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Research article

On combinatorial properties of one type of Hessenberg matrix and the Leonardo numbers

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Abstract: In this paper, we introduced and investigated a Hessenberg matrix T_n, which incorporates both local interactions of tridiagonal matrices and fixed-distance non-local couplings, thereby exhibiting a hybrid Hessenberg–tridiagonal structure. We established a recurrence relation for its determinant and demonstrate that the determinant of T_n coincides with the Leonardo numbers. As a consequence, alternative representations involving Fibonacci numbers and Chebyshev polynomials are also obtained. The characteristic polynomial of T_n is derived through a Schur complement approach, and its spectral properties are examined in detail. The eigenvalue distribution reveals non-Hermitian features, including the presence of complex conjugate pairs and oscillatory instabilities. Moreover, an explicit LU factorization is obtained, enabling efficient computation of determinants and shedding light on the sparse structural properties of the matrix. Numerical experiments and visualization of eigenvalue behavior confirm the theoretical findings and highlight the potential of T_n as a model for structured non-Hermitian operators relevant in dynamical systems, wave propagation, and transport phenomena.

Keywords: Hessenberg matrix; Leonardo numbers; structured matrices; characteristic polynomial; spectral analysis; *LU* decomposition
Mathematics Subject Classification: 05A15, 05A19, 11B39, 15A15, 15B36

1. Introduction

Structured matrices such as tridiagonal and Hessenberg matrices play a fundamental role in numerical linear algebra due to their well-defined algebraic properties and computational efficiency. These matrices commonly arise in the discretization of differential equations, signal processing, eigenvalue computations, and the numerical solution of partial differential equations. Their banded and sparse structure allows for fast algorithms, reduced storage complexity, and predictable spectral characteristics, making them indispensable in both theoretical and applied contexts. Among these structured forms, Hessenberg matrices occupy a particularly important place. An upper Hessenberg matrix is a nearly triangular matrix where all entries below the first subdiagonal are zero. They naturally arise during the reduction of general matrices into simpler forms such as in the QR algorithm and serve as intermediate steps in solving eigenvalue problems. In the symmetric case, Hessenberg reduction yields tridiagonal matrices, which can be solved or analyzed with even greater efficiency using dedicated methods such as the Thomas algorithm.

In this study, we introduce and investigate a novel structured matrix T_n, which combines features from both tridiagonal and Hessenberg matrices but does not fall neatly into either category. Specifically, the matrix is defined by the following pattern of non-zero entries:

$$t_{11} = t_{12} = t_{13} = 1, \quad t_{m,m} = 2, \quad t_{m,m-1} = 1, \quad t_{m,m+2} = -1 \quad \text{for } m = 2, 3, \ldots, n,$$

with all other entries equal to zero. The resulting $n \times n$ matrix takes the form:

$$T_n = \begin{bmatrix} 1 & 1 & 1 & & & 0 \\ 1 & 2 & 0 & -1 & & \\ & 1 & 2 & 0 & \ddots & \\ & & 1 & 2 & \ddots & -1 \\ & & & \ddots & \ddots & 0 \\ 0 & & & & 1 & 2 \end{bmatrix}. \tag{1.1}$$

This matrix structure exhibits both *local interactions* (through subdiagonal and diagonal elements) and *non-local coupling* (through the $t_{m,m+2} = -1$ term), making it a compelling object of study both algebraically and numerically.

Motivation and related work

The determinant and spectral analysis of tridiagonal and Hessenberg matrices have been the subject of extensive research. In particular, there has been significant progress in expressing the determinants of such matrices in terms of recurrence relations, generating functions, and combinatorial sequences.

Qi and Kouba [1] provided elegant determinant formulas for certain classes of Hessenberg matrices using binomial coefficients and falling factorials. Leerawat and Daowsud [2] analyzed specific lower Hessenberg matrices whose entries are related to number sequences, expressing their determinants via generating functions. Goy and Shattuck [3] explored Toeplitz–Hessenberg matrices with generalized Leonardo sequences using Trudi's formula. Mirashe et al. [4] derived determinant formulas for matrices with specified diagonal and off-diagonal patterns. Expanding the combinatorial landscape, Merca [5] expressed determinants of Toeplitz–Hessenberg matrices through integer partitions and multinomial coefficients. Goy and Shattuck further extended this line of research to Catalan [6] and Motzkin sequences [7]. Additionally, Słowik [8] provided inversion formulas for Toeplitz–Hessenberg matrices, deriving both determinant and inverse expressions.

