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OPTIMAL ISOLATION STRATEGIES OF EMERGING INFECTIOUS DISEASES WITH LIMITED RESOURCES

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In honour of Professor Carlos Castillo-Chavez on his 60th Birthday

Abstract. A classical deterministic SIR model is modified to take into account of limited resources for diagnostic confirmation/medical isolation. We show that this modification leads to four different scenarios (instead of three scenarios in comparison with the SIR model) for optimal isolation strategies, and obtain analytic solutions for the optimal control problem that minimize the outbreak size under the assumption of limited resources for isolation. These solutions and their corresponding optimal control policies are derived explicitly in terms of initial conditions, model parameters and resources for isolation (such as the number of intensive care units). With sufficient resources, the optimal control strategy is the normal Bang-Bang control. However, with limited resources the optimal control strategy requires to switch to time-variant isolation at an optimal rate proportional to the ratio of isolated cases over the entire infected population once the maximum capacity is reached.

1. **Introduction.** Mathematical models are often used to study disease spread, with the susceptible-infectious-recovered(SIR) model being preferred for disease spread via droplet and aerosol. For example, the SIR model has been used to study pandemic flu [6, 9, 11, 12, 17, 19, 25, 26, 27, 28, 36, 38], seasonal flu [7, 10, 15] , SARS [24, 30, 34, 35], and smallpox [14, 16, 22, 31]. These studies use SIR models to simulate the disease outbreak and evaluate the effectiveness of selected control measures under various predefined scenarios. Optimal control theory approaches based on deterministic compartmental models can provide valuable information about how best to control infectious disease outbreaks, and in particular, can determine the optimal distribution of limited resources during epidemics. We refer to [1, 2, 3, 5, 13, 20, 23, 29, 32, 33, 37] for studies of SIR model based optimal control that minimizes a prescribed objective function.

Some of the earliest work in this area was by Abakuks. In [1], Abakuks investigated the optimal control of a simple deterministic SIR model, and determined the

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isolation strategy that minimizes the total number of infected individuals, balanced against a cost associated with using isolation. In [2], Abakuks determined the optimal vaccination strategy for the same model and found that the optimal strategy was to vaccinate N susceptibles at the start of the epidemic, where N depends on the precise form of the objective function. In [3], Abakuks then determined the optimal vaccination strategy for the same model but under the assumption that, at any instant, either all or none of the susceptibles are vaccinated. Shortly after the publication of [2] and [3], Wickwire and Monton studied the same questions but with two notable differences [29, 37]. They found that the optimal isolation strategy was to use either maximal control for the entire epidemic or to use no control at all and that the optimal vaccination policy is a Bang-Bang control from maximal vaccination to no vaccination. In 2000, Behncke [?] expanded Wickwire's results to models with more general contact rates. Sethi derived optimal closed-form results for isolation and immunization policies [32, 33] using an SI model. The control is to either isolate and vaccinate at a maximum rate or do nothing. Clancy [13] studied the properties of optimal policies for isolation and immunization assuming that all infectious individuals can be immediately isolated and all susceptible individuals can be immediately immunized. The policy takes no action when the number of infectious is below an optimal threshold and immediately isolates and/or immunizes when the number exceeds the threshold. Lin, Muthuraman and Lawley [23] used an expanded SIR model to develop triggers for NPI implementation to minimize expected person-days lost resulting from influenza related deaths and NPI implementation. NPI policies are derived for the control model using a linear NPI implementation cost. However, none of these studies provided results for a combined isolation-vaccination model, and no analytical solution for an optimal control problem of epidemics was discussed. In 2010, Hansen and Day [20] extended that of [1, 2, 3, 5, 29, 37] by examining the kind of resource constraints mentioned earlier. Specifically, the simple SIR model with mass action contact is revisited, and the analytic solutions rather than numerical ones are obtained. For the isolation model (Problem 2 in [20]), under an assumption of total isolation resources being limited, the optimal policy is proved to be either maximum isolation or any isolation with constraint boundary .

But, it is not the case that all infectious individuals can be immediately isolated owing to time delay for rapid response and limitation of bed capacity provided by a hospital, and so the assumption in [20] of total isolation capacity is unreasonable. In this work, we use an expanded SIR model to minimize infectious size under limited isolation resources. In this model, we use the variable(D) in the basic SIR model in [4, 20] to represent bed capacity and modify the constraint condition of the total isolation capacity in [20] to be time-variant. This can be regarded as a subclass of the recovery compartment. Changing of constraint conditions complicates the optimal control problem based on the expanded SIR model. We find, for this new optimal control problem, that there are four different scenarios (in comparison with three scenarios in the case of without resource constraints) for optimal isolation treatment strategies, and we get the analytic solutions for each of the four scenarios.

