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Research article

Finite-time boundedness of delayed Takagi-Sugeno quasi-one-sided Lipschitz systems

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Abstract: This paper investigates the finite-time boundedness (FBss) of perturbed nonlinear systems with interval time-varying delay. First, the system's nonlinear dynamics are represented by a Takagi–Sugeno (TS) fuzzy model, while quasi-one-sided Lipschitz (QL) terms account for either intrinsic nonlinearities or lumped uncertainties. This formulation captures a broad class of systems, encompassing both natural system nonlinearities and aggregated perturbations. Second, to ensure the FBss of the proposed system, an observer-based controller (OC) is employed under the assumption that the premise variables are measurable, even if the QL condition itself involves unmeasurable states. Then, a single-step procedure based on linear matrix inequalities (LMIs) is proposed to design the OC gains. To this end, a Lyapunov–Krasovskii (LK) functional is employed, comprising a delay-free quadratic term, which incorporates the matrix $\mathscr{P}$ used to verify the QL condition, along with three integral terms that capture the effects of the interval time-varying delay, including its lower and upper bounds. In addition, we employ a suitable decoupling approach that allows $\mathscr{P}$ to be regarded as a full decision-variable matrix and enabling the LMI conditions to be derived in a single step. The effectiveness and feasibility of the proposed methods are finally verified through a simulation study.

Keywords: time delay; quasi-one-sided Lipschitz nonlinearity; finite-time boundedness; observer-

based controller
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1. Introduction

The concept of Lyapunov asymptotic stability (LAS) has been extensively investigated for nonlinear systems across various modeling approaches. This notion analyzes the system's long-term behavior, a key consideration for systems expected to function indefinitely. The Takagi-Sugeno (TS) fuzzy approach is among the most widely used methods for analyzing LAS in nonlinear systems, as it represents the system through a weighted combination of local linear submodels. For instance, in [1], the authors investigated H_∞ control for TS fuzzy systems using an enhanced method for aligning membership functions, and they provided sufficient conditions to guarantee asymptotic stability. Kahouli et al. [2] investigated the design of observers for TS fuzzy systems subject to specific disturbances, ensuring that the error dynamics converge within a bounded region. An alternative body of research has dealt with nonlinear systems characterized by Lipschitz nonlinearities. The authors in [3] studied the observer-based control (OC) problem for a class of Lipschitz nonlinear systems subject to noise. Although classical Lipschitz-based methods ensure the convergence of the observer error for systems with small Lipschitz constants, they become conservative when dealing with systems characterized by larger constants. To reduce the conservatism of classical Lipschitz methods, Hu [4] proposed the one-sided Lipschitz (OL) condition as a generalized framework. LAS, especially regarding observer design, has been extensively investigated for systems exhibiting OL nonlinearities [5–7]. Building on the OL condition, Hu [8] developed the quasi-one-sided Lipschitz (QL) condition, offering additional reduction in conservatism. Subsequent works have leveraged the QL condition to develop observers for nonlinear systems [9, 10].

Alongside nonlinear dynamics, time delays are a primary factor influencing system behavior. This has motivated significant research efforts to extend delay-free theoretical frameworks to delayed systems. These extensions originated in 1956, when Krasovskii [11] generalized Lyapunov's methods, establishing a rigorous framework for analyzing the stability of time-delay systems. The TS fuzzy model was extended in [12] to handle delayed nonlinear systems. Since then, considerable research has been devoted to reducing the conservatism in the LAS analysis of delayed TS fuzzy systems. Key advances include techniques such as improved Lyapunov–Krasovskii (LK) functionals [13], the reciprocally convex inequality [14], and the delay-partitioning method [15]. Several researchers have investigated delayed nonlinear systems described by Lipschitz conditions and their generalizations, including the OL and the more general QL models. Leveraging the generalized Lipschitz framework, most of these researchers have centered on the observer design problem. In [16], the authors tackled the observer design problem for a nonlinear Lipschitz system with sampled measurements, employing a reset integrator operator to compensate for the errors introduced by sampling. The design of reduced-order OC has been studied for systems satisfying the OL condition in [17] and the QL condition in [18]. In [19, 20], the OC problem for delayed QL systems was investigated with consideration of input saturation.

It is worth noting that the aforementioned works primarily address LAS. However, an alternative concept, referred to by Dorato in 1961 [21] as 'short-time stability,' emphasizes the behavior of the system over a finite time interval rather than its asymptotic properties. This concept was further formalized in [22], introducing finite-time stability (FS) for unperturbed systems and finite-time boundedness (FBss) for systems subject to perturbations. Subsequently, results have been developed for various types of systems, both with and without time delays. The case of delay-free linear systems has been principally studied by Amato et al. [23–25]. With the choice of an appropriate LK functionnal and by using free matrix based integral inequalities, the problem of FS of a delayed linear system is addressed in [26]. For delayed TS fuzzy models, sufficient conditions for FS and FBss were established in [27, 28], respectively. Specifically, the conditions in [27] were derived using the free-weighting matrices technique, while those in [28] were obtained via the reciprocally convex combination method. Further developments in this area have considered some classes of nonlinear systems in which the nonlinear part satisfies Lipschitz or OL conditions. For example, the finite-time OC problem is addressed in [29] for time-delay systems satisfying OL conditions, while [30] extends the analysis to switched systems, providing a more general framework involving subsystem switching. In a recent study [31], the separation principle was employed to design an observer-based tracking controller for OL nonlinear systems with sensor perturbations. However, we observe that, for delayed nonlinear systems with QL conditions, finite-time results remain largely unexplored, which motivates our focused investigation of this problem.

Motivated by the preceding discussion, this paper investigates the FBss of delayed Takagi-Sugeno quasi-one-sided Lipschitz (TSQL) systems. This work advances the existing literature through the following contributions:

- A novel aspect of this work is the assumption that the nonlinear function satisfies a QL condition, a property previously unaddressed in the FS literature. Unlike the classical OL assumption commonly adopted in prior analyses [29, 30], this condition offers a broader framework.
- To enhance the generality of the proposed methodology, the system dynamics not governed by the QL condition are not restricted to linearity as in [19–35]. Instead, these residual nonlinear dynamics are approximated within a TS fuzzy approach.
- In previous works addressing the LAS of delayed systems with QL conditions, the time delay is typically assumed to be either constant $\theta(t) = \theta_2$ [18–32], or time-varying within the range $0 \leq \theta(t) \leq \theta_2$ [19–35]. In contrast, in this work, we consider an interval time-varying delay, $0 \leq \theta_1 \leq \theta(t) \leq \theta_2$ which provides information not only on the upper bound but also on the lower bound of the delay.
- The proposed controller design does not assume full state measurability. In particular, the states involved in the QL nonlinearity are estimated via an observer. In this work, we highlight the significance of the observer's initial conditions. While these initial conditions do not affect LAS, they play a critical role in FBss, directly influencing the parameters that characterize the finite time behavior, namely the finite time and the prescribed bounds. Unlike the observer initial conditions considered in [29, 30], we demonstrate that a more judicious selection can significantly improve the overall performance.
- A central challenge in dealing with QL conditions lies in the role of the matrix $\mathscr{P}$, which appears both in the verification of the condition itself and in the Lyapunov function. Previous studies have dealt with this by imposing specific structures on $\mathscr{P}$: In [18–32], it is taken as a constant matrix within the Lyapunov function, while in [20–35], it adopts a fixed sparse structure where only certain entries are treated as decision variables. In contrast, the present work treats all entries of $\mathscr{P}$ as free

decision variables, thereby granting significantly more flexibility in the design procedure.

