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PERMANENCE FOR TWO-SPECIES LOTKA-VOLTERRA COOPERATIVE SYSTEMS WITH DELAYS*

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Abstract. In this paper, a two-species Lotka-Volterra cooperative delay system is considered, and the relationships between the delays and the permanence are obtained. Some sufficient conditions for the permanence under the assumption of smallness of the delays are obtained. Two examples are given to illustrate the theorems.

1. **Introduction.** In this paper, we consider the following general two-species Lotka-Volterra cooperative system with discrete delays:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1 - \sum_{l=1}^{l_{11}} a_{11l}x_1(t-\tau_{11l}) + \sum_{l=1}^{l_{12}} a_{12l}x_2(t-\tau_{12l})], \\ \dot{x}_2(t) &= x_2(t)[r_2 + \sum_{l=1}^{l_{21}} a_{21l}x_1(t-\tau_{12l}) - \sum_{l=1}^{l_{22}} a_{22l}x_2(t-\tau_{22l})], \end{aligned} \tag{1.1}$$

with initial conditions

$$x_i(t) = \phi_i(t) \geq 0, \quad t \in [-\tau_0, 0]; \quad \phi_i(0) > 0 (i = 1, 2), \tag{1.2}$$

where $r_i > 0$, $a_{ijl} > 0$ and $\tau_{ijl} \geq 0$ are constants and $\tau_0 = \max\{\tau_{ijl} : l = 1 \cdots l_{ij}; i, j = 1, 2\}$, $\phi_i(t)$ is continuous on $[-\tau_0, 0]$.

We assume that system (1.1) has a unique positive equilibrium $x^* = (x_1^*, x_2^*)$; that is,

$$\begin{aligned} x_1^* &= \frac{r_1 A_{22} + r_2 A_{12}}{A_{11}A_{22} - A_{12}A_{21}}, \\ x_2^* &= \frac{r_2 A_{11} + r_1 A_{21}}{A_{11}A_{22} - A_{12}A_{21}}, \end{aligned} \tag{1.3}$$

where $A_{ij} = \sum_{l=1}^{l_{ij}} a_{ijl}$ $(i, j = 1, 2)$.

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In this paper, we will focus on the permanence of system (1.1). System (1.1) is permanent if there exist $M_i > 0$, $m_i > 0 (i = 1, 2)$ such that

$$m_i \leq \liminf_{t\to+\infty} x_i(t) \leq \limsup_{t\to+\infty} x_i(t) \leq M_i.$$

Many works are devoted to this problem, such as [1–13]. Lin and Lu [5] consider the following system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1 - a_1 x_1(t) - a_{11} x_1(t-\tau_{11}) + a_{12} x_2(t-\tau_{12})], \\ \dot{x}_2(t) &= x_2(t)[r_2 - a_2 x_2(t) + a_{21} x_1(t-\tau_{12}) - a_{22} x_2(t-\tau_{22})], \end{aligned} \tag{1.4}$$

where $a_i > 0$, $\tau_{ijl} \geq 0 (i, j = 1, 2)$, and obtain a sufficient condition for its permanence. In [13], Saito gives the necessary and sufficient conditions for its global stability in some specific cases. Lu, Lu and Lian [9] show that delays can affect the permanence for system (1.4).

For system (1.4), if $a_{ii} = 0 (i = 1, 2)$, Lu and Wang [10] consider its global stability and obtain the necessary and sufficient conditions, the results are extended to general n by Hofbauer and So [3] and Lu and Lu [6, 7, 8].

If $a_i = 0 (i = 1, 2)$, in [1], two counterexamples are given by Chen, Lu and Wang to show that delays may destroy the permanence of the system. From He [2] and Mukherjee [11], we have known that boundedness and permanence are equivalent for system (1.4).

In the present paper, by modifying the technique of Muroya [12] and Lin and Lu [5], we consider system (1.1), which is an extension of system (1.4).

