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SOCIOLOGICAL PHENOMENA AS MULTIPLE NONLINEARITIES: MTBI'S NEW METAPHOR FOR COMPLEX HUMAN INTERACTIONS

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Abstract. Mathematical models are well-established as metaphors for biological and epidemiological systems. The framework of epidemic modeling has also been applied to sociological phenomena driven by peer pressure, notably in two dozen dynamical systems research projects developed through the Mathematical and Theoretical Biology Institute, and popularized by authors such as Gladwell (2000). This article reviews these studies and their common structures, and identifies a new mathematical metaphor which uses multiple nonlinearities to describe the multiple thresholds governing the persistence of hierarchical phenomena, including the situation termed a "backward bifurcation" in mathematical epidemiology, where established phenomena can persist in circumstances under which the phenomena could not initially emerge.

1. **Introduction.** Humans have long used metaphors to communicate, explain, and understand ideas across all fields of endeavor. Their expressiveness is so efficient that their use has become engrained in everyday language (e.g., "it's raining cats and dogs"). Metaphor's ability to encapsulate essential features of an idea makes it a powerful tool despite its limitations (e.g., the rain metaphor is not meant to extend to the falling of solid, let alone living, bodies). Metaphor is commonly identified as a term in literature, with many notable examples in the works of Shakespeare, but has been used for centuries in humanities more broadly as well as the physical and social sciences (see, e.g., [18] for a discussion of the latter). Biology, for instance, has long relied on metaphors to describe and explain ideas such as evolution; Darwin's "tree of life" metaphor transformed our understanding of evolution by replacing (superseding) all other evolutionary metaphors (like Lamarck's).

The forms metaphors may take also go well beyond that of a single image described in words. Mathematics, a language developed to describe complex relationships concisely through symbols packed densely with meaning, lends itself to this sort of use, and under the label of models, mathematical objects and systems have been fruitfully used to describe phenomena observed in the world around us. Population biology and theoretical epidemiology are examples of related fields in which various types of mathematical models, most notably dynamical systems, have offered seminal insights into the nature and persistence of populations, be they

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Dedicated to my friend and colleague Carlos Castillo-Chavez, who follows the maxim, attributed to Gandhi, "Be the change you wish to see in the world."

cells, infections, or collections of individuals. The quantitative nature of mathematical metaphors invites, for some, the same kind of overextrapolation that any metaphor offers—in this case, a focus on quantitative predictions when the aim of the metaphor is qualitative—but the qualitative descriptions at the heart of the metaphor remain powerful nevertheless. (Caveat: there are, of course, also many mathematical models intended to be quantitatively accurate as well; those should be distinguished from the notion of model as metaphor discussed here.) A mathematical model as a qualitative metaphor deliberately oversimplifies and idealizes the phenomenon it describes, in order to highlight insights about the behavior of the phenomenon's central feature(s). Toward such a qualitative end, these models should be what one author calls "generic and robust" [38, p. 37]; that is, one limits oneself to describing certain essential features of the system, and if one changes the precise mathematical functions and expressions used in the model, to others which still have those same essential features, the resulting qualitative behavior should remain the same as well.

One such qualitative insight that has developed as a central concept in the fields of population biology and theoretical epidemiology mentioned above is that of thresholds: points at which the nature of a system's behavior changes in a wholesale, qualitative way. This notion has become central to an understanding of the persistence of infectious diseases: the threshold quantity denoted R_0, the *basic reproductive number* of an infection, describes an infection's ability to invade a susceptible population, by quantifying the average number of secondary infections produced by a single infectious individual introduced into that naïve population. If this number is less than one, then the infection is reproducing poorly, and the invasion should fail; if it exceeds one, then each infection is on average more than replacing itself, and an epidemic results. Ronald Ross, a British physician studying malaria during the days of Britain's involvement in the Panama Canal, notably discovered the threshold phenomenon and tied it to mosquito population density [31], work which later won him the Nobel Prize in medicine. The notion of R_0 was developed more explicitly by Kermack and McKendrick some decades later [21, 22, 23] and continues to drive the mathematical analysis of epidemiological models to this day. In this case, the threshold at $R_0 = 1$ distinguishes persistence from eradication in simple infection dynamics, but the basic idea is generally applicable to many situations in nature and human interactions (see [20] for one review).

The mathematical phenomenon at the heart of the metaphor is bifurcation, a point where solutions of a dynamical system diverge in their long-term behavior. The bifurcations exhibited by dynamical systems make them nice metaphors for phenomena which undergo such threshold changes; Figure 1(a) illustrates graphically the epidemic bifurcation at $R_0 = 1$ by describing the equilibrium (eventual) prevalence of the infection in terms of the basic reproductive number. When $R_0 < 1$, the equilibrium prevalence is zero; when $R_0 > 1$, it is positive, and grows as R_0 grows. Bifurcations are driven by nonlinearities in the model, terms which reflect a faster-than-linear growth in a particular rate of change as a function of the population(s) undergoing the change (cf. [38, p. 116]). Perhaps the best-known such nonlinearity is the mass-action law in chemistry, which states that the rate of creation of a new compound in a bimolecular elementary reaction is proportional to the product AB of the amounts of the two reactants A and B used to create it. Although this law gives a literal description in its original context, it has been successfully extrapolated in population biology and epidemiology to describe rates at

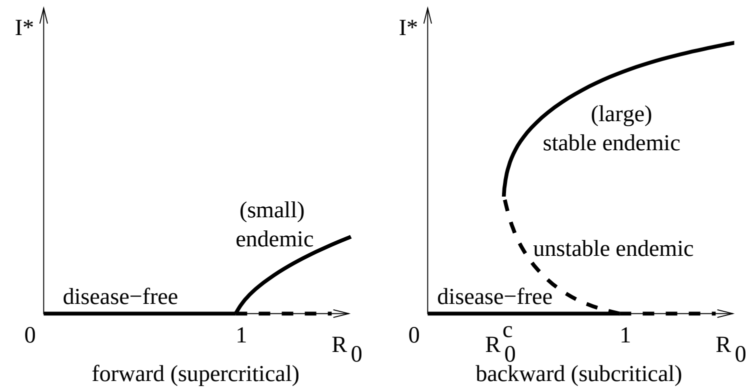

FIGURE 1. Bifurcation diagrams depicting (a) the traditional "forward" bifurcation in an epidemiological model at $R_0 = 1$ and (b) the corresponding "backward" bifurcation. Here I^* is the number of infected individuals at epidemiological equilibrium, and R_0 is a composite function of multiple system parameters.

which two distinct populations (say, susceptibles and infectives, or predators and prey) come into contact. These intergroup contact rates control the eventual nature of the interactions between populations.

