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GLOBAL EXISTENCE AND UNIQUENESS OF CLASSICAL SOLUTIONS FOR A GENERALIZED QUASILINEAR PARABOLIC EQUATION WITH APPLICATION TO A GLIOBLASTOMA GROWTH MODEL

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Abstract. This paper studies the global existence and uniqueness of classical solutions for a generalized quasilinear parabolic equation with appropriate initial and mixed boundary conditions. Under some practicable regularity criteria on diffusion item and nonlinearity, we establish the local existence and uniqueness of classical solutions based on a contraction mapping. This local solution can be continued for all positive time by employing the methods of energy estimates, L^p-theory, and Schauder estimate of linear parabolic equations. A straightforward application of global existence result of classical solutions to a density-dependent diffusion model of in vitro glioblastoma growth is also presented.

1. Introduction.

In this paper, we consider the existence and uniqueness of classical solutions for a generalized quasilinear parabolic equation of the form

$$u_t - \operatorname{div}[\mathbf{A}(x,t,u,\nabla u)] = F(x,t,u,\nabla u) \quad \text{in} \quad \Omega \times (0,\infty) \tag{1}$$

with initial and mixed boundary conditions

$$\begin{aligned} &u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \\ &\frac{\partial u}{\partial \nu} = \sigma(x,t,u) \quad \text{on} \quad \partial\Omega \times (0,\infty), \end{aligned} \tag{2}$$

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where $\Omega \subset \mathbb{R}^n$ is a bounded open domain with the boundary $\partial\Omega$ being $C^{2+\alpha}(0 < \alpha < 1)$, ν is the unit outward normal vector on $\partial\Omega$ and $\nabla u = (u_{x_1}, \cdots, u_{x_n})$.

Throughout this paper, we assume that $\mathbf{A} : \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $F : \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, and $\mathbf{A} = \mathbf{A}(x_1, \cdots, x_n, t, u, p_1, \cdots, p_n)$ satisfies the strongly uniformly parabolic condition

$$\mu_2|\xi|^2 \geq \sum_{i,j=1}^{n} \frac{\partial A_j}{\partial p_i}(x, t, u, p)\xi_i\xi_j \geq \mu_1|\xi|^2 > 0$$

for all $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$.

In the last few decades, many quasilinear parabolic equations, equipped with appropriate initial or boundary conditions, have been widely investigated to explain and predict the real-world phenomena in areas such as chemistry, physics, biology, ecology. For example, some of these mathematical models can be applied in depicting various biological processes, such as bacterial growth process, development and growth of tumors, immune response of the body, see [5, 14] and [18].

Mathematically, quasilinear parabolic systems have been extensively studied by the methods in nonlinear analysis and theory of PDEs. Besides the existence of time-dependent solutions which have been discussed in [2, 3, 4, 11, 13, 15, 17, 20] and [21], research efforts have also focused on spatial or spatio-temporal patterns in [9, 12] and [16]. While [7] and [8] investigated traveling waves or other types of entire solutions, and in addition to numerical simulations by finite element methods with small diffusion found in [6] and [10].

In [11], three types of generalized quasilinear equations, i.e., (1) and

$$u_t - a_{ij}(x, t, u, u_x)u_{x_i x_j} = F(x, t, u, u_x) \tag{3}$$

for the first boundary value problem, and

$$u_t - a_{ij}(x, t, u)u_{x_i x_j} = F(x, t, u, u_x) \tag{4}$$

for other boundary value problem are considered. Based on Leray-Schauder principle, the local solvability of classical solutions to these systems are established provided that $|u|$ and $|u_x|$ are both bounded on $\Omega \times (0, T)$, F and the derivatives of $\mathbf{A}$ (or the second derivative of a_{ij}) satisfy some regularity or growth restrictions.

In a series of papers on dynamic theory for quasilinear parabolic equations ([2], [3], [4]), Amann discussed the local and global existence of classical solution for a general second order quasilinear parabolic systems. According to Amann's results, equation

$$u_t - \operatorname{div}[a(x, t, u)\nabla u] = F(x, t, u, u_x) \tag{5}$$

with initial value of $W^{s,p}$ class and homogeneous Neumann boundary condition has a unique solution

$$u \in C^1((0, T), C(\overline{\Omega})) \cap C((0, T), C^2(\overline{\Omega})) \cap C^1([0, T), W^{s,p}(\Omega))$$

if s, p are chosen properly,

$$a \in C^2(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}), F \in C^2(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R}), \tag{6}$$

and

$$(|F| + |\partial_2 F| + |\partial_3 F| + (1 + |p|)|\partial_4 F|)(x, t, u, p) \leq c(|u|)(1 + |p|^\kappa) \tag{7}$$

for some constant $\kappa \geq 1$ and an increasing function c. Moreover, the regularity assumption for F and condition (7) have only been used in the proof of the existence and regularity of u. If we are already in possession of a classical solution u, it suffices to assume $F \in C^1(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R})$,

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$$

and

$$|F(\mathbf{x}, t, u, \mathbf{p})| \leq c(|u|)(1 + |\mathbf{p}|^\kappa)$$

for the global existence of u. Based on these results and comparison theorem, [22] gave some conditions on the nonlinear part F for the global-in-time existence of classical solution to equation (5) under homogeneous Neumann boundary condition. However, for some concrete forms of equations (1) and (3)-(5), diffusion and nonlinearity functions may have low regularity than those in (6) or can not meet so many assumptions in §5.6, §5.7 and §6.4 of [11]. Consequently, the results available in the literature are not readily applicable to these cases. Therefore, there are practical needs for additional studies for establishing both local and global existence of classical solutions to the generalized quasilinear parabolic equations under weaker regularity or fewer growth restrictions.

In this paper, we investigate the global existence and uniqueness of classical solution to problem (1)-(2). In Section 2, under some continuity conditions on diffusion and growth terms, the local existence and uniqueness of classical solution are established by using the contraction mapping theory. Next, we perform some priori estimates, and then show that the local solution can be extended to entire time interval in Section 3. In Section 4, we apply the main results to two specific examples, including a recently data-validated glioblastoma growth model.

