \triangleq \prod_{i=1}^{4} D(A_i)$ and hence we can write (1) as

$$\partial_t u = Au + F(u), \quad u(0) = u_0 = \phi.$$

We recall some relevant definitions

Definition 3.1. (pg. 2, 3, 11 [38]) Let (Y, d) be any metric space and $f : Y \mapsto Y$ a continuous map. A bounded set A is said to attract a bounded set $B \subset Y$ if $\lim_{n \to \infty} \sup_{x \in B} d(f^n(x), A) = 0$. A subset $A \subset Y$ is an attractor for f if A is nonempty, compact and invariant ($f(A) = A$), and A attracts some open neighborhood of itself. A global attractor for f is an attractor that attracts every point in Y. Moreover, f is said to be point dissipative if there exists a bounded set B_0 in Y such that B_0 attracts each point in Y. Finally, a nonempty invariant subset M of Y is isolated for $f : Y \mapsto Y$ if it is the maximal invariant set in some neighborhood of itself.

Definition 3.2. (pg. 38, 40, 46, [38]) Let E be an ordered Banach space with positive cone P such that $\text{int}(P) \neq \emptyset$. For $x, y \in E$, we write $x \geq y$ if $x - y \in P, x > y$ if $x - y \in P \setminus \{0\}$, and $x \gg y$ if $x - y \in \text{int}(P)$.

A linear operator L on E is said to be positive if $L(P) \subset P$, strongly positive if $L(P \setminus \{0\}) \subset \text{int}(P)$. For any subset U of E, $f : U \mapsto U$, a continuous map, f is said to be monotone if $x \geq y$ implies $f(x) \geq f(y)$, strictly monotone if $x > y$ implies $f(x) > f(y)$, and strongly monotone if $x > y$ implies $f(x) \gg f(y)$.

Let $U \subset P$ be nonempty, closed, and order convex. Then a continuous map $f : U \mapsto U$ is said to be subhomogeneous if $f(\lambda x) \geq \lambda f(x)$ for any $x \in U$ and $\lambda \in [0, 1]$, strictly subhomogeneous if $f(\lambda x) > \lambda f(x)$ for any $x \in U$ with $x \gg 0$ and $\lambda \in (0, 1)$, and strongly subhomogeneous if $f(\lambda x) \gg \lambda f(x)$ for any $x \in U$ with $x \gg 0$ and $\lambda \in (0, 1)$.

Definition 3.3. (pg. 56, 129, [21]) An $n \times n$ matrix $M = (M_{ij})$ is irreducible if $\forall\, I \subsetneq N = \{1, \ldots, n\}$, $I \neq \emptyset$, there exists $i \in I$ and $j \in J = N \setminus I$ such that $M_{ij} \neq 0$. Moreover, $F : [0, 1] \times \Lambda \mapsto \mathbb{R}^n, \Lambda$ any nonempty, closed, convex subset of $\mathbb{R}^n$, is cooperative if $\frac{\partial F_i}{\partial u_j}(x, u) \geq 0$, $\forall\, (x, u) \in [0, 1] \times \Lambda, i \neq j$.

Lemma 3.4. (*Theorem 7.3.1, Corollary 7.3.2, [21]*) *Suppose that* $F : [0, 1] \times \mathbb{R}_+^4 \mapsto \mathbb{R}^4$ *has the property that*

$$F_i(x, u) \geq 0 \quad \forall\, x \in [0, 1], u \in \mathbb{R}_+^4 \text{ and } u_i = 0.$$

Then $\forall \psi \in X^+$,

$$\begin{cases} \partial_t u_i(x,t) = D_i \partial_{xx}^2 u_i(x,t) + F_i(x, u(x,t)), & t > 0, x \in (0,1), \\ \alpha_i(x) u_i(x,t) + \delta_i \partial_x u_i(x,t) = 0, & t > 0, x = 0, 1, \\ u_i(x,0) = \psi_i(x), & x \in (0,1), \end{cases}$$

has a unique noncontinuable mild solution $u(x,t,\psi) \in X^+$ *defined on* $[0,\sigma)$ *where* $\sigma = \sigma(\psi) \leq \infty$ *such that if* $\sigma < \infty$, *then* $\|u(t)\|_{C([0,1])} \to \infty$ *as* $t \to \sigma$ *from below. Moreover,*

1. *u is continuously differentiable in time on* $(0,\sigma)$,
2. *it is in fact a classical solution,*
3. *if* $\sigma(\psi) = +\infty \ \forall \ \psi \in X^+$, *then* $\Psi_t(\psi) = u(t,\psi)$ *is a semiflow on* X^+,
4. *if* $Z \subset X^+$ *is closed and bounded,* $t_0 > 0$ *and* $\cup_{t \in [0,t_0]} \Psi_t(Z)$ *is bounded, then* $\Psi_{t_0}(Z)$ *has a compact closure in* X^+.

Remark 2. This lemma remains valid even if the Laplacian is replaced by a general second order differentiation operator; in fact, all results from Chapter 7, [21] remain valid for a general second order differentiation operator (see pg. 121, [21]). In relevance we also refer readers to Theorem 1.1, [15], Corollary 8.1.3 [36] for similar general well-posedness results.

The following result was obtained in [37]:

Lemma 3.5. *(Theorems 2.1, 2.2, [37])* $\forall \phi \in X^+$ *the system (1) subjected to (2) and (3) admits a unique nonnegative mild solution on the interval of existence* $[0,\sigma)$ *where* $\sigma = \sigma(\phi)$. *If* $\sigma < \infty$, *then* $\|u(t)\|_{C([0,1])}$ *becomes unbounded as* t *approaches* σ *from below.*

Moreover, if $D_1 = D_2 = D_3$, *then* $\sigma = +\infty$. *Therefore,* $\Phi_t(\phi) = u(t,\phi)$ *is a semiflow on* X^+.

Remark 3. In the statement of Theorems 2.1, 2.2 of [37], we required the initial regularity to be in $X^+ \cap H^1([0,1])$ where $H^1([0,1]) = \{f : f, \partial_x f \in L^2([0,1])\}$ and obtained higher regularity beyond $C([0,1], \mathbb{R}^4)$; here we point out that to show the global existence of the solution $u(t) \in X^+ \ \forall \ t \geq 0$, it suffices that the initial data is in X^+. For completeness, in the Appendix we describe the estimate more carefully than that of Proposition 1 in [37] that is needed to verify this claim.

Lemma 3.6. *(Theorem 2.3.2, [38])* *Let* E *be an ordered Banach space with positive cone* P *such that* $int(P) \neq \emptyset$, $U \subset P$ *be nonempty, closed and order convex set. Suppose* $f : U \mapsto U$ *is strongly monotone, strictly subhomogeneous and admits a nonempty compact invariant set* $K \subset int(P)$. *Then* f *has a fixed point* $e \gg 0$ *such that every nonempty compact invariant set of* f *in* $int(P)$ *consists of* e.

Lemma 3.7. *(Theorem 3.4.8, [10])* *If there exists* $t_1 \geq 0$ *such that the* C^r-*semigroup* $T(t) : Y \mapsto Y, t \geq 0$, Y *any metric space, is completely continuous for* $t > t_1$ *and point dissipative, then there exists a global attractor* A. *If* Y *is a Banach space, then* A *is connected and if* $t_1 = 0$, *then there is an equilibrium point of* $T(t)$.

Lemma 3.8. *(Lemma 3, [22])* *Let* Y *be a metric space,* Ψ *a semiflow on* Y, $Y_0 \subset Y$ *an open set,* $\partial Y_0 = Y \setminus Y_0$, $M_\partial = \{y \in \partial Y_0 : \Psi_t(y) \in \partial Y_0 \ \forall \ t \geq 0\}$ *and* q *be a generalized distance function for semiflow* Ψ. *Assume that*

1. Ψ *has a global attractor* A,
2. *there exists a finite sequence* $K = \{K_i\}_{i=1}^n$ *of pairwise disjoint, compact and isolated invariant sets in* ∂Y_0 *with the following properties*

- $\cup_{y \in M_\partial} \omega(y) \subset \cup_{i=1}^n K_i$,
- *no subset of K forms a cycle in ∂Y_0,*
- K_i *is isolated in Y,*
- $W^s(K_i) \cap q^{-1}(0, \infty) = \emptyset \ \forall \ i = 1, \ldots, n.$

Then there exists $\delta > 0$ such that for any compact chain transitive set L that satisfies $L \not\subset K_i \ \forall \ i = 1, \ldots, n$, $\min_{y \in L} q(y) > \delta$ holds.