2. **Formulation of the optimal control problem.**

We consider the optimal control issue for the following modification of the classical SIR model examined in [4, 20] to incorporate the class of isolated individuals explicitly

$$\begin{cases} \dot{S} = -\beta SI, \\ \dot{I} = \beta SI - (\mu + u)I, \\ \dot{D} = uI - \alpha D, \end{cases}$$

where, S, I and D are the numbers of susceptible, infected/infectious and isolated individuals, β is the transmission rate, μ is the removal rate of the infected individuals due to either mortality or recovery, u is the rate of isolation, and α is the recovery or mortality rate of isolated individuals. In the model, we do not differentiate infected and infectious individuals, and we assume isolation is effective that isolated individuals can no longer infect others.

The control is through the time-varying isolation rate u. So, the main difference from the study of [20] is the isolation rate which is determined by the available resources such as hospital beds and ICUs instead of total numbers of isolated individuals in the entire course of an outbreak. So, the condition, $\int_{t_0}^{T} u(t)I(t)dt \leq \omega$ in [20] must be replaced by the state variable constraint

$$D(t) \leq \omega,$$

with ω being the maximum capacity to accommodate the isolated individuals at any give time. Our objective is also to determine an optimal isolation rate that minimizes

$$\int_0^T \beta(t)S(t)I(t)dt,$$

where T is a fixed or free parameter (duration of the control interval). Without loss of generality, we will always assume, in the remaining part of this paper, that the initial time $t_0 = 0$.

Since $\int_0^T \beta SIdt = S(0) - S(T)$, minimizing the objective function is equivalent to minimizing $-S(T)$. Therefore, our optimal control problem can be formulated as follows:

$$\begin{cases} \min \quad J = -S(T), \\ \qquad \dot{S} = -\beta SI, \\ \qquad \dot{I} = \beta SI - (\mu + u)I, \\ \qquad \dot{D} = uI - \alpha D, \\ \qquad S(0) = S_0, I(0) = I_0, D(0) = D_0, u \in [0, u_{max}], \\ \qquad D(t) \leq \omega, \end{cases} \tag{1}$$

where u_{max} is a maximum isolation rate. We note that in practice the control is implemented by either controlling the rate u or controlling the total number of newly treated infected individuals $v(t) = u(t)I(t)$.

3. **Optimal isolation strategies.** Our main results are as follows:

Theorem 3.1. *Depending on the initial conditions, model parameters and constraint conditions, we have the following optimal isolation strategies:*

(i). If the resource for isolation is unlimited or sufficient in the sense that $D(t) \leq \omega$ for all possible $t \in [0, T]$, then the optimal control is with maximal effort, i.e.,

$$u^*(t) = u_{max} \quad \textit{for all} \quad t \in [0, T];$$

(ii). Let $D(t)$ be optimal path in (i) *(without the constrain $D(t) \leq \omega$). If $D_0 = \omega$, and there exists a $c \in (0, T]$ such that $D(t) \leq \omega$ for all $t \in [c, T]$, then the optimal control is given by $v^*(t) = \alpha\omega$ for $0 \leq t < c$ and $u^*(t) = u_{max}$ for $c \leq t \leq T$.*

(iii). Let $D(t)$ be optimal path in (i) *(without the constrain $D(t) \leq \omega$). If $D_0 < \omega$, then there exists a subinterval $[c_1, c_2] \subset [0, T]$ such that $u^*(t) = u_{max}$ for $0 \leq t < c_1$ and $c_2 \leq t \leq T$;and $v^*(t) = \alpha\omega$ for $c_1 \leq t < c_2$.*

4. **Proof of optimal policies.** Firstly, for the sake of convenience, we state the maximum principle to be used in the sequel and we refer to [8, 18, 21] for more details. We consider the following optimal control problem with state inequality conditions:

$$\begin{cases} \min \quad J = \varphi(x(T), T) + \int_{t_0}^{T} L(x(t), u(t), t)dt, \\ \qquad\quad \dot{x} = f(x(t), u(t), t), \\ \qquad\quad x(t_0) = x_0, \ u \in U, \\ \qquad\quad N(x(T), T) = 0, \\ \qquad\quad g(x(t), t) \leq \omega, \end{cases} \tag{2}$$

where, $x \in R^n$ is the state vector, $u \in R^m$ is the control vector, φ and L are scalar functions of corresponding variables, f, N and g are vector functions of their respective variables, ω is a constant vector with appropriate dimension, U is an admissible and time-invariant set, φ, L, f, N and g are at least once continuously differentiable with respect to all of their arguments.