2. **Local existence and uniqueness.** In this section, we explore the local existence and uniqueness of classical solution to problem (1)-(2). The approach is based on the theory of fixed point.

In the following, let $Q_T = \Omega \times (0,T), \Gamma_T = \partial\Omega \times (0,T)$, where $0 < T < +\infty$. Let $W_q^{2,1}(Q_T)$ and $V_2(Q_T)$ denote Banach spaces with the usual norms

$$\|u\|_{W_q^{2,1}(Q_T)} = \|u\|_{L^q(Q_T)} + \|D_t u\|_{L^q(Q_T)} + \|D_x u\|_{L^q(Q_T)} + \|D_x^2 u\|_{L^q(Q_T)}$$

and

$$\|u\|_{V_2(Q_T)} = \sup_{0\leq t\leq T} \|u(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)},$$

and $C^{l,l/2}(\overline{Q}_T) = \{u | D_t^r D_x^s u \in C(\overline{Q}_T), 0 \leq 2r + s < l, l > 0\}$ denotes Hölder space with the norm

$$\begin{aligned}\|u\|_{C^{l,l/2}(\overline{Q}_T)} =& \sum_{j=1}^{\lfloor l\rfloor} \sum_{2r+s=j} \max_{Q_T} |D_t^r D_x^s u| + \max_{Q_T} |u| + \sum_{2r+s=\lfloor l\rfloor} \langle D_t^r D_x^s u\rangle_{x,Q_T}^{(l-\lfloor l\rfloor)} \\ &+ \sum_{0<l-(2r+s)<2} \langle D_t^r D_x^s u\rangle_{t,Q_T}^{\frac{l-(2r+s)}{2}},\end{aligned}$$

where

$$\begin{aligned}\langle v\rangle_{x,Q_T}^\alpha &= \sup_{(x_1,t),(x_2,t)\in\overline{Q}_T, |x_1-x_2|\leq\rho_0} \frac{|v(x_1,t) - v(x_2,t)|}{|x_1 - x_2|^\alpha}, \quad 0 < \alpha < 1, \\ \langle v\rangle_{t,Q_T}^\alpha &= \sup_{(x,t_1),(x,t_2)\in\overline{Q}_T, |t_1-t_2|\leq\rho_0} \frac{|v(x,t_1) - v(x,t_2)|}{|t_1 - t_2|^\alpha}, \quad 0 < \alpha < 1.\end{aligned} \tag{8}$$

In order to establish the local solvability of problem (1)-(2), we make the following assumptions.

(H1) $A(x,t,u,p)$ is differentiable in the variables x, u and p. $\frac{\partial A_j}{\partial x_i}$, $\frac{\partial A_j}{\partial u}$, $\frac{\partial A_j}{\partial p_i}$ and F are locally $C^{\alpha,\alpha/2}$ continuous with respect to (x,t) and locally Lipschitz continuous with respect to (u,p), uniformly with respect to the other variables.

(H2) $u_0(x) \in C^{2+\alpha}(\overline{\Omega})$.

(H3) σ is twice differentiable in the variable x and differentiable in the variable t, each of these derivatives is locally $C^{\alpha,\alpha/2}$ continuous with respect to (x,t) while locally Lipschitz continuous with respect to u. Further, σ satisfies the compatibility condition of 0-order:

$$\frac{\partial u}{\partial \nu}(x,0) = \sigma(x,0,u_0(x)), \quad x \in \partial\Omega.$$

Theorem 2.1. *Assume that (H1)-(H3) hold true. Then, there exists a unique solution $u(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ of the system* (1)-(2) *for some $T > 0$ depending on $\|u_0\|_{C^{2+\alpha}(\overline{\Omega})}$.*

Proof. Define

$$X = \{\phi | \phi \in C^{1+\alpha,\alpha/2}(\overline{Q}_T), 0 < T < 1\},$$
$$X_B = \{\phi \in X | \phi(x,0) = u_0(x), \|\phi\|_{C^{1+\alpha,\alpha/2}(\overline{Q}_T)} \le B\},$$

where $B = \|u_0\|_{C^{2+\alpha}(\overline{\Omega})} + 1$.

For any $u \in X_B$, define a mapping $G : u \mapsto \tilde{u}$, where $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_t - \sum\limits_{i,j=1}^{n} b_{ij}(x,t)\tilde{u}_{x_i x_j} + \sum\limits_{j=1}^{n} b_j(x,t)\tilde{u}_{x_j} = f_1(x,t) \quad \text{in} \quad \Omega \times (0,\infty), \\ \tilde{u}(x,0) = u_0(x) \quad \text{in} \quad \Omega, \\ \frac{\partial \tilde{u}}{\partial \nu} = \sigma_1(x,t) \quad \text{on} \quad \partial\Omega \times (0,\infty), \end{cases} \tag{9}$$

with

$$\begin{aligned} b_{ij}(x,t) &= \frac{\partial A_j}{\partial p_i}(x,t,u(x,t),\nabla u(x,t)), \\ b_j(x,t) &= -\frac{\partial A_j}{\partial u}(x,t,u(x,t),\nabla u(x,t)), \\ f_1(x,t) &= F(x,t,u(x,t),\nabla u(x,t)) + \sum_{j=1}^{n} \frac{\partial A_j}{\partial x_j}(x,t,u(x,t),\nabla u(x,t)), \\ \sigma_1(x,t) &= \sigma(x,t,u(x,t)). \end{aligned} \tag{10}$$

From $u \in X_B$, (H1), (H3) and the definition of Hölder spaces, it is easy to verify that $b_{ij}, b_j, f_1 \in C^{\alpha,\alpha/2}(\overline{Q}_T), \sigma_1 \in C^{2+\alpha,1+\alpha/2}(\overline{\Gamma}_T)$. By Theorem 5.3 (pp.320-321) in [11], (9) possesses a unique solution $\tilde{u} \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ satisfying

$$\begin{aligned} \|\tilde{u}\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)} \le& C_1 \left(\|u_0\|_{C^{2+\alpha}(\overline{\Omega})} + \|f_1\|_{C^{\alpha,\alpha/2}(\overline{Q}_T)} + \|\sigma_1\|_{C^{2+\alpha,1+\alpha/2}(\overline{\Gamma}_T)}\right) \\ \le& C_2(B). \end{aligned} \tag{11}$$