For system (1.1), if

$$\tau_{iil} = 0 \quad \text{for} \quad \text{some} \quad l = 1 \cdots \bar{l}_{ii} < l_{ii} (i = 1, 2), \tag{H$_1$}$$

then system (1.1) simplifies to the form

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1 - \sum_{l=1}^{\bar{l}_{11}} a_{11l} x_1(t) - \sum_{l=\bar{l}_{11}+1}^{l_{11}} a_{11l} x_1(t-\tau_{11l}) + \sum_{l=1}^{l_{12}} a_{12l} x_2(t-\tau_{12l})], \\ \dot{x}_2(t) &= x_2(t)[r_2 - \sum_{l=1}^{\bar{l}_{22}} a_{22l} x_2(t) + \sum_{l=1}^{l_{21}} a_{21l} x_1(t-\tau_{12l}) - \sum_{l=\bar{l}_{22}+1}^{l_{22}} a_{22l} x_2(t-\tau_{22l})]. \end{aligned} \tag{1.5}$$

Like Muroya [12] and Lin and Lu [5], we have

Theorem 1.1. *System* (1.5) *is permanent if it satisfies that*

$$(\sum_{l=1}^{\bar{l}_{11}} a_{11l})(\sum_{l=1}^{\bar{l}_{22}} a_{22l}) - (\sum_{l=1}^{l_{12}} a_{12l})(\sum_{l=1}^{l_{21}} a_{21l}) > 0. \tag{H$_2$}$$

If condition (H$_1$) or (H$_2$) fails, Muroya's technique seems difficult to apply to the proof. In this case, we suppose the following:

When condition (H$_1$) or (H$_2$) fails, there exist a large enough time $T > 0$ and constants $C_i > 0$, D_i $(i = 1, 2)$ such that for each $t > T$,

$$\dot{x}_i(t) < C_i x_i(t) + D_i. \tag{H$_3$}$$

We have the following theorem.

Theorem 1.2. *System* (1.1) *is permanent if it satisfies conditions* (H_3) *and*

$$\bar{A} \text{ is an M-matrix.} \tag{H_4}$$

Here

$$\bar{A} = \begin{pmatrix} \sum_{l=1}^{l_{11}} a_{11l}(1 - C_1\tau_{11l}) & -\sum_{l=1}^{l_{12}} a_{12l} \\ -\sum_{l=1}^{l_{21}} a_{21l} & \sum_{l=1}^{l_{22}} a_{22l}(1 - C_2\tau_{22l}) \end{pmatrix}.$$

From Theorem 1.2, we get relations between delays and permanence under some assumptions.

Subsequently, we apply Theorem 1.2 to the following specific form of system (1.1); that is,

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[r_1 - a_{111}x_1(t-2\tau) - a_{11}x_1(t-\tau) + a_{12}x_2(t-\tau)], \\ \dot{x}_2(t) &= x_2(t)[r_2 + a_{21}x_1(t-\tau) - a_{221}x_2(t) - a_{22}x_2(t-\tau)], \end{aligned} \tag{1.6}$$

with a unique positive equilibrium and initial conditions (1.2), where $a_{11} > a_{21}, a_{22} > a_{12}, a_{111} \geq a_{21}, a_{221} \geq a_{12}$.

By Theorem 1.2 and as with those in Lu, Lu, and Lian [9], we have

Theorem 1.3. *System* (1.6) *is permanent if it satisfies*

$$[a_{111}(1 - 2r_1\tau) + a_{11}(1 - r_1\tau)]a_{221} - a_{12}a_{21} > 0. \tag{H_5}$$

2. **Proof of our Main Results.** Firstly, we give some lemmas. Lemma 2.1 is a direct result from system (1.1), Lemma 2.2 is a modified one from Muroya [12].

Lemma 2.1. *Every solution* $x(t)$ *of system* (1.1) *with initial condition* (1.2) *exists in the interval* $[0, +\infty)$ *and remains positive for all* $t \geq 0$.