Lemma 3.9. *(pg. 3, [38]) Suppose the Kuratowski's measure of non-compactness for any bounded set B of Y, any metric space, is denoted by*

$$\alpha(B) = \inf\{r : B \text{ has a finite cover of diameter } r\}.$$

Firstly, $\alpha(B) = 0$ if and only if $\overline{B}$ is compact.

Moreover, a continuous mapping $f : Y \mapsto Y, Y$ any metric space, is α-condensing (α-contraction of order $0 \le k < 1$) if f takes bounded sets to bounded sets and $\alpha(f(B)) < \alpha(B)$ ($\alpha(f(B)) \le k\alpha(B)$) for any nonempty closed bounded set $B \subset Y$ such that $\alpha(B) > 0$. Moreover, f is asymptotically smooth if for any nonempty closed bounded set $B \subset Y$ for which $f(B) \subset B$, there exists a compact set $J \subset B$ such that J attracts B.

It is well-known that a compact map is an α-contraction of order 0, and an α-contraction or order k is α-condensing. Moreover, by Lemma 2.3.5, [10], any α-condensing maps are asymptotically smooth.

Lemma 3.10. *(Theorem 3.7, [14]) Let (M, d) be a complete metric space, and $\rho : M \to [0, \infty)$ a continuous function such that $M_0 = \{x \in M : \rho(x) > 0\}$ is nonempty and convex. Suppose that $T : M \mapsto M$ is continuous, asymptotically smooth, ρ-uniformly persistent, T has a global attractor A and satisfies $T(M_0) \subset M_0$. Then $T : (M_0, d) \mapsto (M_0, d)$ has a global attractor A_0.*

Remark 4. (Remark 3.10, [14]) Let (M, d) be a complete metric space. A family of mappings $\Psi_t : M \mapsto M, t \ge 0$, is called a continuous-time semiflow if $(x, t) \mapsto \Psi_t(x)$ is continuous, $\Psi_0 = Id$ and $\Psi_t \circ \Psi_s = \Psi_{t+s}$ for $t, s \ge 0$. Lemma 3.10 is valid even if replaced by a continuous-time semiflow Ψ_t on M such that $\Psi_t(M_0) \subset M_0 \ \forall \ t \ge 0$.

Lemma 3.11. *(Theorem 4.7, [14]) Let M be a closed convex subset of a Banach space $(X, \|\cdot\|)$, $\rho : M \to [0, \infty)$ a continuous function such that $M_0 = \{x \in M : \rho(x) > 0\}$, where M_0 is nonempty and convex, and Ψ_t a continuous-time semiflow on M such that $\Psi_t(M_0) \subset M_0 \ \forall \ t \ge 0$. If either Ψ_t is α-condensing $\forall \ t > 0$ or Ψ_t is convex α-contracting for $t > 0$, and $\Psi_t : M_0 \mapsto M_0$ has a global attractor A_0, then Ψ_t has an equilibrium $a_0 \in A_0$.*

4. **Key proposition.** Many authors found Lemma 1 of [13] to be very useful in various proofs (see e.g. Lemma 3.2, [28]). The key to the proof of our claim is the following extension of Lemma 1 of [13] to consider the case with convection:

Proposition 1. *Consider in a spatial domain with $x \in [0, 1]$, the following scalar reaction-convection-diffusion equation*

$$\begin{cases} \partial_t w(x, t) = \overline{D}\partial_{xx}^2 w(x, t) - \overline{U}\partial_x w(x, t) + g(x) - \lambda w(x, t), \\ \overline{D}\partial_x w(x, t) - \overline{U}w(x, t)|_{x=0} = \partial_x w(x, t)|_{x=1} = 0, \quad w(x, 0) = \psi(x), \end{cases} \tag{11}$$

where $\overline{D} > 0, \lambda > 0, \overline{U} \ge 0$, and $g(x) > 0$ is a continuous function. Then $\forall \psi \in C([0, 1], \mathbb{R}_+)$, there exists a unique positive steady state w^* which is globally attractive in $C([0, 1], \mathbb{R})$. Moreover, in the case $\overline{U} = 0$ and $g(x) \equiv g$, it holds that $w^* = \frac{g}{\lambda}$.

Proof. The case $\overline{U} = 0$ is treated in Lemma 1 of [13]; we assume $\overline{U} > 0$ here. By continuity we know that there exists

$$0 < \min_{x \in [0,1]} g(x) \leq g(x) \leq \max_{x \in [0,1]} g(x) \triangleq \overline{g} \; \forall \; x \in [0,1].$$

We define $F(x, w) \triangleq g(x) - \lambda w(x, t)$. It is immediate that (e.g. by Lemma 3.4 and Remark 2) $\forall \psi \in C([0, 1], \mathbb{R}_+)$, there exists a unique solution $w = w(x, t, \psi) \in C([0, 1], \mathbb{R}_+)$ on some time interval $[0, \sigma), \sigma = \sigma(\psi)$.

We fix $\psi \in C([0, 1], \mathbb{R}_+)$ so that by continuity there exists $\max_{x \in [0,1]} \psi(x)$. Now if $v \equiv M$ for M sufficiently large such that $M > \max\{\max_{x \in [0,1]} \psi(x), \frac{\overline{g}}{\lambda}\}$, then by Theorem 7.3.4 of [21] and the blow up criterion from Lemma 3.4 and Remark 2, we immediately deduce the existence of a unique solution on $[0, \infty)$.

Hence, there exists the solution semiflow P_t such that $P_t(\psi) = w(t, \psi), \psi \in C([0, 1], \mathbb{R}_+)$. It follows that

$$\omega(\psi) \subset \{\varphi : \frac{\min_{x \in [0,1]} g(x)}{\lambda} \leq \varphi \leq \frac{\max_{x \in [0,1]} g(x)}{\lambda}\}$$

by comparison principle (e.g. Theorem 7.3.4 [21]); we emphasize here again that as stated on pg. 121, [21], Theorem 7.3.4 [21] is applicable to the general second-order differentiation operator such as $\overline{D}\partial_{xx}^2 - \overline{U}\partial_x$. By comparison principle again (e.g. Corollary 7.3.5, Theorem 7.4.1, [21]), it also follows that

$$P_t(\psi_1) \gg P_t(\psi_2) \quad \forall \; t > 0$$

if $\psi_1 > \psi_2$; this implies that P_t is strongly monotone (see Definition 3.2). Moreover, F is strictly subhomogeneous (see Definition 3.2) in a sense that $F(x, \alpha w) > \alpha F(x, w) \; \forall \; \alpha \in (0, 1)$ as $g(x) > 0$. We now follow the idea from pg. 348 [9] to complete the proof. Let $L(t) \triangleq w(t, \alpha\psi) - \alpha w(t, \psi)$ so that

$$\partial_t L = \overline{D}\partial_{xx}^2 L - \overline{U}\partial_x L + (1 - \alpha)g(x) - \lambda L,$$
$$L(0) = 0, \quad \overline{D}\partial_x L - \overline{U}L|_{x=0} = \partial_x L|_{x=1} = 0.$$

Let $\Psi(t, s), t \geq s \geq 0$ be the evolution operator of

$$\begin{cases} \partial_t N = \overline{D}\partial_{xx}^2 N - \overline{U}\partial_x N - \lambda N, \\ \overline{D}\partial_x N - \overline{U}N|_{x=0} = \partial_x N|_{x=1} = 0. \end{cases} \tag{12}$$

Then $\Psi(t, 0)(0) = 0$. Thus, by Theorem 7.4.1 [21], which is applicable to the general second-order differentiation operator such as $\overline{D}\partial_{xx}^2 - \overline{U}\partial_x$, we see that $\forall \; \psi > 0, \Psi(t, s)\psi \gg 0$. Hence by Comparison Principle as $g(x)(1 - \alpha) \geq 0$, we obtain $\forall \; \psi > 0, L(x, t, \psi) \gg 0$. Therefore, $\forall \; \psi > 0, w(t, \alpha\psi) > \alpha w(t, \psi)$; i.e. P_t is strictly subhomogeneous (see Definition 3.2).