By the norm definition in (8) and differential mean value theorem, one has

$$\begin{aligned}
&\|\tilde{u}-u_0\|_{C^{1+\alpha,\alpha/2}(\overline{Q}_T)}\\
=&\|\tilde{u}-u_0\|_{C^{1,0}(\overline{Q}_T)}+\|\tilde{u}-u_0\|_{C^{\alpha,0}(\overline{Q}_T)}+\|\nabla(\tilde{u}-u_0)\|_{C^{\alpha,0}(\overline{Q}_T)}\\
&+\|\tilde{u}-u_0\|_{C^{0,\alpha/2}(\overline{Q}_T)}+\|\nabla(\tilde{u}-u_0)\|_{C^{0,\alpha/2}(\overline{Q}_T)}\\
\le&\|\tilde{u}-u_0\|_{C^{1,0}(\overline{Q}_T)}+\Big(2\|\tilde{u}-u_0\|_{C(\overline{Q}_T)}+\|\nabla(\tilde{u}-u_0)\|_{C(\overline{Q}_T)}\Big)\\
&+\Big(2\|\nabla(\tilde{u}-u_0)\|_{C(\overline{Q}_T)}+\|\nabla^2(\tilde{u}-u_0)\|_{C(\overline{Q}_T)}\Big)\\
&+\|\tilde{u}-u_0\|_{C^{0,\alpha/2}(\overline{Q}_T)}+\|\nabla(\tilde{u}-u_0)\|_{C^{0,\alpha/2}(\overline{Q}_T)}\\
\le&T^{\alpha/2}\Big(\|\tilde{u}\|_{C^{0,\alpha/2}(\overline{Q}_T)}+\|\nabla\tilde{u}\|_{C^{0,\alpha/2}(\overline{Q}_T)}\Big)+T^{\alpha/2}\Big(2\|\tilde{u}\|_{C^{0,\alpha/2}(\overline{Q}_T)}\\
&+\|\nabla\tilde{u}\|_{C^{0,\alpha/2}(\overline{Q}_T)}+2\|\nabla\tilde{u}\|_{C^{0,\alpha/2}(\overline{Q}_T)}+\|\nabla^2\tilde{u}\|_{C^{0,\alpha/2}(\overline{Q}_T)}\Big)\\
&+T\|\tilde{u}\|_{C^{0,1+\alpha/2}(\overline{Q}_T)}+T^{1/2}\|\nabla\tilde{u}\|_{C^{0,(1+\alpha)/2}(\overline{Q}_T)}\\
\leqslant&\max\{3T^{\alpha/2},T^{1/2}\}\|\tilde{u}\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)}\\
\leqslant&\max\{3T^{\alpha/2},T^{1/2}\}C_2(B),
\end{aligned}$$

which implies that for sufficiently small T relations

$$\begin{aligned}
\|\tilde{u}\|_{C^{1+\alpha,\alpha/2}(\overline{Q}_T)}\le&\|\tilde{u}-u_0\|_{C^{1+\alpha,\alpha/2}(\overline{Q}_T)}+\|u_0\|_{C^{1+\alpha}(\overline{\Omega})}\\
<&1+\|u_0(x)\|_{C^{2+\alpha}(\overline{\Omega})}=B
\end{aligned}\tag{12}$$

hold true. Hence, $\tilde{u}\in X_B$, i.e., G maps X_B into itself.

Next, we show that G is contractive. Let $u,v\in X_B$, $\tilde{u}=Gu,\tilde{v}=Gv$. We only need to verify

$$\|\tilde{u}-\tilde{v}\|_{X_B}\leqslant\delta\|u-v\|_{X_B}\quad\text{for some}\quad\delta\in(0,1).\tag{13}$$

Denote $\tilde{w}=\tilde{u}-\tilde{v}$. Then, $\tilde{w}$ satisfies

$$\begin{cases}
\tilde{w}_t-\sum\limits_{i,j=1}^{n}b_{ij}(x,t)\tilde{w}_{x_ix_j}+\sum\limits_{j=1}^{n}b_j(x,t)\tilde{w}_{x_j}=f_2(x,t)\quad\text{in}\quad\Omega\times(0,\infty),\\
\tilde{w}(x,0)=0\quad\text{in}\quad\Omega,\\
\frac{\partial\tilde{w}}{\partial\nu}=\sigma_2(x,t)\quad\text{on}\quad\partial\Omega\times(0,\infty),
\end{cases}\tag{14}$$

where b_{ij} and b_j are given in (10),

$$\begin{aligned}
f_2(x,t)=&\sum_{i,j=1}^{n}\left[\frac{\partial A_j}{\partial p_i}(x,t,u(x,t),\nabla u(x,t))-\frac{\partial A_j}{\partial p_i}(x,t,v(x,t),\nabla v(x,t))\right]\tilde{v}_{x_ix_j}\\
&+\sum_{j=1}^{n}\left[\frac{\partial A_j}{\partial u}(x,t,u(x,t),\nabla u(x,t))-\frac{\partial A_j}{\partial u}(x,t,v(x,t),\nabla v(x,t))\right]\tilde{v}_{x_j}\\
&+\sum_{j=1}^{n}\left[\frac{\partial A_j}{\partial x_j}(x,t,u(x,t),\nabla u(x,t))-\frac{\partial A_j}{\partial x_j}(x,t,v(x,t),\nabla v(x,t))\right]\\
&+F(x,t,u(x,t),\nabla u(x,t))-F(x,t,v(x,t),\nabla v(x,t)),\\
\sigma_2(x,t)=&\sigma(x,t,u(x,t))-\sigma(x,t,v(x,t)).
\end{aligned}$$