Solving an optimal control problem with state constraints is normally achieved by the so-called "differentiation approach" that converts state constraints into control constraints. Here, we assume that the state inequality constraint is q-th order (for an integer q) and so the original state constraint can be replaced by a control equality constraint $g^{(q)} = 0$ and a set of point constraints

$$g^{(0)}|_{t=c} = g^{(1)}|_{t=c} = \cdots = g^{(q-1)}|_{t=c} = 0,$$

where $g^{(0)} := g(x(t), t) - \omega$. For these constraints, three cases must be considered:

- If the optimal path begins with a boundary subarc, the point conditions must be applied at the initial point, and the prescribed initial conditions must satisfy these constraints;
- If the optimal path ends with a boundary subarc, the point conditions must applied at the final point, and the prescribed final conditions must be consistent with these constraints;
- If the boundary subarc occurs between the endpoints, the point conditions need be applied only at the beginning of the subarc.

To illustrate the approach, we state the technical Lemma for the case where the boundary subarc occurs between the endpoints.

Lemma 4.1. *For optimal control problem (2), we assume that the state constraint is active in a subinterval $[t_1, t_2]$ of $[t_0, T]$. Let $H = L + \lambda^T f, \bar{H} = L + \lambda^T f + \gamma^T g^{(q)}$. If $u^*(t)$ is an optimal control with $x^*(t)$ being the corresponding optimal path, then there exist nontrivial vector functions λ, γ and nontrivial constant vectors μ and ξ*

such that the following conditions are satisfied:

$$
\begin{aligned}
& t_0 \le t \le t_1 : \\
& \qquad \dot{x} = f, \\
& \qquad \dot{\lambda} = -\tfrac{\partial H}{\partial x},\ H(x^*(t), u^*(t), t, \lambda) = \min_{u \in U, t_0 \le t \le t_1} H(x^*(t), u(t), t, \lambda); \\
& t_1 \le t \le t_2 : \\
& \qquad \dot{x} = f,\ \dot{\lambda} = -\tfrac{\partial \bar{H}}{\partial x}, \\
& \qquad u^* \text{ is determined from } g^{(q)}(x, u, t) = 0, \\
& \qquad \gamma \text{ is determined from } \tfrac{\partial \bar{H}}{\partial u} = 0; \\
& t_2 \le t \le T : \\
& \qquad \dot{x} = f, \\
& \qquad \dot{\lambda} = -\tfrac{\partial H}{\partial x},\ H(x^*(t), u^*(t), t, \lambda) = \min_{u \in U, t_2 \le t \le T} H(x^*(t), u(t), t, \lambda); \\
& t_0 : \\
& \qquad x(t_0) = x_0; \\
& t_1 : \\
& \qquad g^{(0)}(x(t_1), t_1) = \cdots = g^{(q-1)}(x(t_1), t_1) = 0,\ H|_{t=t_1-} = H|_{t=t_1+} - \tfrac{\partial \bar{\varphi}}{\partial t_1}, \\
& \qquad \lambda(t_1-) = \lambda(t_1+) + \tfrac{\partial \bar{\varphi}}{\partial x(t_1)}; \\
& t_2 : \\
& \qquad H|_{t=t_1-} = H|_{t=t_1+},\ \lambda(t_1-) = \lambda(t_1+); \\
& T : \\
& \qquad N(x(T), T) = 0,\ \lambda(T) = \tfrac{\partial H}{\partial x(T)}, \\
& \qquad H|_{t=T} = -\tfrac{\partial \bar{\varphi}}{\partial T} \text{ with } \bar{\varphi} = \varphi + \mu^T N + \xi^T \theta, \theta = (g^{(0)}, \cdots, g^{(q-1)})^T.
\end{aligned}
$$

With the above preparation, we can now give the following

Proof. It is not difficult to see that the optimal control problem (1) admits an optimal solution (see [8, 18, 21]). So, we need only find the necessary conditions for the optimal control under the constraints.