By Lemma 3.6 we now conclude that P_t has a fixed point $w^*(x) \gg 0$ such that $\omega(\psi) = w^* \in C([0, 1], \mathbb{R}_+) \; \forall \; \psi \in C([0, 1], \mathbb{R}_+)$. $\square$

5. **Proof of Theorem 2.1.** Firstly, by Lemma 3.5, we know that given $\phi \in X^+$, there exists a unique global nonnegative solution to the system (1) subjected to (2), (3).

Now, from the proof of Theorem 2.3 (1) [37], we know that if we linearize (1) about the DFE $(S, I, R, B) = (m^*, 0, 0, 0)$, we obtain

$$\begin{cases} \partial_t S = D\partial_{xx}^2 S - m^* \left(\beta_1 I + \frac{\beta_2}{K}B \right) - d\,S + \sigma R, \\ \partial_t I = D\partial_{xx}^2 I + m^* \left(\beta_1 I + \frac{\beta_2}{K}B \right) - I(d+\gamma), \\ \partial_t R = D\partial_{xx}^2 R + \gamma I - R(d+\sigma), \\ \partial_t B = D_4\partial_{xx}^2 B - U\partial_x B + \xi I + gB - \delta B, \end{cases} \tag{13}$$

so that substituting $(S, I, R, B) = (e^{\lambda t}\psi_1(x), e^{\lambda t}\psi_2(x), e^{\lambda t}\psi_3(x), e^{\lambda t}\psi_4(x))$ in (13) gives us the eigenvalue problem of

$$\begin{cases} \lambda\psi_1 = D\partial_{xx}^2\psi_1 - m^* \left(\beta_1\psi_2 + \frac{\beta_2}{K}\psi_4 \right) - d\psi_1 + \sigma\psi_3, \\ \lambda\psi_2 = D\partial_{xx}^2\psi_2 + m^* \left(\beta_1\psi_2 + \frac{\beta_2}{K}\psi_4 \right) - \psi_2(d+\gamma), \\ \lambda\psi_3 = D\partial_{xx}^2\psi_3 + \gamma\psi_2 - \psi_3(d+\sigma), \\ \lambda\psi_4 = D_4\partial_{xx}^2\psi_4 - U\partial_x\psi_4 + \xi\psi_2 + g\psi_4 - \delta\psi_4. \end{cases} \tag{14}$$

We define

$$\tilde{\Theta}(\psi_1, \psi_2, \psi_3, \psi_4) \triangleq \begin{pmatrix} D\partial_{xx}^2\psi_1 - m^* \left(\beta_1\psi_2 + \frac{\beta_2}{K}\psi_4 \right) - d\psi_1 + \sigma\psi_3 \\ D\partial_{xx}^2\psi_2 + m^* \left(\beta_1\psi_2 + \frac{\beta_2}{K}\psi_4 \right) - \psi_2(d+\gamma) \\ D\partial_{xx}^2\psi_3 + \gamma\psi_2 - \psi_3(d+\sigma) \\ D_4\partial_{xx}^2\psi_4 - U\partial_x\psi_4 + \xi\psi_2 + g\psi_4 - \delta\psi_4 \end{pmatrix}. \tag{15}$$

It is shown in the proof of Theorem 2.3 (1) [37] that defining

$$\Theta \begin{pmatrix} \psi_2 \\ \psi_4 \end{pmatrix} \triangleq \left(\begin{pmatrix} D\partial_{xx}^2 - (d+\gamma) & 0 \\ 0 & D_4\partial_{xx}^2 - U\partial_x - \delta \end{pmatrix} + \begin{pmatrix} m^*\beta_1 & m^*\frac{\beta_2}{K} \\ \xi & g \end{pmatrix} \right) \begin{pmatrix} \psi_2 \\ \psi_4 \end{pmatrix}$$

$$= (\Theta_2 + \Theta_1) \begin{pmatrix} \psi_2 \\ \psi_4 \end{pmatrix}, \tag{16}$$

we have the spectral bound of Θ_2, $s(\Theta_2)$, to satisfy $s(\Theta_2) < 0$. Thus, by Theorem 3.5 [24], $s(\Theta)$, the spectral bound of Θ, and hence $s(\tilde{\Theta})$, due to the independence of Θ from the first and third equations of $\tilde{\Theta}(\psi_1, \psi_2, \psi_3, \psi_4)$ in (15) , has the same sign as

$$r(-\Theta_1\Theta_2^{-1}) - 1 = \mathcal{R}_0 - 1.$$

That is, $\mathcal{R}_0 - 1$ and the principal eigenvalue of $\tilde{\Theta}$, $\lambda = \lambda(m^*)$, have same signs. Now by hypothesis, $\mathcal{R}_0 < 1$ and hence $\mathcal{R}_0 - 1 < 0$ so that $\lambda(m^*) < 0$. This implies

$$\lim_{\epsilon \to 0} \lambda(m^* + \epsilon) = \lambda(m^*) < 0$$

and therefore, there exists $\epsilon_0 > 0$ such that $\lambda(m^* + \epsilon_0) < 0$. Let us fix this $\epsilon_0 > 0$.

By [37] (see (14a), (14b), (14c) of [37]), we know that defining $V \triangleq S + I + R$, we obtain

$$\partial_t V = D\partial_{xx}^2 V + b - dV, \quad \partial_x V|_{x=0,1} = 0, \quad V(x, 0) = V_0(x) \tag{17}$$

where $V_0(x) \triangleq \phi_1(x) + \phi_2(x) + \phi_3(x), D > 0, b > 0, d > 0$. By Proposition 1 with $\overline{U} = 0, g(x) \equiv b, \lambda = d$, we see that (17) admits a unique positive steady state $m^* = \frac{b}{d}$ which is globally attractive in $C([0, 1], \mathbb{R}_+)$. Therefore, due to the non-negativity of S, I, R, for the fixed $\epsilon_0 > 0$, there exists $t_0 = t_0(\phi)$ such that $\forall t \geq t_0, x \in [0, 1], S(t, x) \leq m^* + \epsilon_0$. Thus, $\forall t \geq t_0, x \in [0, 1]$,

$$\partial_t I \leq D\partial_{xx}^2 I + \beta_1(m^* + \epsilon_0)I + \frac{\beta_2 B}{K}(m^* + \epsilon_0) - I(d+\gamma) \tag{18}$$

by (1) as $B \geq 0$ and

$$\partial_t B \leq D_4 \partial_{xx}^2 B - U \partial_x B + \xi I + B(-\delta) + gB \tag{19}$$

by (1) as $B^2 \geq 0, g > 0, K_B > 0$. As we will see, it was crucial above how we take these upper bounds carefully. Thus, we now consider for $x \in [0,1], t \geq t_0$,

$$\begin{cases} \partial_t V_2 = D \partial_{xx}^2 V_2 + \beta_1 (m^* + \epsilon_0) V_2 + \frac{\beta_2 V_4}{K} (m^* + \epsilon_0) - V_2 (d + \gamma), \\ \partial_t V_4 = D_4 \partial_{xx}^2 V_4 - U \partial_x V_4 + \xi V_2 + V_4 (-\delta) + g V_4, \end{cases} \tag{20}$$

for which its corresponding eigenvalue problem obtained by substituting $(V_2, V_4) = (e^{\lambda t} \psi_2(x), e^{\lambda t} \psi_4(x))$ in (20) is

$$\begin{cases} \lambda \psi_2 = D \partial_{xx}^2 \psi_2 + \beta_1 (m^* + \epsilon_0) \psi_2 + \frac{\beta_2 \psi_4}{K} (m^* + \epsilon_0) - \psi_2 (d + \gamma), \\ \lambda \psi_4 = D_4 \partial_{xx}^2 \psi_4 - U \partial_x \psi_4 + \xi \psi_2 + \psi_4 (-\delta) + g \psi_4. \end{cases} \tag{21}$$

We may write this right hand side as

$$\begin{pmatrix} D \partial_{xx}^2 \psi_2 \\ D_4 \partial_{xx}^2 \psi_4 - U \partial_x \psi_4 \end{pmatrix} + \begin{pmatrix} \beta_1 (m^* + \epsilon_0) - (d + \gamma) & \frac{\beta_2}{K} (m^* + \epsilon_0) \\ \xi & g - \delta \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_4 \end{pmatrix} \tag{22}$$

$$\triangleq \begin{pmatrix} D \partial_{xx}^2 \psi_2 \\ D_4 \partial_{xx}^2 \psi_4 - U \partial_x \psi_4 \end{pmatrix} + M(x) \begin{pmatrix} \psi_2 \\ \psi_4 \end{pmatrix}$$

so that $M_{ij} \geq 0 \; \forall \; i \neq j$ as $\xi, \frac{\beta_2}{K} (m^* + \epsilon_0) > 0$. Moreover, it is also clear that M is irreducible as $M_{12}, M_{21} > 0$ (see Definition 3.3). Therefore, by Theorem 7.6.1 [21], the eigenvalue problem of (21) has a real eigenvalue $\overline{\lambda}$ and its corresponding positive eigenfunction ψ_0.

