MATHEMATICAL BIOSCIENCES
AND ENGINEERING
Volume **11**, Number **1**, February **2014**

doi:10.3934/mbe.2014.11.1

pp. **1–10**

A SIMPLE ALGORITHM TO GENERATE FIRING TIMES FOR LEAKY INTEGRATE-AND-FIRE NEURONAL MODEL

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Abstract. A method to generate first passage times for a class of stochastic processes is proposed. It does not require construction of the trajectories as usually needed in simulation studies, but is based on an integral equation whose unknown quantity is the probability density function of the studied first passage times and on the application of the hazard rate method. The proposed procedure is particularly efficient in the case of the Ornstein-Uhlenbeck process, which is important for modeling spiking neuronal activity.

1. **Introduction.** The problem of the first passage time is known as a difficult task in the theory of stochastic processes and it plays an important role in various applications, e.g., quantitative finance, theoretical biology, engineering, chemistry, epidemiology and others.

Here we present it in the context of theoretical neuroscience, namely in modeling the firing of a single neuron. The membrane potential of a neuron can often be described by an Ornstein-Uhlenbeck stochastic process whose excursion is limited from above by an absorbing barrier, called firing threshold. Reaching the threshold is identified with generation of an action potential (spike). Determination of the distribution of the inter-spike intervals or at least its statistical properties is an important task from a neuronal coding point of view. More details on the first passage time problem for the Ornstein-Uhlenbeck process in computational neuroscience can be found, for example, in [19].

One approach to the problem is based on simulating trajectories of the membrane potential by using available numerical methods for stochastic differential equations (see, for instance, [10]). However, having discretized trajectories, it is necessary to evaluate the probability that a spike (crossing) occurs within each interval of the temporal mesh and not being observed, otherwise the actual first passage time is overestimated. For different methods based on simulations of the trajectory, see [11], [6], [7], [8] and [17].

2010 *Mathematics Subject Classification.* Primary: 60J60; Secondary: 60H35.

Key words and phrases. Ornstein-Uhlenbeck process, first passage time, spike train generation, instantaneous firing rate, hazard rate method.

This work has been performed under partial support by F.A.R.O Project (Finanziamenti per l'Avvio di Ricerche Originali, III tornata) "Controllo e stabilità di processi diffusivi nell'ambiente", Polo delle Scienze e Tecnologie, Università degli Studi di Napoli Federico II.

In the present paper we propose a simulation method based on the hazard function and on the use of the integral equation ([1], [5], [9]), known in the literature as the singularity removed probability current equation ([4]).

Taking into account that the application of the method of the hazard function can also lead to the evaluation of the above density for large values of its argument, the proposed method is particularly operative when the hazard function tends, as time increases, to a positive constant. We prove that it happens, asymptotically with respect to the firing threshold, in the case of the leaky integrate-and-fire (LIF) model ([2]) with constant parameters.

In Sections 2 and 4 we resume the theoretical essentials; in Section 3 the algorithm with some warnings are provided as well as the main result of the paper; finally, in Section 5 the proposed method is applied to the stochastic LIF model with constant parameters.

2. **Preliminary remarks.**

Let I be a non empty subset of $\mathbb{R}$. The transition probability density function $f_X(x,t|y,\tau)$ of a diffusion process $\{X(t),\ t \in [0,+\infty[\}$ with state space I is the solution of the following Fokker-Planck equation

$$\frac{\partial}{\partial t} f_X(x,t|y,\tau) = -\frac{\partial}{\partial x}\left[A_1(x,t) f_X(x,t|y,\tau)\right] + \frac{\partial^2}{\partial x^2}\left[A_2(x,t) f_X(x,t|y,\tau)\right] \qquad (x, y \in I; 0 \le \tau < t), \tag{1}$$

in which the coefficients $A_1(x,t)$ and $A_2(x,t)$ represent drift and infinitesimal variance of the process, respectively.

