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GLOBAL STABILITY FOR EPIDEMIC MODEL WITH CONSTANT LATENCY AND INFECTIOUS PERIODS

Gang Huang

School of Mathematics and Physics
China University of Geosciences, Wuhan, 430074, China

Edoardo Beretta

CIMAB, University of Milano
via C. Saldini 50, I 20133 Milano, Italy

Yasuhiro Takeuchi

Graduate School of Science and Technology
Shizuoka University, Hamamatsu, 4328561, Japan

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Abstract. In recent years many delay epidemiological models have been proposed to study at which stage of the epidemics the delays can destabilize the disease free equilibrium, or the endemic equilibrium, giving rise to stability switches. One of these models is the SEIR model with constant latency time and infectious periods [2], for which the authors have proved that the two delays are harmless in inducing stability switches. However, it is left open the problem of the global asymptotic stability of the endemic equilibrium whenever it exists. Even the Lyapunov functions approach, recently proposed by Huang and Takeuchi to study many delay epidemiological models, fails to work on this model. In this paper, an age-infection model is presented for the delay SEIR epidemic model, such that the properties of global asymptotic stability of the equilibria of the age-infection model imply the same properties for the original delay-differential epidemic model. By introducing suitable Lyapunov functions to study the global stability of the disease free equilibrium (when $\mathcal{R}_0 \leq 1$) and of the endemic equilibria (whenever $\mathcal{R}_0 > 1$) of the age-infection model, we can infer the corresponding global properties for the equilibria of the delay SEIR model in [2], thus proving that the endemic equilibrium in [2] is globally asymptotically stable whenever it exists.

Furthermore, we also present a review of the SIR, SEIR epidemic models, with and without delays, appeared in literature, that can be seen as particular cases of the approach presented in the paper.

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1. **Previous results.** In the paper [2], Beretta and Breda study a two-delays SEIR epidemic model as follows:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \mu_1 S(t) - g(I(t))S(t),\\
\frac{dE(t)}{dt} &= g(I(t))S(t) - g(I(t-\tau_1))S(t-\tau_1)e^{-\mu_2\tau_1} - \mu_2 E(t),\\
\frac{dI(t)}{dt} &= g(I(t-\tau_1))S(t-\tau_1)e^{-\mu_2\tau_1}\\
&\quad - g(I(t-\tau_1-\tau_2))S(t-\tau_1-\tau_2)e^{-\mu_2(\tau_1+\tau_2)} - \mu_2 I(t),\\
\frac{dR(t)}{dt} &= g(I(t-\tau_1-\tau_2))S(t-\tau_1-\tau_2)e^{-\mu_2(\tau_1+\tau_2)} - \mu_3 R(t).
\end{aligned} \tag{1.1}$$

In the SEIR model (1.1), they have assumed that the exposed individuals $E(t)$, who are infected but not yet infectious, have a latency time τ_1, after which they become infectious $I(t)$ (i.e. capable to infect susceptibles $S(t)$) and have an infectious period τ_2. Both exposed and infectious individuals are assumed to have the same death rate "μ_2", whereas the susceptibles have a natural death rate constant "μ_1". At the end of the infectious period, i.e. at the infection age $\hat{a} = \tau_1 + \tau_2$, it is assumed that the infected individuals are removed from the infection, thus entering in the class of the removed individuals $R(t)$, which have a proper death rate constant "μ_3". Furthermore, for the nonlinear incidence rate $g(I)$, the structure $g(I) = \frac{\beta I}{1+\alpha I}$, $\alpha, \beta \in \mathbf{R}_+$ was chosen.

The initial conditions for model (1.1), by biological reasons, are the positive continuous functions:

$$S(\theta) = \psi_1(\theta), \quad I(\theta) = \psi_2(\theta), \quad \theta \in [-(\tau_1+\tau_2), 0], \tag{1.2}$$

with $S(0),\ R(0) \geq 0$. Moreover, since $E(t), I(t)$ in (1.1) can be written as:

$$\begin{aligned}
E(t) &= \int_0^{\tau_1} g(I(t-a))S(t-a)e^{-\mu_2 a}da,\\
I(t) &= \int_{\tau_1}^{\tau_1+\tau_2} g(I(t-a))S(t-a)e^{-\mu_2 a}da,
\end{aligned}$$

then, by continuity $E(0),\ I(0)$ must satisfy that :

$$E(0) = \int_0^{\tau_1} g(I(-a))S(-a)e^{-\mu_2 a}da, \quad I(0) = \int_{\tau_1}^{\tau_1+\tau_2} g(I(-a))S(-a)e^{-\mu_2 a}da. \tag{1.3}$$

The delay model (1.1) has two equilibria: the disease-free equilibrium (DFE) $\mathbf{E}_0 = (S_0 = \frac{\Lambda}{\mu_1}, E_0 = 0, I_0 = 0, R_0 = 0)$, which exists for all the parameter values, and the positive equilibrium $\mathbf{E}^* = (S^*, E^*, I^*, R^*)$ which exists if and only if the basic reproduction number

$$\mathcal{R}_0 = \frac{1}{\mu_2}\frac{\Lambda}{\mu_1} \cdot g'(0)e^{-\mu_2\tau_1}(1 - e^{-\mu_2\tau_2}),$$

satisfies $\mathcal{R}_0 > 1$, since it can be proven that its components are:

$$\begin{aligned}
S^* &= \frac{\Lambda}{\mu_1 + g(I^*)}, \qquad E^* = \frac{1}{\mu_2} g(I^*)S^*(1 - e^{-\mu_2\tau_1}),\\
I^* &= \frac{1}{\mu_2} g(I^*)S^* e^{-\mu_2\tau_1}(1 - e^{-\mu_2\tau_1}), \qquad R^* = \frac{1}{\mu_3} g(I^*)S^* e^{-\mu_2(\tau_1+\tau_2)},
\end{aligned}$$

In [2], both by the analysis of the characteristic equations and by the iterative schemes coupled with the comparison principle for differential equations, the following dynamical properties for model (1.1), with i.c. (1.2), were obtained:

Theorem 1.1. *If $\mathcal{R}_0 \leq 1$ the DFE $\mathbf{E}_0$ is globally attractive, i.e. for all initial conditions* $\lim_{t\to\infty}(S(t), E(t), I(t), R(t)) = (\frac{\Lambda}{\mu_1}, 0, 0, 0)$.