Now we recall that $\lambda(m^*)$ is the principal eigenvalue of (15) and make a key observation that the second and fourth equations are independent of the first and third equations and therefore, $\lambda(m^*)$ must also be the eigenvalue of

$$\begin{pmatrix} D \partial_{xx}^2 \psi_2 + m^* \left(\beta_1 \psi_2 + \frac{\beta_2}{K} \psi_4 \right) - \psi_2 (d + \gamma) \\ D_4 \partial_{xx}^2 \psi_4 - U \partial_x \psi_4 + \xi \psi_2 + g \psi_4 - \delta \psi_4. \end{pmatrix} \tag{23}$$

$$= \begin{pmatrix} D \partial_{xx}^2 \psi_2 \\ D_4 \partial_{xx}^2 \psi_4 - U \partial_x \psi_4 \end{pmatrix} + \begin{pmatrix} m^* \beta_1 - (d + \gamma) & m^* \frac{\beta_2}{K} \\ \xi & g - \delta \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_4 \end{pmatrix}.$$

Moreover, we observe that replacing m^* with $m^* + \epsilon_0$ gives us the eigenvalue problem (21). Hence, $\overline{\lambda} = \lambda(m^* + \epsilon_0) < 0$ is the principal eigenvalue of (21) which therefore has a solution of

$$e^{\lambda(m^* + \epsilon_0)(t - t_0)} \psi_0(x), \quad t \geq t_0.$$

Now we find $\eta > 0$ sufficiently large so that

$$(I(x, t_0), B(x, t_0)) \leq \eta \psi_0(x)$$

which is possible as ψ_0 is positive. Considering (9), we may define

$$F_2^+ \triangleq \beta_1 (m^* + \epsilon_0) I + \frac{\beta_2 B}{K} (m^* + \epsilon_0) - I(d + \gamma), \tag{24a}$$

$$F_4^+ \triangleq \xi I + B(-\delta) + gB, \tag{24b}$$

so that

$$\frac{\partial F_2^+}{\partial B} = \frac{\beta_2}{K} (m^* + \epsilon_0) \geq 0, \quad \frac{\partial F_4^+}{\partial I} = \xi \geq 0,$$

and hence $\begin{pmatrix} F_2^+ \\ F_4^+ \end{pmatrix}$ is cooperative (see Definition 3.3). By comparison principle, or specifically Theorem 7.3.4 [21], due to (18), (19), (24), we obtain $\forall\, t \geq t_0, x \in [0,1]$,

$$(I(x,t), B(x,t)) \leq \eta e^{\lambda(m^* + \epsilon_0)(t - t_0)} \psi_0(x)$$

where $\eta e^{\lambda(m^* + \epsilon_0)(t - t_0)} \psi_0(x) \to 0$ as $t \to \infty$ because $\lambda(m^* + \epsilon_0) < 0$.

Thus, the equation for R, by (1), is asymptotic to

$$\partial_t V_3 = D \partial_{xx}^2 V_3 - V_3 (d + \sigma)$$

and hence by the theory of asymptotically autonomous semiflows (see Corollary 4.3 [23]), we have $\lim_{t \to \infty} R(x,t) = 0$. As we noted already, (17) admits a unique positive steady state m^* which is globally attractive, and we just showed that $\forall\, x \in [0,1], \lim_{t \to \infty} I(x,t) = \lim_{t \to \infty} R(x,t) = 0$, and therefore we obtain $\lim_{t \to \infty} S(x,t) = m^*$. This completes the proof of Theorem 2.1.

6. **Proof of Theorem 2.2.** We need the following proposition:

Proposition 2. *Let $u(x,t,\phi)$ be the solution of the system (1) with $D = D_1 = D_2 = D_3$, subjected to (2), (3) such that $u(x,0,\phi) = \phi \in X^+$. If there exists some $t_0^I \geq 0$ such that $I(\cdot, t_0^I) \not\equiv 0$, then $I(x,t) > 0\ \forall\, t > t_0^I, x \in [0,1]$. Similarly, if there exists some $t_0^R \geq 0$ such that $R(\cdot, t_0^R) \not\equiv 0$, then $R(x,t) > 0\ \forall\, t > t_0^R, x \in [0,1]$. Finally, if there exists some $t_0^B \geq 0$ such that $B(\cdot, t_0^B) \not\equiv 0$, then $B(x,t) > 0\ \forall\, t > t_0^B, x \in [0,1]$.*

Moreover, for any $\phi \in X^+$, it always holds that $S(x,t) > 0\ \forall\, x \in [0,1], t > 0$ and

$$\liminf_{t \to \infty} S(\cdot, t, \phi) \geq \frac{b}{\beta_1 2m^* + \beta_2 + d}.$$

Proof. We observe that by (1),

$$\partial_t I \geq D \partial_{xx}^2 I - I(d + \gamma). \tag{25}$$

$$\partial_t R \geq D \partial_{xx}^2 R - R(d + \sigma). \tag{26}$$

Thus, we consider

$$\begin{cases} \partial_t V_2 = D \partial_{xx}^2 V_2 - V_2(d + \gamma) \triangleq D \partial_{xx}^2 V_2 + \tilde{F}_2, \\ \partial_x V_2(x,t)|_{x=0,1} = 0, \end{cases} \tag{27}$$

$$\begin{cases} \partial_t V_3 = D \partial_{xx}^2 V_3 - V_3(d + \sigma) \triangleq D \partial_{xx}^2 V_3 + \tilde{F}_3, \\ \partial_x V_3(x,t)|_{x=0,1} = 0, \end{cases} \tag{28}$$

such that $V_2(\cdot, t_0^I) \not\equiv 0, I(\cdot, t_0^I) \geq V_2(\cdot, t_0^I)$, and $V_3(\cdot, t_0^R) \not\equiv 0, R(\cdot, t_0^R) \geq V_3(\cdot, t_0^R)$ respectively. By Lemma 3.4, the solutions to (27), (28) exist locally in time. For both systems (27), (28), we may repeat the argument in the proof of Proposition 1 for the system (11) at $\overline{U} = 0, g(x) \equiv 0, \lambda = d + \gamma, \lambda = d + \sigma$ respectively to obtain the sup-norm bounds of both V_2, V_3; therefore, these solutions exist globally in time by the blowup criterion from Lemma 3.4.

Now since $x \in [0,1]$, a one-dimensional space, we may denote

$$LV_2 \triangleq -D \partial_{xx}^2 V_2 + (d + \gamma) V_2$$

so that

$$\partial_t V_2 + LV_2 = 0 \text{ in } [0,1] \times (0,T], \ \forall\, T > 0$$

by (27). Therefore, if $V_2(x^*, t^*) = 0$ for some $(x^*, t^*) \in (0,1) \times (t_0^I, T]$, then it has a nonpositive minimum in $[0,1] \times [t_0^I, T]$ and therefore, V_2 is a constant on

$(0,1) \times (0,t^*]$ by Maximum Principle (see e.g. Theorem 7.1.12, pg. 367 [8]). Hence as $V_2(x^*,t^*) = 0$ for $x^* \in (0,1)$, we must have $V_2(\cdot,\cdot) \equiv 0$ on $(0,1) \times (0,t^*]$. Since $t^* \in (t_0^I, T]$, this implies $V_2(\cdot, t_0^I) \equiv 0$ on $(0,1)$, and hence by continuity in x, on $[0,1]$. This is a contradiction to $V_2(\cdot, t_0^I) \not\equiv 0$.