For $t_0 \ge 0$, $X(t_0) = x_0 \in I$ and for a pre-assigned $S(t) \in C^2_-(t_0,+\infty[$, such that $S(t_0) > x_0$, with

$$T_{X,S}(x_0,t_0) := \inf\{t > t_0 : X(t) \ge S(t)\}$$

we denote the random variable *first passage time* of $\{X(t),\ t \in [t_0,+\infty[\}$ through the *threshold* $S(t)$ and with

$$g_X[S(t),t|x_0,t_0] = \frac{d}{dt}\mathbb{P}\left(T_{X,S}(x_0,t_0) \le t\right) =: \frac{d}{dt} G_X[S(t),t|x_0,t_0]. \tag{2}$$

its probability density function.

The process and the threshold properties ensure that $T_{X,S}(x_0,t_0)$ is an absolutely continuous random variable.

In the present paper we consider thresholds and diffusion processes admitting a space-time transformation in the sense indicated in [14]. In such a case, indeed, $g_X[S(t),t|x_0,t_0]$ is solution of the following integral equation with non-singular kernel:

$$g_X[S(t),t|x_0,t_0] = -\psi_X[S(t),t|x_0,t_0] + \int_{t_0}^{t} \psi_X[S(t),t|S(\tau),\tau]\, g_X[S(\tau),\tau|x_0,t_0]\, d\tau. \tag{3}$$

Such a result obtained in [9] is an extension of the one in [1] for the Wiener and for the homogeneous Ornstein-Uhlenbeck process and successively proved in [5] for temporally homogeneous diffusion processes with $A_2(x)$ of $C^1(I)$ class.

In Eq. (3) the free term and the kernel can be evaluated by means of the following:

$$\begin{aligned}\psi_X[S(t),t|z,s] &:= \left\{ \frac{dS(t)}{dt} + \frac{3}{4}\frac{\partial A_2(x,t)}{\partial x}\Big|_{x=S(t)} - A_1[S(t),t] \right\} f_X[S(t),t|z,s] \\ &\quad + A_2(x,t)\frac{\partial f_X[x,t|z,s]}{\partial x}\Big|_{x=S(t)} \qquad (z \in I, s \geq t_0).\end{aligned} \tag{4}$$

In particular, for the kernel of Eq. (3) one has

$$\lim_{\tau \to t^-} \psi_X[S(t),t|S(\tau),\tau] = 0,$$

and this allows, for example, to apply explicit composed quadrature formulas in order to obtain approximations of the first passage time probability density function (see, for instance, [1] and [3]) on a suitable mesh $\{t_n = t_0 + n\Delta t\}_{n\in\mathbb{N}}$. Note that, such formulas, although easy to implement, have computational complexity of $O(n^2)$. Furthermore, via a suitable numerical procedure one can obtain an approximation for the first passage time distribution function $G_X[S(t),t|x_0,t_0]$.

3. **The algorithm.** Thereafter, at least in principle, the Hazard Rate Method (HRM) can be applied in order to have a generator for $T_{X,S}(x_0,t_0)$ (see, for instance, [15]):

HRM-Algorithm

1. determine an upper bound Λ of $\lambda_X(t) := \dfrac{g_X[S(t),t|x_0,t_0]}{1 - G_X[S(t),t|x_0,t_0]}$ $(t \geq t_0)$;
2. make sure that $\int_{t_0}^{+\infty} \lambda_X(t)\,dt = \infty$;
3. set $T = 0$, $n = 0$, $Y_0 = 1$, $U_0 = 1$;
4. $n \leftarrow n + 1$;
5. get $Y_n \overset{d}{=} \text{Exp}(\Lambda)$ independent from $\sigma(Y_0,\ldots,Y_{n-1})$;
6. $T \leftarrow T + Y_n$;
7. get $U_n \overset{d}{=} \text{U}(0,1)$ independent from Y_n and $\sigma(U_0,\ldots,U_{n-1})$;
8. if $\dfrac{\lambda_X(T)}{\Lambda} < U_n$ return to `step 4`;
9. save $T_{X,S}(x_0,t_0) = T$.