By Cauchy inequality, (H1), (H3) and (11), we have

$$\begin{aligned}\|f_2\|_{L^\infty(Q_T)} \leqslant & C_3\big[C_2(B)nL\|u-v\|_{C(Q_T)} + C_2(B)n\sqrt{n}L\|u-v\|_{C(Q_T)} \\ & + C_2(B)n\|D(u-v)\|_{C(Q_T)} + C_2^2(B)nL\|u-v\|_{C(Q_T)} \\ & + L\|D(u-v)\|_{C(Q_T)} + L\|u-v\|_{C(Q_T)}\big] \\ \leqslant & C_4(B)\|u-v\|_{C^{1,0}(Q_T)}, \\ \|D^k\sigma_2\|_{L^\infty(\Gamma_T)} \leqslant & C_5L\|u-v\|_{C^{1,0}(\Gamma_T)}, \quad 0 \le k \le 2,\end{aligned}$$

where L is the maximum of Lipschitz constants. Notice that, $b_{ij}(x,t)$ is bounded continuous in Q_T. By Theorem 9.1 (pp.341-342) in [11], one has

$$\|\tilde{w}\|_{W_q^{2,1}(Q_T)} \leqslant C_6\left(\|f_2\|_{L^\infty(Q_T)} + \|\sigma_2\|_{W_q^{2-1/q,1-1/(2q)}(\Gamma_T)}\right) \quad \text{for any} \quad q > 3. \quad (15)$$

Lemma 3.3 (p.80) in [11] implies that

$$\|\tilde{w}\|_{C^{1+\beta,(1+\beta)/2}(\overline{Q}_T)} \leqslant C_7\|\tilde{w}\|_{W_q^{2,1}(Q_T)},$$

where $\beta = 1-(n+2)/q$. It is clear that $\beta > \alpha$ when q is sufficiently large. Hence,

$$\|\tilde{w}\|_{C^{1+\beta,(1+\beta)/2}(\overline{Q}_T)} \leqslant C_8(B)\|u-v\|_{C^{1,0}(\overline{Q}_T)},$$

and then

$$\begin{aligned}&\|\tilde{w}\|_{C^{1+\alpha,\alpha/2}(\overline{Q}_T)} \\ =&\|\tilde{w}\|_{C^{\alpha,0}(\overline{Q}_T)} + \|\tilde{w}\|_{C^{0,\alpha/2}(\overline{Q}_T)} + \|D\tilde{w}\|_{C^{\alpha,0}(\overline{Q}_T)} + \|D\tilde{w}\|_{C^{0,\alpha/2}(\overline{Q}_T)} \\ &+ \|\tilde{w}\|_{C^{1,0}(\overline{Q}_T)} \\ \leqslant&|\Omega|^{\beta-\alpha}(\|\tilde{w}\|_{C^{\beta,0}(\overline{Q}_T)} + \|D\tilde{w}\|_{C^{\beta,0}(\overline{Q}_T)}) + T^{(1+\beta-\alpha)/2}(\|\tilde{w}\|_{C^{0,(1+\beta)/2}(\overline{Q}_T)} \\ &+ \|D\tilde{w}\|_{C^{0,(1+\beta)/2}(\overline{Q}_T)}) + \|\tilde{w}\|_{C^{1,0}(\overline{Q}_T)} \\ \leqslant&C_9(B)\|u-v\|_{C^{1,0}(\overline{Q}_T)} \\ \leqslant&C_9(B)T^{\alpha/2}\|u-v\|_{C^{1+\alpha,\alpha/2}(\overline{Q}_T)}.\end{aligned} \quad (16)$$

Take $T = T_0$ small enough such that

$$\max\{3T_0^{\alpha/2}, T_0^{1/2}\}C_2(B) < 1, \quad C_9(B)T_0^{\alpha/2} < 1/2,$$

then inequality (13) holds. By the classical contraction mapping theorem, G has a unique fixed point $u \in X_B$, which is the unique local solution of (1)-(2).

Taking T_0 as initial time and $u(\cdot, T_0)$ as initial value, one can continue the solution to a larger time interval. The procedure may be repeated indefinitely leading to the construction of a maximally defined solution $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T))$ for some $T > 0$. □

3. **Global existence for the case** $A(x,t,u,\nabla u) = a(x,t,u)\nabla u$**.** In this section, we investigate the global existence of classical solution to system (1)-(2) by extending the existence interval of the local solution to $[0,+\infty)$. Let $a : \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$. We only consider the special case of $A(x,t,u,\nabla u) = a(x,t,u)\nabla u$ and $\sigma \equiv 0$, i.e.,

$$u_t - \operatorname{div}[a(x,t,u)\nabla u] = F(x,t,u,\nabla u) \quad \text{in} \quad \Omega \times (0,\infty) \quad (17)$$

with initial and boundary conditions

$$u(x,0) = u_0(x) \geq 0 \quad \text{in} \quad \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty). \quad (18)$$

According to the discussion in Section 2, we assume that

(H1)* $a(x,t,u)$ is differentiable in the variables x,t and u. a, $\frac{\partial a}{\partial x_i}$, $\frac{\partial a}{\partial u}$ and F are locally $C^{\alpha,\alpha/2}$ continuous with respect to (x,t) and locally Lipschitz continuous with respect to u, uniformly with respect to the other variables. Further, there exist positive constants $\tilde{\mu}_1, \tilde{\mu}_2$ such that

$$\tilde{\mu}_1 \leq a(x,t,u) \leq \tilde{\mu}_2, \quad \forall (x,t,u) \in \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}.$$

In order to establish the non-negativity and L^p-estimate of solutions for system (17)-(18), we further assume that the nonlinear part satisfies

(H4) $|F(x,t,u,p)| \leqslant h(x,t,u)(1+|p|)$ for $\forall (x,t,u,p) \in \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$ with some $h \in C(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+, (0,\infty))$.

(H5) $F(x,t,0,0) \geq 0$ for $\forall (x,t) \in Q_T$.

Now, we investigate the non-negativity of solution to problem (17)-(18). Note that, if u is a classical solution to (17)-(18), then we have $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$. Assume that $u(x,t)$ is a classical solution to (17)-(18). Then, u also satisfies the following differential inequality

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial^2 u}{\partial x_i x_j} + \sum_{j=1}^{n} a_j(x,t) \frac{\partial u}{\partial x_j} \geq 0,$$

where

$$\begin{aligned} &a_{ij}(x,t) = a(x,t,u(x,t))\delta_{ij}, \quad \delta_{ij} \text{ is Kronecker delta function,} \\ &a_j(x,t) = -\frac{\partial a}{\partial x_j}(x,t,u(x,t)) - \frac{\partial a}{\partial u}(x,t,u(x,t)) \frac{\partial u}{\partial x_j}(x,t). \end{aligned} \tag{19}$$

Considering the assumptions (H1)*, one can easily verify that a_{ij}, a_j are all continuous. By the maximum principle for linear parabolic equation, one has $u \geq 0$ in $\overline{Q}_T$.