Therefore, we must have $V_2(x,t) > 0 \ \forall \ (x,t) \in (0,1) \times (t_0^I, T]$ and hence $V_2(x,t) > 0 \ \forall \ t > t_0^I, x \in (0,1)$ due to the arbitrariness of $T > 0$. By Comparison Principle (e.g. Theorem 7.3.4 [21]), we conclude that due to (25),

$$I(\cdot,t) \geq V_2(\cdot,t) > 0 \quad \forall \ t > t_0^I, x \in (0,1).$$

Making use of the boundary values in (3), we conclude that $I(\cdot,t) > 0 \ \forall \ t > t_0^I, x \in [0,1]$.

The proof that $R(\cdot,t) > 0 \ \forall \ t > t_0^R, x \in (0,1)$ is done very similarly. We may denote

$$LV_3 \triangleq -D\partial_{xx}^2 V_3 + (d+\sigma)V_3$$

so that

$$\partial_t V_3 + LV_3 = 0 \quad \text{in } [0,1] \times (0,T] \quad \forall \ T > 0$$

by (28). An identical argument as in the case of V_2 using Maximum Principle (e.g. Theorem 7.1.12, pg. 367 [8]) deduces that $V_3(x,t) > 0 \ \forall \ (x,t) \in (0,1) \times (t_0^R, T]$ and hence $V_3(x,t) > 0 \ \forall \ t > t_0^R, x \in (0,1)$ due to the arbitrariness of $T > 0$. By Comparison Principle (e.g. Theorem 7.3.4 [21]), we conclude that due to (26)

$$R(\cdot,t) \geq V_3(\cdot,t) > 0 \quad \forall \ t > t_0^R, x \in (0,1).$$

Relying on the boundary values in (3) allows us to conclude that $R(\cdot,t) > 0 \ \forall \ t > t_0^R, x \in [0,1]$.

Finally, we fix t_0^B such that $B(\cdot, t_0^B) \not\equiv 0$ on $[0,1]$ and then $t > t_0^B$ arbitrary. We know B exists globally in time due to Lemma 3.5 and thus fix $T > t_0^B$ so that $t \in [0,T]$. Then by continuity of B in $(x,t) \in [0,1] \times [0,T]$, there exists $M \triangleq \max_{(x,t) \in [0,1] \times [0,T]} B(x,t)$.

Now $\forall \ (x,t) \in [0,1] \times [0,T]$,

$$\partial_t B \geq D_4 \partial_{xx}^2 B - U\partial_x B + (g-\delta)B - \frac{gMB}{K_B} \tag{29}$$

by (1). Thus, we consider

$$\begin{cases} \partial_t V_4 = D_4 \partial_{xx}^2 V_4 - U\partial_x V_4 + (g-\delta-\frac{gM}{K_B})V_4 \triangleq D_4 \partial_{xx}^2 V_4 - U\partial_x V_4 + \tilde{F}_4, \\ D_4 \partial_x V_4(x,t) - UV_4(x,t)|_{x=0} = \partial_x V_4(x,t)|_{x=1} = 0, \end{cases} \tag{30}$$

such that $V_4(\cdot, t_0^B) \not\equiv 0, B_4(\cdot, t_0^B) \geq V_4(\cdot, t_0^B)$.

It follows that the solution V_4 exists locally in time by Lemma 3.4, Remark 2. Again, repeating the argument in the proof of Proposition 1 for the system (11) at $\overline{U} = U, g(x) \equiv 0, \lambda = \frac{gM}{K_B} + \delta - g > 0$ due to the hypothesis that $g < \delta$ leads to the sup-norm bound so that the solution exists globally in time by the blowup criterion of Lemma 3.4. Now we may denote

$$LV_4 \triangleq -D_4 \partial_{xx}^2 V_4 + U\partial_x V_4 + \left(\frac{gM}{K_B} + \delta - g\right)V_4$$

where $\frac{gM}{K_B} + \delta - g \geq \delta - g > 0$ by the hypothesis so that

$$\partial_t V_4 + LV_4 = 0 \quad \text{in } [0,1] \times (0,T].$$

Therefore, if $V_4(x^*,t^*) = 0$ for some $(x^*,t^*) \in (0,1) \times (t_0^B, T]$, then it has a non-positive minimum in $[0,1] \times [t_0^B, T]$ and hence V_4 is a constant on $(0,1) \times (0,t^*]$ by

Maximum Principle (e.g. Theorem 7.1.12, pg. 367, [8]). Hence, as $V_4(x^*, t^*) = 0$ for $x^* \in (0, 1)$, we must have $V_4(\cdot, \cdot) \equiv 0$ on $(0, 1) \times (0, t^*]$. Since $t^* \in (t_0^B, T]$, this implies that $V_4(\cdot, t_0^B) \equiv 0$ on $(0, 1)$ and hence by continuity in x, on $[0, 1]$. But this contradicts that $V_4(\cdot, t_0^B) \not\equiv 0$.

Therefore, we must have $V_4(x, t) > 0 \; \forall \, (x, t) \in (0, 1) \times (t_0^B, T]$. By Comparison Principle (e.g. Theorem 7.3.4 [21]), we conclude that due to (29)

$$B(\cdot, t) \geq V_4(\cdot, t) > 0 \quad \forall \, t \in (t_0^B, T], x \in (0, 1).$$

We conclude that by arbitrariness of $T > t_0$ and arbitrariness of $t \in [t_0^B, T]$, this inequality holds for all $t > t_0^B$. Making use of the boundary values in (3) allows us to conclude that $B(\cdot, t) > 0 \; \forall \, t > t_0^B, x \in [0, 1]$.

Finally, from the proof of Theorem 2.1, specifically due to (17) and an application of Proposition 1, we know that there exists $t_1 = t_1(\phi)$ such that $\forall \, x \in [0, 1], t \geq t_1$, $I(x, t, \phi) \leq 2m^*$. Thus, from (1) $\forall \, x \in [0, 1], t \geq t_1$,

$$\partial_t S \geq D \partial_{xx}^2 S + b - S(\beta_1 2m^* + \beta_2 + d). \tag{31}$$

Hence, we consider

$$\begin{cases} \partial_t V_1 = D \partial_{xx}^2 V_1 + b - V_1(\beta_1 2m^* + \beta_2 + d) \triangleq D \partial_{xx}^2 V_1 + \tilde{F}_1, \\ \partial_x V_1(x, t)|_{x=0,1} = 0. \end{cases} \tag{32}$$

Firstly, by Lemma 3.4, the existence of the unique nonnegative local solution follows. Again, repeating the argument in the proof of Proposition 1 for the system (11) at $\overline{U} = 0, g(x) \equiv 0, \lambda = \beta_1 2m^* + \beta_2 + d$ leads to the sup-norm bound so that the global existence of the solution follows due to the blowup criterion of Lemma 3.4. Now we may denote by

$$LV_1 \triangleq -D \partial_{xx}^2 V_1 + (\beta_1 2m^* + \beta_2 + d) V_1$$

so that $\partial_t V_1 + LV_1 = b \geq 0$. Therefore, if $V_1(x^*, t^*) = 0$ for some $(x^*, t^*) \in (0, 1) \times (0, T]$ for any $T > 0$, then V_1 attains a nonpositive minimum over $[0, 1] \times [0, T]$ at $(x^*, t^*) \in (0, 1) \times (0, T]$, then by Maximum Principle (e.g. Theorem 7.1.12, pg. 367, [8]), $V_1 \equiv c$ on $(0, 1) \times (0, t^*]$. Since $V_1(x^*, t^*) = 0$, this implies $V_1 \equiv 0$ on $(0, 1) \times (0, t^*]$. But by (32), we see that this implies $0 = b$ which is a contradiction because $b > 0$. Therefore, we must have $V_1(x, t, \phi) > 0 \; \forall \, x \in [0, 1], t \in [0, T]$ and hence by the arbitrariness of $T > 0$, $\forall \, t > 0$. By (31) and Comparison Principle (e.g. Theorem 7.3.4 [21]), we conclude that $\forall \, t > 0, x \in [0, 1]$,

$$S(x, t, \phi) \geq V_1(x, t, \phi) > 0.$$

Finally, since (32) has a unique positive steady state of $\frac{b}{\beta_1 2m^* + \beta_2 + d}$ by Proposition 1 with $\overline{U} = 0, g(x) \equiv b, \lambda = \beta_1 2m^* + \beta_2 + d$, we obtain

$$\liminf_{t \to \infty} S(\cdot, t, \phi) \geq \frac{b}{\beta_1 2m^* + \beta_2 + d}.$$