The symbols $\overset{d}{=}$, $\text{Exp}(\cdot)$ and $\sigma(\cdot)$ indicate the equality in distribution, the exponential distribution and the generated σ-algebra, respectively.

The implementation of HRM-Algorithm can be quite affected by the following arguments:

Warnings for the HRM-Algorithm

(A) the function $f_X(x,t|y,\tau)$ should be evaluated via a numerical procedure by means of Eq. (1);
(B) a numerical method is also required in order to obtain the first derivative of $f_X(x,t|y,\tau)$ with respect to x;
(C) the functions $g_X[S(t),t|x_0,t_0]$ and $G_X[S(t),t|x_0,t_0]$ have to be calculated via a numerical procedure by means of Eq. (3);
(D) it could be required to evaluate the function $\lambda_X(t)$ for large values of its argument.

Furthermore, since $\lambda_X(t)$ is evaluated for less than an error δ, Δt-dependent, when we apply the HRM-Algorithm we have to consider the case in which U_k is in the interval

$$\left(\frac{\lambda(T)}{\Lambda} - \frac{\delta}{2}, \frac{\lambda(T)}{\Lambda} + \frac{\delta}{2}\right). \tag{5}$$

Such a situation occurs with probability δ and therefore the generated first passage time is exact with probability $(1-\delta)^N$ at least, where N is the number of the required iterations to terminate the HRM-Algorithm. On the other hand, the Wald's equality provides $\mathbb{E}(N) = \mathbb{E}[T_{X,S}(x_0, t_0)] \cdot \Lambda$ and, in the majority of circumstances, the two factors have reciprocal magnitude. However, whenever U_k is in the interval given in (5), in the reported calculations (Tables 1–3), we reject the instance of the algorithm.

Recalling that $f_1(t) \sim f_2(t)$, in the neighborhood of $+\infty$, means that $\lim_{t\to+\infty} \frac{f_1(t)}{f_2(t)}$ $= 1$, for the first passage time hazard rate function $\lambda_X(t)$ the following result holds.

Proposition 3.1. *We set*

$$\begin{aligned} \psi(t) &:= \psi_X[S(t), t|x_0, t_0], & \psi(t,\tau) &:= \psi_X[S(t), t|S(\tau), \tau] \\ \psi &:= \lim_{t\to+\infty} \psi(t), & \varepsilon(t,\tau) &:= \psi(t,\tau) - \psi, \\ g(t) &:= g_X[S(t), t|x_0, t_0], & G(t) &:= G_X[S(t), t|x_0, t_0]. \end{aligned} \tag{6}$$

Now, (i) *if* $\psi \neq 0$ *and,* (ii) *if for all* $t > t_0$ *a constant* $d > 0$ *such that the function* $\varepsilon(t, \tau)$ *does not change its sign within the interval* $(t_0, t - d)$ *exists, then*

$$\lambda_X(t) \sim \frac{-\psi + \varepsilon\,[t, t - (1-\xi_t)d]}{1 - \int_{t_0}^{t-d} \varepsilon(t,\tau)\, d\tau}, \tag{7}$$

where ξ_t *and* η_t *are real numbers in* $(0, 1)$.

Proof. The following holds:

$$\begin{aligned} g(t) &\sim -\psi + \int_{t_0}^{t} \psi g(\tau)\, d\tau + \int_{t_0}^{t} \varepsilon(t,\tau) g(\tau)\, d\tau \\ &\sim -\psi[1 - G(t)] + \int_{t_0}^{t-d} \varepsilon(t,\tau) g(\tau)\, d\tau + \int_{t-d}^{t} \varepsilon(t,\tau) g(\tau)\, d\tau. \end{aligned}$$