From a computational standpoint, classical algorithms such as the Thomas algorithm solve tridiagonal systems in $O(n)$ time, while Hessenberg matrices form the basis for QR-based eigenvalue solvers. Special Hessenberg matrices like the Frank and Grcar matrices are often used as benchmark examples and are known for their complex spectral behavior [9]. Recent developments have also highlighted the role of algebraic matrix methods in the analysis of logical systems and finite state machines. In particular, the semi-tensor product (STP) framework provides an algebraic state-space representation that enables the systematic study of structured matrices arising in logical networks and discrete-event systems. A comprehensive survey of these developments is given in [10]. Although the present work is based on classical matrix analysis rather than the STP framework, both approaches demonstrate the broad applicability of algebraic matrix methods to structured systems.

Despite this rich body of work, there is a noticeable gap in the literature regarding matrices like T_n that contain both tridiagonal-like local structure and fixed-distance non-local interactions. To the best of our knowledge, no prior study has examined the algebraic and spectral properties of a matrix that includes a fixed super-superdiagonal while retaining a Hessenberg-type sparsity.

To facilitate the presentation of our main results, we briefly recall the Leonardo numbers (see, e.g., [3]). The Leonardo sequence $\{Le_n\}_{n\geq 1}$ is defined recursively by

$$Le_1 = 1, \qquad Le_2 = 1, \qquad Le_n = Le_{n-1} + Le_{n-2} + 1, \quad n \geq 3.$$

As will be shown in the next section, this sequence naturally arises as the determinant sequence of the structured Hessenberg matrix T_n.

Scope and contributions of this study

The main contributions of this paper are as follows:

- We define the matrix T_n and explore its structure in detail.
- We derive a recurrence relation for $\det(T_n)$ and present closed-form expressions for small values of n.
- We analyze invertibility conditions, rank, and nullity under various parameter settings.
- We study the spectral properties of T_n, including potential symmetries and eigenvalue behavior.
- We discuss numerical and algorithmic implications, especially with regard to solving systems involving T_n.

The remainder of the paper is organized as follows. In Section 2, we present the main results concerning the modified tridiagonal matrix T_n. This includes determinant formulas expressed in terms of Leonardo numbers, Fibonacci numbers, and Chebyshev polynomials, as well as the derivation of the characteristic polynomial, spectral properties, and eigenvalue distribution. In the same section, we also provide the LU factorization of T_n and illustrate the theoretical findings with explicit examples. Section 3 contains computational implementations and numerical verifications, including MATLAB codes for characteristic polynomial evaluation, eigenvalue computation, and visualization of spectral behavior. Finally, the paper ends with concluding remarks that summarize the main contributions and highlight potential directions for future work.

2. Main results

In this section, we investigate the main algebraic and spectral properties of the structured Hessenberg matrix T_n defined in Eq (1.1). We first establish a determinant formula in terms of the Leonardo numbers, and then derive several related results, including equivalent representations, the characteristic polynomial, spectral properties, and an explicit LU factorization.

Proposition 1. *For the matrix T_n defined in Equation (1.1), the determinant satisfies*

$$\det(T_n) = Le_n,$$

where Le_n denotes the nth Leonardo number.

Proof. For $n = 1$ and $n = 2$, the following equalities hold $D_1 = \det(T_1) = 1$, $D_2 = \det(T_2) = 1$. Assume that $n \geq 3$. By expanding the determinant of T_n along the first row, we obtain

$$D_n = \det(M_{11}) - \det(M_{12}) + \det(M_{13}),$$

where M_{1j} denotes the $(n-1) \times (n-1)$ minor obtained by deleting the first row and the jth column of T_n. The minors M_{11} and M_{12} have identical last $n-2$ columns and differ only in their first column. Let M denote this common block and let b and c denote the first columns of M_{11} and M_{12}, respectively. Hence,

$$M_{11} = [\, b \mid M \,], \qquad M_{12} = [\, c \mid M \,],$$

where

$$b = \begin{bmatrix} 2 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Using the linearity of the determinant with respect to the first column,