This is an optimal control problem with a state inequality constraint. The key is to determine whether there exist any corner points for the problem (1). From the third state equation in (1), we have

$$D(t) = e^{-\alpha t} \left(D_0 + \int_0^t e^{\alpha \tau} u I d\tau \right) \to 0.$$

Therefore, $D(t)$ is bounded and the state inequality constraint in (1) is inactive if ω is large enough. In this case, the control control problem (1) is a problem without any constraint, that is the case *(i)* in the Theorem 3.1. In what follows, we will show that the optimal control is to engage the maximal effort for the control without any constraint.

We, now, suppose that $D(t)$ is the optimal path in *(i)* of the Theorem 3.1. As $D(t) = e^{-\alpha t} \left(D_0 + \int_0^t e^{\alpha \tau} u I d\tau \right)$, the optimal control problem (1) must fall into one of following four cases:

Case I. $D(t) \le \omega$ for all $t \in [0, T]$, problem (1) is equivalent to an optimal control problem without any constraint. This case is illustrated by the numerical simulation shown in the top-left panel of Figure 1, where the parameters in (1) are taken as $\beta = 0.1,\ \mu = 0.3,\ \alpha = 0.2,\ S(0) = 5,\ I(0) = 3,\ D(0) = 1,\ \omega = 3,\ T = 12$, and $u_{max} = 0.7$.

Case II. There exists a $c \in (0, T]$ such that $D(t) \le \omega$ does hold for all $t \in [c, T]$, and $D_0 = \omega$ is satisfied. This case is shown in top-right panel Figure 1, where

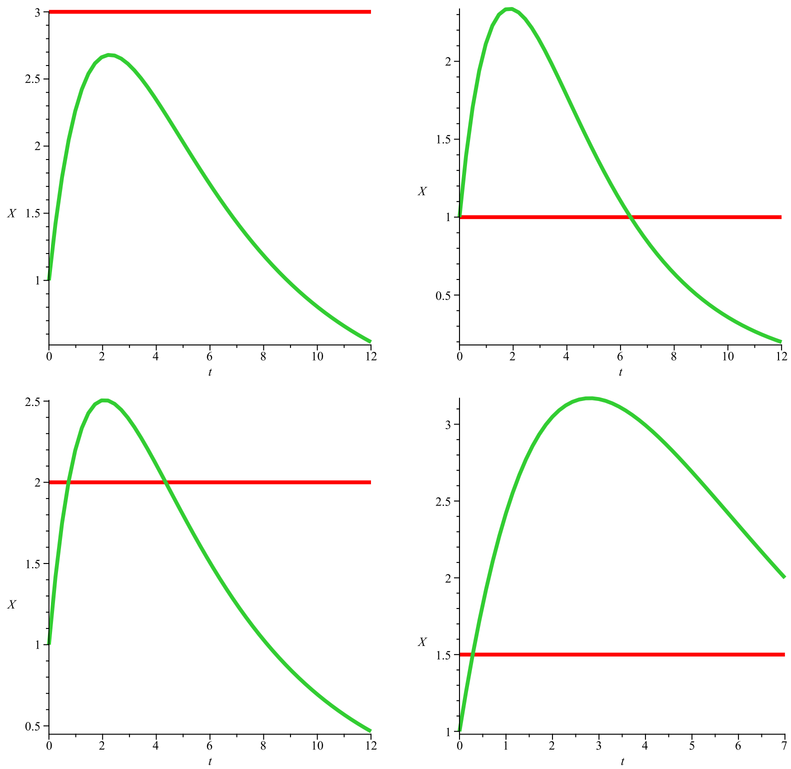

FIGURE 1. Four different cases corresponding to different parameter values

the parameters in (1) are taken as $\beta = 0.1,\ \mu = 0.3,\ \alpha = 0.3,\ S(0) = 5,\ I(0) = 3,\ D(0) = 1,\ \omega = 1,\ T = 12$, and $u_{max} = 0.7$.

Case III. There exist t_1 and t_2 with $0 < t_1 < t_2 < T$ such that $D(t) \leq \omega$ hold only for all $t \in [0, t_1]$ and $t \in [t_2, T]$. The case occurs with the following parameters and is shown in the bottom-left panel of Figure 1: $\beta = 0.1,\ \mu = 0.4,\ \alpha = 0.2,\ S(0) = 5,\ I(0) = 3,\ D(0) = 1,\ \omega = 2,\ T = 12$, and $u_{max} = 0.7$.