By applying the mean value theorem for integration to both integrals at right hand-side one has:

$$\begin{aligned} g(t) \sim -\psi[1 - G(t)] + g(t) \frac{g[t_0 + (t - d - t_0)\eta_t]}{g(t)} \int_{t_0}^{t-d} \varepsilon(t,\tau)\, d\tau \\ + \varepsilon(t, t - d + d\xi_t)\big[G(t) - G(t-d)\big]. \end{aligned}$$

The result follows by dividing both sides of the above relation for $1 - G(t)$ and by considering the following:

$$\lim_{t\to+\infty} \frac{g[t_0 + (t - d - t_0)\eta_t]}{g(t)} = 1, \quad \lim_{t\to+\infty} \frac{G(t) - G(t-d)}{1 - G(t)} = 1.$$

The last limit is obtained dividing by d both the numerator and the denominator, substituting 1 with $G(t + d)$ and making use of Eq. (2). $\square$

4. **The Ornstein-Uhlenbeck processes.** Here we consider the class of stochastic processes, largely used in neuronal modelling, having

$$A_1(x,t) = xa(t) + b(t) \quad \text{and} \quad A_2(x,t) \equiv A_2(t) = \sigma^2(t),$$

so that the Warnings (A) and (B) for the application of HRM-Algorithm are immediately overcome, since the transition probability density function $f_X(x,t|y,\tau)$ of the process $\{X(t),\ t \in [t_0,+\infty[\}$ is Gaussian with mean and variance, given respectively by:

$$M_X(t|y,\tau) = m_X(t) + \frac{v_X(t)}{v_X(\tau)}[y - m_X(\tau)], \tag{8}$$

$$(t_0 \leq \tau < t)$$

$$D^2_X(t|\tau) = \frac{v_X(t)}{v_X(\tau)}[u_X(t)v_X(\tau) - u_X(\tau)v_X(t)]. \tag{9}$$

It is convenient to note that it is easy to obtain the derivative of $f_X(x,t|y,\tau)$ with respect to x.

In Eqs. (8) and (9) $m_X(t)$, $u_X(t)$ and $v_X(t)$ represent, respectively, the mean

$$m_X(t) = e^{\int_{t_0}^t a(s)\,ds}\left[x_0 + \int_{t_0}^t b(\xi)e^{-\int_{t_0}^\xi a(s)\,ds}\,d\xi\right] \quad (t_0 \leq t), \tag{10}$$

and the two covariance factors

$$\begin{aligned} c_X(\tau,t) &= u_X(\tau)v_X(t) \\ &= \left[e^{\int_{t_0}^\tau a(s)\,ds}\int_{t_0}^\tau \sigma^2(\xi)e^{-2\int_{t_0}^\xi a(s)\,ds}\,d\xi\right]e^{\int_{t_0}^t a(s)\,ds} \quad (t_0 \leq \tau \leq t), \end{aligned} \tag{11}$$

of the process $\{X(t),\ t \in [t_0,+\infty[\}$.

The above functions can be obtained in closed form or via quadrature formulas by means of the infinitesimal coefficients.

In the following, we show that the difficulties in the application of the HRM-Algorithm (Warnings (C) and (D)) are greatly reduced in the case of Ornstein-Uhlenbeck with constant coefficients.

5. **The LIF model with constant parameters.** We consider the homogeneous Ornstein-Uhlenbeck process $\{U(t),\ t \in [t_0,+\infty[\}$ for which

$$A_1(x,t) = -\frac{x}{\theta} + \frac{\rho + \mu\theta}{\theta} \quad \text{and} \quad A_2(t) = \sigma^2,$$

largely known playing a key rule in the LIF stochastic model for the membrane potential dynamics of some kind of neurons (see, for instance, [12] and [16]). In such a context θ, ρ and μ represent the time constant of the membrane potential, the resting potential and a constant current, respectively. The quantity μ, although indicated as current, it actually is a current divided by the membrane potential capacitance.