Lemma 2.2. [12] *For any* $0 \leq t_p^i \leq t$

$$\begin{aligned} x_i(t) = x_i(t_p^i) &\exp(\textstyle\int_{t_p^i}^t \{r_i + \sum_{l=1}^{l_{ii}} a_{iil}x_i(s) - \sum_{l=1}^{l_{ii}} a_{iil}x_i(s-\tau_{iil}) + \sum_{l=1}^{l_{ij}} a_{ijl}x_j(s-\tau_{ijl})\}ds) \\ &/[1 + (\textstyle\sum_{l=1}^{l_{ii}} a_{iil})x_i(t_p^i)\int_{t_p^i}^t \exp(\int_{t_p^i}^s \{r_i + \sum_{l=1}^{l_{ii}} a_{iil}x_i(s) - \sum_{l=1}^{l_{ii}} a_{iil}x_i(\sigma-\tau_{iil}) \\ &+ \textstyle\sum_{l=1}^{l_{ij}} a_{ijl}x_j(\sigma-\tau_{ijl})\}d\sigma)ds] \end{aligned}$$

$$\begin{aligned} &x_i(t) - x_i^* \\ &= [1 - (\textstyle\sum_{l=1}^{l_{ii}} a_{iil})k_i(t_p^i, t)](x_i(t_p^i) - x_i^*) + [x_i(t_p^i)\int_{t_p^i}^t \{\sum_{l=1}^{l_{ii}} a_{iil}x_i(s) - \sum_{l=1}^{l_{ii}} a_{iil}x_i(s-\tau_{iil}) \\ &\quad + \textstyle\sum_{l=1}^{l_{ij}} a_{ijl}(x_j(s-\tau_{ijl}) - x_j^*)\} \times \exp(\int_{t_p^i}^s \{r_i + \sum_{l=1}^{l_{ii}} a_{iil}x_i(\sigma) - \sum_{l=1}^{l_{ii}} a_{iil}x_i(\sigma-\tau_{iil}) \\ &\quad + \textstyle\sum_{l=1}^{l_{ij}} a_{ijl}x_j(\sigma-\tau_{ijl})\}d\sigma)ds] \\ &\quad /[1 + (\textstyle\sum_{l=1}^{l_{ii}} a_{iil})x_i(t_p^i)\int_{t_p^i}^t \exp(\int_{t_p^i}^s \{r_i + \sum_{l=1}^{l_{ii}} a_{iil}x_i(\sigma) - \sum_{l=1}^{l_{ii}} a_{iil}x_i(\sigma-\tau_{iil}) \\ &\quad + \textstyle\sum_{l=1}^{l_{ij}} a_{ijl}x_j(\sigma-\tau_{ijl})\}d\sigma)ds], \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} k_i(t_p^i, t) = \quad & x_i(t_p^i) \int_{t_p^i}^t \exp(\int_{t_p^i}^s \{r_i + \sum_{l=1}^{l_{ii}} a_{iil} x_i(\sigma) - \sum_{l=1}^{l_{ii}} a_{iil} x_i(\sigma - \tau_{iil}) \\ & + \sum_{l=1}^{l_{ij}} a_{ijl} x_j(\sigma - \tau_{ijl})\} d\sigma) ds / [1 + (\sum_{l=1}^{l_{ii}} a_{iil}) x_i(t_p^i) \int_{t_p^i}^t \exp(\int_{t_p^i}^s \{r_i \\ & + \sum_{l=1}^{l_{ii}} a_{iil} x_i(\sigma) - \sum_{l=1}^{l_{ii}} a_{iil} x_i(\sigma - \tau_{iil}) + \sum_{l=1}^{l_{ij}} a_{ijl} x_j(\sigma - \tau_{ijl})\} d\sigma) ds] \end{aligned} \tag{2.4}$$

with $i \neq j$, $i, j = 1, 2$.