Case IV. There exists a $c \in (0, T]$ such that $D(t) \leq \omega$ holds only for all $t \in [0, c]$. This can take place with the following parameter values and is shown in the bottom-right panel of Figure 1.: $\beta = 0.1,\ \mu = 0.1,\ \alpha = 0.2,\ S(0) = 5,\ I(0) = 3,\ D(0) = 1,\ \omega = 1.5,\ T = 7$, and $u_{max} = 0.3$.

To complete the proof for each of the four scenarios, we set $g(x(t)) = D(t) - \omega$ with $x = (S, I, D)^T$ (the superscript T denotes transpose). Then we have the constrain $g(x(t)) \leq 0$. Also, we have $\dot{g} = uI - \alpha D$, i.e, the state inequality constraint in (1) is a 1-order inequality constraint. In what follows, we also denote that

$$\bar{\varphi}(x(T), r) = -S(T) + r(D(c) - \omega), c \in [0, T],$$
Hamiltonian: $H = -\lambda_S \beta SI + \lambda_I \beta SI - \lambda_I(\mu + u)I + \lambda_D uI - \lambda_D \alpha D$,
Expended Hamiltonian: $\bar{H} = H + \gamma(uI - \alpha D)$.

For Case I, the maximum principle implies that

(I1). $H(x^*(t), u^*(t), \lambda) = \min_u H(x^*(t), u, \lambda)$ and so

$$u^*(t) = \begin{cases} u_{max}, & \lambda_I > \lambda_D, \\ \text{to be determined}, & \lambda_I = \lambda_D, \\ 0, & \lambda_I < \lambda_D; \end{cases}$$

(I2). $\dot{S} = -\beta SI,\ \dot{I} = \beta SI - (\mu + u)I,\ \dot{D} = uI - \alpha D;$
(I3). $\dot{\lambda}_S = \beta I(\lambda_S - \lambda_I),\ \dot{\lambda}_I = \beta S(\lambda_S - \lambda_I) + (\mu + u)\lambda_I - u\lambda_D,\ \dot{\lambda}_D = \alpha\lambda_D;$
(I4). $S(0) = S_0,\ I(0) = I_0,\ D(0) = D_0;$
(I5). $\lambda_S(T) = -1,\ \lambda_I(T) = 0,\ \lambda_D(T) = 0.$

Obviously, from (I3) and (I5), we have that $\lambda_D(t) \equiv 0$. We now prove that the optimal control for Case I is with the maximal effort, i.e, $u^*(t) = u_{max}$.

First of all, we note that if there exists a $c \in (0, T]$ such that $u^*(t) = 0$ for all $t \in [c, T]$, or $\lambda_I < 0$ for all $t \in [c, T]$, then the costate equations with regard to λ_I and λ_S become

$$\begin{aligned} &\dot{\lambda}_S = \beta I(\lambda_S - \lambda_I), \\ &\dot{\lambda}_I = \beta S(\lambda_S - \lambda_I) + \mu\lambda_I. \end{aligned}$$

By continuity, $\lambda_S(T) = -1 < 0 = \lambda_I(T)$ implies that there exists an $\varepsilon > 0$, such that $\lambda_I > \lambda_S$ for all $T - \varepsilon < t \le T$, thus $\dot{\lambda}_I < 0$, for all $T - \varepsilon < t \le T$, and so, $\lambda_I(t) > \lambda_I(T) = 0$, for all $T - \varepsilon < t \le T$, which is a contradiction. This implies that there exists a $c \in (0, T]$ such that $\lambda_I \ge 0$ for all $t \in [c, T]$.

Next, we show that $\lambda_I > 0$ for all $t \in [0, T)$. In fact, if this is not so, then there exists a $c \in [0, T)$ such that $\lambda_I(c) = 0$. By the costate equation with regard to λ_I, we have

$$\left(e^{\int_c^t (\mu(\tau)+u(\tau)-\beta(\tau)S(\tau))d\tau}\lambda_I(t)\right)' = e^{\int_c^t (\mu(\tau)+u(\tau)-\beta(\tau)S(\tau))d\tau}\beta(t)S(t)\lambda_S(t).$$

Integrating the above equation over $[c, T]$, we get

$$\begin{aligned} &\int_c^T e^{\int_c^t (\mu(\tau)+u(\tau)-\beta(\tau)S(\tau))d\tau}\beta(t)S(t)\lambda_S(t)dt \\ &= \left[e^{\int_c^t (\mu(\tau)+u(\tau)-\beta(\tau)S(\tau))d\tau}\lambda_I(t)\right]_c^T = 0. \end{aligned}$$

On the other hand, it is impossible that the left-hand side of above equation is 0 owing to S being non-negative and strictly monotonically decreasing, and so we obtain a contradiction. Therefore, the optimal control in Case I must be given by $u^*(t) = u_{max}$.