For $\{U(t),\ t \in [t_0,+\infty[\}$ it results

$$\begin{aligned} m_U(t) &= x_0 e^{-(t-t_0)/\theta} + (\rho + \mu\theta)\left[1 - e^{-(t-t_0)/\theta}\right], \\ u_U(t) &= \frac{\sigma\theta}{2}\left[e^{(t-t_0)/\theta} - e^{-(t-t_0)/\theta}\right], \\ v_U(t) &= \sigma e^{-(t-t_0)/\theta}, \end{aligned}$$

for which, by using (4), (8), (9) and for a constant threshold S, one obtains:

$$\psi_U[S,t|y,\tau] = \left\{ \frac{S-(\rho+\mu\theta)}{\theta} - \frac{2}{\theta}\frac{S-(\rho+\mu\theta)[1-e^{-(t-\tau)/\theta}]-ye^{-(t-\tau)/\theta}}{1-e^{-2(t-\tau)/\theta}} \right\} \times \frac{1}{\sqrt{\pi\sigma^2\theta[1-e^{-2(t-\tau)/\theta}]}} e^{-\dfrac{\{S-(\rho+\mu\theta)[1-e^{-(t-\tau)/\theta}]-ye^{-(t-\tau)/\theta}\}^2}{\sigma^2\theta[1-e^{-2(t-\tau)/\theta}]}}. \tag{12}$$

Now, it can be proved that, for any assigned setting of parameters $S, x_0, \theta, \rho, \mu$ and σ^2:

$$\lim_{t\to+\infty} \lambda_U(t) = \lambda_U. \tag{13}$$

Indeed, in the case in which the membrane potential evolves in a subthreshold regime, i.e. $\lim_{t\to+\infty} m_U(t) = \rho+\mu\theta < S$, it first results:

$$\psi = \lim_{t\to+\infty} \psi_U[S,t|x_0,t_0] = -\frac{S-(\rho+\mu\theta)}{\theta}\frac{1}{\sqrt{\pi\sigma^2\theta}} e^{-\dfrac{[S-(\rho+\mu\theta)]^2}{\sigma^2\theta}} < 0,$$

so that the *(i)* of Proposition 3.1 is satisfied. With reference to the function

$$\varepsilon(t,\tau) \equiv \varepsilon(t-\tau) = \psi_U[S,t|S,\tau] - \psi$$

introduced in Eq. (6), a value S^* of the firing threshold exists such that, for $S < S^*$ it is positive for all $t_0 < \tau \le t$; while, for $S \ge S^*$, $\varepsilon(t,\tau) > 0$ for τ in the neighborhood of t and negative otherwise. Therefore, the *(ii)* of Proposition 3.1 is also satisfied and Eq. (7) holds. However, in this specific case, the function $\varepsilon(s)$ is a continuous function such that in the neighborhood of $+\infty$ is $O\left(e^{-s/\theta}\right)$, by which

$$\begin{aligned} \lim_{t\to+\infty} \int_{t_0}^{t-d} \varepsilon(t,\tau)\,d\tau &= \lim_{t\to+\infty} \int_{t_0}^{t-d} \varepsilon(t-\tau)\,d\tau = \lim_{t\to+\infty} \int_{d}^{t-t_0} \varepsilon(s)\,ds \\ &= \int_d^{+\infty} \varepsilon(s)\,ds = \text{const.} \end{aligned} \tag{14}$$

The Eq. (13) follows from Proposition 3.1 taking in account (14) and noting that the ξ_t value in Eq. (7) does not depend on t:

$$\begin{aligned} \int_{t-d}^{t} \varepsilon(t,\tau)g(\tau)\,d\tau &= \int_0^d \varepsilon(s)g(t-s)\,ds = \varepsilon(\xi d)\int_0^d g(t-s)\,ds \\ &= \varepsilon(\xi d)\int_{t-d}^{t} g(\tau)\,d\tau, \quad \xi \in (0,d). \end{aligned} \tag{15}$$

We note that the asymptotic firing rate λ_U does not depend from reset value x_0.