$$\det(M_{11}) - \det(M_{12}) = \det([\, b - c \mid M \,]).$$

Since

$$b - c = \begin{bmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

expanding the resulting determinant along the first row yields

$$\det(M_{11}) - \det(M_{12}) = D_{n-1}.$$

Next, consider the minor M_{13}. Its first column contains exactly two nonzero entries. Expanding the determinant along this column produces two contributions. The first contribution is precisely the determinant D_{n-2}, while the second contribution corresponds to a lower triangular block whose determinant equals 1. Consequently,

$$\det(M_{13}) = D_{n-2} + 1.$$

Therefore,

$$D_n = D_{n-1} + D_{n-2} + 1, \qquad n \geq 3.$$

Since this recurrence together with the initial conditions $D_1 = 1$, $D_2 = 1$, coincides with the defining recurrence of the Leonardo sequence, it follows by the uniqueness of solutions of linear recurrence relations that $D_n = Le_n$. Hence, $\det(T_n) = Le_n$. This completes the proof. □

The following connection provides a direct combinatorial interpretation of the determinant through the well-studied Fibonacci sequence, establishing a clear link between the matrix structure and classical integer sequences. Since the Leonardo numbers Le_n can be expressed in terms of the Fibonacci numbers F_n by the relation $Le_n = 2F_{n+1} - 1$, it follows that the determinant of the matrix T_n satisfies

$$\det(T_n) = 2F_{n+1} - 1.$$

The following explicit formula elegantly expresses $\det(T_n)$ in terms of Chebyshev polynomials of the second kind, confirming the deep connection between the spectral properties of T_n and classical orthogonal polynomials. For the matrix T_n defined above, the determinant satisfies

$$\det(T_n) = 2i^{1-n}U_n\left(\frac{i}{2}\right) - 1,$$

where i denotes the imaginary unit and U_{n-1} denotes the $(n-1)-th$ Chebyshev polynomial of second kind.

These formulas enable the representation of the determinant in terms of Fibonacci numbers, Leonardo numbers and Chebyshev polynomials, providing a powerful analytical tool for understanding the matrix structure and computing its determinant. The following theorem is derived through a synthesis of the Schur complement approach and the recursive structure of the associated determinants, and it plays a critical role in the spectral analysis of the matrix.

Theorem 1. *Assume that the characteristic equation $r^3 - ar^2 + 1 = 0$ has three distinct roots r_1, r_2, r_3, where $a = \lambda - 2$. Then the characteristic polynomial of T_n admits the representation*

$$p_n(\lambda) = \sum_{j=1}^{3} \alpha_j \left((a+1)r_j^{n-1} - r_j^{n-2} + r_j^{n-3}\right),$$

where the constants $\alpha_1, \alpha_2, \alpha_3$ are determined by the initial conditions.

Proof. Using the definition of the characteristic polynomial $p_n(\lambda) = \det(\lambda I - T_n)$. To proceed, we define $a = \lambda - 2$ and introduce the auxiliary $k \times k$ tridiagonal matrix

$$D_k = \begin{bmatrix} a & 0 & 1 & 0 & \cdots & 0 \\ -1 & a & 0 & 1 & \ddots & \vdots \\ 0 & -1 & a & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & a & 0 \\ 0 & \cdots & \cdots & 0 & -1 & a \end{bmatrix},$$

where $a = \lambda - 2$, and define $d_k = \det(D_k)$. These auxiliary determinants satisfy a third-order recurrence relation given by

$$d_k = ad_{k-1} - d_{k-3}, \quad \text{and for} \quad k \geq 3,$$

with initial conditions $d_0 = 1$, $d_1 = a$, $d_2 = a^2$. We partition the matrix $\lambda I_n - T_n$ into the block form

$$\lambda I_n - T_n = \begin{bmatrix} \lambda - 1 & B \\ C & D_{n-1} \end{bmatrix},$$

where $B = (-1, -1, 0, \dots, 0)$ and $C = (-1, 0, \dots, 0)^T$. Using the Schur complement formula, we obtain

$$p_n(\lambda) = \det(D_{n-1}) \cdot \det\Big((\lambda - 1) - BD_{n-1}^{-1}C\Big).$$