For Case II, the control is made up of the following two parts (II1) and (II2):

(II1). On $[0, c](\gamma > 0)$, u is determined by $\dot{g} = 0$, and so, $u^* = \frac{\alpha D^*}{I^*} = \frac{\alpha\omega}{I^*}$. Also, γ is determined by $0 = \frac{\partial \bar{H}}{\partial u} = -\lambda_I I + \lambda_D I + \gamma I$, and so, $\gamma = \lambda_I - \lambda_D$. Thus the state equations and the costate equations become

$$\begin{aligned} &\dot{S} = -\beta SI, \\ &\dot{I} = \beta SI - \mu I - \alpha\omega, \\ &\dot{D} = 0, \\ &\dot{\lambda}_S = \beta I(\lambda_S - \lambda_I), \\ &\dot{\lambda}_I = \beta S(\lambda_S - \lambda_I) + \mu\lambda_I, \\ &\dot{\lambda}_D = 2\alpha\lambda_D - \alpha\lambda_I. \end{aligned}$$

We can then get the optimal control u^* by solving the above equations.

(II2). Using a similar argument as utilized in the proof of Case I, we know that the optimal control is $u^* = u_{max}$ on $[c, T]$.

For Case III and Case IV, the discussions are similar. In other words, the constraint boundary must be in the middle of the defined interval, that is, there exist c_1 and c_2 so that $0 < c_1 < c_2 < T$ and that the constrain boundary $[c_1, c_2] \subset [0, T]$.

Again, $D_0 < \omega$ implies that the optimal control $u^* \neq 0$ at the beginning of the disease outbreak. In fact, if there exists a $c > 0$ such that $u(t) = 0$ for all $t \in [0, c]$, then $D(t) = D_0 e^{-\alpha t}$ is deceasing on $[0, c]$(see also below the state equation on $[0, c_1]$), and so $D(t)$ cannot take the number ω again, for otherwise we would obtain a contradiction to the continuity of the state variable D(noted that $D(c_1) = \omega$). Therefore, on $[0, c_1]$, the optimal control u^* is u_{max} or undetermined.

We now complete the detailed analyses on the optimal solutions for Case III and Case IV. By Lemma 1, we have the following results:

A: When $t \in [0, c_1]$, we have

$$
\begin{aligned}
&u^*(t) = \begin{cases} u_{max}, & \lambda_I > \lambda_D, \\ \text{to be determined}, & \lambda_I = \lambda_D, \end{cases} \\
&\dot{S} = -\beta S I,\ \dot{I} = \beta S I - (\mu + u) I,\ \dot{D} = uI - \alpha D, \\
&\dot{\lambda}_S = \beta I(\lambda_S - \lambda_I),\ \dot{\lambda}_I = \beta S(\lambda_S - \lambda_I) + (\mu + u)\lambda_I - u\lambda_D, \\
&\dot{\lambda}_D = \alpha \lambda_D;
\end{aligned}
$$

B: When $t \in [c_1, c_2]$, using the proof for Case II, we can conclude that $u^* = \frac{\alpha\omega}{I^*}$, γ is determined by $\frac{\partial \bar{H}}{\partial u} = 0$ which implies that $\gamma = \lambda_I - \lambda_D$, and

$$
\begin{aligned}
&\dot{S} = -\beta S I, \\
&\dot{I} = \beta S I - \mu I - \alpha D, \\
&\dot{D} = 0,
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\dot{\lambda}_S = \beta I(\lambda_S - \lambda_I), \\
&\dot{\lambda}_I = \beta S(\lambda_S - \lambda_I) + (\mu + u)\lambda_I - u\lambda_D - \delta u, \\
&\dot{\lambda}_D = \alpha \lambda_D - \alpha\gamma;
\end{aligned}
$$