Theorem 1.2. *If $\mathcal{R}_0 > 1$, the DFE $\mathbf{E}_0$ is unstable.*

Theorem 1.3. *The positive equilibrium $\mathbf{E}^*$ is globally attractive if "$\frac{\beta}{\alpha} < \mu_1$".*

Theorem 1.4. *The system (1.1) is permanent if $\mathcal{R}_0 > 1$ and "$\frac{\beta}{\alpha} < \mu_1$".*

Theorem 1.5. *Whenever it exists, the positive equilibrium $\mathbf{E}^*$ is locally asymptotically stable.*

As already noticed in [2], the results in Theorems 1.1.-1.5. show that both the delays are harmless in inducing stability switches at both the equilibria $\mathbf{E}_0$ and $\mathbf{E}^*$. However, since the basic reproductive number $\mathcal{R}_0$ depends upon both the latency time τ_1 and on the infectivity period τ_2, even the existence condition $\mathcal{R}_0 > 1$ of the positive equilibrium $\mathbf{E}^*$, jointly with the stability properties of $\mathbf{E}_0$, will be dependent upon both delays. The latency time and the infectivity period however play an opposite role on the existence of the endemic equilibrium $\mathbf{E}^*$ (we say $\mathbf{E}^*$ to be endemic iff we can prove that whenever $\mathcal{R}_0 > 1$ then $\mathbf{E}^*$ is globally asymptotically stable): while the condition $\mathcal{R}_0 > 1$ requires that the latency time τ_1 must be sufficiently small in order that

$$\tau_1 < h(\tau_2) := \frac{1}{\mu_2} \ln\left[\frac{\beta\Lambda}{\mu_1\mu_2}(1 - e^{-\mu_2\tau_2})\right],$$

on the opposite side, the infectivity period must be sufficiently large to ensure that:

$$\tau_2 > \tau_2^* := \frac{1}{\mu_2} \ln\left[\frac{\beta\Lambda}{\beta\Lambda - \mu_1\mu_2}\right].$$

Furthermore, it is evident from the condition $\tau_1 < h(\tau_2)$ that there is a threshold for the latency time, say

$$\tau_1^* := \frac{1}{\mu_2} \ln\left[\frac{\beta\Lambda}{\mu_1\mu_2}\right],$$

such that if $\tau_1 > \tau_1^*$ the condition $\mathcal{R}_0 > 1$ cannot be realized whatever large the infectivity period τ_2 is.

Returning to the asymptotic stability of the positive equilibrium $\mathbf{E}^*$, we see that Theorems 1.3. and 1.5. only imply its global asymptotic stability if "$\frac{\beta}{\alpha} < \mu_1$". Hence, it is interesting to see wether it is possible to prove the global asymptotic stability of the endemic equilibrium $\mathbf{E}^*$ whenever it exists ($\mathcal{R}_0 > 1$). This is one of the main targets of this paper.

Recently, many researchers studied delay epidemic models by using Lyapunov approach and achieve nice results on global stability of equilibria (e.g., [6, 7, 8, 18]). In [7, 8], Huang and Takeuchi employed a class of Goh-type Lyapunov functions that integrate over past states to establish global stability for delay SIR, SEIR, SEI, SIS epidemiological models with a general incidence rate. However, in the models appearing in the above mentioned papers the delay terms appeared with positive signs and this is an essential feature to ensure that the related Lyapunov functionals work well. However, in the model (1.1) two delay terms appear with

negative signs. Correspondingly, the typical Lyapunov functions in [7, 8] do not work for the model (1.1). Hence, the global asymptotic stability of the positive equilibrium $\mathbf{E}^*$ by Lyapunov functionals is left as open question in [2].

In this paper, in Section 2 we reformulate the above delay differential equations model with nonlinear incidence as an equivalent age-infection model with nonlinear boundary conditions, which implies the stability properties of (1.1). In Section 3, by using Lyapunov function approach for age-structured models [9, 14, 15], we establish the global stability of the endemic equilibrium of the delay model (1.1). Furthermore, in Section 4, we prove that our approach by Lyapunov functions applied to infection age-structured models can be applied to a wide class of delay-differential models.

2. **Reformulating the delay model as an age-structured model.** To reformulate the delay-differential model (1.1) we introduce the density $i(t,a)$ at time t of infected individuals with infection age a, where the variable a measures, at time t, the duration for which the individuals have been infected. According to the SEIR model (1.1), we define:

$$E(t) = \int_0^{\tau_1} i(t,a)da, \quad I(t) = \int_{\tau_1}^{\tau_1+\tau_2} i(t,a)da, \tag{2.1}$$

whereas, the balance equation for the removed individuals $R(t)$ is

$$\frac{dR(t)}{dt} = i(t,\tau_1+\tau_2) - \mu_3 R(t). \tag{2.2}$$

Thus, the age-structured model corresponding to the delay-differential model (1.1) is:

$$\begin{cases} \frac{d}{dt}S(t) = \Lambda - \mu_1 S(t) - i(t,0), \\ \frac{\partial}{\partial t}i(t,a) + \frac{\partial}{\partial a}i(t,a) = -\mu_2 i(t,a), \\ +b.c.\ i(t,0) = g(I(t))S(t). \\ +i.c.\ i(0,a) = \psi(a),\ a \in [0,\hat{a}], \end{cases} \tag{2.3}$$

jointly with the equations (2.1) and (2.2).

It is to be noticed that, once known the initial condition (1.2) for system (1.1), the initial condition $\psi(a),\ a \in [0,\hat{a}]$ for the age model (2.3) is also given and, according to (1.2), satisfies the continuity condition (1.3) for $E(0), I(0)$. This initial condition is:

$$\psi(a) := g(\psi_2(-a))\psi_1(-a)\exp(-\mu_2 a), \quad a \in [0,\hat{a}]. \tag{2.4}$$

Now , it is well known (see [5] [13]) that the solution of the Lotka-McKendric equation:

$$\begin{cases} \frac{\partial}{\partial t}i(t,a) + \frac{\partial}{\partial a}i(t,a) = -\mu(a)i(t,a), \\ i(t,0) = B(t) := g(I(t))S(t), \\ i(0,a) = \psi(a),\ a \in [0,\hat{a}], \end{cases} \tag{2.5}$$

where $\psi(a),\ a \in [0,\hat{a}]$, is the initial condition (2.4), exists and it is unique for all $(t,a) \in (0,+\infty) \times [0,\hat{a}]$.

Once introduced the "survival probability $\Pi(a) = \exp(-\int_0^a \mu(\nu)d\nu)$", the solution is given by:

$$i(t,a) = \begin{cases} \psi(a-t)\frac{\Pi(a)}{\Pi(a-t)} & \text{if } a \geq t, \\ B(t-a)\Pi(a) & \text{if } a < t. \end{cases}$$

Since for system (2.3) we have $\Pi(a) = \exp(-\mu_2 a)$, then the solution to system (2.3) with i.c. (2.4) becomes:

$$i(t,a) = \begin{cases} g(\psi_2(-(a-t)))\psi_1(-(a-t))\Pi(a-t)\frac{\Pi(a)}{\Pi(a-t)} & \text{if } a \geq t, \\ g(I(t-a))S(t-a)\Pi(a) & \text{if } a < t. \end{cases} \tag{2.6}$$

Hence, in synthesis, the solution (2.6) can be written as:

$$i(t,a) = g(I(t-a))S(t-a)\exp(-\mu_2 a) \quad \text{for all } t \geq 0 \text{ and } a \in [0, \hat{a}], \tag{2.7}$$

where, according to the initial conditions for the delay system (1.1), it is:

$$S(t-a) = \psi_1(t-a), \quad I(t-a) = \psi_2(t-a), \quad \text{whenever } t-a \in [-\hat{a}, 0].$$

Now, thanks to (2.7), we want to show that the age model (2.3) with (2.1), (2.2) is equivalent to the delay-differential SEIR model (1.1) in the sense that, any of its solution $(S(t), E(t), I(t), R(t)), t \geq 0$, is also solution of system of the delay-differential system (1.1) with initial conditions (1.2).