$\square$

We also need the following proposition:

Proposition 3. *Suppose $D = D_1 = D_2 = D_3, \phi \in X^+$ and $g < \delta$. Then the system (1) subjected to (2), (3) admits a unique nonnegative solution $u(x, t, \phi)$ on $[0, 1] \times [0, \infty)$, and its solution semiflow $\Phi_t : X^+ \mapsto X^+$ has a global compact attractor A.*

Proof. Firstly, by Lemma 3.5, the unique nonnegative solution $u(t, \phi)$ exists on $[0, \infty)$. As already used in the proof of Theorem 2.1, we know that (17) admits a unique positive steady state $m^* = \frac{b}{d}$. This implies that, as $S, I, R \geq 0$, there exists $t_1 > 0$ such that $\forall t \geq t_1, S(t), I(t), R(t) \leq 2m^*$. Therefore, $\forall t \geq t_1$,

$$\partial_t B \leq D_4 \partial_{xx}^2 B - U \partial_x B + \xi 2m^* + (g - \delta)B$$

by (1). Thus, by Proposition 1 with $\overline{U} > 0, g(x) = \xi 2m^* + x, \lambda = \delta - g$, we see that there exists $t_2 = t_2(\phi) > 0$ large so that $B(t, \phi) \leq \frac{\xi 4m^* + 1}{\delta - g}$; here we used the hypothesis that $g < \delta$. Hence, the solution semiflow Φ_t is point dissipative (see Definition 3.1).

As noted in the Preliminaries section, T is compact. From the definitions of (10), it is clear that $F = (F_1, F_2, F_3, F_4)$ is continuously differentiable and therefore locally Lipschitz in $C([0, T], X^+)$. Moreover, our diffusion operators including the convection operator $T(t)$ is analytic (see the Preliminaries Section) and thus strongly continuous. It follows that the solution semiflow $\Phi_t : X^+ \mapsto X^+$ is compact $\forall t > 0$. Therefore, by Lemma 3.7, we may conclude that Φ_t has a global compact attractor. $\square$

Now we let

$$\mathbb{W}_0 \triangleq \{\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in X^+ : \psi_2(\cdot) \not\equiv 0 \text{ or } \psi_4(\cdot) \not\equiv 0\}$$

and observe that $\mathbb{W}_0 \subset X^+$ is an open set. Moreover, we define

$$\partial \mathbb{W}_0 \triangleq X^+ \setminus \mathbb{W}_0$$
$$= \{\psi = (\psi_1, \psi_2, \psi_3, \psi_4) \in X^+ : \psi_2(\cdot) \equiv 0 \text{ and } \psi_4(\cdot) \equiv 0\}.$$

By Proposition 2, it follows that $\Phi_t(\mathbb{W}_0) \subset \mathbb{W}_0 \forall t \geq 0$ because if $\psi \in \mathbb{W}_0$ is such that $\psi_2(\cdot) \not\equiv 0$, then by Proposition 2, $I(x, t, \psi) > 0 \forall x \in [0, 1], t > 0$ whereas if $\psi \in \mathbb{W}_0$ is such that $\psi_2 \equiv 0$, then by the definition of $\mathbb{W}_0$ we must have $\psi_4(\cdot) \not\equiv 0$ so that by Proposition 2, $B(x, t, \psi) > 0 \forall x \in [0, 1], t > 0$.

We now define

$$M_\partial \triangleq \{\psi \in \partial \mathbb{W}_0 : \Phi_t(\psi) \in \partial \mathbb{W}_0 \forall t \geq 0\}$$

and let $\omega(\phi)$ be the ω-limit set of the orbit $\gamma^+(\phi) \triangleq \{\Phi_t(\phi)\}_{t \geq 0}$.

Proposition 4. *Suppose $D = D_1 = D_2 = D_3$ and $g < \delta$. For any $\phi \in X^+$, let $u(x, t, \phi)$ be the unique nonnegative solution to the system (1) subjected to (2), (3) such that $u(x, 0, \phi) = \phi$. Then $\forall \psi \in M_\partial, \omega(\psi) = \{(m^*, 0, 0, 0)\}$.*

Proof. We fix $\psi \in M_\partial$ so that by definition of M_∂, we have $\Phi_t(\psi) \in \partial \mathbb{W}_0 \forall t \geq 0$; i.e.

$$I(\cdot, t) \equiv 0 \text{ and } B(\cdot, t) \equiv 0 \quad \text{on } [0, 1], \forall t \geq 0.$$

Then S, R-equations in (1) reduce to

$$\partial_t S = D\partial_{xx}^2 S + b - dS + \sigma R,$$
$$\partial_t R = D\partial_{xx}^2 R - R(d + \sigma),$$

which leads to $\forall x \in [0, 1]$,

$$\lim_{t \to \infty} R(x, t, \psi) = 0.$$

Hence, the S-equation in (1) is asymptotic to

$$\partial_t V_1 = D\partial_{xx}^2 V_1 + b - dV_1$$

and therefore by Proposition 1 with $\overline{U} = 0, g(x) \equiv b, \lambda = d$, we obtain

$$\lim_{t \to \infty} S(x, t, \psi) = \frac{b}{d} = m^* \ \forall \ x \in [0, 1].$$

$\square$

Next, we show that $(m^*, 0, 0, 0)$ is a weak repeller for $\mathbb{W}_0$:

Proposition 5. *Suppose $D = D_1 = D_2 = D_3, \phi \in \mathbb{W}_0$ and $g < \delta$. Let $u(x, t, \phi)$ be the unique global nonnegative solution of the system (1) subjected to (2), (3) such that $u(x, 0, \phi) = \phi(x)$ and $\Phi_t(\phi) = u(t, \phi)$ be its solution semiflow. If $\mathcal{R}_0 > 1$, then there exists $\delta_0 > 0$ such that*

$$\limsup_{t \to \infty} \|\Phi_t(\phi) - (m^*, 0, 0, 0)\|_{C([0,1])} \geq \delta_0. \tag{33}$$

Proof. By hypothesis $\mathcal{R}_0 > 1$ so that $\mathcal{R}_0 - 1 > 0$ and as discussed in the proof of Theorem 2.1, we have $\lambda(m^*) > 0$ where $\lambda(m^*)$ is the principal eigenvalue of $\tilde{\Theta}$ in (15). To reach a contradiction, suppose that there exists some $\psi_0 \in \mathbb{W}_0$ such that $\forall \ \delta_0 > 0$ and hence in particular for $\delta_0 \in (0, m^*)$,

$$\limsup_{t \to \infty} \|\Phi_t(\psi_0) - (m^*, 0, 0, 0)\|_{C([0,1])} < \delta_0. \tag{34}$$

This implies that there exists $t_1 > 0$ sufficiently large such that in particular

$$m^* - \delta_0 < S(x, t), \quad B(x, t) < \delta_0 \quad \forall \ t \geq t_1, x \in [0, 1],$$

as $\Phi_t(\psi_0) = (S, I, R, B)(t)$. Thus, we see that due to (1),

$$\partial_t I \geq D \partial_{xx}^2 I + \beta_1 (m^* - \delta_0) I + (m^* - \delta_0) \frac{\beta_2}{(\delta_0 + K)} B - I(d + \gamma), \tag{35}$$

$$\partial_t B \geq D_4 \partial_{xx}^2 B - U \partial_x B + \xi I + g B \left(1 - \frac{\delta_0}{K_B}\right) - \delta B \tag{36}$$