The remaining term is evaluated by expressing the relevant cofactors of D_{n-1} recursively in terms of the auxiliary determinants d_k. Consequently,

$$p_n(\lambda) = (a + 1)d_{n-1} - d_{n-2} + d_{n-3}.$$

Furthermore, the recurrence relation for d_k can be solved in closed form. Since the characteristic equation has three distinct roots r_1, r_2, r_3, the general solution of the recurrence is

$$d_k = \alpha_1 r_1^k + \alpha_2 r_2^k + \alpha_3 r_3^k.$$

The constants $\alpha_1, \alpha_2, \alpha_3$ are determined from the initial conditions

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 1, \\ \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 = a, \\ \alpha_1 r_1^2 + \alpha_2 r_2^2 + \alpha_3 r_3^2 = a^2. \end{cases}$$

Thus, the characteristic polynomial admits the explicit closed form

$$\begin{aligned} p_n(\lambda) &= (a + 1)(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1} + \alpha_3 r_3^{n-1}) - (\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2} + \alpha_3 r_3^{n-2}) + (\alpha_1 r_1^{n-3} + \alpha_2 r_2^{n-3} + \alpha_3 r_3^{n-3}) \\ &= \sum_{j=1}^{3} \alpha_j \left((a + 1) r_j^{n-1} - r_j^{n-2} + r_j^{n-3}\right). \end{aligned}$$

This completes the proof. □

Remark 1. *The representation obtained in Theorem 1 is valid whenever the characteristic equation $r^3 - ar^2 + 1 = 0$ has three distinct roots. If repeated roots occur, the corresponding generalized solution containing polynomial factors associated with the repeated roots should be used.*

2.1. Numerical spectral analysis

To illustrate the theoretical results established in Theorem 1, we numerically investigate the characteristic polynomial and the corresponding eigenvalue distribution of the matrix T_n. The purpose of this subsection is to validate the theoretical analysis and to visualize the evolution of the spectrum as the matrix dimension increases.

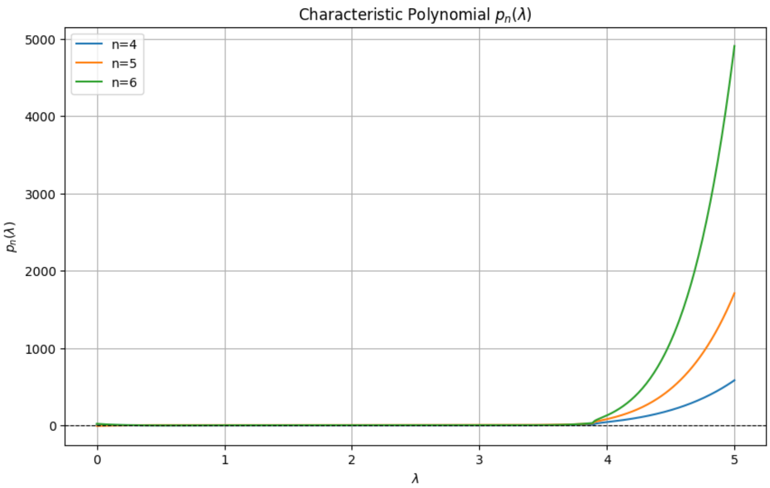

Figure 1. Numerical plots of the characteristic polynomial of T_n for $n = 4, 5$, and 6.

Figure 1 illustrates the characteristic polynomial for three different matrix dimensions. The intersections of the curves with the horizontal axis correspond to the eigenvalues of T_n. As the matrix dimension increases, the degree of the characteristic polynomial also increases, producing a larger number of real and complex zeros. These numerical results are fully consistent with the characteristic polynomial derived in Theorem 1.