This paper is organized into six sections. Section 2 presents the modeling of a delayed nonlinear system using the TSQL approach. An algorithm based on the mean value theorem is also provided to verify the QL conditions. Section 3 introduces the finite-time bounded (FB) control framework, where the premise variables are assumed to be measurable, while the state involved in the QL function is estimated through an observer. In Section 4, sufficient conditions are derived using an appropriate LK functional that accounts for both the lower and upper bounds of the time-varying delay. During the derivation process, certain sources of unnecessary conservatism commonly encountered in the literature are avoided. A numerical example illustrating the effectiveness of the proposed method is presented in Section 5. Finally, Section 6 concludes the paper and outlines directions for future research.

2. System modeling via TSQL approach

Notations 1.

i. 0 denotes the scalar zero or a zero matrix of appropriate dimensions. $\mathbb{R} \setminus \{0\}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}^{o_1}$, and $\mathbb{R}^{o_1 \times o_2}$ denote, respectively, the set of all real numbers except zero, the set of strictly positive real numbers, the set of nonnegative real numbers, the o_1-dimensional real vector space, and the set of real matrices of size $o_1 \times o_2$. $\mathbb{S}^{o_1}$ denotes the subset of $\mathbb{R}^{o_1 \times o_1}$ consisting of symmetric matrices. The superscript T denotes the transpose. $\mathbb{I}^u := \{1, \dots, u\}$.

ii. Given $(\mu, \nu) \in \mathbb{R}^{o_1} \times \mathbb{R}^{o_1}$, $\langle \mu, \nu \rangle = \mu^T \nu$.

iii. Given $\mathscr{P} \in \mathbb{S}^{o_1}$, $\mathscr{P} \geq 0$ (> 0) means that $\mathscr{P}$ positive semidefinite (positive definite). The symbols $>$, $\geq$, $<$, and $\leq$ are used for scalar inequalities.

iv. Given $\theta_2 \in \mathbb{R}_{>0}$, $C\left(\begin{bmatrix} -\theta_2 & 0 \end{bmatrix}, \mathbb{R}^{o_1}\right)$ denotes the space of continuous functions with domain $\begin{bmatrix} -\theta_2 & 0 \end{bmatrix}$ and codomain $\mathbb{R}^{o_1}$.

Definition 2.1. [32] Let $\psi : \mathbb{R}^{o_1} \times \mathbb{R}^{o_1} \to \mathbb{R}^{o_1}$. ψ is called QL if there exist $\mathscr{P}, \mathscr{U}, \mathscr{V} \in \mathbb{S}^{o_1}$ such that

$$\mathscr{P} > 0, \ \mathscr{V} \geq 0, \tag{2.1}$$

$$\begin{aligned} \langle \mathscr{P}(\psi(\mu_1, \mu_2) - \psi(\nu_1, \nu_2)), \mu_1 - \nu_2 \rangle \ \leq \ & \begin{bmatrix} \mu_1 - \nu_1 \\ \mu_2 - \nu_2 \end{bmatrix}^T \begin{bmatrix} \mathscr{U} & 0 \\ 0 & \mathscr{V} \end{bmatrix} \begin{bmatrix} \mu_1 - \nu_1 \\ \mu_2 - \nu_2 \end{bmatrix}, \\ & \forall (\mu_j, \nu_j) \in \mathbb{R}^{o_1} \times \mathbb{R}^{o_1}, \ j \in \{1, 2\}. \end{aligned} \tag{2.2}$$

Noting that $\psi(0, 0) = 0$, we have

$$\langle \mathscr{P}\psi(\mu_1, \mu_2), \mu_1 \rangle \ \leq \ \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}^T \begin{bmatrix} \mathscr{U} & 0 \\ 0 & \mathscr{V} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}. \tag{2.3}$$

Algorithm 1. In the following steps, we illustrate how to verify whether ψ is QL and how to determine the associated matrices.

Step 1: Let $p_1, p_2, \dots, p_o$ be the independent entries of $\mathscr{P} \in \mathbb{S}^{o_1}$ serving as decision variables, with $o = \frac{o_1(o_1+1)}{2}$. Apply the mean value theorem to obtain the structure of the matrices $\mathscr{U} \in \mathbb{S}^{o_1}$ and $\mathscr{V} \in \mathbb{S}^{o_1}$, whose entries are linear combinations of the decision variables $p_1, p_2, \dots, p_o$.

Step 2: If feasible solutions exist for the LMIs $\mathscr{P} > 0$ and $\mathscr{V} \geq 0$, then ψ is QL, and numerical values of the decision variables can be obtained.

Remark 1. For a given function ψ, any triplet $(\mathscr{P}, \mathscr{U}, \mathscr{V})$ satisfying Step 1 in Algorithm 1 will be referred to as a ψ-QL matrix triplet.

Consider the delayed nonlinear system given below:

$$\begin{cases} \dot{\zeta}(t) = \aleph_1\Big(\zeta(t), \zeta(t-\theta(t)), \xi(t), \varpi(t)\Big), \\ \eta(t) = \aleph_2\Big(\zeta(t), \zeta(t-\theta(t))\Big), \\ \zeta(t) = f(t), t \in [-\theta_2, 0], \end{cases} \tag{2.4}$$

where $\zeta(t) \in \mathbb{R}^{o_1}$ is the state vector, $\zeta(t-\theta(t))$ its delayed value, and $f(t) \in C\Big(\begin{bmatrix} -\theta_2 & 0 \end{bmatrix}, \mathbb{R}^{o_1}\Big)$ its initial condition, with $\theta(t)$ being an interval time-varying delay satisfying

$$0 \leq \theta_1 \leq \theta(t) \leq \theta_2, \ \dot{\theta}(t) \leq \theta_3 < 1; \tag{2.5}$$

$\xi(t) \in \mathbb{R}^{o_2}$ and $\eta(t) \in \mathbb{R}^{o_3}$ represent the control input and the measured output, respectively; $\varpi(t) \in \mathbb{R}^{o_4}$ corresponds to an exogenous input disturbance; $\aleph_1$ and $\aleph_2$ are nonlinear functions representing the system dynamics and sensor behavior, respectively.

In this paper, system (2.4) is represented by a TSQL model consisting of u rules, with the ℓ-th rule defined as follows:

Plant rule ℓ: If $\iota_1(t)$ is $I_{\ell 1}$ and $\cdots$ and $\iota_p(t)$ is $I_{\ell p}$ THEN

$$\begin{cases} \dot{\zeta}(t) = A_\ell \zeta(t) + B_\ell \zeta(t-\theta(t)) + C_\ell \xi(t) + D_\ell \varpi(t) + \psi(\zeta(t), \zeta(t-\theta(t))), \\ \eta(t) = Q_\ell \zeta(t) + R_\ell \zeta(t-\theta(t)), \\ \zeta(t) = f(t), t \in [-\theta_2, 0], \end{cases} \tag{2.6}$$

where $\iota_r(t)$ and $I_{\ell r}$ are the premise variables along with the fuzzy sets, for $(r, \ell) \in \{1, \ldots, p\} \times \{1, \ldots, u\}$; the system matrices A_ℓ, B_ℓ, C_ℓ D_ℓ, Q_ℓ, and R_ℓ are of appropriate dimensions; $\psi(\zeta(t), \zeta(t-\theta(t)))$ is a QL function.