Figure 1 shows the behavior of $\lambda_U(t)$ for some values of S while Figure 2 shows the behavior of $\lambda_U(t)$ for some values of x_0. The other parameters are chosen as in [12], in which the authors consider experimental recordings of cortical neurons membrane potential of Guinea pigs subject to a spontaneous activity. In the above described model, the authors take ρ equal to x_0. They determine estimators of $1/\theta$, μ and σ^2 based on each experimental recording, to which an initial (about 10 ms) and a final part (about 10 ms) are eliminated. The initial one (final one) is used to obtain an empirical estimation of x_0 (S). The median of the estimation obtained from recording is taken as estimator of each parameter. The estimated values are the following: $\theta = 38.7534$ ms, $\mu = 0.2846$ mV/ms and $\sigma^2 = 0.1824$ mV^2/ms; while S results 13 mV over ρ. In the present paper we prefer to keep different x_0 and ρ, in order to preserve their specific physiological significance, and to rescale the values

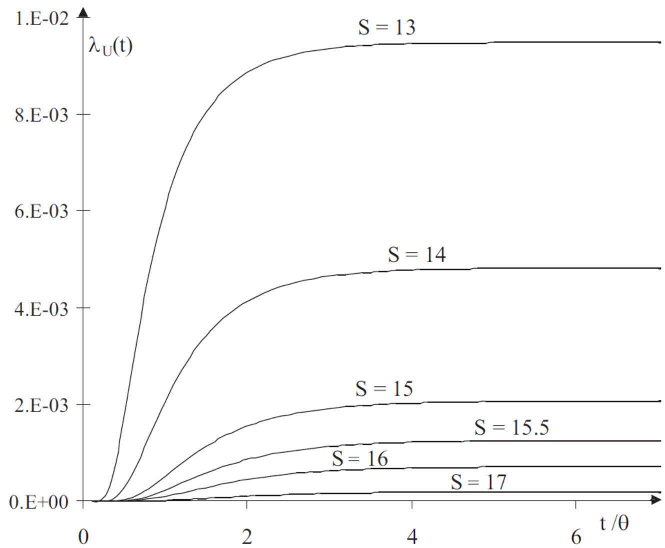

FIGURE 1. Plot of $\lambda_U(t)$ (in ms^{-1}), in which the time is given in θ units, for some values of S specified by the corresponding labels and with $x_0 = 7.5$ mV. The instantaneous firing rate decreases when the threshold increases and after few time units it reaches the asymptotic value. Other parameters are: $\theta = 38.7534$ ms, $\rho = 0$ mV, $\mu = 0.2846$ mV/ms and $\sigma^2 = 0.1824$ mV^2/ms.

of the membrane potential with respect to $\rho = -65$ mV; furthermore, we consider either x_0 or S to be variable.

From Figure 2 we argue that when x_0 is less than the asymptotic mean membrane potential $\rho + \mu\theta$, a rather few iterations in the HRM-Algorithm will be sufficient in order to obtain $\lambda_U(T)/\Lambda$ next to 1, so that the stop criterion will be quickly satisfied. Instead, when x_0 is greater than $\rho + \mu\theta$, the plot of the instantaneous firing rate shows a maximum value considerably larger than the asymptotic firing rate λ_U so that, for large T, a greater number of iterations in the HRM-Algorithm will be required (the stop criterion will be satisfied with probability λ_U/Λ).

In Tables 1, 2 and 3 we set Λ as the maximum of $1.01 \cdot \lambda_U(t)$ evaluated until 20θ. Here, with "Euler's method" we denote the first passage time generation by means of trajectories simulated using the Euler stochastic discretization of the Langevin equation, i.e.

$$x(t_{n+1}) = x(t_n) + [a(t_n)x(t_n) + b(t_n)]\,h + \sigma(t_n)z\sqrt{h},$$

where $h > 0$, $t_n = t_0 + nh$ and z is a standard gaussian number. Instead, with "Quadrature" we refer to a numerical method, having time step Δt, for solving the integral equation (3) for the homogeneous Ornstein-Uhlenbeck process $\big\{U(t),\ t \in [t_0, +\infty[\big\}$ and constant threshold S, in order to obtain an approximation for the instantaneous firing rate $\lambda_U(t)$.