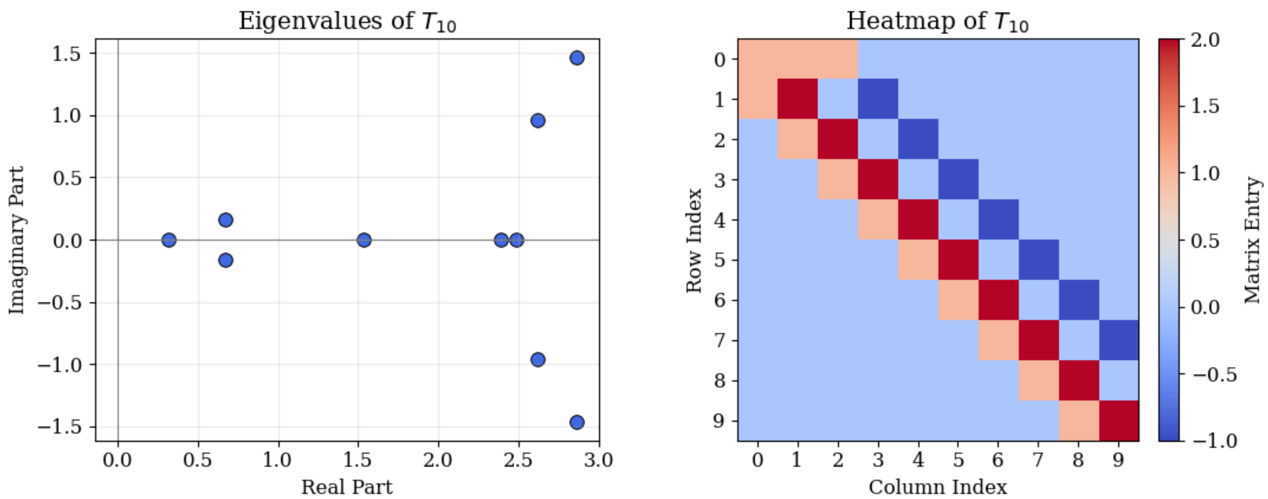

Figure 2. Eigenvalue distribution (left) and heatmap (right) of the matrix T_{10}.

The eigenvalues are displayed in the left hand side of Figure 2. Besides several real eigenvalues, complex conjugate pairs also appear, indicating that the matrix T_n is generally non-symmetric. The right hand side presents the heatmap of T_{10}, illustrating its structured Hessenberg form. Considering the matrices in the same order in Figure 2, it provides a direct comparison between the spectral properties and the underlying matrix structure, thereby offering a numerical illustration of the theoretical results established in Theorem 1.

Remark 2. *(The norms of the T_n matrix) The matrix norm provides additional information about the spectral behavior of the matrix family $\{T_n\}$. In particular, the infinity norm of T_n is given by*

$$\|T_n\|_\infty = 4,$$

since the maximum absolute row sum of T_n equals 4. Furthermore, the spectral radius satisfies the classical inequality

$$\rho(T_n) \leq \|T_n\|_\infty,$$

where $\rho(T_n)$ denotes the spectral radius of T_n. Therefore, the infinity norm provides an upper bound for the magnitude of the eigenvalues and offers additional insight into their localization, complementing the numerical spectral analysis presented in Figure 2.

2.2. LU Factorization of T_n

The LU factorization of the matrix T_n provides an efficient framework for computing its determinant. In addition, it reveals the algebraic structure of the matrix and facilitates its numerical analysis.

Let $T_n = (t_{i,j})_{1\leq i,j\leq n}$, where $t_{i,j}$ denotes the entry of T_n in the ith row and the jth column. We seek a factorization

$$T_n = L\,U,$$

where $L = (\ell_{i,j})_{n\times n}$ is a unit lower triangular matrix and $U = (u_{i,j})_{n\times n}$ is an upper triangular matrix with two superdiagonals. The nonzero entries of these factors are determined recursively by the scalar sequences

$$\{d_i\}_{i=1}^{n}, \qquad \{\ell_i\}_{i=2}^{n}, \qquad \{e_i\}_{i=1}^{n-1}, \qquad \{f_i\}_{i=1}^{n-2},$$

where d_i denote the pivots of U, ℓ_i are the multipliers defining L, and e_i and f_i correspond to the first and second superdiagonals of U, respectively. Accordingly, the factors take the form

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \ell_1 & 1 & 0 & \cdots & 0 \\ 0 & \ell_2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ell_{n-1} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} d_1 & u_{1,2} & u_{1,3} & 0 & \cdots & 0 \\ 0 & d_2 & u_{2,3} & u_{2,4} & \cdots & 0 \\ 0 & 0 & d_3 & u_{3,4} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & u_{n-2,n} \\ 0 & \cdots & & 0 & d_{n-1} & u_{n-1,n} \\ 0 & \cdots & & & 0 & d_n \end{bmatrix}.$$

Thus, the banded structure of T_n uniquely determines all entries of the factorization through these recurrences, ensuring that L is unit lower triangular and U has bandwidth two.