Lemma 2.3. *For system* (1.6)*, there exists a large enough time* $T > 0$ *and constant* D_1 *such that for each* $t > T$

$$\dot{x}_1(t) < r_1 x_1(t) + D_1.$$

Proof. Since $a_{11} > a_{21}, a_{22} > a_{12}$. Construct a continuous functional

$$V_1(t) = x_1(t) x_2(t);$$

then we obtain

$$\begin{aligned} \dot{V}_1(t) \quad = \quad & V_1(t)[r_1 + r_2 - (a_{11} - a_{21}) x_1(t-\tau) - (a_{22} - a_{12}) x_2(t-\tau) \\ & -a_{111} x_1(t - 2\tau) - a_{221} x_2(t)] \\ \leq \quad & V_1(t)[r_1 + r_2 - (a_{11} - a_{21}) x_1(t-\tau) - (a_{22} - a_{12}) x_2(t-\tau)] \\ \leq \quad & V_1(t)[r_1 + r_2 - k V_1^{\frac{1}{2}}(t-\tau)], \end{aligned}$$

where $k = 2[(a_{11} - a_{21})(a_{22} - a_{12})]^{\frac{1}{2}} > 0$.

Let $u_1(t) = V_1^{\frac{1}{2}}(t)$; then we have

$$\dot{u}_1(t) \leq \tfrac{1}{2} u_1(t)[r_1 + r_2 - k u_1(t-\tau)].$$

This implies that

$$\limsup_{t \to +\infty} u_1(t) \leq \frac{r_1 + r_2}{k} e^{\frac{1}{2}(r_1 + r_2)\tau},$$

and then, there exists a large enough T, such that for $t > T$,

$$u_1(t) \leq \frac{r_1 + r_2}{k} e^{\frac{1}{2}(r_1 + r_2)\tau},$$

which implies that

$$V_1(t) \leq (\tfrac{r_1 + r_2}{k})^2 e^{(r_1 + r_2)\tau} = N_1.$$

For $a_{111} \geq a_{21}, a_{221} \geq a_{12}$, we construct a continuous functional

$$V_2(t) = x_1(t) x_2(t - \tau).$$

By calculating the derivative along system (2.1), we have

$$\begin{aligned} \dot{V}_2(t) \quad = \quad & V_2(t)[r_1 + r_2 - a_{11} x_1(t-\tau) - (a_{111} - a_{21}) x_1(t - 2\tau) \\ & -(a_{221} - a_{12}) x_2(t-\tau) - a_{22} x_2(t - 2\tau)] \\ \leq \quad & V_2(t)[r_1 + r_2 - a_{11} x_1(t-\tau) - a_{22} x_2(t - 2\tau)], \end{aligned}$$

and as in the above proof, we obtain

$$V_2(t) \le N_2.$$

From the above proof, we obtain

$$\frac{\dot{x}_1(t)}{x_1(t)} + \frac{\dot{x}_2(t)}{x_2(t)} \le r_1 + r_2.$$

Therefore, there exists a large enough time $T > 0$ such that for each $t > T$

$$\begin{aligned} \frac{\dot{x}_1(t)}{x_1(t)} &\le r_1 - a_{21}x_1(t-\tau) + a_{221}x_2(t) + a_{22}x_2(t-\tau) \\ &< r_1 + a_{221}x_2(t) + a_{22}x_2(t-\tau). \end{aligned}$$

So,

$$\begin{aligned} \dot{x}_1(t) &< r_1x_1(t) + a_{221}x_1(t)x_2(t) + a_{22}x_1(t)x_2(t-\tau) \\ &< r_1x_1(t) + a_{221}N_1 + a_{22}N_2 \\ &= r_1x_1(t) + D_1. \end{aligned}$$

□

Proof of Theorem 1.1. The proof is similar to those in Muroya [12] and Lin and Lu [5]. □

Proof of Theorem 1.2. Since $\bar{A}$ is an M-matrix, there is a diagonal matrix $\bar{D} = \mathrm{diag}(\bar{\mathrm{d}}_1, \bar{\mathrm{d}}_2)$ such that $\bar{d}_i > 0 \quad (i = 1,2)$ and $\bar{A}\bar{D}$ is a diagonally dominant matrix. Therefore, we may assume, without loss of generality, that $\bar{A}$ is diagonally dominant; that is,

$$\sum_{l=1}^{l_{ii}} a_{iil}(1 - C_i\tau_{iil}) > \sum_{l=1}^{l_{ij}} a_{ijl}, \quad i \ne j, \quad i,j = 1,2.$$

Subsequently, we show that each solution $x_i(t)$ is eventually uniformly bounded above by a positive constant.