Applying fuzzy blending, the resulting global TSQL model is derived as follows:

$$\begin{cases} \dot{\zeta}(t) = \sum\limits_{\ell=1}^{u} g_\ell(\iota(t))[A_\ell \zeta(t) + B_\ell \zeta(t-\theta(t)) + C_\ell \xi(t) + D_\ell \varpi(t)] + \psi(\zeta(t), \zeta(t-\theta(t))), \\ \eta(t) = \sum\limits_{\ell=1}^{u} g_\ell(\iota(t))[Q_\ell \zeta(t) + R_\ell \zeta(t-\theta(t))], \\ \zeta(t) = f(t), t \in [-\theta_2, 0], \end{cases} \tag{2.7}$$

where $g_\ell(\iota(t)) = \dfrac{\prod\limits_{r=1}^{p} I_{\ell r}(\iota_r(t))}{\sum\limits_{\ell=1}^{u} \prod\limits_{r=1}^{p} I_{\ell r}(\iota_r(t))}$ in which $\iota(t) = \begin{bmatrix} \iota_1(t) & \ldots & \iota_p(t) \end{bmatrix}$.

Based on the characteristics of the membership functions, the subsequent relationships hold true:

$$\sum_{\ell=1}^{u} \prod_{r=1}^{p} I_{\ell r}(\iota_r(t)) > 0 \text{ and } \prod_{r=1}^{p} I_{\ell r}(\iota_r(t)) \geq 0, \ \text{for } \ell \in \mathbb{I}^u, \tag{2.8}$$

thus yielding

$$g_\ell(\iota(t)) \geq 0, \ \sum_{\ell=1}^{u} g_\ell(\iota(t)) = 1. \tag{2.9}$$

Assumption 1. *Suppose that*

i. the TSQL system (2.4) is both controllable and observable,
ii. $\iota(t)$ is measurable,
iii. $\psi(0,0) = 0$.

Remark 2. This paper introduces, for the first time, a class of models termed the delayed TSQL model (2.6). This framework generalizes the models considered in [19–32], which are restricted to QL models without a TS representation. By incorporating the TS structure, our approach can capture a broader range of delayed nonlinear behaviors. It is important to note that while the premise variables are assumed to be measurable, the QL function itself may depend on unmeasurable states. Consequently, this function can effectively represent either intrinsic nonlinearities or lumped uncertainties within the system.

Remark 3. In the existing literature on delayed QL models, delay assumptions are typically restricted to two cases: Constant delays, as considered in [18–32], or time-varying delays satisfying $0 \leq \theta(t) \leq \theta_2$ as in [19–35]. This paper considers, instead, an interval time-varying delay $0 \leq \theta_1 \leq \theta(t) \leq \theta_2$ which provides information not only on the upper bound but also on the lower bound of the delay. Furthermore, unlike previous studies where delays affect only the system dynamics, this paper also considers delays occurring in the sensor.

3. FB control structure

Notations 2.

i. The time argument t is omitted in non-delayed vectors, while vectors subject to a time-varying delay $\theta(t)$ are denoted with an index θ, such that the delayed state $\zeta(t-\theta(t))$ is concisely written as ζ_θ.
ii. A hat on a vector, such as $\hat{\zeta}$, denotes its estimate. $g_\ell(\iota(t))$ is simply denoted by g_ℓ.
iii. Given $\zeta \in \mathbb{R}^{o_1}$, we define $\|\zeta\|^2 = \zeta^T\zeta$.
iv. We define the intervals $\Omega_1 := \begin{bmatrix} -\theta_2 & 0 \end{bmatrix}$ and $\Omega_2 := \begin{bmatrix} 0 & \mathcal{T} \end{bmatrix}$ where $\mathcal{T} \in \mathbb{R}_{>0}$.
v. $\varpi \in \mathcal{L}_2(\Omega_2) \ \Leftrightarrow \ \left(\int_{\Omega_2} \|\varpi(s)\|^2 ds \right)^{\frac{1}{2}} < +\infty$.

Definition 3.1. Let $h_1, \ h_2 \in \mathbb{R}_{>0}$, with $h_1 < h_2$, a class of signals $\mathscr{C}$, and Ω_2 as given in notations 2. The system (2.7) is FB wrt $(h_1, \ h_2, \ \mathscr{C}, \ \Omega_2)$ if

$$\sup_{s \in \Omega_1} \|f(s)\|^2 \leq h_1 \ \Rightarrow \ \|\zeta\|^2 < h_2, \ \forall t \in \Omega_2, \ \forall \varpi(.) \in \mathscr{C}, \tag{3.1}$$

where we consider the following class of signals:

$$\mathscr{C} := \{\varpi \in \mathcal{L}_2(\Omega_2) \mid \int_{\Omega_2} \|\varpi(s)\|^2 ds \leq b, \ b \in \mathbb{R}_{>0}\}. \tag{3.2}$$

The following fuzzy observer-based controller (FOC) is adopted to guarantee that system (2.7) is FB:
FOC rule: If ι_1 is $I_{\ell 1}$ and $\cdots$ and ι_p is $I_{\ell p}$ THEN

$$\begin{cases} \dot{\hat{\zeta}} = A_\ell \hat{\zeta} + B_\ell \hat{\zeta}_\theta + C_\ell \xi + L_\ell(\eta - \hat{\eta}) + \psi(\hat{\zeta}, \hat{\zeta}_\theta), \\ \hat{\eta} = Q_\ell \hat{\zeta} + R_\ell \hat{\zeta}_\theta, \\ \xi = K_\ell \hat{\zeta}, \\ \hat{\zeta} = \bar{f}, t \in [-\theta_2, 0], \end{cases} \tag{3.3}$$

where $\bar{f} \in C\Big(\Big[\begin{array}{cc} -\theta_2 & 0 \end{array}\Big], \mathbb{R}^{o_1}\Big)$ is the observer's initial condition such that

$$\sup_{s \in \Omega_1} \|\bar{f}(s)\|^2 = \sigma_1, \tag{3.4}$$

while $L_\ell \in \mathbb{R}^{o_1 \times o_3}$ and $K_\ell \in \mathbb{R}^{o_2 \times o_1}$ represent the FOC gains to be designed.
The resulting FOC is as follows:

$$\begin{cases} \dot{\hat{\zeta}} = \sum_{\ell=1}^{u} g_\ell [A_\ell \hat{\zeta} + B_\ell \hat{\zeta}_\theta + C_\ell \xi + L_\ell(\eta - \hat{\eta})] + \psi(\hat{\zeta}, \hat{\zeta}_\theta), \\ \hat{\eta} = \sum_{\ell=1}^{u} g_\ell [Q_\ell \hat{\zeta} + R_\ell \hat{\zeta}_\theta], \\ \xi = \sum_{\ell=1}^{u} g_\ell [K_\ell \hat{\zeta}], \\ \hat{\zeta} = \bar{f},\ t \in [-\theta_2, 0]. \end{cases} \tag{3.5}$$