The techniques developed for population dynamics, and more particularly for the transmission of infectious diseases, have already been applied to systems outside the purview of the hard sciences. As long ago as 1950, mathematical models were being applied to the dynamics of economics and finance [2, 27, 28, 29] to describe what Weidlich called "analogies between economic and biologic evolution" [38, p. 203]. The epidemiological metaphor (casting the spread of ideas as the spread of infections) also applies to sociological phenomena in which peer pressure or the force of opinion convinces individuals to change their behaviors. In such cases, the pressure exerted on one group by the beliefs, priorities or opinions of others forms a nonlinear contact process that drives the spread of the behavior(s) that follow those beliefs. This peer-pressure contact metaphor can be used to describe a wide variety of phenomena including both directly cooperative processes, good and bad—say, learning environments or gang activity—and processes where contacts occur indirectly, often via the media—say, political beliefs or ideals of health and beauty. At the beginning of this century, Gladwell [13] popularized the single-nonlinearity metaphor of a "tipping point" to explain trends in everything from shoe sales to sexually transmitted diseases and crime rates. In research literature, mathematical models had been used during the last three decades of the twentieth century to study thresholds in collective behaviors in general, as well as applied to specific phenomena such as segregation, dropout rates, teen pregnancy, and political beliefs [8, 16, 17, 33, 37], fads and fashion demand, and urban growth [38, p. 274]. In each case, the metaphor from mathematical epidemiology could be used to identify thresholds governing the persistence of the phenomenon.

It was during this same time that Carlos Castillo-Chavez developed the Mathematical and Theoretical Biology Institute (MTBI), a student research program in which, unlike most such programs, the students are free to choose the research topic, even when it goes well beyond the anticipated range of subjects. Although the vast majority of the over 150 dynamical systems research projects produced since MTBI's inception in 1996 have studied problems in population biology and theoretical ecology and epidemiology, fully two dozen have instead applied the epidemiological metaphor to questions of collective behaviors, many of which have subsequently been published in journals. The problems studied include drug use [35], which has led to more detailed studies of alcohol and tobacco use [26, 32]; eating disorders [15, 12] and depression driven by unhealthy societal ideals [11]; political behaviors and the growth of grassroots movements [30]; cooperative learning environments [9] and (on the other hand) teacher and student burnout [5]; delinquency, gang and criminal activity, and the effects of "three strikes" laws; immigration and immigration policy; the spread of rumors and gossip [4]; and even competing advertisements. In these studies[1], the manifestation of peer pressure's driving force as nonlinear contact rates in systems of differential equations led to behavior thresholds and threshold quantities like those reviewed by Gladwell and Weidlich, in a direct line from Ross and Kermack and McKendrick. However, the relative intensity of this research line over the past fifteen years has also led to a more detailed examination of hierarchical participation in these activities—for instance, those who organize and promote grassroots movements, as opposed to those who simply vote for them, or those who train others in collaborative learning, as opposed to those who simply collaborate. When advancement to the higher levels of participation involves further contact (training, debate, discussion), the resulting dynamical model contains multiple nonlinearities in series, and the subsequent collective behaviors become much more complex than the single-level nonlinearity metaphor can account for. Put another way, the real complexity of human interactions may produce multiple-level nonlinearities, and thus lead to an altogether different qualitative metaphor.

Some relatively simple examples of this multi-level interaction have already been seen in mathematical epidemiology, the best-known of which is the so-called "backward" bifurcation which can arise when individuals change their level of infection risk during an outbreak, say by gaining or losing vaccination protection, or by changing the rate at which they make potentially infectious contacts, even when these changes have nothing to do with awareness of the outbreak or danger, e.g., [19, 25]. Figure 1(b) depicts such a bifurcation graphically, illustrating the resultant possibility of an infection to persist at a high level even when its basic reproductive number R_0 remains below 1. The consequences of this behavior continue to have major repercussions in disease control policy, and the corresponding implications about the robustness of hierarchical collective behaviors are equally groundbreaking.

It is only in reviewing these MTBI studies, however, that the new behaviors emerge, and the new metaphor becomes clear. This is the aim of the present article. The review begins in the following section by examining the effect of spontaneous (linear) relapse by those "recovered" from participation, an extension of the single-level epidemiological metaphor, which visibly augments the phenomenon's basic reproductive number (threshold quantity). The next section describes the common structure of a simplified multi-level metaphor, in which automatic advancement to higher levels of participation creates a less-than-additive effect (in the reproductive number) of joint pressure by intermediate and advanced participants. Finally, we

[1]Research reports for all MTBI projects can be found at the MTBI webpage http://mtbi.asu.edu/research/archive.

examine the effects of the fully developed hierarchical metaphor, in which multiple nonlinear transitions create multiple thresholds, enabling backward bifurcations and even more complex behavior to occur, reflecting more deeply the broad, intricate spectrum of behaviors that result from simultaneous interaction among several groups of individuals, even when restricted to a single phenomenon or topic.

2. Spontaneous relapse: Extending the single-level metaphor.

Several MTBI projects have described the dynamics of a system in which participants "recover" or quit but may then "relapse" or resume participation spontaneously, i.e., without further pressure to do so. In mathematical terms, both recovery and relapse are linear under this hypothesis: they occur at rates that can be described in terms of (actually as the reciprocals of) average "residence" times in each status, completely independently of how many individuals are still participating. This sets up a three-class model (those who have never participated, current participants, and former participants), which has been considered for a wide variety of situations including juvenile delinquency, drug use of various sorts, and even sex workers and their clients. In each case the common structure involves a sort of addiction which prompts relapses independently of the kind of peer pressure that recruited the individuals in the first place.

One of the first studies to present a system with spontaneous (linear) relapse was Castillo-Garsow et al. [7], which examined the system in its simplest form, with only the three basic transitions (recruitment, recovery, relapse) in addition to demographic renewal. The model, of which a flow chart is given in Figure 2(a), was designed to describe drug and tobacco use, and has been used in other subsequent projects such as one studying crack cocaine use. Given a per capita pressure contact rate into the D user class of $\beta D/N$ and per capita departure rate of $\mu+\gamma$, the basic reproductive number without relapse is $\beta/(\mu+\gamma)$. However, the per capita relapse rate δ adds to this number, because the average number of times that an individual visits the user class is no longer 1. Instead, a fraction $(\frac{\gamma}{\mu+\gamma})$ of those who leave the user class go into the recovering class, of whom a fraction $(\frac{\delta}{\mu+\delta})$ then relapse back into the user class. In all, a fraction $\phi = \frac{\gamma}{\mu+\gamma} \cdot \frac{\delta}{\mu+\delta}$ of users relapse. Of those who relapse, a further fraction ϕ—that is, a fraction ϕ^2 of all users—eventually relapse a second time. Likewise a fraction ϕ^3 of all users eventually relapse at least three times, etc. The average number of times a user enters the user class is therefore given by a geometric series, $\sum_{k=1}^{\infty} \phi^k$, making the basic reproductive number

$$R_0 = \frac{\beta}{\mu+\gamma} \sum_{k=1}^{\infty} \left(\frac{\gamma}{\mu+\gamma} \cdot \frac{\delta}{\mu+\delta} \right)^k = \frac{\beta(\mu+\delta)}{\mu(\mu+\gamma+\delta)}.$$