Suppose that there is a subset P of {1, 2} such that for each $j \in P$,

$$\limsup_{t\to+\infty} x_j(t) = +\infty. \qquad (\mathrm{H}_6)$$

Then, for each $j \in P$, there is a sequence $\{\bar{t}_p^j\}_{p=1}^{\infty}$ such that

$$x_j(t) \le x_j(\bar{t}_p^j) \quad \text{for} \quad t_0 \le t \le \bar{t}_p^j \quad \text{and} \quad \lim_{t\to+\infty} x_j(\bar{t}_p^j) = +\infty.$$

For each sufficiently large positive integer l, we can take $i \in P$ such that

$$x_i(\bar{t}_{p+1}^i) - x_i^* = \max_{1\le j\le 2}(x_j(\bar{t}_{p+1}^i) - x_j^*) \quad \text{and} \quad x_i(t) \le x_i(\bar{t}_{p+1}^i) \text{ for any } t_0 \le t \le \bar{t}_{p+1}^i.$$

Since $\sum_{l=1}^{l_{ii}} a_{iil} > 0 (i = 1,2)$, by lemma 2.2, we have that $k_i(\bar{t}_p^i, \bar{t}_{p+1}^i) > 0$ and $1 - (\sum_{l=1}^{l_{ii}} a_{iil})k_i(\bar{t}_p^i, \bar{t}_{p+1}^i) > 0$. By condition ($\mathrm{H}_3$), we obtain that there exists a large $T > 0$ such that for $t > T$,

$$\dot{x}_i(t) \le C_ix_i(t) + D_i.$$

By integrating it from $t-\tau_{iil}$ to t, we obtain

$$x_i(t)-x_i(t-\tau_{iil})\le C_i\int_{t-\tau_{iil}}^{t}x_i(s)ds+D_i\tau_{iil}.$$