Theorem 2. *(Existence and recurrence of the LU factorization) Let $T_n = (t_{i,j})_{1\leq i,j\leq n}$ be the matrix defined in* (1.1). *Then T_n admits a unique factorization*

$$T_n = L\,U,$$

where L is a unit lower triangular matrix and U is an upper triangular matrix with two superdiagonals. Furthermore, the nonzero entries of L and U satisfy recursive relations that uniquely determine the factors.

Proof. Assume that $T_n = LU$, where $L = (\ell_{i,j})_{n\times n}$ is a unit lower triangular matrix and $U = (u_{i,j})_{n\times n}$ is an upper triangular matrix with two superdiagonals. By equating the corresponding entries of the matrices T_n and LU, we obtain recurrence relations for the sequences $\{\ell_i\}$, $\{d_i\}$, $\{e_i\}$, and $\{f_i\}$, which determine the nonzero entries of the factors L and U. From the first row of T_n, the initial values are obtained as

$$d_1 = t_{1,1} = 1, \qquad e_1 = t_{1,2} = 1, \qquad f_1 = t_{1,3} = 1.$$

These values serve as the initial conditions for the recursive construction. The recurrence relations derived below are applied only for admissible indices, so no quantities with non-positive indices are required. For $i \geq 2$, considering the $(i, i-1)$ entry of LU gives

$$(LU)_{i,i-1} = \ell_i d_{i-1} = t_{i,i-1} = 1,$$

which yields

$$\ell_i = \frac{1}{d_{i-1}}.$$

For $i \geq 2$, the diagonal entries satisfy

$$(LU)_{i,i} = \ell_i e_{i-1} + d_i = t_{i,i} = 2,$$

so that

$$d_i = 2 - \ell_i e_{i-1}.$$

For $i \geq 2$, the $(i, i+1)$ entries of T_n are zero. Thus,

$$(LU)_{i,i+1} = e_i + \ell_i f_{i-1} = t_{i,i+1} = 0,$$

which implies

$$e_i = -\ell_i f_{i-1} = -\frac{f_{i-1}}{d_{i-1}}.$$

Finally, since the second superdiagonal entries of T_n are prescribed, we obtain $f_i = t_{i,i+2}$ where the values of f_i are prescribed by the second superdiagonal of T_n. Substituting $e_{i-1} = -f_{i-2}/d_{i-2}$ and $\ell_i = 1/d_{i-1}$ into the diagonal relation for $i \geq 3$ gives

$$d_i = 2 - \frac{1}{d_{i-1}}\left(-\frac{f_{i-2}}{d_{i-2}}\right) = 2 + \frac{f_{i-2}}{d_{i-1}d_{i-2}}.$$

For $i = 2$, this simplifies to

$$d_2 = 2 - \frac{1}{d_1} e_1.$$

Hence, the full set of recurrence relations reads

$$\begin{cases} d_1 = 1, \quad e_1 = 1, \quad f_1 = 1, \\[2ex] \ell_i = \dfrac{1}{d_{i-1}}, \qquad e_i = -\dfrac{f_{i-1}}{d_{i-1}}, \\[2ex] d_i = \begin{cases} 2 - \dfrac{1}{d_1} e_1, & i = 2, \\ 2 + \dfrac{f_{i-2}}{d_{i-1}d_{i-2}}, & i \geq 3, \end{cases} \end{cases}$$

where the sequence $\{f_i\}$ is determined by the prescribed second superdiagonal of T_n (in the standard case $f_1 = 1$, $f_i = -1$ for $2 \le i \le n-2$, and $f_{n-1} = 0$).