It is, of course, also possible to derive R_0 in a less *ad hoc* way, using one of the two next generation operator (NGO) methods developed for this purpose in an epidemiological context [10, 36]. A word is in order, however, regarding the interpretive first step of any of these methods, in which one must identify which classes are considered "infected" (in behavioral terms, committed to participation or the ideas under study), or which transitions are considered new "infections." Because relapse from the recovering class occurs spontaneously and without any need for "reinfection," this class should correctly be considered an infected class, meaning that relapses do not constitute new infections, merely reactivation of existing infections. This corresponds to the terminology of addiction, in which those who quit are considered recovering and not permanently recovered. This interpretation leads

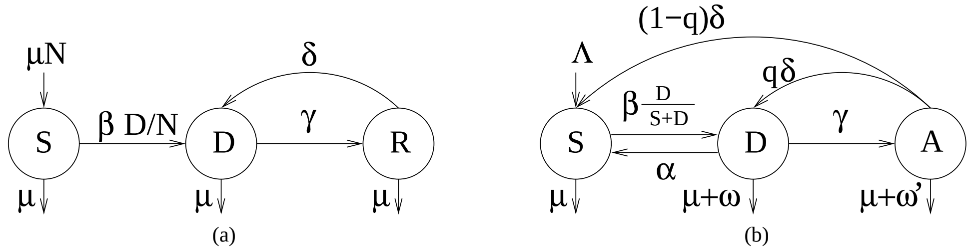

FIGURE 2. Flow charts for models depicting spontaneous (linear) relapse: (a) at left, the model for drug use with linear relapse of [7]; (b) at right, the juvenile delinquency model of [6]. All labels represent per capita rates. Both models compartmentalize the "at-risk" population into "susceptible" non-participants S, participants/users D, and former (recovering R or arrested A) participants.

to the expression for R_0 given above. If one were instead to consider the R class as recovered and uninfected, qualifying relapses as new infections despite their spontaneous nature, a different expression for R_0 would result, namely $\frac{\beta}{\mu+\gamma} + \frac{\delta}{\mu+\delta} \cdot \frac{\gamma}{\mu+\gamma}$ (which, as it differs only in interpretation, serves the same mathematical purpose, since it is always on the same side of 1 as the true R_0).

At the same time, another study [6] used a similar form to study juvenile delinquency as a collective behavior driven by peer pressure. The model developed, illustrated in Figure 2(b), adds two transitions to the basic structure, with the third (removed) class representing individuals who are arrested (and therefore unable to influence others to participate). Some proportion $(1-q)$ of individuals released from arrest are assumed to be rehabilitated, while others (q) resume delinquency; likewise delinquents may choose to quit (reverting to susceptible non-participants) while others are caught and arrested. The change in interpretation of the removed class leads to these additional transitions back to the susceptible class, adding incrementally to the complexity of the phenomenon's basic reproductive number, but not changing the qualitative behavior any. Calculation of the threshold quantity, again considering the removed (arrested) class as still "infected," yields

$$R_0 = \frac{\beta}{\mu+\alpha+\omega+(1-k)\gamma}, \quad \text{where} \quad k = \frac{q\delta}{\mu+\delta+\omega'},$$

which is clearly greater than the value would be without relapse ($q = k = 0$). Note this expression can also be written in the other forms given for the R_0 of the previous model; in particular, the geometric series form uses the probability $\frac{\gamma k}{\mu+\alpha+\omega+\gamma}$ of a delinquent returning to the delinquent class after leaving it.

In mathematical terms, the addition of linear terms for new (i.e., beyond basic recruitment to participate) transitions does not change the qualitative behavior of the system, which remains driven by the sole nonlinearity. In sociological terms, inevitable relapses or resumptions of collective behaviors strengthen the behaviors' ability to persist, but remain part of the single-threshold metaphor. That is, relapse which is inherent to an individual's nature (prompted by means such as addiction or economic need), rather than driven by contact with others' ideas, can't sustain a phenomenon by itself, although it can enable weak peer pressure to do so.

3. **Automatic advancement: Toward the multi-level metaphor.** The heart of the multi-level metaphor is a hierarchical structure which identifies multiple levels of participation in a collective behavior, distinguishing novices from experts, passive agreement from activism, average participants from organizers. In some phenomena, however, studies have viewed the transition from the first to the second stage as automatic, with the first stage representing adoption of a certain belief or idea, and the second representing implementation; here the transition between stages corresponds to a delay in beginning to live out the belief in question (or to see the consequences of doing so). This structure has been applied (in MTBI projects) to phenomena such as eating disorders [15], depression [11], teen smoking [7], and even U.S.-Mexico immigration; in the latter case, the first stage of participation represented residents of Mexico who had developed an interest in moving to the U.S. (because of reports from others), while the second stage represented those who had actually immigrated. A review of these projects illustrates how the resulting automatic-transition structure affects the phenomenon's reproductive number in a different way, but leaves the qualitative nature of the collective behavior the same.

A 2004 study [12] described the development of obesity in the U.S. in terms of an epidemic (1 in 3 Americans is now clinically obese) fueled by the adoption of a lifestyle involving increased caloric intake (dominated by fast food restaurants) and reduced physical activity levels. The underlying hypothesis is that individuals (and families) adopt this lifestyle through interaction with others who have already done so (especially since meals are common settings for social interaction). In this study, the two stages of participation in a fast-food lifestyle lead initially to being overweight (class O_1) and eventually to being obese (class O_2), the latter transition occurring naturally as the effects of the new lifestyle accumulate. Individuals in both stages may quit the lifestyle and return to healthier habits either spontaneously (by noticing the lifestyle's effects on themselves) or by observing the lifestyle's effects (heart attacks and other health problems) on the obese, but in either case it is assumed in the simplest model (three are considered) that there is no relapse, the lesson learned having been sufficiently severe. The fast-food lifestyle is thus described as a two-stage phenomenon with automatic transition to the second stage.

The corresponding mathematical model (illustrated in Figure 3(a)) exhibits only the single threshold driven by the initial nonlinearity: persistence of the lifestyle is driven by the strength of the initial recruitment interaction, measured in the model by the parameter β. However, in contrast to single-stage phenomena, the corresponding reproductive number has a more complex form, reflecting the two-stage nature of the "infection":

$$R_0 = \frac{1}{2}\left(R_1 + \sqrt{R_1^2 + 4r_2^2}\right), \text{ where } R_1 = \frac{\beta}{\mu+\gamma+\alpha_1},\ r_2 = \sqrt{R_1 \frac{\gamma}{\mu+\alpha_2}}.$$

Here R_1 gives the behavior's reproductive number for the influence of individuals in the first stage only, and the term r_2 (written in lowercase since it is not properly speaking a reproductive number) deals with the influence of individuals in the second stage. The latter term resembles the reproductive number for a vector-borne infection, as the geometric mean (square root of the product) of two single-stage reproductive ratios: R_1 and $\frac{\gamma}{\mu+\alpha_2}$, each of which gives the rate into the respective stage over the rate out. One can also, however, interpret r_2^2 instead as the proportion

$\frac{\gamma}{\mu+\gamma+\alpha_1}$ of stage one (overweight) individuals who eventually become obese, multiplied by the reproductive effect $\frac{\beta}{\mu+\alpha_2}$ of stage two (they influence non-participants at a rate of β, for an average duration of $1/(\mu+\alpha_2)$). One also observes that R_0 is independent of the factor α_0 denoting the strength of the influence of seeing the negative consequences of others' obesity, since the context of the basic reproductive number involves the phenomenon at its outset, when only a few (thus first-stage) individuals are promulgating the lifestyle.