From (2.1) and (2.2), we have

$$\begin{aligned} \frac{dE(t)}{dt} &= \frac{d}{dt}\int_0^{\tau_1} i(t,a)da = \int_0^{\tau_1} \frac{\partial}{\partial t} i(t,a)da \\ &= -\int_0^{\tau_1} \left(\frac{\partial}{\partial a} i(t,a) + \mu_2 i(t,a)\right) da \\ &= -i(t,a)\,|_0^{\tau_1} - \mu_2 \int_0^{\tau_1} i(t,a)da \\ &= i(t,0) - i(t,\tau_1) - \mu_2 E(t), \end{aligned}$$

that thanks to (2.7) gives

$$\frac{dE(t)}{dt} = g(I(t))S(t) - e^{-\mu_2\tau_1} \cdot g(I(t-\tau_1))S(t-\tau_1) - \mu_2 E(t).$$

Similarly, we have

$$\begin{aligned} \frac{dI(t)}{dt} &= \frac{d}{dt}\int_{\tau_1}^{\tau_1+\tau_2} i(t,a)da = i(t,\tau_1) - i(t,\tau_1+\tau_2) - \mu_2 I(t) \\ &= e^{-\mu_2\tau_1} \cdot g(I(t-\tau_1))S(t-\tau_1) \\ &\quad - e^{-\mu_2(\tau_1+\tau_2)} \cdot g(I(t-\tau_1-\tau_2))S(t-\tau_1-\tau_2) - \mu_2 I(t), \end{aligned}$$

and finally

$$\begin{aligned} \frac{dR(t)}{dt} &= i(t,\tau_1+\tau_2) - \mu_3 R(t) = e^{-\mu_2(\tau_1+\tau_2)} i(t-(\tau_1+\tau_2),0) - \mu_3 R(t) \\ &= e^{-\mu_2(\tau_1+\tau_2)} \cdot g(I(t-(\tau_1+\tau_2)))S(t-(\tau_1+\tau_2)) - \mu_3 R(t). \end{aligned}$$

The above three equations are identical to the last three delay-differential equations in system (1.1). Hence, we have the following:

Proposition 1. *The model (1.1) is implied by the age-structured model (2.3) with (2.1) and (2.2) in the sense that, for all $t \geq 0$, any of its solutions $(S(t), E(t), I(t), R(t))$ is also solution of the delay-differential system (1.1).*

3. **Global stability of the age-structured model (2.3).** The equilibria $\mathcal{E} = (S^*,\ i^*(a)), a \in [0, \hat{a}]$ of the age-structured model (2.3) with (2.1) and (2.2) are solutions of

$$\begin{cases} \Lambda - \mu_1 S - i(0) = 0, \\ \frac{d}{da} i(a) = -\mu_2 i(a), \quad i(0) = g(I)S,\ a \in [0, \hat{a}], \end{cases} \tag{3.1}$$

where, according to (2.1), the other equilibrium components are given by

$$E^* = \int_0^{\tau_1} i^*(a)da,\ I^* = \int_{\tau_1}^{\tau_1+\tau_2} i^*(a)da,\ R^* = \frac{i^*(\tau_1+\tau_2)}{\mu_3}. \tag{3.2}$$

We see that the trivial equilibrium of model (2.3) is the disease-free equilibrium

$$\mathcal{E}_0 = \ (S_0,\ i_0(a)) = \left(\frac{\Lambda}{\mu_1},\ 0\right),\quad a \in [0, \hat{a}],$$

which, according to (3.2), corresponds to the disease-free equilibrium $\mathbf{E}_0 = (\Lambda/\mu_1, 0, 0, 0)$ of system (1.1).

Besides $\mathcal{E}_0$, system (2.3) has the positive equilibrium

$$\mathcal{E}^* = (S^* = \frac{\Lambda}{\mu_1 + g(I^*)},\ i^*(a)),\quad a \in [0, \hat{a}],$$

where

$$i^*(a) = g(I^*)S^* e^{-\mu_2 a},\quad a \in [0, \hat{a}].$$

According to (3.1) and (3.2), S^*, I^* is the unique positive solution of

$$\begin{cases} \Lambda - \mu_1 S^* - g(I^*)S^* = 0, \\ \mu_2 I^* = g(I^*)S^*(e^{-\mu_2\tau_1} - e^{-\mu_2(\tau_1+\tau_2)}), \end{cases} \tag{3.3}$$

which exist if and only if the basic reproduction number $\mathcal{R}_0$ of system (1.1) is: $\mathcal{R}_0 > 1$. Of course, according (3.2) the equilibrium $\mathcal{E}^*$ corresponds to the positive equilibrium $\mathbf{E}^*$ of system (1.1).

For the sake of simplicity, in the following, by referring to the equilibria $\mathcal{E}_0$ and $\mathcal{E}^*$ we will leave out the information $a \in [0, \hat{a}]$.

Proposition 2. *The age-infection model (2.3) has always the disease-free equilibrium $\mathcal{E}_0(\Lambda/\mu_1, i_0(a))$. In addition, there exists a unique positive equilibrium $\mathcal{E}^*(S^*, i^*(a))$ when $\mathcal{R}_0 > 1$.*

In the following, we would study the stability of equilibria of the age-infection model (2.3) by Lyapunov functions.

Theorem 3.1. *When $\mathcal{R}_0 \le 1$, the equilibrium $\mathcal{E}_0 = (\Lambda/\mu_1, 0)$ is globally asymptotically stable.*

Proof. Firstly, we define two non-negative functions:

(i), $\phi(a) = (1 - e^{-\mu_2\tau_2})e^{\mu_2(a-\tau_1)}$ for $a \in [0, \tau_1]$. We have

$$\begin{aligned} \phi(0) &= (1 - e^{-\mu_2\tau_2})e^{-\mu_2\tau_1}, \\ \phi(\tau_1) &= 1 - e^{-\mu_2\tau_2}, \\ \phi'_a(a) &= \frac{d}{da}\phi(a) = \mu_2\phi(a). \end{aligned}$$

(ii), $\varphi(a) = \int_a^{\tau_1+\tau_2} \mu_2 \cdot e^{\mu_2(a-\theta)}\, d\theta$ for $a \in [\tau_1, \tau_1+\tau_2]$. We have

$$\begin{aligned} \varphi(\tau_1) &= 1 - e^{-\mu_2\tau_2}, \\ \varphi(\tau_1+\tau_2) &= 0, \\ \varphi'_a(a) &= \frac{d}{da}\varphi(a) = \mu_2\varphi(a) - \mu_2. \end{aligned}$$

Note that $\phi(\tau_1) = \varphi(\tau_1)$.