Then, similar to the proof of Theorem 2.1, we get a local existence conclusion for problem (17)-(18).

Theorem 3.1. *Assume that (H1)*, (H2), (H4) and (H5) hold. Then, there exists a unique non-negative solution $u(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ of the system* (17)-(18) *for some $T > 0$ depending on $\|u_0\|_{C^{2+\alpha}(\overline{\Omega})}$.*

In the following discussion, denote by $C(T)$, $C_i(T)(i = 1,2,\cdots)$ the constants depending not only on the parameters and initial value in (17)-(18) but also on time span T, and by C, $C_i(i = 1,2,\cdots)$ the constants only depending on the parameters and initial value in (17)-(18).

3.1. **A priori estimates.** To continue the local solution established in Theorem 2.1, we need to perform some a priori estimates for the unknown function and its derivative.

First, suppose that a bounded function u is the classical solution of (17)-(18), i.e., there exists a positive constant M such that $\max_{Q_T} |u| \leq M$. We then develop the L^p estimate of u.

Lemma 3.2. *(L^2-estimate) Let $u(x,t)$ be a bounded function satisfying* (17)-(18). *Then, $u \in W_2^{2,1}(Q_T)$ and*

$$\|u(\cdot,t)\|_{L^2(\Omega)}, \|\nabla u(\cdot,t)\|_{L^2(\Omega)} \leqslant C(M,T), \ 0 < t < T.$$

Proof. Multiplying (17) by $2u$ and integrating it over Q_T, we have

$$\int_\Omega u^2(\cdot,t)dx - \int_\Omega u_0^2(\cdot)dx = -2\int_{Q_T} a|\nabla u|^2 dxdt + 2\int_{Q_T} uF dxdt.$$

Then by (H1)*, (H2), (H4), (H5) and Young inequality, one has

$$\begin{aligned}&\int_\Omega u^2(\cdot,t)dx + 2\tilde{\mu}_1\int_{Q_T}|\nabla u|^2 dxdt\\ \leqslant &\int_\Omega u_0^2(\cdot)dx + 2\int_{Q_T} uh\, dxdt + \frac{1}{\epsilon_1}\int_{Q_T} u^2h^2 dxdt + \epsilon_1\int_{Q_T}|\nabla u|^2 dxdt.\end{aligned}$$

Let $0 < \epsilon_1 \le \tilde{\mu}_1$, then

$$\int_\Omega u^2(\cdot,t)dx + \tilde{\mu}_1\int_{Q_T}|\nabla u|^2 dxdt \leqslant C_{10}(M,T)\left(1+\int_{Q_T} u^2 dxdt\right). \tag{20}$$

By Gronwall's inequality, we have

$$\|u\|_{L^2(Q_T)} \leqslant C_{11}(M,T). \tag{21}$$

From (20) and (21), it follows that

$$\|u(\cdot,t)\|_{L^2(\Omega)}, \|\nabla u(\cdot,t)\|_{L^2(\Omega)} \leqslant C(M,T),\ 0<t<T. \tag{22}$$

By (H1)*, (H2) and (22), the standard L^p-estimate for parabolic equation, and aid of Theorem 9.1 (pp.341-342) in [11], this implies that $u \in W_2^{2,1}(Q_T)$. □

Remark 1. From the proof of Lemma 3.2, one can easily conclude that $u \in V_2(Q_T)$ and $\|u\|_{V_2(Q_T)} \leqslant C(M,T)$. Moreover, the Hölder continuity of solution to problem (17)-(18) yields that $u \in V_2^{1,0}(Q_T)$.

Lemma 3.3. *(L^p-estimate) Let $u(x,t)$ be a bounded function satisfying* (17)-(18). *Then, for any $p > 1$ and $0 < t < T$,*

$$\|u(\cdot,t)\|_{L^p(\Omega)}, \|u\|_{L^p(Q_T)} \leqslant C(M,T).$$

Proof. Multiplying (17) by pu^{p-1} and integrating it over Q_T leads to

$$\int_\Omega u^p(\cdot,t)dx - \int_\Omega u_0^p(\cdot)dx = -p(p-1)\int_{Q_T} au^{p-2}|\nabla u|^2 dxdt + p\int_{Q_T} u^{p-1}F dxdt.$$

From (H2), (H4) and Young inequality, it follows that

$$\begin{aligned}&\int_\Omega u^p(\cdot,t)dx + p(p-1)\tilde{\mu}_1\int_{Q_T} u^{p-2}|\nabla u|^2 dxdt\\ \leqslant &\int_\Omega u_0^p(\cdot)dx + p\int_{Q_T} hu^{p-1}(1+|\nabla u|)dxdt\\ \leqslant &\int_\Omega u_0^p(\cdot)dx + p\int_{Q_T} hu^{p-1}dxdt + \frac{p^2}{4\epsilon_2}\int_{Q_T} h^2u^p dxdt\\ &+ \epsilon_2\int_{Q_T} u^{p-2}|\nabla u|^2 dxdt.\end{aligned} \tag{23}$$

Choose an ϵ_2 with $0 < \epsilon_2 \le p(p-1)\tilde{\mu}_1/2$. By (23), one has

$$\int_\Omega u^p(\cdot,t)dx \leqslant C_{12}(M,T)\left(1+\int_{Q_T} u^p dxdt\right). \tag{24}$$

Then, Gronwall's lemma implies that $\|u\|_{L^p(Q_T)} \leqslant C(M,T)$. Therefore, $\|u(\cdot,t)\|_{L^p(\Omega)} \leqslant C(M,T)$ for any $t \in [0,T)$. □

In order to achieve higher regularity of $|\nabla u|$, we introduce the following lemma.