Combining (2.7) and (3.5) yields

$$\begin{cases} \dot{\varsigma} = \sum_{\ell=1}^{u} \sum_{m=1}^{u} g_\ell g_m [X_{\ell m} \varsigma + Y_{\ell m} \varsigma_\theta + Z_\ell \varpi] + \Psi(\tilde{\zeta}), \\ \varsigma = \mathcal{F}, t \in [-\theta_2, 0], \end{cases} \tag{3.6}$$

where

$$\begin{aligned} \varsigma &= \begin{bmatrix} \zeta \\ \zeta - \hat{\zeta} \end{bmatrix},\ X_{\ell m} = \begin{bmatrix} A_\ell + C_\ell K_m & -C_\ell K_m \\ 0 & A_\ell - L_\ell Q_m \end{bmatrix}, \\ Y_{\ell m} &= \begin{bmatrix} B_\ell & 0 \\ 0 & B_\ell - L_\ell R_m \end{bmatrix},\ Z_\ell = \begin{bmatrix} D_\ell \\ D_\ell \end{bmatrix}, \\ \Psi(\tilde{\zeta}) &= \begin{bmatrix} \psi(\zeta, \zeta_\theta) \\ \psi(\zeta, \zeta_\theta) - \psi(\hat{\zeta}, \hat{\zeta}_\theta) \end{bmatrix},\ \mathcal{F} = \begin{bmatrix} f \\ f - \bar{f} \end{bmatrix}, \end{aligned}$$

in which $\tilde{\zeta} = \begin{bmatrix} \zeta^T & \zeta_\theta^T & \hat{\zeta}^T & \hat{\zeta}_\theta^T \end{bmatrix}^T$.

Remark 4. Definition 3.1 extends the notion of FS. In particular, when $\varpi = 0$, a system that is FB wrt $(h_1,\ h_2,\ \mathscr{C},\ \Omega_2)$ becomes FS wrt $(h_1,\ h_2,\ \Omega_2)$.

Remark 5. FS and LAS are conceptually different and mutually independent notions. The former concerns boundedness over a prescribed finite time interval, whereas the latter describes convergence as $t \to \infty$. Consequently, LAS does not imply FS, nor does FS ensure asymptotic convergence.

Remark 6. Existing works on OC of delayed QL systems have primarily focused on Lyapunov-based asymptotic [18–32] and exponential stability analyses [35]. However, ensuring that the system is FB in general, and FS in particular, remains an open problem for this class of systems. To the best of our knowledge, the present paper is the first to establish FBss results for delayed QL systems within an OC framework.

4. FB control design

Notations 3.

i. Entries marked $\bullet$ are symmetric; I represents the identity matrix of suitable size; $diag(.)$ denotes a block diagonal matrix.

ii. Given $\mathscr{P} \in \mathbb{R}^{o_1 \times o_1}$, $\underline{\lambda}(\mathscr{P})$ ($\overline{\lambda}(\mathscr{P})$) denotes the smallest (largest) eigenvalue of $\mathscr{P}$; $\{\mathscr{P}\}_\star := \mathscr{P} + \mathscr{P}^T$.

iii. Given $\zeta(t) \in \mathbb{R}^{o_1}$, we define $\zeta_t \in C\Big(\begin{bmatrix} -\theta_2 & 0 \end{bmatrix}, \mathbb{R}^{o_1}\Big)$ by $\zeta_t := \Big\{\zeta(t+\hbar),\ \hbar \in \begin{bmatrix} -\theta_2 & 0 \end{bmatrix}\Big\}$.

Theorem 4.1. *For given $\rho,\ h_1,\ h_2,\ b,\ \mathcal{T} \in \mathbb{R}_{>0}$, the system (2.7) is FB wrt* $(\mathrm{h}_1,\ \mathrm{h}_2,\ \mathscr{C},\ \Omega_2)$ *if there exist ψ-QL matrix triplets $(\mathscr{P}_i, \mathscr{U}_i, \mathscr{V}_i)$, $\mathscr{W}_{ji} \in \mathbb{S}^{o_1}$ with $i \in \mathbb{I}^2$ and $j \in \mathbb{I}^3$, $\mathrm{L}_\ell \in \mathbb{R}^{o_1 \times o_3}$ and $\mathrm{K}_\ell \in \mathbb{R}^{o_2 \times o_1}$ with $\ell \in \mathbb{I}^u$ satisfying the following inequalities:*

$$\mathscr{P}_i > 0,\ \mathscr{V}_i \geq 0,\ \mathscr{W}_{ji} > 0, \tag{4.1}$$

$$\Xi_{\ell m} < 0,\ \forall \ell, m \in \mathbb{I}^u, \tag{4.2}$$

$$\frac{e^{\rho\mathcal{T}}}{\nu_1}(\bar{\nu} + b) < h_2, \tag{4.3}$$

where

$$\Xi_{\ell m} = \begin{bmatrix} \{\mathscr{P}X_{\ell m} + \mathscr{U}\}_\star + \mathscr{W}_1 - \rho\mathscr{P} & \mathscr{P}Y_{\ell m} & 0 & 0 & \mathscr{P}Z_\ell \\ \bullet & \{\mathscr{V}\}_\star + \Lambda_3 & 0 & 0 & 0 \\ \bullet & \bullet & \Lambda_1 & 0 & 0 \\ \bullet & \bullet & \bullet & \Lambda_2 & 0 \\ \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix},$$

$$\bar{\nu} = \nu_2\bar{h}_1 + [\nu_3(e^{\rho\theta_1} - 1) + \nu_4(e^{\rho\theta_2} - e^{\rho\theta_1}) + \nu_5(e^{\rho\theta_2} - e^{\rho\theta_1})]\frac{\bar{h}_1}{\rho},\ \nu_1 = \underline{\lambda}(\mathscr{P}_1),$$

in which

$$\begin{aligned} \nu_2 &= \overline{\lambda}(\mathscr{P}),\ \nu_3 = \overline{\lambda}(\mathscr{W}_1),\ \nu_4 = \overline{\lambda}(\mathscr{W}_2),\ \nu_5 = \overline{\lambda}(\mathscr{W}_3), \\ \bar{h}_1 &= 2h_1 + \sigma_1 + 2\sqrt{h_1\sigma_1}, \\ \Lambda_1 &= e^{\rho\theta_1}(\mathscr{W}_2 + \mathscr{W}_3 - \mathscr{W}_1),\ \Lambda_2 = -e^{\rho\theta_2}\mathscr{W}_2,\ \Lambda_3 = -(1-\theta_3)e^{\rho\theta_1}\mathscr{W}_3, \end{aligned}$$

with

$$\mathscr{P} = diag(\mathscr{P}_1, \mathscr{P}_2),\ \mathscr{W}_j = diag(\mathscr{W}_{j1}, \mathscr{W}_{j2}),\ \mathscr{U} = diag(\mathscr{U}_1, \mathscr{U}_2),\ \mathscr{V} = diag(\mathscr{V}_1, \mathscr{V}_2).$$

Proof. Consider the LK functional of the form

$$\vartheta(\varsigma_t) \;=\; \vartheta_1(\varsigma) + \vartheta_2(\varsigma_t), \tag{4.4}$$

where

$$\begin{aligned}
\vartheta_1(\varsigma) &= \varsigma^T \mathscr{P} \varsigma, \\
\vartheta_2(\varsigma_t) &= \int_{t-\theta_1}^{t} e^{\rho(t-s)} \varsigma(s)^T \mathscr{W}_1 \varsigma(s) ds + \int_{t-\theta_2}^{t-\theta_1} e^{\rho(t-s)} \varsigma(s)^T \mathscr{W}_2 \varsigma(s) ds + \int_{t-\theta(t)}^{t-\theta_1} e^{\rho(t-s)} \varsigma(s)^T \mathscr{W}_3 \varsigma(s) ds.
\end{aligned}$$