Referring again to the experimental setting of parameters as in [12], in Tables 1–2 we give the relative errors of the estimations of some moments firing time obtained by HRM-Algorithm (Euler's method) with respect to the values obtained by means of a series expansions given in [13].

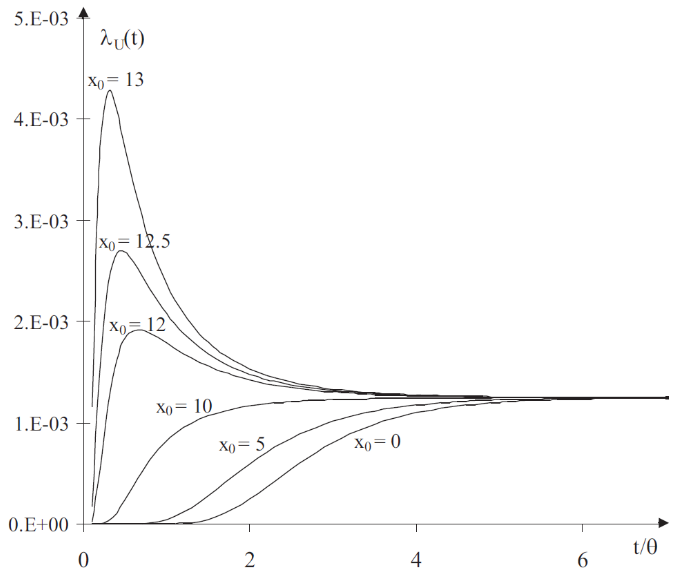

FIGURE 2. As in Figure 1 for the x_0 values specified by the corresponding labels and $S = 15.5$ mV. Note that the asymptotic firing rate does not depend on x_0.

S (mV)	Quadrature ($\Delta t = \theta/100$) Hazard rate method mean	st.dev	skew	Trajectories ($h = \theta/1000$) Euler's method mean	st.dev	skew	Run time ratio
13	$-2.50E{-3}$	$3.13E{-3}$	$8.99E{-3}$	$3.13E{-2}$	$4.69E{-2}$	$1.15E{-2}$	0.66
14	$-3.64E{-3}$	$1.78E{-3}$	$2.06E{-2}$	$4.12E{-2}$	$3.84E{-2}$	$-3.77E{-2}$	0.99
15	$-4.39E{-3}$	$7.49E{-3}$	$2.67E{-2}$	$4.80E{-2}$	$6.01E{-2}$	$-2.59E{-2}$	2.54
15.5	$6.09E{-5}$	$9.40E{-3}$	$2.85E{-2}$	$5.74E{-2}$	$7.71E{-2}$	$-2.35E{-2}$	4.87
16	$1.69E{-3}$	$9.96E{-3}$	$2.99E{-2}$	$7.64E{-2}$	$8.69E{-2}$	$1.39E{-2}$	5.10
17	$-1.75E{-4}$	$5.39E{-3}$	$2.34E{-2}$	$6.49E{-2}$	$6.85E{-2}$	$4.45E{-2}$	11.47

TABLE 1. Relative errors for the mean, standard deviation and skewness of $T_{U,S}(x_0, t_0)$, obtained by means of the methods specified on top, are listed for the S values as indicated in the first column. Reference values, not shown, are obtained by means the series expansions given in [13]. Other parameters as in Figure 1. Times are given in θ units and the samples are both of size 10000. The last column shows the ratio between the run time of the Euler's method and the Hazard rate method.

In Table 3 we compare the estimation provided by HRM-Algorithm (Euler's method) to the mean firing time values given in [18]. In such a paper, a comparison between the discrete Stein model and its diffusion approximation has been performed; we recall that f_e and f_i (a_e and a_i) represent the frequencies (the amplitudes) of excitatory and inhibitory current pulses.