Lemma 3.4. *Let $w \in W_p^{2,1}(Q_T) \cap C^{2,1}(\overline{Q}_T)$ be a nonnegative and bounded function satisfying*

$$\begin{cases} \frac{\partial w}{\partial t} \leqslant \operatorname{div}[a(x,t,w)\nabla w] + r_1(x,t,w)|\nabla w| + r_2(x,t) \quad in \quad \Omega \times (0,\infty), \\ \frac{\partial w}{\partial \nu} \leqslant 0 \quad on \quad \partial\Omega \times (0,\infty), \end{cases} \tag{25}$$

where $r_1 \in C(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}), r_2 \in L^p(Q_T)$. Then, $Dw \in L^{2p}(Q_T)$.

Proof. For any $t \in (0,T)$, multiplying (25) by $w|\nabla w|^{2(p-1)}$ and integrating it over Ω produce

$$\begin{aligned}
&\int_\Omega w_t w |\nabla w|^{2(p-1)} dx \\
\leqslant &\int_{\partial\Omega} aw|\nabla w|^{2(p-1)} \frac{\partial w}{\partial \nu} dx - \int_\Omega a\nabla w \cdot \nabla(w|\nabla w|^{2(p-1)}) dx \\
&+ \int_\Omega w r_1 |\nabla w|^{2p-1} dx + \int_\Omega r_2 w |\nabla w|^{2(p-1)} dx \\
= &\int_{\partial\Omega} aw|\nabla w|^{2(p-1)} \frac{\partial w}{\partial \nu} dx - 2(p-1)\int_\Omega aw|\nabla w|^{2(p-2)} \sum_{i,j=1}^n w_{x_i} w_{x_j} w_{x_i x_j} dx \\
&- \int_\Omega a|\nabla w|^{2p} dx + \int_\Omega w r_1 |\nabla w|^{2p-1} dx + \int_\Omega r_2 w |\nabla w|^{2(p-1)} dx \\
\leq &\frac{\tilde{\mu}_2(p-1)}{\epsilon_3} \int_\Omega w^2 |\nabla w|^{2(p-2)} |D^2 w|^2 dx + \tilde{\mu}_2 n(p-1)\epsilon_3 \int_\Omega |\nabla w|^{2p} dx \\
&- \tilde{\mu}_1 \int_\Omega |\nabla w|^{2p} dx + \int_\Omega w r_1 |\nabla w|^{2p-1} dx + \int_\Omega w r_2 |\nabla w|^{2(p-1)} dx.
\end{aligned}$$

By the Young inequality, one has

$$\begin{aligned}
&\tilde{\mu}_1 \int_\Omega |\nabla w|^{2p} dx \\
\leqslant &C_{13} \int_\Omega [\epsilon_3 |\nabla w|^{2p} + \epsilon_4 |\nabla w|^{2p} + \frac{1}{4\epsilon_4} w^p |w_t|^p + \epsilon_5 |\nabla w|^{2p} + \frac{1}{4\epsilon_5 \epsilon_3^{p/2}} w^p |D^2 w|^p \\
&+ \epsilon_6 |\nabla w|^{2p} + \frac{1}{4\epsilon_6} w^{2p} |r_1|^{2p} + \epsilon_7 |\nabla w|^{2p} + \frac{1}{4\epsilon_7} w^p |r_2|^p] dx,
\end{aligned} \tag{26}$$

where identities $1/p + 2(p-1)/(2p) = 1$, $2/p + 2(p-2)/(2p) = 1$ and $1/(2p) + (2p-1)/(2p) = 1$ are applied. Let $0 < \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7 \leq \tilde{\mu}_1/10$. Then, (26) implies

$$\frac{\tilde{\mu}_1}{2} \int_\Omega |\nabla w|^{2p} dx \leqslant C_{14} \int_\Omega w^p \Big[|w_t|^p + |D^2 w|^p + w^p |r_1|^{2p} + |r_2|^p \Big] dx. \tag{27}$$

Suppose that $\max_{Q_T} |w| \leq \tilde{M}$. Then, integrating (27) with respect to t from 0 to T leads to

$$\|\nabla w\|_{L^{2p}(Q_T)} \leqslant C_{15}(\tilde{M}, T) \Big[\|w\|^{1/2}_{W_p^{2,1}(Q_T)} + \|r_2\|^{1/2}_{L^p(Q_T)} + 1 \Big] \leqslant C_{16}(\tilde{M}, T).$$

□

The following lemma directly follows from Lemma 3.2, Lemma 3.3 and Lemma 3.4.

Lemma 3.5. *Let u be a bounded function satisfying* (17)-(18). *Then, for any positive integer k, $Du \in L^{2^{k+1}}(Q_T)$ and $\|Du\|_{L^{2^{k+1}}(Q_T)} \leqslant C(M,T)$.*

Proof. Applying Lemma 3.4 to problem (17)-(18), one can obtain $Du \in L^4(Q_T)$ due to the result in Lemma 3.2. Since $u(x,t)$ is bounded and a classical solution for (17)-(18), we can rewrite this equation as the following linear form

$$u_t - \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i x_j} + \sum_{j=1}^{n} a_j(x,t)\frac{\partial u}{\partial x_j} = f^*(x,t), \tag{28}$$

where $f^*(x,t) = F(x,t,u(x,t),\nabla u(x,t))$, a_{ij} and a_j are given in (19). Then, (H4), (H5) and Lemma 3.3 indicate that each a_{ij} is bounded continuous function on Q_T, and $a_j, f^* \in L^4(Q_T)$. By employing Theorem 9.1 (pp.341-342) and its remark (p.351) in [11], one has $u \in W_4^{2,1}(Q_T)$ and

$$\|\tilde{w}\|_{W_4^{2,1}(Q_T)} \leqslant C_{17}\left(\|f^*\|_{L^4(Q_T)} + \|u_0\|_{W_4^{2-1/2}(\Omega)}\right) \leqslant C_{18}(M,T). \tag{29}$$

Using Lemma 3.4 again, one has $Du \in L^8(Q_T)$ and $\|Du\|_{L^8(Q_T)} \leqslant C_{19}(M,T)$. By mathematical induction arguments, we conclude that $Du \in L^{2^{k+1}}(Q_T)$ and $\|Du\|_{L^{2^{k+1}}(Q_T)} \leqslant C(M,T)$ for any $k = 1, 2, \cdots$. □