From (3.6), we obtain

$$\dot{\vartheta}_1(\varsigma) \;=\; 2\varsigma^T \mathscr{P}\Big(\sum_{\ell=1}^{u} \sum_{m=1}^{u} g_\ell g_m [X_{\ell m}\varsigma + Y_{\ell m}\varsigma_\theta + Z_\ell \varpi] + \Psi(\tilde{\zeta})\Big). \tag{4.5}$$

Given that $(\mathscr{P}_i, \mathscr{U}_i, \mathscr{V}_i)$, for $i \in \mathbb{I}^2$, form ψ-QL matrix triplets, it follows that

$$\begin{aligned}
2\varsigma^T \mathscr{P}\Psi(\tilde{\zeta}) &= 2\langle \mathscr{P}_1 \psi(\zeta, \zeta_\theta), \zeta\rangle + 2\langle \mathscr{P}_2(\psi(\zeta, \zeta_\theta) - \psi(\hat{\zeta}, \hat{\zeta}_\theta)), \zeta - \hat{\zeta}\rangle \\
&\leq 2\varsigma^T \mathscr{U} \varsigma + 2\varsigma_\theta^T \mathscr{V} \varsigma_\theta.
\end{aligned} \tag{4.6}$$

Using the Leibniz integral rule in conjunction with (2.5), we obtain

$$\begin{aligned}
\dot{\vartheta}_2(\varsigma_t) &= \varsigma^T \mathscr{W}_1 \varsigma + \varsigma(t-\theta_1)^T \Lambda_1 \varsigma(t-\theta_1) + \varsigma(t-\theta_2)^T \Lambda_2 \varsigma(t-\theta_2) \\
&\quad - (1 - \dot{\theta}(t)) e^{\rho\theta(t)} \varsigma_\theta^T \mathscr{W}_3 \varsigma_\theta + \rho \vartheta_2(\varsigma_t),
\end{aligned} \tag{4.7}$$

in which

$$-(1 - \dot{\theta}(t)) e^{\rho\theta(t)} \varsigma_\theta^T \mathscr{W}_3 \varsigma_\theta \leq \varsigma_\theta^T \Lambda_3 \varsigma_\theta. \tag{4.8}$$

From (4.5)–(4.8), it follows that

$$\dot{\vartheta}(\varsigma_t) - \rho\vartheta(\varsigma_t) - \varpi^T \varpi \leq \sum_{\ell=1}^{u} \sum_{m=1}^{u} g_\ell g_m \tilde{\varsigma}^T \Xi_{\ell m} \tilde{\varsigma}, \tag{4.9}$$

in which $\tilde{\varsigma} = \begin{bmatrix} \varsigma^T & \varsigma_\theta^T & \varsigma(t-\theta_1)^T & \varsigma(t-\theta_2)^T & \varpi^T \end{bmatrix}^T$.
As (4.2) holds, we derive

$$\dot{\vartheta}(\varsigma_t) - \rho\vartheta(\varsigma_t) - \varpi^T \varpi < 0, \tag{4.10}$$

leading to

$$\dot{V}(\varsigma_t) < e^{-\rho t} \varpi^T \varpi, \tag{4.11}$$

where $V(\varsigma_t) = e^{-\rho t}\vartheta(\varsigma_t)$.
Upon integrating (4.11) between 0 and t, with $t \in \Omega_2$, we derive

$$V(\varsigma_t) - \vartheta(\varsigma_0) < \int_0^t e^{-\rho s} \varpi(s)^T \varpi(s) ds. \tag{4.12}$$

As $\rho \in \mathbb{R}_{>0}$ and $\varpi \in \mathscr{C}$ defined in (3.2), it follows from (4.12) that

$$V(\varsigma_t) < \vartheta(\varsigma_0) + b, \tag{4.13}$$

leading to

$$\vartheta(\varsigma_t) < e^{\rho t}(\vartheta(\varsigma_0) + b). \tag{4.14}$$

Additionally, we have

$$\begin{aligned} \vartheta(\varsigma_t) &\geq \nu_1 \|\zeta\|^2, && (4.15)\\ \vartheta(\varsigma_0) &\leq \nu_2 \|\mathcal{F}(0)\|^2 + \nu_3 \int_{-\theta_1}^{0} e^{-\rho s} \|\mathcal{F}(s)\|^2 ds + \nu_4 \int_{-\theta_2}^{-\theta_1} e^{-\rho s} \|\mathcal{F}(s)\|^2 ds \\ &\quad + \nu_5 \int_{-\theta(0)}^{-\theta_1} e^{-\rho s} \|\mathcal{F}(s)\|^2 ds. && (4.16) \end{aligned}$$

For any $s \in \Omega_1$, we have

$$\begin{aligned} \|\mathcal{F}(s)\|^2 &= \|f(s)\|^2 + \|f(s) - \bar{f}(s)\|^2 \\ &= 2\|f(s)\|^2 + \|\bar{f}(s)\|^2 - 2 f(s)^T \bar{f}(s). \end{aligned} \tag{4.17}$$

For $\sup\limits_{s \in \Omega_1} \|f(s)\|^2 \leq h_1$ and $\sup\limits_{s \in \Omega_1} \|\bar{f}(s)\|^2 = \sigma_1$, it follows from the Cauchy–Schwarz inequality that

$$\sup_{s \in \Omega_1} \|\mathcal{F}(s)\|^2 \leq \bar{h}_1, \tag{4.18}$$

hence

$$\begin{aligned} \int_{-\theta_1}^{0} e^{-\rho s} \|\mathcal{F}(s)\|^2 ds &\leq \frac{e^{\rho\theta_1} - 1}{\rho} \bar{h}_1, && (4.19)\\ \int_{-\theta_2}^{-\theta_1} e^{-\rho s} \|\mathcal{F}(s)\|^2 ds &\leq \frac{e^{\rho\theta_2} - e^{\rho\theta_1}}{\rho} \bar{h}_1, && (4.20)\\ \int_{-\theta(0)}^{-\theta_1} e^{-\rho s} \|\mathcal{F}(s)\|^2 ds &\leq \frac{e^{\rho\theta(0)} - e^{\rho\theta_1}}{\rho} \bar{h}_1 \leq \frac{e^{\rho\theta_2} - e^{\rho\theta_1}}{\rho} \bar{h}_1. && (4.21) \end{aligned}$$

As a result, (4.16) gives

$$\vartheta(\varsigma_0) \leq \bar{\nu}. \tag{4.22}$$

From (4.14), (4.15), and (4.22), it follows that

$$\nu_1 \|\zeta\|^2 < e^{\rho t}(\bar{\nu} + b), \tag{4.23}$$

which, according to (4.3), implies that

$$\|\zeta\|^2 < h_2, \ \forall t \in \Omega_2, \ \forall \varpi(.) \in \mathscr{C}. \tag{4.24}$$

Hence, the system (2.7) is confirmed to be FB wrt $(h_1, \ h_2, \ \mathscr{C}, \ \Omega_2)$. □

Remark 7. The condition $h_1 < h_2$ in Definition 3.1 ensures that the inner set of radius h_1 is contained within the outer set of radius h_2. Clearly, $\frac{e^{\rho\mathcal{T}}}{v_1}(\bar{v}+b) > h_1$ and therefore (4.3) implies $h_1 < h_2$.