This hybrid (part one-stage, part two-stage) structure for R_0 also merits interpretation. First, from a purely algebraic perspective, one can note that $R_1 < R_0 < R_1 + r_2$; that is, the behavior's reproductive capacity is greater than it would be if only stage one individuals contributed to acceptance of the lifestyle, but less than the sum of the two stages' respective influences, because there is overlap between them (both R_1 and r_2 measure abilities to replenish O_1, but recruitment into O_1 also eventually brings individuals into O_2). We may term this effect "almost additive" or less-than-additive. If we view the two single-stage numbers as measures along different dimensions, the square root term resembles a Euclidean distance expression, as though, in the overall measure R_0, half of R_1 were considered orthogonal to r_2, and the other half parallel to the resultant. For this reason we might also term this form "half-orthogonal." This form for a basic reproductive number has been seen in some models for the transmission of vector-borne diseases (e.g., [24]), in which host-internal transmission (such as vertical transmission from mother to child) appears "half-orthogonal" to transmission between host and vector.

Another study [15] described the spread of the eating disorder bulimia nervosa in a college population in similar terms (see Figure 3(b)). Here, peer pressure to conform to unrealistic ideals of health and beauty prompted some individuals to develop bulimia, further supported within this closed population by eating habits which are public since most students in residential universities eat in groups rather than alone. The two stages here distinguish the severity of the behavior: initially (class B_1) individuals are able to keep their bulimic purges secret, but eventually the behavior and effects progress to a severity in which their health is endangered (B_2), prompting them to enter treatment either by having a severe enough episode that family, friends or healthcare providers become aware of it (considered a spontaneous, non-contact event) or by learning of friends or classmates who have entered treatment (considered a contact-driven event). Like the more complex obesity models (not discussed here) in the previous study, the model also allows for relapse: in this case, spontaneous relapse (treatment failure) back into the more severe class.

As with the obesity model already discussed, the model yields simple threshold behavior driven by the disorder's basic reproductive number, despite the nonlinearity in the treatment rate since the linear treatment rate dominates (even if no one else in the student population is presently in treatment, advanced-stage bulimics will eventually have an episode that causes discovery followed by mandatory treatment). However, relapse does complicate the expression: if we again let $R_1 = \frac{\alpha}{\mu+\gamma}$ be the part of the reproductive number describing the influence of stage one individuals and $r_2 = \sqrt{R_1 \frac{\gamma}{\mu+\epsilon\rho}}$ describe the influence of stage two individuals, and consider relapse to decrease the effective treatment rate by a factor $\epsilon = \frac{\mu}{\mu+\phi}$, then the NGO method of Diekmann et al. [10] again yields $R_0 = \frac{1}{2}\left(R_1 + \sqrt{R_1^2 + 4r_2^2}\right)$. The only significant difference from that of the obesity model is that the second

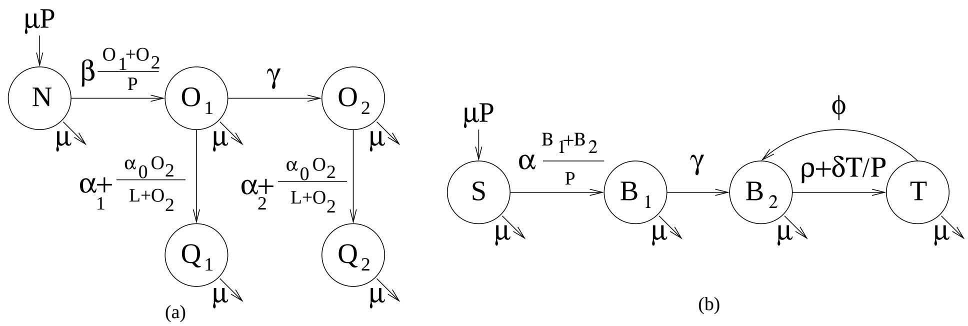

FIGURE 3. Flow charts for models of eating disorders depicting automatic progression to advanced participation: (a) at left, the obesity model of [12]; (b) at right, the bulimia model of [15]. Transition labels represent per capita rates.

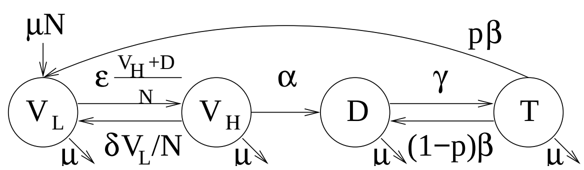

FIGURE 4. Flow chart for the two-stage depression model of [11], with automatic advancement to the second stage. Transition labels represent per capita rates.

stage's influence r_2 is here diluted by relapse. (Note that R_0 is independent of the nonlinear treatment rate δ, which depends on having an established population already in treatment, and that [15] presents a form for R_0 which considers relapses more like reinfections.)

A 2002 study [11] examined depression in young women due to the same ideals which formed the basis for the bulimia study. The resulting model, of which a special case is illustrated in Figure 4, involves four classes: women with low and high vulnerabilities to depression, with depression, and in treatment (which may end with either success or spontaneous relapse). Here the phenomenon is subscription to unhealthy ideals, which initially manifests as increased vulnerability to depression and eventually, under the authors' hypotheses, leads to clinical depression. The two-stage participation with automatic progression again leads to the same behavior as described for the previous two studies, and (including relapse) the same reproductive number as for the bulimia model. Other studies which include this same form and behavior include a 2000 study on teen smoking, education, and lung cancer [1].