Define a Lyapunov function

$$\begin{aligned} V_1 = \phi(0) &\left(S(t) - S_0 - S_0 \ln \frac{S(t)}{S_0} \right) \\ &+ \int_0^{\tau_1} \phi(a) i(t,a) da + \int_{\tau_1}^{\tau_1+\tau_2} \varphi(a) i(t,a) da, \end{aligned} \tag{3.4}$$

which is positive definite with respect to the disease free equilibrium. Taking the time derivative of (3.4), we have

$$\begin{aligned} \frac{dV_1}{dt} &= \phi(0)\left(1 - \frac{S_0}{S(t)}\right) \cdot \frac{dS(t)}{dt} + \int_0^{\tau_1} \phi(a) \cdot \frac{\partial}{\partial t} i(t,a) da + \int_{\tau_1}^{\tau_1+\tau_2} \varphi(a) \cdot \frac{\partial}{\partial t} i(t,a) da \\ &= \phi(0)\left(1 - \frac{S_0}{S(t)}\right) [\mu_1(S_0 - S(t)) - g(I(t)S(t))] \\ &\quad - \int_0^{\tau_1} \phi(a) \left(\frac{\partial i(t,a)}{\partial a} + \mu_2 i(t,a) \right) da \\ &\quad - \int_{\tau_1}^{\tau_1+\tau_2} \varphi(a) \left(\frac{\partial i(t,a)}{\partial a} + \mu_2 i(t,a) \right) da \\ &= \phi(0)\left(1 - \frac{S_0}{S(t)}\right) \mu_1(S_0 - S(t)) - \phi(0) \cdot (S(t) - S_0) g(I(t)) \\ &\quad - \int_0^{\tau_1} \phi(a) \frac{\partial}{\partial a} i(t,a) da - \int_0^{\tau_1} \phi(a) \mu_2 i(t,a) da \\ &\quad - \int_{\tau_1}^{\tau_1+\tau_2} \phi(a) \frac{\partial}{\partial a} i(t,a) da - \int_{\tau_1}^{\tau_1+\tau_2} \varphi(a) \mu_2 i(t,a) da \end{aligned}$$

Then, using integration by parts,

$$\begin{aligned} \frac{dV_1}{dt} &= \phi(0)\left(1 - \frac{S_0}{S(t)}\right) \mu_1(S_0 - S(t)) - \phi(0) \cdot (S(t) - S_0) g(I(t)) \\ &\quad - [\phi(a) i(t,a)] \Big|_{a=0}^{a=\tau_1} + \int_0^{\tau_1} [\phi'(a) - \mu_2 \phi(a)]\ i(t,a) da \\ &\quad - [\varphi(a) i(t,a)] \Big|_{a=\tau_1}^{a=\tau_1+\tau_2} + \int_{\tau_1}^{\tau_1+\tau_2} [\varphi'(a) - \mu_2 \varphi(a)]\ i(t,a) da \end{aligned}$$

Since

$$i(t,0) = S(t) g(I(t)), \tag{3.5}$$

$$\phi(\tau_1) = \varphi(\tau_1), \tag{3.6}$$

$$\phi'(a) - \mu_2 \phi(a) = 0, \tag{3.7}$$

$$\varphi'(a) - \mu_2 \varphi(a) = -\mu_2, \tag{3.8}$$

taking the above into account and that $\varphi(\tau_1+\tau_2)=0$, we obtain

$$\begin{aligned}\frac{dV_1}{dt} &= \phi(0)\left(1-\frac{S_0}{S(t)}\right)\mu_1(S_0-S(t)) - \phi(0)\cdot(S(t)-S_0)g(I(t)) \\ &\quad + \phi(0)g(I(t))S(t) - \mu_2 I(t) \\ &= -\frac{\mu_1}{S(t)}\phi(0)(S(t)-S_0)^2 + \phi(0)S_0 g(I(t)) - \mu_2 I(t) \\ &= -\frac{\mu_1}{S(t)}\phi(0)(S(t)-S_0)^2 + \mu_2 I(t)\left(\frac{\phi(0)S_0}{\mu_2}\frac{g(I(t))}{I(t)} - 1\right) \\ &\le -\frac{\mu_1}{S(t)}\phi(0)(S(t)-S_0)^2 + \mu_2 I(t)\left(\frac{\phi(0)S_0}{\mu_2}g'(0) - 1\right) \\ &= -\frac{\mu_1}{S(t)}\phi(0)(S(t)-S_0)^2 + \mu_2 I(t)(\mathcal{R}_0-1).\end{aligned}$$

where $\mathcal{R}_0 = \frac{S_0}{\mu_2}g'(0)\phi(0)$.

Therefore, $\mathcal{R}_0 \le 1$ ensures that $\frac{dV_1}{dt} \le 0$ holds. Every solution of (2.3) tends to $\mathcal{M}$, where $\mathcal{M}$ is the largest invariant subset in $\frac{dV_1}{dt}=0$. Note that when $\mathcal{R}_0<1$, the equality holds only if $S(t)=S_0$ and $I(t)=0$. From (2.1), we have $i(t,a)=0$ for $t>a$ in $\mathcal{M}$. Hence we have that $\mathcal{M}=\{\mathcal{E}_0\}$.

When $\mathcal{R}_0=1$, $\frac{dV_1}{dt}=0$ only if $S(t)=S_0$. We show that $\mathcal{M}$ also consists only the equilibrium $\mathcal{E}_0$. Let $(S(t),i(t,a))$ be the solution with initial data in $\mathcal{M}$. From the invariance of $\mathcal{M}$, $S(t)=S_0$ for any t. By the first equation of (2.3), it follows that $g(I(t))=0$ for any t, which implies $I(t)=0$ for any t. Similar to the case $\mathcal{R}_0<1$, we have that $\mathcal{M}=\{\mathcal{E}_0\}$. Thus, by Laypunov-LaSalle asymptotic stability theorem for semiflows, the disease free equilibrium $\mathcal{E}_0=(S_0,0)$ is globally asymptotically stable whenever $\mathcal{R}_0\le 1$. □

Theorem 3.2. *Consider (2.3) when $\mathcal{R}_0>1$. The endemic equilibrium $\mathcal{E}^*=(S^*, i^*(a))$, $a\in[0,\hat{a}]$ exists, and if the function $g(I)$ satisfies that*

$$\begin{cases} \frac{I}{I^*} \le \frac{g(I)}{g(I^*)} \le 1 & for \quad 0<I\le I^*, \\ 1 \le \frac{g(I)}{g(I^*)} \le \frac{I}{I^*} & for \quad I>I^*, \end{cases} \tag{3.9}$$

then the endemic equilibrium is globally asymptotically stable.