The results reported in the previous tables make evident that the main advantage of the proposed method is that, on the contrary of methods involving trajectories of the process, it does not show a systematic bias (overestimation). Furthermore, the HRM-Algorithm run time results quite independent from the value of the firing threshold even if it increases when x_0 is placed near to the firing threshold S.

x_0 (mV)	Quadrature ($\Delta t = \theta/100$) Hazard rate method mean	st.dev	skew	Trajectories ($h = \theta/1000$) Euler's method mean	st.dev	skew	Run time ratio
14	$-1.32E-2$	$-1.91E-2$	$-4.53E-2$	$6.63E-2$	$6.81E-2$	$-2.80E-2$	0.56
13	$-1.74E-2$	$-2.73E-2$	$-6.92E-2$	$6.01E-2$	$7.16E-2$	$-2.64E-2$	1.95
12.5	$-6.95E-3$	$-1.06E-2$	$-9.07E-3$	$5.63E-2$	$7.56E-2$	$-2.17E-2$	2.72
12	$9.12E-3$	$1.56E-2$	$2.20E-2$	$5.54E-2$	$7.58E-2$	$-1.56E-2$	3.51
10	$1.85E-2$	$1.03E-2$	$3.27E-2$	$5.72E-2$	$7.62E-2$	$-2.57E-2$	4.80
7.5	$6.09E-5$	$9.40E-3$	$2.85E-2$	$5.74E-2$	$7.71E-2$	$-2.35E-2$	4.87
5	$-2.00E-3$	$8.97E-3$	$2.65E-2$	$5.77E-2$	$7.58E-2$	$-2.73E-2$	4.82
2.5	$-4.06E-3$	$8.24E-3$	$2.54E-2$	$5.68E-2$	$7.65E-2$	$-2.66E-2$	5.11
0	$-3.64E-3$	$8.22E-3$	$2.55E-2$	$5.35E-2$	$7.55E-2$	$-2.62E-2$	5.10

TABLE 2. As in Table 1 for the x_0 values as indicated in the first column. Other parameters as in Figure 2.

f_e (1/ms)	f_i (1/ms)	μ (mV/ms)	σ^2 (mV^2/ms)	Reference values (ms)	Quadrature ($\Delta t = \theta/100$) Hazard rate method	Trajectories ($h = \theta/1000$) Euler's method	run time ratio
2	2	0	4	56.70	$2.37E-3$	$6.83E-2$	6.62
3	2	1	5	9.39	$7.50E-4$	$4.75E-2$	0.78
4	2	2	6	3.69	$-5.42E-3$	$3.94E-2$	0.49
5	2	3	7	2.10	$2.00E-4$	$3.11E-2$	0.43
3	6	-3	9	195.00	$-1.49E-2$	$6.84E-2$	21.87
4	6	-2	10	38.50	$-7.92E-3$	$8.37E-2$	2.50
5	6	-1	11	12.50	$1.42E-2$	$5.63E-2$	0.73
6	6	0	12	5.69	$-3.08E-4$	$5.41E-2$	0.49
7	6	1	13	3.21	$-9.81E-4$	$4.86E-2$	0.46
8	6	2	14	2.09	$-4.50E-3$	$4.07E-2$	0.45

TABLE 3. Relative errors for the mean of $T_{U,S}(x_0, t_0)$ obtained by means of the method specified on top are listed for f_e and f_i values indicated in the first and second columns. Reference values are those given in Table 1 of [18]. The last column shows the ratio between the run time of the Euler's method and the Hazard rate method. Values of μ and σ^2 are listed for convenience. Here $S = 4$ mV, $x_0 = 0$ mV, $a_e = 1$ mV, $a_i = 1$ mV and $\theta = 1$ msec.

Acknowledgments. We thank the reviewers for their careful reading of our manuscript as well as their useful comments.

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Received December 22, 2012; Accepted May 22, 2013.

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