C: When $t \in [c_2, T]$, by Case I, we have : $u^* = u_{max}$, and

$$
\begin{aligned}
&\dot{S} = -\beta S I, \\
&\dot{I} = \beta S I - (\mu + u) I, \\
&\dot{D} = uI - \alpha D,
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\dot{\lambda}_S = \beta I(\lambda_S - \lambda_I), \\
&\dot{\lambda}_I = \beta S(\lambda_S - \lambda_I) + (\mu + u_{max})\lambda_I - u_{max}\lambda_D, \\
&\dot{\lambda}_D = \alpha \lambda_D;
\end{aligned}
$$

D: We have the following boundary conditions:

$$
\begin{aligned}
&t = 0:\ S(0) = S_0,\ I(0) = I_0,\ D(0) = D_0; \\
&t = T:\ \lambda_S(T) = -1,\ \lambda_I(T) = 0,\ \lambda_D(T) = 0; \\
&t = c_1:\ D(c_1) = \omega,\ H|_{t=c_1+} = H|_{t=c_1-} + \left.\frac{\partial \bar{\varphi}}{\partial t}\right|_{t=c_1},
\end{aligned}
$$

$$\begin{aligned}
\lambda_S(c_1+) &= \lambda_S(c_1-) - \frac{\partial\bar{\varphi}}{\partial S}\Big|_{t=c_1} = \lambda_S(c_1-),\\
\lambda_I(c_1+) &= \lambda_I(c_1-) - \frac{\partial\bar{\varphi}}{\partial I}\Big|_{t=c_1} = \lambda_I(c_1-),\\
\lambda_D(c_1+) &= \lambda_D(c_1-) - \frac{\partial\bar{\varphi}}{\partial D}\Big|_{t=c_1} = \lambda_D(c_1-) - r;
\end{aligned}$$

$$\begin{aligned}
t = c_2:\ & D(c_2) = \omega,\ H|_{t=c_2+} = H|_{t=c_2-},\\
& \lambda_S(c_2+) = \lambda_S(c_2-),\ \lambda_I(c_2+) = \lambda_I(c_2-),\ \lambda_D(c_2+) = \lambda_D(c_2-).
\end{aligned}$$

Now, we can determine the optimal control strategies if we can judge whether $\lambda_I = \lambda_D$ on $[0, c_1]$. The analysis is as follows:

If $\lambda_I = \lambda_D$, then the state equations and costate equations become

$$\dot{S} = -\beta SI,\ \dot{I} = \beta SI - (\mu + u)I,\ \dot{D} = uI - \alpha D,$$

and

$$\dot{\lambda}_S = \beta I(\lambda_S - \lambda_I),\ \dot{\lambda}_I = \beta S(\lambda_S - \lambda_I) + \mu\lambda_I,\ \dot{\lambda}_D = \alpha\lambda_D.$$

It follows that

$$\dot{\lambda}_I = \alpha\lambda_I,\ \alpha\lambda_I = \beta S(\lambda_S - \lambda_I) + \mu\lambda_I,$$

thus,

$$(\alpha - \mu + \beta S)\lambda_I = \beta S\lambda_S.$$

Taking derivative on both sides of the above equation and rearranging them, we get

$$\alpha(\alpha - \mu + \beta S)\lambda_I = 0,$$

which implies that either (a). $\lambda_I = 0 \Rightarrow \lambda_D = 0$: again, $(\alpha - \mu + \beta S)\lambda_I = \beta S\lambda_S$, so, $\lambda_S = 0$, it is not possible; or (b). $S = \textit{constant} \Rightarrow 0 = -\beta SI$, it is also impossible.

In summary, the optimal control u^* on $[0, c_1]$ must be with the maximal effort, while the optimal policy, on the whole interval $[0, T]$, is the same as expressed in Theorem 3.1.

This completes the proof. □

5. Conclusion.

We have modified the classical deterministic SIR model to address the issue of of limited resources for diagnostic confirmation/medical isolation. In our model, the D-class (of medically confirmed and isolated individuals) would be the same as the recovered with immunity class in the classical SIR model but the size of the D-class is restricted due to the limited resources, and the optimal control issue is how to control the rate at which infected individuals are isolated within the limited resources to minimize the accumulated number of infectives. The relative simple formulation of the modified SIR model permitted us to obtain analytic solutions for the optimal control problem, and to derive the relevant optimal control policies explicitly in terms of initial conditions, model parameters and resources for isolation (such as the number of intensive care units). How to extend this result to more general model templates which include more disease components and compartments remains an open problem for future studies.

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