Proof. Define a Lyapunov function:

$$\begin{aligned}U(t) &= \phi(0)\left(S(t)-S^*-S^*\ln\frac{S(t)}{S^*}\right) \\ &\quad + \int_0^{\tau_1}\phi(a)i^*(a)\left(\frac{i(t,a)}{i^*(a)} - 1 - \ln\frac{i(t,a)}{i^*(a)}\right)da \\ &\quad + \int_{\tau_1}^{\tau_1+\tau_2}\varphi(a)i^*(a)\left(\frac{i(t,a)}{i^*(a)} - 1 - \ln\frac{i(t,a)}{i^*(a)}\right)da.\end{aligned} \tag{3.10}$$

Here $\phi(.)$ and $\varphi(.)$ are defined as same as (i) and (ii) in proof of Theorem 3.1. Let

$$\begin{aligned}
U_1 &= \phi(0)\left(S(t) - S^* - S^* \ln \frac{S(t)}{S^*}\right),\\
U_2 &= \int_0^{\tau_1} \phi(a) i^*(a) \left(\frac{i(t,a)}{i^*(a)} - 1 - \ln \frac{i(t,a)}{i^*(a)}\right) da,\\
U_3 &= \int_{\tau_1}^{\tau_1+\tau_2} \varphi(a) i^*(a) \left(\frac{i(t,a)}{i^*(a)} - 1 - \ln \frac{i(t,a)}{i^*(a)}\right) da.
\end{aligned}$$

Here the function $U_1 \geq 0$ and has a minimal value at point S^*, and $U_2, U_3 \geq 0$ has a minimal value at $i^*(a)$. Hence, $U(t) = U_1 + U_2 + U_3$ is positive definite and takes its minimal value zero at the equilibrium point $\mathcal{E}^* = (S^*, i^*(a))$.

Calculating the time derivative of $U_i(t)(i = 1, 2, 3)$ along (2.3), we have

$$\begin{aligned}
\frac{dU_1}{dt} &= \phi(0)\left(1 - \frac{S^*}{S(t)}\right)(\mu_1 S^* - \mu_1 S(t))\\
&\quad + \phi(0)\left(1 - \frac{S^*}{S(t)}\right)(g(I^*)S^* - g(I(t))S(t))\\
&= \phi(0)\left(1 - \frac{S^*}{S(t)}\right)\mu_1(S^* - S(t))\\
&\quad + \phi(0)g(I^*)S^* - \phi(0)g(I(t))S(t)\\
&\quad - \frac{S^*}{S(t)}\phi(0)g(I^*)S^* + \phi(0)g(I(t)S^*,
\end{aligned}$$

and

$$\begin{aligned}
\frac{dU_2}{dt} &= \int_0^{\tau_1} \phi(a) i^*(a) \frac{\partial}{\partial t}\left(\frac{i(t,a)}{i^*(a)} - 1 - \ln \frac{i(t,a)}{i^*(a)}\right) da\\
&= \int_0^{\tau_1} \phi(a)\left(1 - \frac{i^*(a)}{i(t,a)}\right)\frac{\partial i(t,a)}{\partial t} da\\
&= \int_0^{\tau_1} \phi(a)\left(1 - \frac{i^*(a)}{i(t,a)}\right)\left(-\frac{\partial i(t,a)}{\partial a} - \mu_2 i(t,a)\right) da\\
&= -\int_0^{\tau_1} \phi(a)\left(1 - \frac{i^*(a)}{i(t,a)}\right)\frac{\partial i(t,a)}{\partial a} da - \int_0^{\tau_1} \mu_2 \phi(a) i(t,a)\left(1 - \frac{i^*(a)}{i(t,a)}\right) da.
\end{aligned}$$

Note that

$$\frac{\partial}{\partial a}\left(\frac{i(t,a)}{i^*(a)} - 1 - \ln \frac{i(t,a)}{i^*(a)}\right) = \left(1 - \frac{i^*(a)}{i(t,a)}\right) \times \left(\frac{i_a(t,a)}{i^*(a)} - \frac{i(t,a) \cdot i_a^*(a)}{[i^*(a)]^2}\right),$$

here $i_a(a,t) = \frac{\partial}{\partial a} i(t,a)$ and $i_a^*(a) = \frac{d}{da} i^*(a)$.

From the second equation of (2.3), we know

$$i_a^*(a) = -\mu_2 i^*(a).$$

hence,

$$\begin{aligned}\frac{dU_2}{dt} =& -\int_0^{\tau_1}\phi(a)i^*(a)d\left(\frac{i(t,a)}{i^*(a)}-1-\ln\frac{i(t,a)}{i^*(a)}\right)\\ =& -\phi(a)i^*(a)\left(\frac{i(t,a)}{i^*(a)}-1-\ln\frac{i(t,a)}{i^*(a)}\right)|_{a=0}^{a=\tau_1}\\ &+\int_0^{\tau_1}\left(\frac{i(t,a)}{i^*(a)}-1-\ln\frac{i(t,a)}{i^*(a)}\right)d(\phi(a)i^*(a))\end{aligned}$$

Further, from $\phi_a'(a)=\mu_2\phi(a)$ and $i_a^*(a)=-\mu_2 i^*(a)$, we have

$$d(\phi(a)i^*(a))=[\phi_a'(a)i^*(a)+\phi(a)i_a^*(a)]da=[\mu_2-\mu_2]\phi(a)i^*(a)da=0.$$

Thus,

$$\begin{aligned}\frac{dU_2}{dt} =& \phi(0)i(t,0)-\phi(0)i^*(0)-\phi(0)i^*(0)\ln\frac{i(t,0)}{i^*(0)}\\ &-\phi(\tau_1)i(t,\tau_1)+\phi(\tau_1)i^*(\tau_1)-\phi(\tau_1)i^*(\tau_1)\ln\frac{i(t,\tau_1)}{i^*(\tau_1)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\frac{dU_3}{dt} =& -\int_{\tau_1}^{\tau_1+\tau_2}\varphi(a)i^*(a)d\left(\frac{i(t,a)}{i^*(a)}-1-\ln\frac{i(t,a)}{i^*(a)}\right)\\ =& -\varphi(a)i^*(a)\left(\frac{i(t,a)}{i^*(a)}-1-\ln\frac{i(t,a)}{i^*(a)}\right)|_{a=\tau_1}^{a=\tau_1+\tau_2}\\ &+\int_{\tau_1}^{\tau_1+\tau_2}\left(\frac{i(t,a)}{i^*(a)}-1-\ln\frac{i(t,a)}{i^*(a)}\right)d\varphi(a)i^*(a)\end{aligned}$$

By using

$$d(\varphi(a)i^*(a))=\varphi(a)di^*(a)+i^*(a)d\varphi(a)=-\mu_2 i^*(a)da,$$

we have

$$\begin{aligned}\frac{dU_3}{dt} =& \varphi(\tau_1)i(t,\tau_1)-\varphi(\tau_1)i^*(\tau_1)-\varphi(\tau_1)i^*(\tau_1)\ln\frac{i(t,\tau_1)}{i^*(\tau_1)}\\ &-\varphi(\tau_1+\tau_2)i^*(\tau_1+\tau_2)\left(\frac{i(t,\tau_1+\tau_2)}{i^*(\tau_1+\tau_2)}-1-\ln\frac{i(t,\tau_1+\tau_2)}{i^*(\tau_1+\tau_2)}\right)\\ &-\int_{\tau_1}^{\tau_1+\tau_2}\mu_2 i^*(a)\left(\frac{i(t,a)}{i^*(a)}-1-\ln\frac{i(t,a)}{i^*(a)}\right)da.\end{aligned}$$