Now for $\bar{k}_{ip}=k_i(\bar{t}_p^i,\bar{t}_{p+1}^i)>0$, we have

$$\begin{aligned}
&x_i(\bar{t}_{p+1}^i)-x_i^*\\
&=[1-(\sum_{l=1}^{l_{ii}}a_{iil})\bar{k}_{ip}](x_i(\bar{t}_p^i)-x^*)+[x_i(\bar{t}_p^i)\textstyle\int_{\bar{t}_p^i}^{\bar{t}_{p+1}^i}\{\sum_{l=1}^{l_{ii}}a_{iil}x_i(s)-\sum_{l=1}^{l_{ii}}a_{iil}x_i(s-\tau_{iil})\\
&\quad+\sum_{l=1}^{l_{ij}}a_{ijl}(x_j(s-\tau_{ijl})-x_j^*)\}\times\exp(\textstyle\int_{t_p^i}^{s}\{r_i+\sum_{l=1}^{l_{ii}}a_{iil}x_i(\sigma)-\sum_{l=1}^{l_{ii}}a_{iil}x_i(\sigma-\tau_{iil})\\
&\quad+\sum_{l=1}^{l_{ij}}a_{ijl}x_j(\sigma-\tau_{ijl})\}d\sigma)ds]/[1+(\sum_{l=1}^{l_{ii}}a_{iil})x_i(\bar{t}_p^i)\textstyle\int_{\bar{t}_p^i}^{\bar{t}_{p+1}^i}\exp(\int_{\bar{t}_p^i}^{s}\{r_i+\sum_{l=1}^{l_{ii}}a_{iil}x_i(\sigma)\\
&\quad-\sum_{l=1}^{l_{ii}}a_{iil}x_i(\sigma-\tau_{iil})+\sum_{l=1}^{l_{ij}}a_{ijl}x_j(\sigma-\tau_{ijl})\}d\sigma)ds]\\
&\le[1-(\sum_{l=1}^{l_{ii}}a_{iil})\bar{k}_{ip}](x_i(\bar{t}_p^i)-x^*)+[x_i(\bar{t}_p^i)\textstyle\int_{\bar{t}_p^i}^{\bar{t}_{p+1}^i}\{\sum_{l=1}^{l_{ii}}a_{iil}C_i\int_{s-\tau_{iil}}^{s}x_i(\sigma)d\sigma\\
&\quad+\sum_{l=1}^{l_{ii}}a_{iil}D_i\tau_{iil}+\sum_{l=1}^{l_{ij}}a_{ijl}(x_j(s-\tau_{ijl})-x_j^*)\}\times\exp(\textstyle\int_{\bar{t}_p^i}^{s}\{r_i+\sum_{l=1}^{l_{ii}}a_{iil}x_i(\sigma)\\
&\quad-\sum_{l=1}^{l_{ii}}a_{iil}x_i(\sigma-\tau_{iil})+\sum_{l=1}^{l_{ij}}a_{ijl}x_j(\sigma-\tau_{ijl})\}d\sigma)ds]\\
&\quad/[1+(\sum_{l=1}^{l_{ii}}a_{iil})x_i(\bar{t}_p^i)\textstyle\int_{\bar{t}_p^i}^{\bar{t}_{p+1}^i}\exp(\int_{\bar{t}_p^i}^{s}\{r_i+\sum_{l=1}^{l_{ii}}a_{iil}x_i(\sigma)-\sum_{l=1}^{l_{ii}}a_{iil}x_i(\sigma-\tau_{iil})\\
&\quad+\sum_{l=1}^{l_{ij}}a_{ijl}x_j(\sigma-\tau_{ijl})\}d\sigma)ds]\\
&\le[1-(\sum_{l=1}^{l_{ii}}a_{iil})\bar{k}_{ip}](x_i(\bar{t}_{p+1}^i)-x^*)+\bar{k}_{ip}[\sum_{l=1}^{l_{ii}}a_{iil}C_i\tau_{iil}x_i(\bar{t}_{p+1}^i)+\sum_{l=1}^{l_{ii}}a_{iil}D_i\tau_{iil}\\
&\quad+\sum_{l=1}^{l_{ij}}a_{ijl}(x_i(\bar{t}_{p+1}^i)-x^*)].
\end{aligned}$$

This implies that

$$\bar{k}_{ip}\{[\sum_{l=1}^{l_{ii}}a_{iil}(1-C_i\tau_{iil})-\sum_{l=1}^{l_{ij}}a_{ijl}](x_i(\bar{t}_{p+1}^i)-x_i^*)-\sum_{l=1}^{l_{ii}}a_{iil}C_i\tau_{iil}x_i^*-\sum_{l=1}^{l_{ii}}a_{iil}D_i\tau_{iil}\}\le 0.$$

By the definition of $\bar{k}_{ip}$, we get

$$[\sum_{l=1}^{l_{ii}}a_{iil}(1-C_i\tau_{iil})-\sum_{l=1}^{l_{ij}}a_{ijl}](x_i(\bar{t}_{p+1}^i)-x_i^*)\le\sum_{l=1}^{l_{ii}}a_{iil}C_i\tau_{iil}x_i^*+\sum_{l=1}^{l_{ii}}a_{iil}D_i\tau_{iil}.$$

Therefore,

$$x_i(\bar{t}_{p+1}^i)\le x_i^*+\frac{\sum_{l=1}^{l_{ii}}a_{iil}C_i\tau_{iil}x_i^*+\sum_{l=1}^{l_{ii}}a_{iil}D_i\tau_{iil}}{\sum_{l=1}^{l_{ii}}a_{iil}(1-C_i\tau_{iil})-\sum_{l=1}^{l_{ij}}a_{ijl}}<+\infty,$$

which contradicts to our assumption (H_6). Hence, we have

$$\limsup_{t\to+\infty}x_i(t)<+\infty,\quad i=1,2.$$

Moreover, by Lemma 2.2, similarly to the above discussion, we can get that there exist $M_i > 0$ such that

$$\limsup_{t\to+\infty} x_i(t) < M_i < +\infty, \quad i = 1, 2.$$

Therefore, for each sufficiently large t, any solution $x(t) = (x_1(t), x_2(t))$ to (1.1) satisfies $0 < x_i(t) \le M_i$.