These recurrence relations uniquely determine the sequences $\{\ell_i\}$, $\{d_i\}$, $\{e_i\}$, $\{f_i\}$, and hence the factors L and U. The existence of the factorization follows since the construction is consistent and requires only that the pivots d_k remain nonzero. Positivity (and hence nonvanishing) of the pivots d_k can be verified inductively from the recurrence, ensuring that no breakdown occurs in the factorization. For uniqueness, suppose that

$$T_n = LU = \widetilde{L}\widetilde{U},$$

where both L and $\widetilde{L}$ are unit lower triangular. Then

$$L^{-1}\widetilde{L} = U\widetilde{U}^{-1}$$

is simultaneously unit lower triangular and upper triangular, and therefore it must be the identity matrix. Consequently, $L = \widetilde{L}$ and $U = \widetilde{U}$, which proves the uniqueness of the factorization. This completes the proof. □

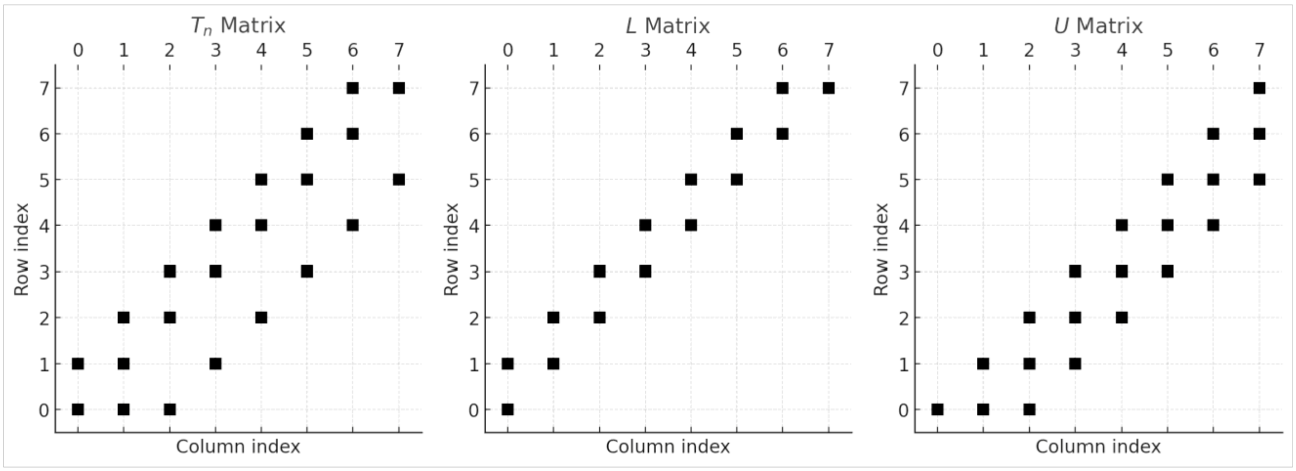

Figure 3. The sparsity patterns of the matrices T_n, L, and U for $n = 8$.

The sparsity patterns of the matrices T_n, L, and U for $n = 8$ reveal the underlying structural simplicity of the LU factorization (see Figure 3). The matrix T_n exhibits a banded structure, characterized by nonzero entries along the main diagonal, the first subdiagonal, and the second superdiagonal. The lower triangular factor L is unit lower-bidiagonal, containing nonzero entries only on the main diagonal and the first subdiagonal, which reflects the minimal fill-in during factorization. Meanwhile, the upper triangular factor U is upper-banded, with nonzeros confined to the main diagonal, the first superdiagonal, and the second superdiagonal. This structured sparsity not only facilitates efficient storage and computation but also offers analytical insight into the behavior of the matrix and its spectral properties.

Remark 3. *(Numerical stability) The LU factorization described above is performed without pivoting. Since the pivots $\{d_k\}$ remain nonzero throughout the recursive construction, no breakdown occurs during the elimination process. As with any Gaussian elimination procedure without pivoting, floating-point rounding errors may accumulate for sufficiently large matrix dimensions. Nevertheless, our* *numerical experiments revealed no significant loss of accuracy or numerical instability for the matrix sizes considered in this work.*