Remark 2. We can establish the boundedness of u on Q_T by using Theorem 7.1 (pp.181-182) in [11]. In fact, suppose further that

(H6) $F(x,t,u,p) \leqslant h^*(x,t,u)(u+|p|)$ for $\forall(x,t,u,p) \in \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$ with some bounded function $h^* \in C(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}_+, (0,\infty))$. Similar to the proof of Lemma 3.2 and 3.3, one can obtain

$$u \in W_2^{2,1}(Q_T) \cap L^p(Q_T) \cap V_2^{1,0}(Q_T),$$

and the corresponding norms are bounded by some positive constant $C_{20}(T)$. On the other hand, by Gagliardo-Nirenberg interpolation inequality for $n \leq 3$ ([1]), one can conclude that

$$\|Du(\cdot,t)\|_{L^4(\Omega)} \leq C_{21}\|Du(\cdot,t)\|_{L^2(\Omega)}^{1-\theta}\|D^2u(\cdot,t)\|_{L^2(\Omega)}^{\theta} \leq C_{22}(T) \quad (\theta = n/4).$$

Thus, $Du \in L^4(Q_T)$. A standard L^p estimate for linear parabolic equation shows that $u \in W_4^{2,1}(Q_T)$. Similar to estimate (27), one has

$$\begin{aligned}\frac{\tilde{\mu}_1}{2}\int_\Omega |\nabla u|^6 dx \leqslant & C_{23}\int_\Omega \left(u^3|u_t|^3 + u^3|D^2u|^3 + u^6\right)dx \\ \leqslant & C_{24}\int_\Omega \left(u^{12} + |u_t|^4 + |D^2u|^4 + u^6\right)dx.\end{aligned}$$

And then,

$$\|\nabla u\|_{L^6(Q_T)} \leqslant C_{25}\left(\|u\|_{L^{12}(Q_T)}^2 + \|u\|_{W_4^{2,1}(Q_T)}^{2/3} + \|u\|_{L^6(Q_T)}\right) \leqslant C_{26}(T).$$

Now, we are at the right position to establish the boundedness of u. Since the coefficients a_{ij}^2, a_j^2, f^* of equation (28) are in $L^3(Q_T)$, all conditions of Theorem 7.1 (pp.181-182) [11] are fulfilled for $n \leq 3$. Therefore, there must be a positive constant $C_{27}(T)$ such that $\|u\|_{L^\infty(Q_T)} \leq C_{27}(T)$.

By Remark 2 and the proof of Lemma 3.4, we can easily prove the following result similar to Lemma 3.5.

Lemma 3.6. *Let u be a classical solution of* (17)-(18) *and $n \le 3$. Then, for any positive integer k, $Du \in L^{2^{k+1}}(Q_T)$ and $\|Du\|_{L^{2^{k+1}}(Q_T)} \leqslant C(T)$.*

□

3.2. **Continuation of local solution.** Now, we extend the local solution of (17)-(18) obtained in Theorem 2.1 to $[0, \infty)$.

Lemma 3.7. *Let u be a bounded function satisfying* (17)-(18)*. Then*

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)} \leqslant C(M,T).$$

Proof. From the proof of Lemma 3.5, we can conclude that there must be some positive integer k_0 such that $2^{k_0+1} > n+2$. Then, we can stop the regularity argument at this k_0 and draw a conclusion that

$$\|u\|_{W^{2,1}_{q_0}(Q_T)} \leqslant C_{28}(M,T), \quad q_0 = 2^{k_0+1}.$$

By the embedding relation (for example, Lemma 3.3, p.80 in [11]) $W^{2,1}_q(Q_T) \hookrightarrow C^{1+\beta,(1+\beta)/2}(\overline{Q}_T)$ with $\beta = 1-(n+2)/q > \alpha$ for $q > n+2$, we have

$$\begin{aligned}\|u\|_{C^{1+\alpha,(1+\alpha)/2}(\overline{Q}_T)} \leqslant & C_{29}\|u\|_{C^{1+\beta,(1+\beta)/2}(\overline{Q}_T)} \\ \leqslant & C_{30}\|u\|_{W^{2,1}_{q_0}(Q_T)} \leqslant C_{31}(M,T).\end{aligned}$$

Then, $a_{ij}, a_j, f^* \in C^{\alpha,\alpha/2}(\overline{Q}_T)$. Applying Schauder theory (for example, Theorem 5.3, pp.320-321 in [11]) to (17), we can obtain

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)} \leqslant C(M,T).$$

□

Theorem 3.8. *Assume that (H1)*, (H2), (H4) and (H5) hold. Then, for any given $T > 0$, there exists a unique non-negative solution $u(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ of the system* (17)-(18)*. Moreover, if u is bounded on Q_T, then $T = +\infty$.*

Proof. We prove it by contradiction arguments. Assume that $[0, T^*)$ is the maximum existence interval of the solution to (17)-(18). For any $\epsilon \in (0, T^*)$, take $u(x, T^* - \epsilon)$ as the new initial value. By Theorem 2.1, one can extend the solution to $Q_{(T^*-\epsilon)+T_1}$ for some $T_1 > 0$, here T_1 only depends on the upper bound of $\|u(x,T^*-\epsilon)\|_{C^{2+\alpha}(\overline{\Omega})}$. By the a priori estimate in Lemma 3.7, T_1 only depends on T^* but does not depend on ϵ. Hence, we can choose an appropriate ϵ satisfying $\epsilon < \max\{T^*, T_1\}$ and then $(T^*-\epsilon)+T_1 > T^*$, which contradicts to the definition of T^*. The proof is complete. □

Theorem 3.9. *Assume that (H1)*, (H2), (H4)-(H6) hold and $n \le 3$. Then, there exists a unique non-negative solution $u(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,+\infty))$ of the system* (17)-(18)*.*

Remark 3. It is easy to see that $a \in C^2(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ and $F \in C^1(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ are sufficient to ensure (H1)* holds true.