The remaining parts are similar to those in Lin and Lu [5].

This completed the proof of the theorem. □

Proof of Theorem 1.3. Since (H_5) holds, we may assume, without loss of generality, that

$$a_{111}(1 - 2r_1\tau) + a_{11}(1 - r_1\tau) > a_{12}, \quad a_{221} > a_{21};$$

otherwise, a transformation $x_i(t) = \beta_i \bar{x}_i(t)$ will work.

Suppose that there is a subset P of {1, 2} such that for each $j \in P$,

$$\limsup_{t\to+\infty} x_j(t) = +\infty.$$

Then, for each $j \in P$, there is a sequence $\{\bar{t}^j_p\}_{p=1}^{\infty}$ such that

$$x_j(t) \le x_j(\bar{t}^j_p) \quad \text{for} \quad t_0 \le t \le \bar{t}^j_p \quad \text{and} \quad \lim_{t\to+\infty} x_j(\bar{t}^j_p) = +\infty.$$

For each sufficiently large positive integer l, we can take $i \in P$ such that

$$x_i(\bar{t}^i_{p+1}) - x^*_i = \max_{1\le j\le 2}(x_j(\bar{t}^i_{p+1}) - x^*_j) \quad \text{and} \quad x_i(t) \le x_i(\bar{t}^i_{p+1}) \quad \text{for} \quad \text{any} \quad t_0 \le t \le \bar{t}^i_{p+1}.$$

If $i = 1$, by lemma 2.3, there exist a large $T > 0$ such that for $t > T$,

$$\dot{x}_1(t) < r_1 x_1(t) + D_1.$$

The remaining parts are similar to those in Theorem 1.2. For $i = 2$, the remaining parts are similar to those in Muroya [12] and Lin and Lu [5].

This completes the proof of Theorem 1.3. □

3. **Concluding Remarks.** In this section, we give some examples to illustrate our theorems.

Example 3.1

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[1 - x_1(t - 2\tau) - e x_1(t - \tau) + 2e^{-\frac{11}{4}} x_2(t - \tau)], \\ \dot{x}_2(t) &= x_2(t)[1 + \tfrac{2}{3} e^{\frac{1}{4}} x_1(t - \tau) - \tfrac{1}{2} x_2(t) - \tfrac{1}{6} e^2 x_2(t - \tau)]. \end{aligned} \tag{3.1}$$

From condition (H_5), we have known that if τ satisfies $0 < \tau < \tau^* = \dfrac{1 + e - \frac{8}{3} e^{-\frac{5}{2}}}{2 + e}$ $\doteq 0.74$, system (4.1) is permanent.

The following example shows that for some general delays, system (3.2) can have an unbounded solution.

Example 3.2

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[1 - x_1(t - 3) - e x_1(t - 4) + 2e^{-\frac{11}{4}} x_2(t - \tfrac{1}{4})], \\ \dot{x}_2(t) &= x_2(t)[1 + \tfrac{2}{3} e^{\frac{1}{4}} x_1(t - \tfrac{1}{4}) - \tfrac{1}{2} x_2(t) - \tfrac{1}{6} e^2 x_2(t - 2)]. \end{aligned} \tag{3.2}$$

System (3.2) with initial condition (1.2) has an unbounded solution $x(t) = (x_1(t), x_2(t)) = (e^t, e^t)$, if $\phi_i(t) = e^t (i = 1, 2)$.

We have obtained sufficient conditions for a class of two-species Lotka-Volterra cooperative system to be permanent when the delays are small. For a specific case, we shown that the delays can destroy the permanence.

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