One non-example serves to clarify that order matters. Boyd et al. [5] developed a model for a school with teacher and student populations each functioning at three different levels. Teachers (see Figure 5) were classified as unprepared, prepared, and master teachers: Unprepared teachers are those who enter the profession with emergency or temporary credentials and without any background preparation for teaching. Prepared teachers are those who are able to manage classrooms well enough to deliver lessons, because of either a formal background in teacher education

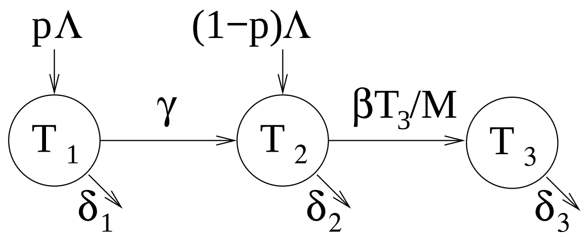

FIGURE 5. Flow chart for the teacher system in the education model of [5], where advancement to the first stage is automatic. Transition labels represent per capita rates.

or sufficient teaching experience; this qualification addresses basic logistical competence but not pedagogical quality. Master teachers are defined by pedagogical excellence in helping students learn and construct their own knowledge. Consequently, new teachers arrive at the school in one of the first two classes (since the study concerns schools with high drop-out rates, it is assumed that the school is only able to recruit teachers new to the profession, many of them on emergency credentials). The study also assumed that, although unprepared teachers have a higher burnout rate, those who persevere eventually master the logistical aspects of the job and become prepared. However, it was also assumed that master teachers served as mentors, to help prepared teachers develop into master teachers themselves (unprepared teachers were assumed to be too busy dealing with logistical and classroom management issues to attend to finer pedagogical questions). These assumptions lead to a mathematical model in which "recruitment" into the body of competent teachers occurs automatically (a linear transition) but advancement to master level requires mentoring (a nonlinear, contact-based transition). Here the order of the two transition types is reversed from that of the other two-stage models reviewed in this section, with the result that there will always be a population of teachers with baseline competence (even if all new arrivals are unprepared), and the only question is whether a master/mentor class will survive; this latter question is answered with a single threshold quantity, measuring the master teachers' average ability to mentor their prepared colleagues into master teachers themselves before retiring.

Therefore partly hierarchical collective behaviors, in which initial recruitment is driven by peer-pressure contacts but progression is automatic, manifests in the phenomenon's basic reproductive number, which takes on an "almost-additive" or "half-orthogonal" form reflecting a component for each stage, but not in the qualitative description of the phenomenon's ability to persist, which can still be described by the single-threshold metaphor. Distinguishing the levels of participation allows one to quantify the contribution of each level to sustaining the collective behavior (and to see that their influence is limited to initial persuasion only) but does not change our understanding of how, or under what conditions, it persists. This conclusion changes when advancement requires further contacts, as the following section explains.

4. **Multi-level cooperation: A new metaphor for complex behavior.** Collective behaviors in which either cooperation or the influence of others' opinions are key to progression to advanced levels of participation have emerged as phenomena

with a more complex structure, in which a genuinely multi-level metaphor with multiple thresholds is necessary to describe the surprising robustness they exhibit. This structure has been observed in a variety of studies covering topics such as peer-pressure-based alcohol, drug and tobacco use for which simpler models described in earlier sections painted a simpler picture, as well as other, more directly collaborative endeavors such as cooperative learning environments and grassroots political movements. One of the simplest consequences of a fully hierarchical model is the so-called backward bifurcation described in the introduction, although it will be shown in this section that backward bifurcations in sociological models can be much more complicated than those seen more typically in epidemiological models. In the review that follows, we consider two separate versions of the metaphor: one in which the two consecutive contact-based transitions are aligned in the same direction, leading individuals to initial and then advanced participation, and another in which the transitions occur in opposite directions, both leading into the initial participation level, in which the latter transition represents a so-called nonlinear relapse brought about by the influence of peer pressure. We begin with the former.

In the summer of 2001, already having given rise to an unexpected diversity of research, MTBI itself became the focus of one project, and the motivation behind a more general model of a cooperative learning environment. The model, depicted in Figure 6(a), retains the three-class structure of earlier models but (in extending the epidemiological metaphor) stratifies participation into novices and mentors, with recruitment into each level dependent on contact with those at the new level or higher. That is, novices "teach" by example, while mentors are experienced enough that they can not only train individuals in collaborative learning (and more efficiently than novices do), but also train novices to become mentors. Although mathematical models had previously been used to describe cooperation (e.g., [14] applied game theory, casting it as a binomial version of the prisoner's dilemma), this was the first extension of the epidemiological metaphor to describe cooperative learning as a hierarchically structured phenomenon.

As one might anticipate, two distinct reproductive numbers quickly emerge from the analysis of such a model, representing the respective abilities of the novice and mentor classes to replace themselves from the class immediately below:

$$R_1 = \frac{\beta_1 q}{\mu}, \quad R_2 = \frac{\beta_2}{\mu+\gamma}\left(1 - \frac{1}{R_1}\right).$$

Here $q \in (0,1)$ describes the effectiveness of novices' "training" ability relative to that of mentors, and the term $\left(1 - \frac{1}{R_1}\right)$ in R_2 represents the proportion of the population in the novice class when $R_1 > 1$, i.e., the fraction of the population available to train as mentors, which multiplies the mentors' basic efficiency $r_2 = \frac{\beta_2}{\mu+\gamma}$ in doing so. These two numbers serve as threshold quantities determining the persistence of each class: if $R_1 > 1$, then a population of novice cooperative learners is guaranteed to survive, and if in addition $R_2 > 1$ then there will always be a population of mentors as well (spontaneous "burnout" relapse from mentors to passive participants has only a minor effect on R_2). To this extent the hierarchical metaphor may resemble two concatenated copies of the single-level metaphor. But there is more.

The multi-level metaphor features a third threshold, beyond the two predictable ones, with its own threshold quantity, a sort of reproductive number describing the resilience of the hierarchical structure—and in particular incorporating mentors' ability to train/recruit new novices (which does not factor into R_1 or R_2):

$$R_3 = 1\Big/\left(\sqrt{\frac{q}{R_1}}+\sqrt{\frac{1-q}{r_2}}\right)^2.$$

If $R_3 > 1$, then a backward bifurcation occurs, but with potentially much greater impact than the simple backward bifurcation depicted in the introduction for epidemic models. The bifurcation at $R_1 = 1$ is always forward, but when the bifurcation at $R_2 = 1$ turns backward (i.e., if $R_3 > 1$), it has the potential to reach back below the prior bifurcation point. That is, if $R_1 < 1$, so that normally not even the novice class could persist, but $R_3 > 1$, then a large enough "seed" community can persist (once introduced) at all three levels including mentors, as illustrated in Figure 6(b). Thus backward bifurcations in general provide a measure of robustness to the phenomena they describe; here that phenomenon is the persistence of the mentor class, but by virtue of mentors' work training individuals to become novices, the robustness extends to sustain the entire collaborative learning community, novices included, in cases where it would never arise on its own.

The lesson here is that dedicated mentors can sustain hierarchically structured communities once they're established. This result contradicts Gladwell's [13] single-level "tipping point" metaphor, which offers a limited view of the complex impact of multi-level social forces on community resilience, by failing to predict this third threshold yielded by two nonlinear transitions in series.