This work presents the first study addressing the FBss problem for delayed TSQL systems via an OC. While this problem has been investigated for a few other classes of systems, certain limitations in these approaches are identified and addressed in the present study. In the following remarks, we discuss these improvements.

Remark 8. In recent works [30–34], terms of the form $\int_{-\theta_1}^{0} e^{-\rho s}\mathcal{F}(s)^T\mathscr{W}_1\mathcal{F}(s)ds$ are often upper-bounded as

$$\int_{-\theta_1}^{0} e^{-\rho s}\mathcal{F}(s)^T\mathscr{W}_1\mathcal{F}(s)ds \quad \leq \quad v_3\theta_1 e^{\rho\theta_1}\bar{h}_1, \tag{4.25}$$

which can introduce unnecessary conservatism, avoided in this work by applying the bounding in (4.19)–(4.21).

Remark 9. We note that the feasibility of (4.3) depends on the choice of the observer's initial condition, specifically σ_1. This differs from LAS, where the observer's initial conditions do not explicitly appear in the design conditions. In [29,30], the authors select the initial condition such that $\sigma_1 = h_1$. While this is a valid choice, in principle the initial condition of the observer can be freely chosen. Nevertheless, increasing σ_1 reduces the feasibility of inequality (4.3), and therefore, the optimal choice is $\sigma_1 = 0$.

In Theorem 4.1, the matrix inequalities are nonlinear and therefore difficult to solve directly. To address this issue, the following lemma is employed to convert them into LMIs.

Lemma 4.2. *[33] Consider matrices $\tilde{\Xi}$, Γ_1, Γ_2, and Γ_3 with suitable dimensions and $\tau \in \mathbb{R}\setminus\{0\}$. Then*

$$\begin{bmatrix} \tilde{\Xi} & \tau\Gamma_1 + \Gamma_2^T\Gamma_3^T \\ * & -\tau(\{\Gamma_3\}_\star) \end{bmatrix} \prec 0 \;\Rightarrow\; \tilde{\Xi} + \{\Gamma_1\Gamma_2\}_\star \prec 0. \tag{4.26}$$

The following theorem presents the LMIs, from which the gain matrices can be readily obtained.

Theorem 4.3. *For given $\tau \in \mathbb{R}\setminus\{0\}$, $\rho,\ h_1,\ h_2,\ b,\ \mathcal{T} \in \mathbb{R}_{>0}$, the system (2.7) is FB wrt $(\mathrm{h}_1,\ \mathrm{h}_2,\ \mathscr{C},\ \Omega_2)$ if there exist $w_k \in \mathbb{R}$ with $k \in \mathbb{I}^5$, ψ-QL matrix triplets $(\mathscr{P}_i, \mathscr{U}_i, \mathscr{V}_i)$, $\Gamma_3 \in \mathbb{R}^{o_2\times o_2}$, $\mathscr{W}_{ji} \in \mathbb{S}^{o_1}$ with $i \in \mathbb{I}^2$ and $j \in \mathbb{I}^3$, $\tilde{L}_\ell \in \mathbb{R}^{o_1\times o_3}$ and $\tilde{K}_\ell \in \mathbb{R}^{o_2\times o_1}$ with $\ell \in \mathbb{I}^u$ satisfying (4.1) and the following inequalities:*

$$w_k > 0,\ k \in \mathbb{I}^5, \tag{4.27}$$

$$\mathscr{P}_1 - w_1 I \succ 0,\ w_2 I - \mathscr{P} \succ 0,\ w_3 I - \mathscr{W}_1 \succ 0,\ w_4 I - \mathscr{W}_2 \succ 0,\ w_5 I - \mathscr{W}_3 \succ 0, \tag{4.28}$$

$$\begin{bmatrix} \tilde{\Xi}_{\ell m} & \tilde{\Gamma}_{\ell m} \\ * & -\tau(\{\Gamma_3\}_\star) \end{bmatrix} \prec 0,\ \forall \ell, m \in \mathbb{I}^u, \tag{4.29}$$

$$e^{\rho\mathcal{T}}(\bar{w}+b) - h_2 w_1 < 0, \tag{4.30}$$

where

$$\tilde{\Xi}_{\ell m} = \begin{bmatrix} \tilde{\Xi}_{\ell m}(1,1) & -C_\ell \tilde{K}_m & \mathscr{P}_1 B_\ell & 0 & 0 & 0 & \mathscr{P}_1 D_\ell \\ \bullet & \tilde{\Xi}_{\ell m}(2,2) & 0 & \mathscr{P}_2 B_\ell - \tilde{L}_\ell R_m & 0 & 0 & \mathscr{P}_2 D_\ell \\ \bullet & \bullet & \{\mathscr{V}_1\}_\star + \Lambda_{31} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \{\mathscr{V}_2\}_\star + \Lambda_{32} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \Lambda_1 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \Lambda_2 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix},$$

$$\tilde{\Gamma}_{\ell m} = \begin{bmatrix} \tau \mathcal{P}_1 C_\ell - \tau C_\ell \Gamma_3 + \tilde{K}_m^T \\ -\tilde{K}_m^T \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\bar{w} = w_2 \bar{h}_1 + [w_3(e^{\rho\theta_1} - 1) + w_4(e^{\rho\theta_2} - e^{\rho\theta_1}) + w_5(e^{\rho\theta_2} - e^{\rho\theta_1})]\frac{\bar{h}_1}{\rho},$$

in which

$$\begin{aligned} \tilde{\Xi}_{\ell m}(1,1) &= \{\mathscr{P}_1 A_\ell + C_\ell \tilde{K}_m + \mathscr{U}_1\}_\star + \mathscr{W}_{11} - \rho \mathscr{P}_1, \\ \tilde{\Xi}_{\ell m}(2,2) &= \{\mathscr{P}_2 A_\ell - \tilde{L}_\ell Q_m + \mathscr{U}_2\}_\star + \mathscr{W}_{12} - \rho \mathscr{P}_2, \\ \Lambda_{3i} &= -(1-\theta_3)e^{\rho\theta_1}\mathscr{W}_{3i},\ i \in \mathbb{I}^2. \end{aligned}$$

In this case, the OC gains are

$$K_m = \Gamma_3^{-1}\tilde{K}_m,\ L_\ell = \mathscr{P}_2^{-1}\tilde{L}_\ell. \tag{4.31}$$

Proof. The proof is carried out in two stages.
In the first part, we show that (4.29) implies (4.2).
From (4.29), we have $-\tau(\{\Gamma_3\}_\star) < 0$ for $\tau \in \mathbb{R} \setminus \{0\}$, which ensures that Γ_3 is invertible.
$\tilde{\Gamma}_{\ell m}$ takes the following form:

$$\tilde{\Gamma}_{\ell m} = \tau \Gamma_{\ell m1} + \Gamma_{m2}^T \Gamma_3^T, \tag{4.32}$$

where

$$\Gamma_{\ell m1} = \begin{bmatrix} \mathcal{P}_1 C_\ell - C_\ell \Gamma_3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \Gamma_{m2} = \Gamma_3^{-1}\begin{bmatrix} \tilde{K}_m & -\tilde{K}_m & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