$\forall \ t \geq t_1, x \in [0, 1]$. We thus consider for $t \geq t_1, x \in [0, 1]$,

$$\begin{cases} \partial_t V_2 = D \partial_{xx}^2 V_2 + \beta_1 (m^* - \delta_0) V_2 + (m^* - \delta_0) \frac{\beta_2}{(\delta_0 + K)} V_4 - V_2 (d + \gamma), \\ \partial_t V_4 = D_4 \partial_{xx}^2 V_4 - U \partial_x V_4 + \xi V_2 + g V_4 \left(1 - \frac{\delta_0}{K_B}\right) - \delta V_4. \end{cases} \tag{37}$$

We may write the right hand side as

$$\begin{pmatrix} D \partial_{xx}^2 V_2 \\ D_4 \partial_{xx}^2 V_4 - U \partial_x V_4 \end{pmatrix} + M \begin{pmatrix} V_2 \\ V_4 \end{pmatrix} \tag{38}$$

where

$$M \triangleq \begin{pmatrix} \beta_1 (m^* - \delta_0) - (d + \gamma) & (m^* - \delta_0) \frac{\beta_2}{(\delta_0 + K)} \\ \xi & g(1 - \frac{\delta_0}{K_B}) - \delta \end{pmatrix}$$

and therefore, $M_{ij} \geq 0 \ \forall \ i \neq j$ as $\xi, (m^* - \delta_0) \frac{\beta_2}{(\delta_0 + K)} > 0$ because $\delta_0 < m^*$ by assumption. This also implies that it is irreducible as in fact $M_{ij} > 0 \ \forall \ i \neq j$ (see Definition 3.3). Therefore, by Theorem 7.6.1 [21], we may find a real eigenvalue $\lambda(m^*, \delta_0)$ and its corresponding positive eigenfunction ϕ_0 so that this system has a solution

$$(V_2, V_4)(x, t) = e^{\lambda(m^*, \delta_0)(t - t_1)} \phi_0(x)$$

for $x \in [0, 1], t \geq t_1$.

Now by assumption, $\psi_0 \in \mathbb{W}_0$ and hence $\psi_2(\cdot) \not\equiv 0$ or $\psi_4(\cdot) \not\equiv 0$. If $\psi_2(\cdot) \not\equiv 0$, then by Proposition 2, we know that $I(x, t, \psi_0) > 0 \ \forall \ x \in [0, 1], t > 0$. If for any $t_0 > 0$,

$B(\cdot, t_0) \equiv 0 \ \forall \ x \in [0,1]$, then by (1), $0 = \xi I(x, t_0)$ which is a contradiction because $\xi > 0$. Therefore, $B(\cdot, t_0) \not\equiv 0$ and it follows that by Proposition 2, $B(x, t, \psi_0) > 0 \ \forall x \in [0,1], t > t_0$ and hence $\forall \ t > 0$ by arbitrariness of $t_0 > 0$.

On the other hand, if $\psi_4(\cdot) \not\equiv 0$, then by Proposition 2, we know that $B(x, t, \psi_0) > 0 \ \forall \ x \in [0,1], t > 0$. Now if for any $t_0 > 0$, $I(\cdot, t_0) \equiv 0 \ \forall \ x \in [0,1]$, then by (1), $0 = \beta_2 S(x, t_0) \left(\frac{B(x, t_0)}{B(x, t_0) + K} \right)$ which is a contradiction because $\beta_2 > 0$ and $S(x, t) > 0 \ \forall x \in [0,1], t > 0$ by Proposition 2 as $\psi_0 \in \mathbb{W}_0 \subset X^+$. Therefore, $I(\cdot, t_0) \not\equiv 0$ and it follows that by Proposition 2, $I(x, t, \psi_0) > 0 \ \forall x \in [0,1], t > t_0$ and hence $\forall \ t > 0$ by arbitrariness of $t_0 > 0$. Thus, we conclude that $\forall \ \psi_0 \in \mathbb{W}_0$, $I(x, t, \psi_0) > 0, B(x, t, \psi_0) > 0 \ \forall \ x \in [0,1], t > 0$ and hence in particular $\forall \ t \geq t_1$.

Hence, we may obtain

$$(I(x, t_1, \psi_0), B(x, t_1, \psi_0)) \geq \eta \phi_0(x) \tag{39}$$

for $\eta > 0$ sufficiently small. Therefore, by Comparison Principle, specifically Theorem 7.3.4 [21] with (9),

$$F_2^- \triangleq \beta_1(m^* - \delta_0)I + (m^* - \delta_0)\frac{\beta_2}{(\delta_0 + K)}B - I(d + \gamma),$$

$$F_4^- \triangleq \xi I + gB \left(1 - \frac{\delta_0}{K_B} \right) - \delta B,$$

so that

$$\frac{\partial F_2^-}{\partial B} = (m^* - \delta_0)\frac{\beta_2}{(\delta_0 + K)} \geq 0, \quad \frac{\partial F_4^-}{\partial I} = \xi > 0,$$

we obtain for $t \geq t_1, x \in [0,1]$,

$$(I(x, t, \psi_0), B(x, t, \psi_0)) \geq (V_2(x, t, \eta\phi_0), V_4(x, t, \eta\phi_0)) = \eta e^{\lambda(m^*, \delta_0)(t - t_1)}\phi_0(x)$$

due to linearity of (37). Now $\lambda(m^*) > 0$ and in comparison of the second and fourth equations of (15) and (37), we see that $\lim_{\delta_0 \to 0} \lambda(m^*, \delta_0) = \lambda(m^*) > 0$ so that taking $\delta_0 \in (0, m^*)$ even smaller if necessary, we have $\lambda(m^*, \delta_0) > 0$.

Thus, we see that $\eta e^{\lambda(m^*, \delta_0)(t - t_1)}\phi_0(x) \to \infty$ as $t \to \infty$ because $\phi_0(x) \gg 0$ and $\eta > 0$. This implies $(I, B)(x, t, \psi_0)$ and hence $(S, I, R, B)(x, t, \psi_0)$ is unbounded, contradicting

$$\limsup_{t \to \infty}(\|S(t) - m^*\|_{C([0,1])} + \|I(t)\|_{C([0,1])} + \|R(t)\|_{C([0,1])} + \|B(t)\|_{C([0,1])}) < \delta_0$$

by (6) and (34). Therefore, we have shown that for $\delta_0 \in (0, m^*)$ sufficiently small, (33) holds. $\qquad \square$

Now we define a function $p : X^+ \mapsto \mathbb{R}_+$ by

$$p(\psi) \triangleq \min\{ \min_{x \in [0,1]} \psi_2(x), \min_{x \in [0,1]} \psi_4(x)\}$$

It immediately follows that $p^{-1}((0, \infty)) \subset \mathbb{W}_0$.

Now suppose $p(\psi) = 0$ and $\psi \in \mathbb{W}_0$. The hypothesis that $\psi \in \mathbb{W}_0$ implies that

$$\psi_2(\cdot) \not\equiv 0 \text{ or } \psi_4(\cdot) \not\equiv 0.$$

This deduces that by the argument in the proof of Proposition 5, $I(x, t, \psi) > 0$ and $B(x, t, \psi) > 0 \ \forall \ t > 0, x \in [0,1]$. Thus, in this case we deduce that

$$\min\{ \min_{x \in [0,1]} I(x, t, \psi), \min_{x \in [0,1]} B(x, t, \psi)\} > 0 \quad \forall \ t > 0$$

which implies that $p(\Phi_t(\psi)) > 0 \ \forall \ t > 0$.

Next, suppose $p(\psi) > 0$ so that $\psi_2(\cdot) \not\equiv 0$ and $\psi_4(\cdot) \not\equiv 0$. Thus, by Proposition 2, this implies $p(\Phi_t(\psi)) > 0 \ \forall \, t > 0$. Hence, we have shown that p is a generalized distance function for the semiflow $\Phi_t : X^+ \mapsto X^+$.