A 2005 study (published as [30]) of the growth of grassroots political movements and third parties used a similar structure to describe the recruitment of partisans (voters) and party organizers. The primary difference in model structure is that "relapse" is considered not from party members/organizers to voters, but from party voters back into the general voting population, with both spontaneous (linear) and contact-based (nonlinear) transitions there (see Figure 7(a) for a sketch of the model). However, the behavior is again driven by the two contact processes connected in series: party voters (V) and members (M) convincing voters not affiliated with the movement S to subscribe to party ideas, and members recruiting more organizers from among those who already share the movement's ideals. Like the cooperative education model, this application of the multi-level contact metaphor involves not two but three thresholds, each with its own associated reproductive number:[2]

$$R_1 = \frac{\beta-\phi}{\mu+\epsilon},\quad R_2 = r_2\left(1-\frac{1}{R_1}\right),\quad R_3 = r_3\left(1-\frac{1+q+2\sqrt{q(1-R_2)}}{r_2}\right),$$

where R_1, $r_2 = \frac{\gamma}{\mu}$, and $r_3 = \frac{\alpha\beta}{\mu}$ measure the basic recruitment efficiencies, respectively, of party voters V on general voters S, party activists M on party voters V, and party activists M on general voters S. The terms by which r_2 and r_3 are multiplied to yield R_2 and R_3 represent the proportion of the population available to be recruited in each case. Also, $q = (\beta-\phi)/\alpha\beta < 1$ describes the influence of party voters' influence on the general voting populace relative to that of party activists. It is worth noting that any and all of q, R_1, R_2, R_3 can be negative, which may appear to defy their definitions as average reproductive numbers (or relative strength in the case of q), but negative values for these quantities may more properly be interpreted as the corresponding recruitment ability being moot due to a

[2]Note the R_3 given here differs slightly from that given in [30]: $R_3 > 1$ here is equivalent to $R_{3b} > 1$ in the article.

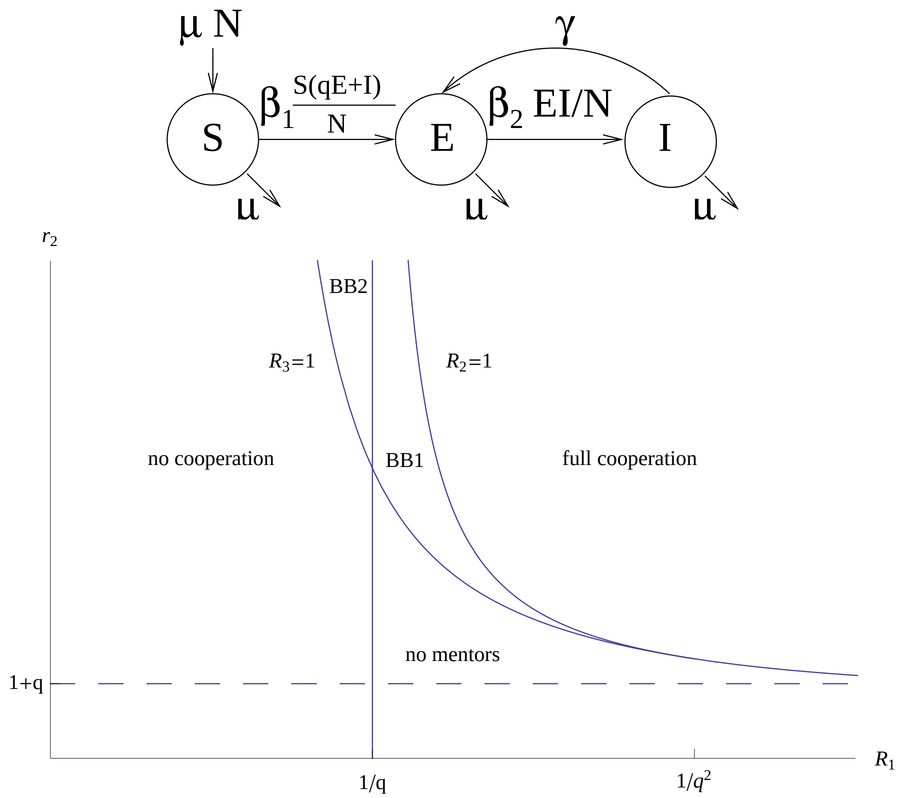

FIGURE 6. (a) Above, a flow chart for the collaborative learning environment model of [9], with two nonlinear transitions in series (all transition rates are per capita); (b) below, a bifurcation diagram in parameter space depicting the behavior of this model. BB1 refers to the situation where a backward bifurcation allows the mentor class to persist when it would not naturally arise, BB2 (where $R_1 < 1$) to the more extreme scenario where the backward bifurcation allows novices and mentors both to persist under conditions where neither would normally arise.

recruitment failure at a lower level (R_2 depends visibly on R_1, and R_3 on r_2 and R_2, even though it only applies when $R_2 \leq 1$).

The three thresholds which determine the movement's (party's) fate are depicted graphically in Figure 7(b), illustrating again the robustness imported to a collective behavior with a fully cooperative hierarchy, through the multi-level metaphor. Party members' active recruitment of "susceptible" individuals into the group of party voters provides the movement a third sustaining force which can be critical to its survival when strictly hierarchical recruitment is not enough. From [30]: "This activism, which sidesteps the traditional hierarchical structure of a party, is a key characteristic of growing grassroots movements, which often lack a political environment favorable to their growth in the traditional way." The underlying mathematical structure of this robustness again involves a backward bifurcation (at $R_2 = 1$) which has the capacity to reach back below even the first threshold

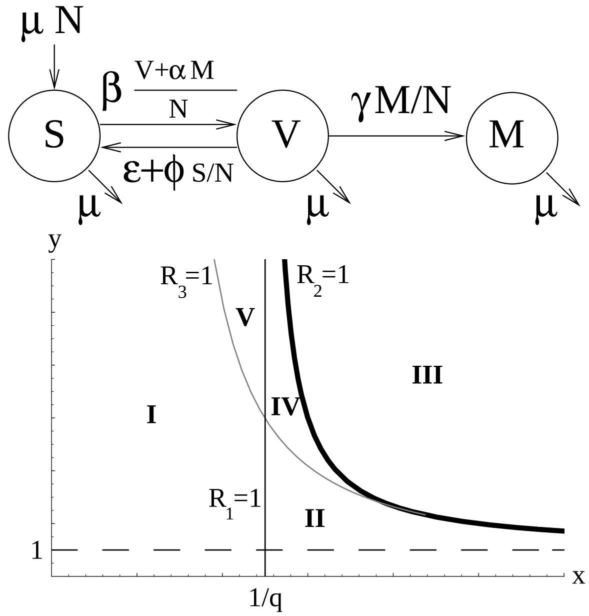

FIGURE 7. (a) Above, flow chart for the grassroots political movement model of [30], with two nonlinear transitions in series; (b) below, a bifurcation diagram in parameter space depicting the behavior of this model (x and y are measures of the strength of recruiting into the V and M classes, respectively)

($R_1 = 1$, where $R_2 = 0$) and allow an existing movement to survive where it would never be able to arise. This can explain the survival of dedicated movements under shifting political winds.