According to Lemma 4.2, condition (4.29) implies

$$\tilde{\tilde{\Xi}}_{\ell m} + \{\Gamma_{\ell m1}\Gamma_{m2}\}_\star \prec 0. \tag{4.33}$$

The preceding inequality coincides with (4.2), with $\Xi_{\ell m}$ written in the expanded form as follows:

$$\Xi_{\ell m} = \begin{bmatrix} \Xi_{\ell m}(1,1) & -\mathscr{P}_1 C_\ell K_m & \mathscr{P}_1 B_\ell & 0 & 0 & 0 & \mathscr{P}_1 D_\ell \\ \bullet & \tilde{\Xi}_{\ell m}(2,2) & 0 & \mathscr{P}_2 B_\ell - \tilde{L}_\ell R_m & 0 & 0 & \mathscr{P}_2 D_\ell \\ \bullet & \bullet & \{\mathscr{V}_1\}_\star + \Lambda_{31} & 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \{\mathscr{V}_2\}_\star + \Lambda_{32} & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \Lambda_1 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \Lambda_2 & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & -I \end{bmatrix}, \tag{4.34}$$

where

$$\Xi_{\ell m}(1,1) = \{\mathscr{P}_1 A_\ell + \mathscr{P}_1 C_\ell K_m + \mathscr{U}_1\}_\star + \mathscr{W}_{11} - \rho \mathscr{P}_1.$$

For the second part, it follows directly from (4.28) and (4.30) that (4.3) holds. □

Remark 10. The singular value decomposition technique, which is widely used in the literature [30] as a decoupling approach, can also be applied in this work. However, in our case, it has two main limitations. First, the input matrices must be identical, i.e., $C_1 = \ldots = C_u = C$. Second, the input matrix C must have full column rank. By using this method, $\mathscr{P}_1$ is constrained to have the block-diagonal form $\mathscr{P}_1 = \mathrm{diag}(\mathscr{P}_{111}, \mathscr{P}_{122})$, where $\mathscr{P}_{111} \in \mathbb{S}^{o_2}$ and $\mathscr{P}_{122} \in \mathbb{S}^{o_1 - o_2}$. Consequently, the number of decision variables in $\mathscr{P}_1$ is reduced to $\frac{o_2(o_2+1)}{2} + \frac{(o_1-o_2)(o_1-o_2+1)}{2}$, which decreases the degrees of freedom in the design and therefore introduces additional conservatism.

Remark 11. Influence of parameters

i. The parameters h_1, h_2, b, $\mathcal{T} \in \mathbb{R}_{>0}$ determine the performance of the FBss controller. In particular, reducing h_2 limits the transient response, for example, to prevent the excessive excitation during transients.
ii. The parameter ρ introduces a trade-off between LMIs (4.29) and (4.30). On one hand, increasing ρ generally enhances the feasibility of (4.29) due to the presence of the negative terms $-\rho\mathscr{P}_1$ in $\Xi_{\ell m}(1,1)$ and $-\rho\mathscr{P}_2$ in $\Xi_{\ell m}(2,2)$, which contribute positively to satisfying the corresponding matrix inequalities. On the other hand, the function $g(\rho) = e^{\rho\mathcal{T}}(\bar{w} + b)$ is strictly increasing with ρ. As a result, increasing ρ forces the optimal value of h_2 to increase in order to satisfy the feasibility of (4.30). This, however, contradicts the goal of FBss, which is to keep h_2 as small as possible so that the system states remain within the prescribed bounds during the transient interval. Consequently, the optimal choice of ρ is the smallest value that still guarantees the feasibility of (4.29).
iii. The tuning parameter τ, introduced as the cost of the decoupling technique, is chosen using a logarithmic search strategy, following the method described in [6].

5. Numerical example

Consider the TSQL model with two fuzzy rules, as detailed below:

$$A_1 = A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.5 & 1 \end{bmatrix},\ B_1 = B_2 = \begin{bmatrix} 0.1 & 0 & 0.2 \\ 0 & 0.1 & 0.2 \\ 0 & 0 & 0.1 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ C_2 = \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix},$$

$$D_1 = D_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ Q_1 = Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9 & 0 \end{bmatrix},\ R_1 = R_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1 & 0 \end{bmatrix},$$

$$\psi(\zeta(t), \zeta(t-\theta(t))) = \begin{bmatrix} 2\sin(\zeta_1) + 0.5\sin(\zeta_1(t-\theta(t)))) \\ -2\zeta_2 + 0.5\sin(\zeta_2(t-\theta(t)))) \\ -2\zeta_3 + 0.5\sin(\zeta_3(t-\theta(t)))) \end{bmatrix},$$

$$\theta(t) = 0.06 + 0.01\sin(t),\ (\theta_1 = 0.05,\ \theta_2 = 0.07,\ \theta_3 = 0.01),$$

$$\varpi(t) = \sqrt{0.5}\cos(\pi t),$$

$$g_1 = \frac{1}{1 + exp(\zeta_1 + 0.7)},\ g_2 = 1 - g_1.$$

We consider the following FB parameters:

$$h_1 = 1,\ h_2 = 2.3,\ \mathcal{T} = 1,\ b = 0.5. \tag{5.1}$$

Figure 1 shows the outer set of radius h_2 and the 3-D phase plot for $\xi = 0$, corresponding to the following three initial conditions within the inner set of radius $h_1 = 1$:

$$f_1(t) = \begin{bmatrix} -0.1 & 0.7 & 0.5 \end{bmatrix}, f_2(t) = \begin{bmatrix} -0.7 & 0.1 & 0.6 \end{bmatrix}, f_3(t) = \begin{bmatrix} 0.8 & -0.1 & 0.3 \end{bmatrix}, t \in \Omega_1. \tag{5.2}$$

This figure clearly illustrates that, for $\xi = 0$, the system is not FB with parameters provided in (5.1) because the trajectories starting from the inner set exit the outer set.

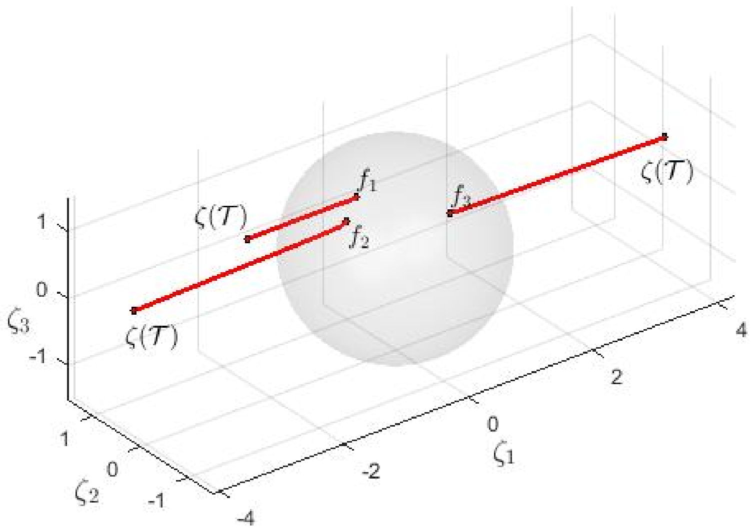

Figure 1. Outer set and state-space trajectories without control input.