We already showed that any forward orbit of Φ_t in M_∂ converges to $(m^*, 0, 0, 0)$ due to Proposition 4. Thus, as $\Phi_t((m^*, 0, 0, 0)) = (m^*, 0, 0, 0)$, $\{(m^*, 0, 0, 0)\}$ is a nonempty invariant set that is also a maximal invariant set in some neighborhood of itself and hence by Definition 3.1, it is also isolated. Thus, if we denote the stable set of $(m^*, 0, 0, 0)$ by $W^s((m^*, 0, 0, 0))$, we see that $W^s((m^*, 0, 0, 0)) \cap \mathbb{W}_0 = \emptyset$ as $\mathbb{W}_0 = \{\psi \in X^+ : \psi_2(\cdot) \not\equiv 0 \text{ or } \psi_4(\cdot) \not\equiv 0\}$. Therefore, making use of Propositions 3 and 4, we may apply Lemma 3.8 to conclude that there exists $\eta > 0$ that satisfies

$$\min_{\psi \in \omega(\phi)} p(\psi) > \eta \quad \forall \, \phi \in \mathbb{W}_0;$$

hence, $\forall \, i = 2, 4$, and $\forall \, x \in [0, 1]$,

$$\liminf_{t \to \infty} u_i(x, t, \phi) \geq \eta \quad \forall \, \phi \in \mathbb{W}_0$$

by (4). By taking η even smaller if necessary to satisfy $\eta \in (0, \frac{b}{\beta_1 2m^* + \beta_2 + d})$, we obtain (9) using Proposition 2.

Finally, we know as shown in the proof of Proposition 3, that Φ_t is compact so that it is asymptotically smooth by Lemma 3.9. Moreover, as we already showed that $\Phi_t(\mathbb{W}_0) \subset \mathbb{W}_0$, by Proposition 5, we see that Φ_t is ρ-uniformly persistent. We also know due to Proposition 3 that $\Phi_t : X^+ \mapsto X^+$ has a global attractor A. Thus, by Lemma 3.10, Remark 4, $\Phi_t : \mathbb{W}_0 \mapsto \mathbb{W}_0$ has a global attractor A_0.

This implies that because we already showed that $\Phi_t(\mathbb{W}_0) \subset \mathbb{W}_0 \ \forall \, t \geq 0$, Φ_t is compact so that it is α-condensing by Lemma 3.9, due to Lemma 3.11, we see that Φ_t has an equilibrium $a_0 \in A_0$. By Proposition 2, it is clear that a_0 is a positive steady state. This completes the proof of Theorem 2.2.

7. **Conclusion.** In this article, we have studied a general reaction-diffusion-convection cholera model, which formulates bacterial and human diffusion, bacterial convection, intrinsic pathogen growth and direct/indirect transmission routes. This general formation of the PDE model allows us to give a thorough investigations on the interactions between the spatial movement of human and bacteria, intrinsic pathogen dynamics and multiple transmission pathways and their contribution of the spatial pattern of cholera epidemics.

The main purpose of this work is to investigate the global dynamics of this PDE model (1). To achieve this goal, we have established the threshold results of global dynamics of (1) using the basic reproduction number R_0. Our analysis shows that if $R_0 > 1$, the disease will persist uniformly; whereas if $R_0 < 1$, the disease will die out and the DFE is globally attractive when the diffusion rate of susceptible, infectious and recovered human hosts are identical. These results shed light into the complex interactions of cholera epidemics in terms of model parameters, and their impact on extinction and persistence of the disease. In turn, these findings may suggest efficient implications for the prevention and control of the disease.

Besides, we would like to mention that there are a number of interesting directions at this point, that haven't been considered in the present work. One direction is to study seasonal and climatic changes. It is well known that these factors can cause fluctuation of disease contact rates, human activity level, pathogen growth and death rates, etc., which in turn have strong impact on disease dynamics. The other direction is to model spatial heterogeneity. For instance, taking the diffusion

and convection coefficients and other model parameters to be space dependent in 2 dimensional spatial domain (instead of constant values in 1 dimensional region) will better reflect the details of spatial variation. These would make for interesting topics in future investigations.

Appendix.

7.1. **Proof of Lemma 3.5.** In this section, we prove Lemma 3.5 for completeness. The local existence of unique nonnegative mild solution on $[0,\sigma), \sigma = \sigma(\phi)$, as well as the blow up criterion that if $\sigma = \sigma(\phi) < \infty$, then the sup norm of the solution becomes unbounded as t approaches σ from below is shown in the Theorem 2.1 [37]. To show that $\sigma = \infty$, we assume that $\sigma < \infty$, fix such σ and show the uniform bound which contradicts the blow up criterion. Specifically we show that by performing energy estimates more carefully, keeping track of the dependence on each constant, we may extend Proposition 1 of [37] to the case $p = \infty$. For brevity, we write L^p to imply $L^p([0,1])$ below for $p \in [1,\infty]$.

Proposition 6. *If $u(x,t,\phi) = (S, I, R, B)(x, t, \phi)$ solves (1) subjected to (2), (3) in $[0,\sigma)$, then*

$$\sup_{t\in[0,\sigma)} \|u(t)\|_{L^\infty} \leq 3(\|\phi_1\|_{L^\infty} + \|\phi_2\|_{L^\infty} + \|\phi_3\|_{L^\infty} + b\sigma)(1 + e^{\sigma g}\xi\sigma) + \|\phi_4\|_{L^\infty}e^{\sigma g}$$

Proof. From (1), we know from the proof of Proposition 1 [37] that defining $V \triangleq S + I + R$, we obtain (17). For $p \in [2,\infty)$, it is shown in the proof of Proposition 1 of [37] that

$$\sup_{t\in[0,\sigma)} \|V(t)\|_{L^p} \leq \|V_0\|_{L^p} + b\sigma.$$

Now as $S, I, R \geq 0$,

$$\|V\|_{L^p}^p \geq \|S\|_{L^p}^p + \|I\|_{L^p}^p + \|R\|_{L^p}^p,$$

$$3(\|S\|_{L^p}^p + \|I\|_{L^p}^p + \|R\|_{L^p}^p)^{\frac{1}{p}} \geq \|S\|_{L^p} + \|I\|_{L^p} + \|R\|_{L^p}$$

and hence together, this implies that $\forall\, p \in [2,\infty)$

$$\sup_{t\in[0,\sigma)} (\|S\|_{L^p} + \|I\|_{L^p} + \|R\|_{L^p})(t) \leq 3 \sup_{t\in[0,\sigma)} \|V(t)\|_{L^p} \leq 3(\|V_0\|_{L^p} + b\sigma).$$

Taking $p \to \infty$ on the right hand side first and then the left hand side shows that

$$\sup_{t\in[0,\sigma)} (\|S\|_{L^\infty} + \|I\|_{L^\infty} + \|R\|_{L^\infty})(t) \leq 3(\|\phi_1\|_{L^\infty} + \|\phi_2\|_{L^\infty} + \|\phi_3\|_{L^\infty} + b\sigma) \quad (40)$$

due to Minkowski's inequalities and (2). Next, a similar procedure shows that, as described in complete in detail in the proof of Proposition 1 of [37], we obtain

$$\partial_t \|B\|_{L^p} \leq \left(\frac{U^2}{4D_4(p-1)} + g \right) \|B\|_{L^p} + \xi\|I\|_{L^p}.$$

Thus, Gronwall's inequality type argument shows that via Hölder's inequality,

$$\|B(t)\|_{L^p} \leq \|\phi_4\|_{L^\infty} e^{t\left(\frac{U^2}{4D_4(p-1)}+g\right)} + \xi \int_0^t \|I(s)\|_{L^\infty} e^{(t-s)\left(\frac{U^2}{4D_4(p-1)}+g\right)} ds$$

Now taking $p \to \infty$ on the left hand side and then on the right hand side gives $\forall\, t \in [0,\sigma)$

$$\|B(t)\|_{L^\infty} \leq \|\phi_4\|_{L^\infty} e^{\sigma g} + \xi 3(\|\phi_1\|_{L^\infty} + \|\phi_2\|_{L^\infty} + \|\phi_3\|_{L^\infty} + b\sigma)e^{\sigma g}\sigma$$

where we used (40). Taking sup over $t \in [0, \sigma)$ on the left hand side completes the proof. □

By continuity in space of the local solution in $[0, \sigma)$, the proof of Lemma 3.5 is complete.

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