4. **Examples.** In order to illustrate the main feature of the this study, in this section, we will explore the global existence and uniqueness of solution to some models, including an application to a data-based density-dependent diffusion model of in vitro glioblastoma growth.

Example 1. In order to model the glioblastoma tumor growth, [19] proposed a density-dependent convective-reaction-diffusion equation, whose one-dimensional Cartesian coordinate version reads

$$u_t = D(u)u_{xx} + D'(u)(u_x)^2 - \gamma u_x + u(1-u), \tag{30}$$

where the behavior of both proliferation and migration processes are incorporated.

In (30), diffusion is large for areas where the amount of cells are small (the migrating tumor cells), but diffusion is small where the cell density is large (the proliferating tumor cells). There are many functions that could serve as the diffusion function $D(u)$, for example,

$$D(u) = D_1 - \frac{D_2 u^n}{a^n + u^n},$$

here D_1, D_2, a, n are all positive constants, $n > 1$, and $D_2 \le D_1$ to avoid negative diffusion.

Although the dynamics of (30) are well explored, the existence and uniqueness of solutions are not investigated since the criteria existing in the literature can not be applied. We claim that the existence and uniqueness of (30) satisfying the boundary condition (18) falls into the framework of this study.

In fact, It is easy to see that (30) is a special case of (17). We have

$$\begin{aligned} D'(u) &= -\frac{na^n D_2 u^{n-1}}{(a^n+u^n)^2}, \\ D''(u) &= na^n D_2 \frac{-(n-1)a^{2n}u^{n-2} + 2a^n u^{2n-2} + (n+1)u^{3n-2}}{(a^n+u^n)^4}. \end{aligned}$$

Then, $D'(0) = D''(0) = 0$. Moreover, if $n > 2$, then, for any $u \geqslant 1$,

$$\begin{aligned} |D'(u)| &= \frac{na^n D_2 u^{n-1}}{(a^n+u^n)^2} \leqslant \frac{na^n D_2 u^n}{2a^n u^n} = \frac{nD_2}{2}, \\ |D''(u)| &\leqslant na^n D_2 \frac{(n-1)a^{2n}u^{n-2} + 2a^n u^{2n-2} + (n+1)u^{3n-2}}{u^{4n}} \\ &\leqslant nD_2 a^n [(n-1)a^{2n} + 2a^n + (n+1)], \end{aligned}$$

and, for any $u \in (0,1)$,

$$\begin{aligned} |D'(u)| &= \frac{na^n D_2 u^{n-1}}{(a^n+u^n)^2} \leqslant \frac{na^n D_2 u^{n-1}}{a^{2n}} < \frac{nD_2}{a^n}, \\ |D''(u)| &\leqslant na^n D_2 \frac{(n-1)a^{2n}u^{n-2} + 2a^n u^{2n-2} + (n+1)u^{3n-2}}{a^{4n}} \\ &< \frac{nD_2[(n-1)a^{2n} + 2a^n + (n+1)]}{a^{3n}}. \end{aligned}$$

Therefore, for any $u \in [0, \infty)$, we have

$$\begin{aligned} |D'(u)| &\leqslant nD_2 \cdot \max\left\{\frac{1}{2}, \frac{1}{a^n}\right\}, \\ |D''(u)| &\leqslant nD_2[(n-1)a^{2n} + 2a^n + (n+1)] \cdot \max\left\{a^n, \frac{1}{a^{3n}}\right\}. \end{aligned}$$

This shows that $D(u)$ and $D'(u)$ have bounded derivatives on $[0, \infty)$. Clearly, $D(u), D'(u)$ are both Lipschitz continuous on $[0, \infty)$, and $F(u,p) = u(1-u) - \gamma p$ is locally Lipschitz continuous with respect to u or p.

From the above discussion, we know that (H1)*, (H2), (H5) and (H6) are satisfied for model (30). Moreover, the maximum principle yields that $0 \leqslant u \leqslant 1$. Then Theorem 3.8 implies that there exists a unique non-negative solution $u(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,+\infty))$ of (30) satisfying the boundary condition (18).

Example 2. Consider the following quasilinear parabolic equation

$$u_t - \operatorname{div}[a(x,t,u)\nabla u] = \min\{ku, K\} \quad \text{in} \quad \Omega \times (0,\infty) \tag{31}$$

where k and K are positive constants. Since $F(u) = \min\{ku, K\}$ is non-differentiable at $u = \frac{K}{k}$, results in [3], [4] and [22] can not be applied to problem (18) and (31).

However, it is obvious that $F(u)$ is Lipschitz continuous on $\mathbf{R}$. Suppose that $a(x,t,u)$ and initial value satisfy assumptions in §3. Then, according to Theorem 3.9, problem (18) and (31) possesses a unique non-negative solution $u(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,+\infty))$.

5. **Discussion.** This paper is devoted to the study of the global-in-time solutions for a generalized quasi-linear parabolic equation with applications in biology and medicine. Under some practical regularity and structure conditions on diffusion term and nonlinearity, we establish the local and global existence and uniqueness of classical solutions for problem (17)-(18). The main results are Theorems 3.1, 3.8 and 3.9, which show that the unique solution of problem (17)-(18) and its derivatives ($u, u_{x_i}, u_{x_ix_j}$ and u_t) are all continuous in $\overline{Q}_T$. One of the main difficulties for global existence is to perform the L^p-estimate of Du. Comparing to the existing results on the global solutions of quasi-linear parabolic equations, our conditions on diffusion item and nonlinearity function represented by $((\text{H1})^*, (\text{H4}) - (\text{H6}))$ are easier to verify and need weaker regularity.

However, we only investigate the existence of classical solutions for a single-equation system in the present paper. In real-world applications, quasi-linear parabolic systems consisting of two or more equations are more common and significant. Thus, we are also interested in the existence of classical solutions for quasi-linear parabolic systems of equations. The method of classical fixed point theory is usually effective for studying the question of local existence of solutions to systems of equations. But the L^p-estimate techniques for the unknowns and their gradients are more difficult due to the coupled diffusion and nonlinear terms. To overcome this difficulty, we may need some novel Sobolev embedding results and new interpolation inequalities. We encourage future efforts along these directions.

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