Following Step 1 in Algorithm 1, $\mathscr{P}_i$, $\mathscr{U}_i$, and $\mathscr{V}_i$ take the following structures, where $p_{i11}, \ldots, p_{i33}$ are the decision variables to be determined:

$$\mathscr{P}_i = \begin{bmatrix} p_{i11} & p_{i12} & p_{i13} \\ p_{i12} & p_{i22} & p_{i23} \\ p_{i13} & p_{i23} & p_{i33} \end{bmatrix}, \ \mathscr{U}_i = \begin{bmatrix} \mathscr{U}_i(1,1) & -p_{i12} & -p_{i13} \\ -p_{i12} & \mathscr{U}_i(2,2) & -2p_{i23} \\ -p_{i13} & -2p_{i23} & \mathscr{U}_i(3,3) \end{bmatrix}, \tag{5.3}$$

$$\mathscr{V}_i = diag\Big(\mathscr{V}_i(1,1), \mathscr{V}_i(2,2), \mathscr{V}_i(3,3)\Big), \tag{5.4}$$

with

$$\begin{aligned}
\mathscr{U}_i(1,1) &= 2.25p_{i11} + 1.25p_{i12} + 1.25p_{i13},\ \mathscr{U}_i(2,2) = 1.25p_{i12} - 1.75p_{i22} + 0.25p_{i23},\\
\mathscr{U}_i(3,3) &= 1.25p_{i13} + 0.25p_{i23} - 1.75p_{i33},\ \mathscr{V}_i(1,1) = 0.25p_{i11} + 0.25p_{i12} + 0.25p_{i13},\\
\mathscr{V}_i(2,2) &= 0.25p_{i12} + 0.25p_{i22} + 0.25p_{i23},\ \mathscr{V}_i(3,3) = 0.25p_{i13} + 0.25p_{i23} + 0.25p_{i33}.
\end{aligned}$$

By solving the LMI conditions with the FB parameters given in (5.1), we obtain the following feasible solution for $\sigma = 0$, $\rho = 10^{-3}$, and $\tau = 100$:

$$\begin{aligned}
&w_1 = 8.8314,\ w_2 = 9.0622,\ w_3 = 11.0814,\ w_4 = 4.7106,\ w_5 = 8.4734,\ \Gamma_3 = 4.2296,\\
&p_{111} = 8.9559,\ p_{112} = -0.0644,\ p_{113} = -0.0359,\ p_{122} = 8.9677,\ p_{123} = -0.0601,\ p_{133} = 8.9630,\\
&p_{211} = 3.6350,\ p_{212} = -1.8910,\ p_{213} = -1.2689,\ p_{222} = 4.9533,\ p_{223} = -1.7267,\ p_{233} = 5.6832,\\
&\mathscr{W}_{11} = \begin{bmatrix} 10.3715 & -0.0341 & -0.0141 \\ -0.0341 & 7.0565 & -1.7269 \\ -0.0141 & -1.7269 & 9.5631 \end{bmatrix},\ \mathscr{W}_{12} = \begin{bmatrix} 10.1317 & 0.0038 & 0 \\ 0.0038 & 10.0955 & -0.0386 \\ 0 & -0.0386 & 10.0151 \end{bmatrix},\\
&\mathscr{W}_{21} = \begin{bmatrix} 2.1235 & -0.0055 & -0.0125 \\ -0.0055 & 0.8694 & -0.9258 \\ -0.0125 & -0.9258 & 1.1400 \end{bmatrix},\ \mathscr{W}_{22} = \begin{bmatrix} 2.6318 & -0.0052 & 0 \\ -0.0052 & 2.5770 & 0.0025 \\ 0 & 0.0025 & 2.5806 \end{bmatrix},\\
&\mathscr{W}_{31} = \begin{bmatrix} 6.2443 & -0.0248 & 0.0119 \\ -0.0248 & 5.4019 & 0.0440 \\ 0.0119 & 0.0440 & 7.3910 \end{bmatrix},\ \mathscr{W}_{32} = \begin{bmatrix} 4.6153 & 0.0202 & 0 \\ 0.0202 & 4.7550 & -0.0461 \\ 0 & -0.0461 & 4.6634 \end{bmatrix},\\
&\tilde{L}_1 = \tilde{L}_2 = \begin{bmatrix} 350.0407 & -0.3838 \\ -4.5295 & 7.4329 \\ -2.5317 & 0.5507 \end{bmatrix},\ \tilde{K}_1 = \begin{bmatrix} -267.6847 & 2.1949 & 1.2262 \end{bmatrix},\\
&\tilde{K}_2 = \begin{bmatrix} -244.0783 & 0.6084 & 0.4086 \end{bmatrix}.
\end{aligned}$$

Accordingly, the OC gains are

$$L_1 = L_2 = \begin{bmatrix} 158.9546 & 1.6714 \\ 80.5130 & 2.5753 \\ 59.5058 & 1.2525 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} -63.2877 & 0.5189 & 0.2899 \end{bmatrix},\ K_2 = \begin{bmatrix} -57.7065 & 0.1438 & 0.0966 \end{bmatrix}. \tag{5.5}$$

Figure 2 shows the outer set of radius h_2 and the 3-D phase plot under the designed OC, corresponding to the three initial conditions given in (5.2). This figure shows that all state trajectories starting within the inner set remain inside the outer set for $t \in \Omega_2$.

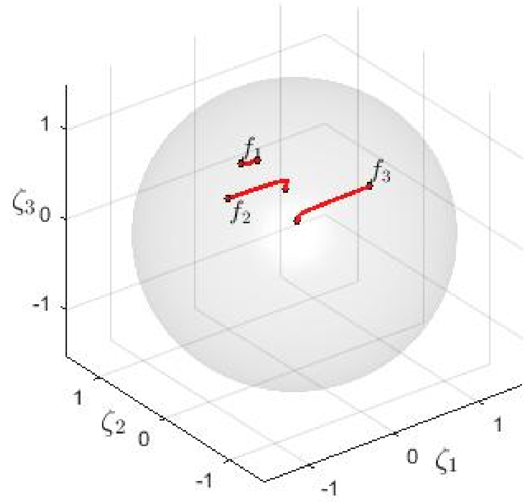

Figure 2. Outer set and state-space trajectories with the designed OC.

For $\rho = 10^{-3}$ and $\tau = 100$, the minimum value of h_2 is 2.2 when $\sigma = 0$. In contrast, selecting $\sigma = h_1$, as proposed in [29,30], results in a minimum h_2 of 5.4, highlighting the significance of choosing $\sigma = 0$.

6. Conclusions

In this work, the FBss problem for nonlinear systems subject to interval time-varying delay is examined. The adopted model incorporates two distinct classes of nonlinearities. The first class is associated with measurable premise variables and is represented through a TS fuzzy model. The second class may depend on unmeasurable variables and is described by a QL function, a formulation particularly suitable for capturing intrinsic nonlinear dynamics or lumped uncertainties within the system. An OC is designed to ensure the FBss of the considered system. An appropriate LK functional is employed to establish sufficient conditions for the design of the OC. These conditions are then transformed, using a decoupling technique, into LMIs that can be solved in a single step. The proposed sufficient conditions are shown to depend on the observer's initial conditions. In particular, selecting a zero initial condition is demonstrated to enhance the performance of the finite-time controller. To demonstrate the applicability of the present results, an illustrative example is included. Extending the current approach to encompass OC for delayed QL singular nonlinear systems is a direction for future study.

Author contributions

O. Kahouli: Methodology, writing original draft preparation; L. El Amraoui: Conceptualization, software; M. Ayari: Visualization, validation; H. Gassara: Methodology, writing original draft preparation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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