Since $\phi(\tau_1) = \varphi(\tau_1)$ and $\varphi(\tau_1 + \tau_2) = 0$, combining $U_2'(t)$ with $U_3'(t)$, we obtain

$$\begin{aligned}
&\frac{dU_2(t)}{dt} + \frac{dU_3(t)}{dt} \\
&= \phi(0)i(t,0) - \phi(0)i^*(0) - \phi(0)i^*(0)\ln\frac{i(t,0)}{i^*(0)} \\
&\quad - \int_{\tau_1}^{\tau_1+\tau_2} \mu_2 i^*(a)\left(\frac{i(t,a)}{i^*(a)} - 1 - \ln\frac{i(t,a)}{i^*(a)}\right)da \\
&= \phi(0)g(I)S - \phi(0)g(I^*)S^* - \phi(0)g(I^*)S^*\ln\frac{g(I)S}{g(I^*)S^*} \\
&\quad - \int_{\tau_1}^{\tau_1+\tau_2} \mu_2\left(i(t,a) - i^*(a) - i^*(a)\ln\frac{i(t,a)}{i^*(a)}\right)da \\
&= \phi(0)g(I)S - \phi(0)g(I^*)S^* - \phi(0)g(I^*)S^*\ln\frac{g(I)S}{g(I^*)S^*} \\
&\quad - \mu_2 I + \mu_2 I^* + \int_{\tau_1}^{\tau_1+\tau_2} \mu_2 i^*(a)\ln\frac{i(t,a)}{i^*(a)}da,
\end{aligned}$$

and

$$\begin{aligned}
\frac{dU}{dt} &= \frac{dU_1}{dt} + \frac{dU_2}{dt} + \frac{dU_3}{dt} \\
&= \phi(0)\left(1 - \frac{S^*}{S(t)}\right)\mu_1(S^* - S(t)) \\
&\quad + \phi(0)g(I^*)S^*\left(-\frac{S^*}{S} + \frac{g(I)}{g(I^*)} + \ln\frac{g(I^*)S^*}{g(I)S}\right) \\
&\quad - \mu_2 I + \phi(0)g(I^*)S^* + \int_{\tau_1}^{\tau_1+\tau_2} \mu_2 i^*(a)\ln\frac{i(t,a)}{i^*(a)}da \\
&= -\frac{\mu_1}{S(t)}\phi(0)\left(S(t) - S^*\right)^2 \\
&\quad + \phi(0)g(I^*)S^*\left(1 - \frac{S^*}{S} + \ln\frac{S^*}{S}\right) \\
&\quad + \mu_2 I^*\left(\frac{g(I)}{g(I^*)} - \frac{I}{I^*} + \ln\frac{g(I^*)}{g(I)}\right) + \int_{\tau_1}^{\tau_1+\tau_2} \mu_2 i^*(a)\ln\frac{i(t,a)}{i^*(a)}da,
\end{aligned}$$

here using

$$\phi(0)g(I^*)S^* = e^{-\mu_2\tau_1}(1 - e^{-\mu_2\tau_2})g(I^*)S^* = \mu_2 I^*,$$

and

$$\ln\frac{g(I^*)S^*}{g(I)S} = \ln\frac{S^*}{S} + \ln\frac{g(I^*)}{g(I)}.$$

Since

$$\mu_2 I^* = \mu_2\int_{\tau_1}^{\tau_1+\tau_2} i^*(a)da,$$

we have

$$\begin{aligned}\frac{dU}{dt} = & -\frac{\mu_1}{S(t)}\phi(0)\left(S(t)-S^*\right)^2 \\ & +\phi(0)g(I^*)S^*\left(1-\frac{S^*}{S}+\ln\frac{S^*}{S}\right) \\ & +\int_{\tau_1}^{\tau_1+\tau_2}\mu_2 i^*(a)\left(\frac{g(I)}{g(I^*)}-\frac{I}{I^*}+\ln\frac{g(I^*)i(t,a)}{g(I)i^*(a)}\right)da \\ = & -\frac{\mu_1}{S(t)}\phi(0)\left(S(t)-S^*\right)^2 \\ & +\phi(0)g(I^*)S^*\left(1-\frac{S^*}{S}+\ln\frac{S^*}{S}\right) \\ & +\int_{\tau_1}^{\tau_1+\tau_2}\mu_2 i^*(a)\left(1-\frac{g(I^*)i(t,a)}{g(I)i^*(a)}+\ln\frac{g(I^*)i(t,a)}{g(I)i^*(a)}\right)da \\ & +\int_{\tau_1}^{\tau_1+\tau_2}\mu_2 i^*(a)\left(\frac{g(I)}{g(I^*)}-\frac{I}{I^*}-1+\frac{g(I^*)i(t,a)}{g(I)i^*(a)}\right)da\end{aligned}$$

Here

$$\begin{aligned}& \int_{\tau_1}^{\tau_1+\tau_2}\mu_2 i^*(a)\left(\frac{g(I)}{g(I^*)}-\frac{I}{I^*}-1+\frac{g(I^*)i(t,a)}{g(I)i^*(a)}\right)da \\ = & \ \mu_2 I^*\left(\frac{g(I)}{g(I^*)}-\frac{I}{I^*}-1+\frac{g(I^*)I}{g(I)I^*}\right) \\ = & \ \mu_2 I^*\left(\frac{I}{I^*}-\frac{g(I)}{g(I^*)}\right)\left(\frac{g(I^*)}{g(I)}-1\right)\end{aligned}$$

Hence,

$$\frac{dU(t)}{dt} = -\frac{\mu_1}{S(t)}\phi(0)\left(S(t)-S^*\right)^2 \tag{3.11}$$

$$+\phi(0)g(I^*)S^*\left(1-\frac{S^*}{S}+\ln\frac{S^*}{S}\right) \tag{3.12}$$

$$+\int_{\tau_1}^{\tau_1+\tau_2}\mu_2 i^*(a)\left(1-\frac{g(I^*)i(t,a)}{g(I)i^*(a)}+\ln\frac{g(I^*)i(t,a)}{g(I)i^*(a)}\right)da \tag{3.13}$$

$$+\mu_2 I^*\left(\frac{I}{I^*}-\frac{g(I)}{g(I^*)}\right)\left(\frac{g(I^*)}{g(I)}-1\right) \tag{3.14}$$

Obviously, the following inequalities always hold for positive $i(t,a)$

$$1-\frac{S^*}{S}+\ln\frac{S^*}{S}\le 0, \tag{3.15}$$

$$1-\frac{g(I^*)i(t,a)}{g(I)i^*(a)}+\ln\frac{g(I^*)i(t,a)}{g(I)i^*(a)}\le 0. \tag{3.16}$$

From the conditions (3.9), we have

$$\left(\frac{I}{I^*}-\frac{g(I)}{g(I^*)}\right)\left(\frac{g(I^*)}{g(I)}-1\right)\le 0. \tag{3.17}$$

That is, positive-definite function $U(t)$ has negative derivative $\frac{d}{dt}U(t)$. Furthermore, the equality $\frac{d}{dt}U(t)=0$ holds if and only if $S(t)=S^*$ and $i(t,a)=i^*(a)$. Hence,

every solution of (2.3) tends to $\mathcal{E}^* = (S^*, i^*(a))$, which is globally asymptotically stable when it exists. □

Finally we have to notice that, concerning the SEIR delay differential model (1.1), the condition (3.9) in Theorem 3.2 simply holds when the nonlinear incidence function $g(I)$ is a concave function with respect to I, condition which is obviously satisfied by the function $g(I) = \frac{\beta I}{1+\alpha I}$, $\alpha, \beta \in \mathbf{R}_+$ chosen in [2].