As mentioned in the introduction to this section, there is another way to connect two contact processes in series: in opposing directions, to describe a phenomenon which uses contacts (peer pressure) to recruit from both the uninitiated and former participants, with different levels of influence on the two groups. The resulting heterogeneity in non-participants' susceptibility to recruitment allows the underlying mathematical structure—nonlinear relapse—to exhibit a backward bifurcation, lending robustness to the phenomenon in question. (Since there is only one level of true participation, however, any backward bifurcation has only one threshold below which to reach, unlike the fully hierarchical behaviors described above.) This structure has been used at MTBI to study the effects of peer pressure on relapse in the use of alcohol, tobacco, and other drugs, as well as gang dynamics [3] and crime rates (including the effects of "three strikes" laws as deterrents [34]), beginning with that first study by Castillo-Garsow et al. in 1997 [7].

The study of nonlinear relapse replaces the spontaneous relapse term of earlier models with a contact-based relapse representing the influence of peer pressure in bringing "recovering" or "reformed" ex-participants back into the behavior in question (setting it clearly apart from the notion of relapse due to addiction). Figure 8(a) sketches the dynamics for the corresponding study considered by Castillo-Garsow et al. (cf. the spontaneous relapse model in Figure 2(a)). Since (to the author's knowledge) these studies have not been published formally before, the details of the equilibrium analysis for this basic model are given briefly in the Appendix. The results indicate two thresholds, one connected to each contact process. The basic reproduction number, given by $R_0 = \frac{\beta}{\mu+\gamma}$, measures the influence of participants' peer pressure in recruiting first-time participants, but remains independent of their influence on former participants. Instead, a secondary reproductive number $R_1 = \frac{\beta'}{\mu+\gamma}$ measures that influence, and the phenomenon persists if either $R_0 > 1$ (as expected) or R_1 exceeds a function $g(R_0)$, given in equation (2) in the Appendix, which depends only on R_0 and the relative strength $a = \frac{\mu}{\mu+\gamma}$ of demographic renewal to quitting, and whose value exceeds 1 (that is, $g(R_0) > 1$, so the condition is stricter than $R_1 > 1$). Thus participants who exert a strong influence on former "members" to rejoin may be able to sustain the phenomenon despite having relatively weak influence on those who have never participated, which supports a common view of gang activity, that reformed gang members feel significant pressure from their former peers to return. The underlying mathematical structure again involves a backward bifurcation, which by offering a second path to stability extends a measure of resilience to the phenomenon.

The studies that followed [7] (in 2006, 2007 and 2011) saw the same structure and behavior, with nonlinear relapse creating a second threshold quantity rather than affecting the phenomenon's basic reproductive number as spontaneous relapse does. Although peer pressure-based relapse does not constitute a truly hierarchical behavior, the "hierarchy" (heterogeneity) in non-participants gives the phenomenon a potential robustness that warrants inclusion here in discussing the limitations of the single-threshold metaphor. There was, however, a 2001 project which incorporated both a fully hierarchical (two-level participation) structure and contact-based relapse; its many embedded nonlinearities stretch the application of the metaphor (as well as the ability of standard techniques to analyze model behavior fully), but the complex behavior it describes warrants inclusion as well.

This project, later published as [35], studied the environment of rave parties and nightclubs as a setting for promulgating the use of the drug ecstasy. The model developed in this study divided a population of young people (ages 13 to 25) into four groups: the majority A who do not attend raves (or frequent clubs), those S who do frequent such gatherings but do not (yet) use ecstasy, ecstasy users I, and former ecstasy users V who are still part of the rave scene. Apart from a linear quitting rate (and demographic renewal), every transition is nonlinear, as seen in the flow chart of Figure 9: six in total, including a contact-based quitting rate. As in the cooperative learning and political movement models, the "advanced" class of ecstasy users I participates in inviting/pressuring others to come to raves (along with non-users S), as well as in introducing rave-goers S to ecstasy use; as in the simpler nonlinear relapse models, ecstasy users also influence reformed users V to resume use. At the same time, all non-users influence users to quit, and non-ravegoers influence non-user ravegoers to stop going to raves. Predictably, this highly complex system exhibits numerous and complicated bifurcations.

The first threshold involves the establishment of a core ravegoing population S from the general population A, which occurs if the core recruitment number (representing the strength of the core's influence) $R_c = \frac{\epsilon-\delta_s}{\mu} > 1$. In this case a proportion $1 - \frac{1}{R_c}$ of the population frequents raves, and a second threshold arises, with $R_0 = \frac{\phi}{\mu+\tau+\gamma}\left(1 - \frac{1}{R_c}\right)$ measuring the ability of ecstasy users to recruit new

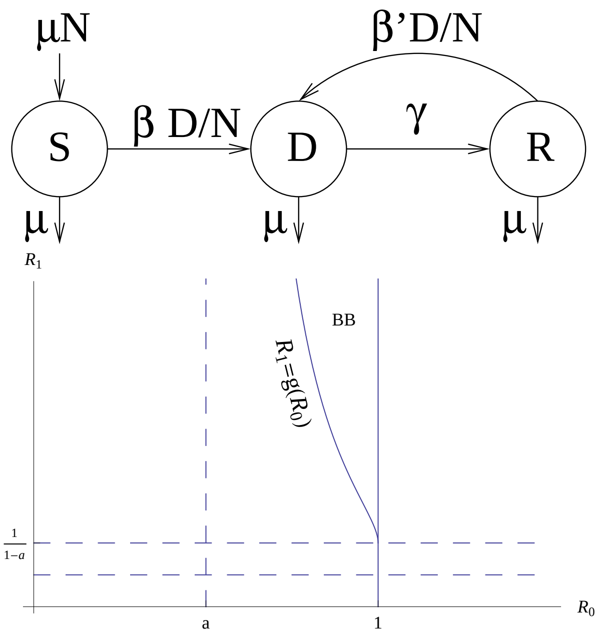

FIGURE 8. (a) Above, flow chart for the nonlinear relapse model, with two nonlinear transitions in inverted series; (b) below, a bifurcation diagram in parameter space depicting the behavior of this model, showing the backward bifurcation region (marked BB) which separates the global endemic state ($R_0 > 1$) from the drug-free scenario ($R_1 < g(R_0)$)

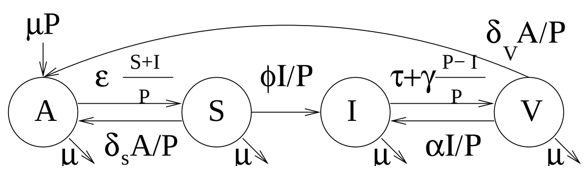

FIGURE 9. Flow chart for the ecstasy/rave model of [35]. Transition rates are per capita.

users from among ravegoers. If $R_0 > 1$, ecstasy use persists in the population. Although the authors of [35] did not identify further threshold quantities analytically, they did verify that for high relapse contact rates α or when the core recruitment rate by users ϵ was high (even when the net recruitment rate by other ravegoers $\epsilon - \delta_s$ was low) backward bifurcations occurred, suggesting the existence of two more threshold quantities, either of which would provide significant resilience to the ravegoing and ecstasy use populations.