Therefore, the age-infection model (2.3) with any given initial condition is such that

$$\big|(S(t), i(t,a)) - (S^*, i^*(a))\big| \to 0 \quad \text{as} \quad t \to \infty$$

for all $a \in [0, \tau_1 + \tau_2]$, thus implying that the solutions of (1.1) satisfy

$$|S(t) - S^*| \to 0,$$
$$|E(t) - E^*| = \left| \int_0^{\tau_1} [i(t,a) - i^*(a)] da \right| \le \int_0^{\tau_1} |i(t,a) - i^*(a)|\, da \to 0,$$
$$|I(t) - I^*| = \left| \int_{\tau_1}^{\tau_1+\tau_2} [i(t,a) - i^*(a)] da \right| \le \int_{\tau_1}^{\tau_1+\tau_2} |i(t,a) - i^*(a)|\, da \to 0,$$

as $t \to \infty$, which in turn implies $|R(t) - R^*| \to 0$ as $t \to \infty$ and therefore the global attractivity for the positive equilibrium $\mathbf{E}^* = (S^*, E^*, I^*, R^*)$. This, jointly with the local asymptotic stability of $\mathbf{E}^*$(see Theorem 1.5), implies the global asymptotic stability of $\mathbf{E}^*$.

Theorem 3.3. *When $\mathcal{R}_0 > 1$, then $\mathbf{E}^*$ is globally asymptotically stable for model (1.1).*

4. **Discussion and conclusion.** We should point out that we can also improve our proof and Lyapunov functions to more general case $f(S)g(I)$ and $f(S,I)$ as in [7, 8]. The reader can prove it easily by using the same approaches here. It only needs that the front part of Lyapunov function $U(t)$ is instead by

$$U_1 = \varphi(0)\left(S - \int_0^S \frac{f(\sigma)}{f(S^*)} d\sigma\right) \quad \text{or} \quad U_1 = \varphi(0)\left(S - \int_0^S \frac{f(\sigma, I^*)}{f(S^*, I^*)} d\sigma\right).$$

A minor open question concerning the delay model (1.1) concerns the possibility that each stage of the epidemic model has different death rate constant (d.r.c.). For example, the exposed E have d.r.c. μ_E and the infectious people have d.r.c. $\mu_I \neq \mu_E$. In such a case the age-infection model (2.3) can easily be generalized by introducing an age dependent death rate:

$$\mu(a) = \begin{cases} \mu_E & \text{if } a \in (0, \tau_1] \\ \mu_I & \text{if } a \in (\tau_1, \tau_1 + \tau_2], \end{cases}$$

i.e.

$$\begin{cases} \frac{d}{dt} S(t) = \Lambda - \mu_1 S(t) - g(I(t))S(t), \\ \frac{\partial}{\partial t} i(t,a) + \frac{\partial}{\partial a} i(t,a) = -\mu(a) i(t,a), \end{cases} \tag{4.1}$$

with the boundary condition $i(t,0) = g(I(t))S(t)$, where $I(t) = \int_{\tau_1}^{\tau_1+\tau_2} i(t,a) da$, and initial condition $i(0,a) = \psi(a)$, $a \in [0, \tau_1 + \tau_2]$.

For the model (4.1), once suitably modified the function $\phi(a)$ and $\varphi(a)$ as

(i) $\phi(a) = (1 - \exp(-\mu_I \tau_2)) \cdot \exp(\mu_E (a - \tau_1))$, for $a \in (0, \tau_1]$,

(ii) $\varphi(a) = \int_a^{\tau_1+\tau_2} \mu_I \exp(\mu_I(a-\theta))d\theta$, for $a \in [\tau_1, \tau_1+\tau_2]$,

the proofs of the global asymptotic stability of the disease-free equilibria $\mathcal{E}_0$ when $\mathcal{R}_0 \leq 1$ and of the endemic equilibrium $\mathcal{E}^*$ when $\mathcal{R}_0 > 1$ can be performed with the same structure of the Lyapunov functions and by using the same technical details as in Theorems 3.1 and 3.2 respectively. It is to be noticed that in this case the basic reproduction number $\mathcal{R}_0$ becomes

$$\mathcal{R}_0 = \frac{\phi(0)S_0}{\mu_I} g'(0),$$

where $S_0 = \Lambda/\mu_1$, and the endemic equilibrium component $i^*(a)$ becomes

$$i^*(a) = \begin{cases} g(I^*)S^* e^{-\mu_E a} & \text{for } a \in (0, \tau_1], \\ g(I^*)S^* e^{-\mu_I a} & \text{for } a \in (\tau_1, \tau_1+\tau_2]. \end{cases}$$

The approach by the infection-age model followed in this paper to study the global stability properties for an SEIR model with two delays, of course, also applies to the delayed SIR model. For example, Xu and Du in [22] considered a delayed SIR epidemic model with constant infectious period. That model they considered is a special case of model (1.1) where $\tau_1 = 0$. The global stability of endemic equilibrium was left as an open question. Here we also solved this problem. Further, comparing with the general age-infection model in [15], the mortality rate of $i(t,a)$ in this nonlinear age-infection model is assumed to be constant μ_2. Our results generalize partly the results in Magal et. al., [15] and McCluskey [16] to nonlinear incidence rate.

On the other hand, in this paper we establish the global stability for the age-infection epidemic model with nonlinear incidence rate. Actually, we find that most epidemiological models with or without time delay can be regarded as specific cases of age-infection model (2.3).

(**I**) When $\tau_1 = 0$ and $\tau_2 = \infty$, we have $I(t) = \int_0^\infty i(t,a)da$, and

$$\begin{aligned} \frac{d}{dt}I(t) = \frac{d}{dt}\int_0^\infty i(t,a)da = \ & i(t,0) - i(t,\infty) - \mu_2 I(t) \\ = \ & g(I(t))S(t) - \mu_2 I(t), \end{aligned}$$

since $g(I(t))S(t)$ is bounded as $t \to \infty$. Hence, age-infection model (2.3) is equivalent to the following ordinary differential equation SIR model,

$$\begin{aligned} S'(t) &= \Lambda - \mu_1 S(t) - g(I(t))S(t), \\ I'(t) &= g(I(t))S(t) - \mu_2 I(t), \\ R'(t) &= \gamma I(t) - \mu_3 R(t). \end{aligned} \tag{4.2}$$

The stability of the above SIR model with nonlinear incidence rate has been established by Korobeinikov et al. [10, 11, 12].