These studies show how multiple nonlinear transitions into a collective behavior yield multiple threshold quantities—and a possibility of backward bifurcations, since the number of thresholds follows the number of influences (one class acting on another), which may be greater than the number of transitions. That is, phenomena in which participants at multiple levels (or varying levels of susceptibility) progress only under the influence of the more advanced have survival thresholds at each level—but, in exchange, established groups may be able to sustain themselves even when they could not (or no longer) arise. In short, multi-level interaction changes the metaphor and creates resilient structures.

5. **Conclusions.**

Metaphors emerge and evolve through use. The epidemiological metaphor describing ideas spreading like infection through contacts between susceptibles and infectives began by lending the notion of a tipping point or threshold to studies of the persistence of collective behaviors, but (as seen in the MTBI studies reviewed above) the complex influences of others' opinions have led to a fundamentally new metaphor, in which hierarchically stratified participation with every level cooperating to sustain the phenomenon makes the whole greater than the sum of its parts (or the sum of single-level metaphors). Mathematically speaking, multiple nonlinear transitions make backward bifurcations possible: cooperation at the highest level can overcome even the most basic failure to recruit participants at the lowest level. It is important to understand backward bifurcations in the proper context: they create scenarios in which the initial size of the phenomenon determines its survivability. In this scenario, short-term efforts to build community, or favorable environmental conditions which disappear due to political or resource shifts in the larger culture can be sufficient to build a critical mass before conditions become less favorable. Eradicating the phenomenon then becomes exceptionally difficult, as it requires reducing the number of participants to quite a low level first. Weidlich, observing such multiple attractors in the evolution of biological and economic systems, wrote [38, p. 204], "In a more imaginative metaphor one may speak of a (multidimensional) 'adaptive landscape' on which evolution proceeds, with a multiplicity of basins of attraction." Here that multidimensional metaphor has emerged, but the dimensions are contact processes, which give rise to thresholds, rather than stages of progression through the process in question.

A secondary conclusion involves the nature of relapse. Spontaneous relapses due to addiction or similar internal forces affect a collective behavior in a fundamentally different way than relapses prompted primarily by contact with current participants (like gang membership): in the former case, the phenomenon's basic reproductive number is increased, but the nature of the metaphor remains single-level, whereas in the latter case, the reproductive number is unaffected, and instead an additional threshold emerges. A similar contrast occurs with multi-stage progression, with automatic or spontaneous advancement affecting the form of the reproductive number, but not the number of threshold quantities (the "dimension" of the metaphor).

One of the most remarkable features of this metaphor (as well as of its predecessor) is the wide variety of behaviors which it can describe: both desirable and undesirable, driven by real person-to-person contacts or the less tangible influence of others' public behavior, whether consciously and deliberately organized or arising naturally from trends in opinions. The notion of mathematical abstraction underlies this property, and although mathematical models serve quantitative purposes quite well in some cases, the robustness of the metaphor across many different specific models remains a critical qualitative hallmark of its utility. Finally, however, it is also important to note that the metaphor does not measure the value of the process(es) studied (whether collaborative learning or drug use) or the accuracy of the models proposed—it just observes the resiliency of structures created through multi-level (collaborative) interactions. For a research program developed at an institution founded on collaboration among undergraduate and graduate students, postdocs, and faculty, with several current faculty who joined as undergraduates, that seems somehow appropriate.

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Appendix. Mathematical analysis for nonlinear relapse model. For the system

$$\begin{aligned} D'(t) &= \beta SD/N + \beta' RD/N - (\mu+\gamma)D, \\ R'(t) &= \gamma D - \mu R - \beta' RD/N, \\ S &= N - D - R, \end{aligned} \tag{1}$$

the equilibrium conditions yield the trivial equilibrium $D^* = R^* = 0$ and a quadratic equation for endemic equilibria,

$$f(d) = Ad^2 + Bd + C = -\beta\beta' d^2 + \beta'\left[\beta\left(1 - \frac{\mu+\gamma}{\beta'}\right) - \mu\right] d + \mu[\beta - (\mu+\gamma)] = 0,$$

where $d = D^*/N$. Since $A < 0$, $f(1) < 0$, and $f'(1) < 0$, all roots of f must be less than 1. In general a quadratic equation has 2 positive solutions iff $B^2 > 4AC > 0$ and $B/A < 0$. Since $A < 0$, $AC > 0 \Leftrightarrow C < 0$.

An NGO approach calculates $R_0 = \frac{\beta}{\mu+\gamma}$; if in addition we define $R_1 = \frac{\beta'}{\mu+\gamma}$ and $a = \frac{\mu}{\mu+\gamma} < 1$, then we can simplify the criteria. First, $C < 0 \Leftrightarrow R_0 < 1$, so that when $R_0 > 1$ there is exactly one positive solution. When $R_0 < 1$, the remaining conditions for [two] positive equilibria are $B > 0$, which becomes $\frac{R_0}{a}\left(1 - \frac{1}{R_1}\right) > 1$, and $B^2 - 4AC > 0$, which becomes

$$(R_0 - a)^2 R_1^2 + 2R_0(2R_0 a - R_0 - a)R_1 + R_0^2 > 0.$$

This is true for R_1 outside (i.e., not between) the values

$$R_0 \cdot \frac{R_0 + a - 2aR_0 \pm 2\sqrt{R_0(1-R_0)a(1-a)}}{(R_0 - a)^2}.$$

Since $R_0(R_0 + a - 2aR_0) > (R_0 - a)^2$, the constraint $R_1 > 1$ implied by $B > 0$ above simplifies the quadratic constraint to

$$R_1 > g(R_0) = R_0 \cdot \frac{R_0 + a - 2aR_0 + 2\sqrt{R_0(1-R_0)a(1-a)}}{(R_0 - a)^2}. \tag{2}$$

We can now rewrite the constraint $B > 0$ as $R_0 > a$ and $R_1 > \frac{R_0}{R_0 - a}$, so that the conditions for two endemic equilibria become $a < R_0 < 1$ and $R_1 > \max\left(\frac{R_0}{R_0-a}, g(R_0)\right)$. Some straightforward algebra shows that $\frac{R_0}{R_0-a} < g(R_0)$ for $a < R_0 < 1$ (with the same vertical asymptote at $R_0 = a$ and equality at $R_0 = 1$), so that, finally, we have the result that the nonlinear relapse model has one endemic equilibrium when $R_0 > 1$, two iff $a < R_0 < 1$ and $R_1 > g(R_0)$, and none otherwise. Note that without demographic renewal (a short-term dynamics model), $a \to 0$, the conditions for two endemic equilibria simplify to $R_0 < 1$ and $R_1 > 1$, highlighting the significance of the secondary (relapse) reproductive number R_1 in making it possible for contact-based (nonlinear) relapse to sustain an established phenomenon (in mathematical terms, for high enough initial conditions even when $R_0 < 1$).

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