(**II**) When τ_1 is a positive constant and $\tau_2 = \infty$, we have $E(t) = \int_0^{\tau_1} i(t,a)da$, $I(t) = \int_{\tau_1}^\infty i(t,a)da$, and

$$\frac{d}{dt}I(t) = \exp(-\mu_2\tau_1)g(I(t-\tau_1))S(t-\tau_1) - \mu_2 I(t).$$

Model (2.3) is equivalent to the following SEIR model with a discrete delay,

$$\begin{aligned} S'(t) &= \Lambda - \mu_1 S(t) - g(I(t))S(t), \\ E'(t) &= g(I(t))S(t) - e^{-\mu_2\tau_1} g(I(t-\tau_1))S(t-\tau_1) - \mu_2 I(t), \\ I'(t) &= e^{-\mu_2\tau_1} g(I(t-\tau_1))S(t-\tau_1) - \mu_2 I(t), \\ R'(t) &= \gamma I(t) - \mu_3 R(t). \end{aligned} \tag{4.3}$$

The global dynamical properties of the above model (4.3) are completely established by Huang et al. in [7].

If we omit the variable $E(t)$ in (4.3) and set new variables as $s(t) = S(t), i(t) = I(t+\tau_1), r(t) = R(t+\tau_1)$, then model (4.3) is transformed to

$$\begin{aligned} s'(t) &= \Lambda - \mu_1 s(t) - g(i(t-\tau_1))s(t), \\ i'(t) &= e^{-\mu_2\tau_1} g(i(t-\tau_1))s(t) - \mu_2 i(t), \\ r'(t) &= \gamma i(t) - \mu_3 r(t). \end{aligned} \tag{4.4}$$

The above delay SIR model with nonlinear incidence was widely studied in [1, 3, 4, 19, 21], and the global stability are established in [6, 8, 18].

This means that the popular epidemic models with or without delays can be regarded as the special cases of age-infection model (2.3), and their global asymptotical properties can be determined by analyzing (2.3).

In conclusion, by the above analysis it appears that time delays in SIR, SEIR models are harmless in generating stability switches delays induced. Of course, we expect that time delays may play a destabilizing role when occurring, for example, in the step by which removed individuals R can (partially) loss their immunity and return to be susceptibles, that is in models like SIS, SIRS and SEIRS.

REFERENCES

[1] J. Arino and P. van den Driessche, *Time delays in epidemic models: Modeling and numerical considerations*, in "Delay Differential Equations and Applications" (eds. O. Arino, M. L. Hbid and E. Ait Dads), Springer, (2006), 539–578.

[2] E. Beretta and D. Breda, *An SEIR epidemic model with constant latency time and infectious period*, Math. Biosci. Eng., **8** (2011), 931–952.

[3] E. Beretta and Y. Takeuchi, *Global stability of an SIR model with time delays*, J. Math. Biol., **33** (1995), 250–260.

[4] E. Beretta, T. Hara, W. Ma and Y. Takeuchi, *Global asymptotic stability of an SIR epidemic model with distributed time delay*, Nonlinear Anal., **47** (2001), 4107–4115.

[5] J. M. Cushing, "An Introduction to Structured Population Dynamics," CBMS-NSF Regional Conference Series in Applied Mathematics, **71**, SIAM, 1998.

[6] Y. Enatsu, Y. Nakata and Y. Muroya, *Global stability of SIR epidemic models with a wide class of nonlinear incidence rates and distributed delays*, Discrete and Continuous Dynamical Systems Ser. B, **15** (2011), 61–74.

[7] G. Huang, Y. Takeuchi, W. Ma and D. Wei, *Global stability for delay SIR and SEIR epidemic models with nonlinear incidence rate*, Bull. Math. Biol., **72** (2010), 1192–1207.

[8] G. Huang and Y. Takeuchi, *Global analysis on delay epidemiological dynamic models with nonlinear incidence*, J. Math. Biol., **63** (2011), 125–139.

[9] G. Huang, X. Liu and Y. Takeuchi, *Lyapunov functions and global stability for age-structured HIV infection model*, SIAM J. Appl. Math., **72** (2012), 25–38.

[10] A. Korobeinikov and P. K. Maini, *A Lyapunov function and global properties for SIR and SEIR epidemiological models with nonlinear incidence*, Math. Biosci. Eng., **1** (2004), 57–60.

[11] A. Korobeinikov, *Lyapunov functions and global stability for SIR and SIRS epidemiological models with non-linear transmission*, Bull. Math. Biol., **68** (2006), 615–626.

[12] A. Korobeinikov, *Global properties of infectious disease models with nonlinear incidence*, Bull. Math. Biol., **69** (2007), 1871–1889.
[13] M. Iannelli, "Mathematical Theory of Age-Structured Population Dynamics," Applied Mathematical Monographs, **7**, Consiglio Nazionale delle Ricerche, CNR, Giardini Editori e Stampatori in Pisa, Aprile, 1995.
[14] A. V. Melnik and A. Korobeinikov, *Lyapunov functions and global stability for SIR and SEIR models with age-dependent susceptibility*, (2011), preprint.
[15] P. Magal, C. C. McCluskey and G. Webb, *Lyapunov functional and global asymptotic stability for an infection-age model*, Appl. Anal., **89** (2010), 1109–1140.
[16] C. C. McCluskey, *Delay versus age-of-infection–global stability*, Appl. Math. Comput., **217** (2010), 3046–3049.
[17] C. C. McCluskey, *Global stability for an SIR epidemic model with delay and nonlinear incidence*, Nonlinear Anal. RWA, **11** (2010), 3106–3109.
[18] C. C. McCluskey, *Global stability of an SIR epidemic model with delay and general nonlinear incidence*, Math. Biosci. Eng., **7** (2010), 837–850.
[19] P. van den Driessche, *Some epidemiological models with delays*, in "Differential Equations and Applications to Biology and To Industry" (eds. M. Martell, K. Cooke, E. Cumberbatch, R. Tang and H. Thieme), World Scientific, (1994), 507–520.
[20] G. F. Webb, "Theory of Nonlinear Age-Dependent Population Dynamics," Marcel Dekker, New York, 1985.
[21] R. Xu and Z. Ma, *Global stability of a SIR epidemic model with nonlinear incidence rate and time delay*, Nonlinear Anal. RWA, **10** (2009), 3175–3189.
[22] R. Xu and Y. Du, *A delayed SIR epidemic model with saturation incidence and a constant infectious period*, J. Appl. Math. Comput., **35** (2011), 229–250.

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E-mail address: huanggang10@yahoo.cn
E-mail address: edoardo.beretta@uniurb.it
E-mail address: takeuchi@sys.eng.shizuoka.ac.jp