Corollary 1. *Let $T_n \in \mathbb{R}^{n\times n}$ be the Hessenberg matrix defined in* (1.1)*, and let $T_n = LU$ be the LU factorization obtained in Theorem 2, where $\{d_k\}_{k=1}^n$ denotes the pivot sequence defining the diagonal entries of the upper triangular factor U. Then,*

$$\det(T_n) = \prod_{k=1}^{n} d_k.$$

Since $\det(T_n) = Le_n$, it follows that

$$Le_n = \prod_{k=1}^{n} d_k.$$

Example 1. *As a concrete instance, consider the matrix $T_5 \in \mathbb{R}_{5\times 5}$ defined by*

$$T_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

This matrix admits a unique LU factorization $T_5 = LU$, where L is a unit lower triangular matrix and U is an upper triangular matrix with bandwidth two. Applying the recurrence relations established in Theorem 2, the initial values are

$$d_1 = 1, \qquad e_1 = 1, \qquad f_1 = 1.$$

The remaining quantities are then obtained recursively as

$$\ell_2 = 1, \quad d_2 = 1, \quad e_2 = -1, \quad \ell_3 = 1, \quad d_3 = 3, \quad \ell_4 = \frac{1}{3}, \quad d_4 = \frac{5}{3}, \quad \ell_5 = \frac{3}{5}, \quad d_5 = \frac{9}{5}.$$

The resulting factors are

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 1 & 0 \\ 0 & 0 & 0 & \frac{3}{5} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 3 & 1 & -1 \\ 0 & 0 & 0 & \frac{5}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{9}{5} \end{bmatrix}.$$

Since L is unit lower triangular, we have $\det(L) = 1$, and the determinant of U equals the product of its diagonal entries, i.e., $\det(U) = 9$. Consequently, $\det(T_5) = \det(L) \cdot \det(U) = 9 = Le_5$.

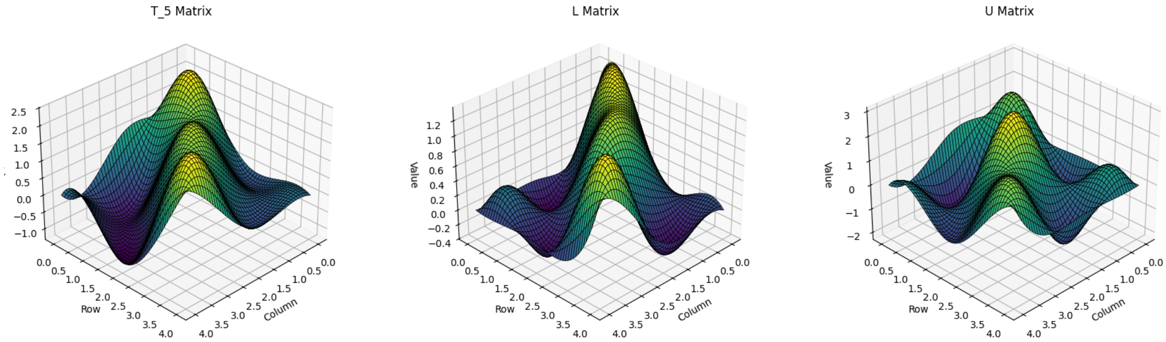

Figure 4. Surface representations of the matrices T_5 (left), L (center), and U (right) for $n = 5$. The plots clearly illustrate the banded structure of T_5, the unit lower-bidiagonal form of L, and the upper-banded structure of U. Surfaces are smoothed via interpolation to enhance visual clarity.

Surface plots of the banded matrix T_n and its LU factors, L and U (see Figure 4). The visualizations highlight how the sparsity pattern of T_n is preserved in the factors, with nonzero entries confined to the necessary diagonals. This structured sparsity minimizes fill-in, reducing both memory usage and computational cost. Consequently, the LU factorization not only provides a theoretical decomposition but also enables efficient determinant calculation, linear system solving, and spectral analysis, demonstrating its practical significance.

3. Conclusions and discussion

This study has presented a comprehensive analysis of the Hessenberg matrix T_n, a novel structured form combining features of Hessenberg systems. We have shown that its determinant is equal to the Leonardo sequence, thereby establishing a clear combinatorial and number-theoretic connection. Furthermore, the determinant was expressed equivalently through Fibonacci numbers and Chebyshev polynomials, enriching the algebraic characterization of the matrix.

The spectral analysis revealed that T_n exhibits a non-Hermitian eigenvalue spectrum with both real and complex conjugate eigenvalues, indicating potential instability and oscillatory dynamics in related systems. The explicit LU decomposition provided both theoretical insights and practical tools for efficient computation. Taken together, these results demonstrate that T_n offers a valuable prototype for exploring the interplay between structured sparsity and spectral complexity.

Future research directions include investigating asymptotic eigenvalue distributions for large matrix sizes, establishing deeper analytical connections with classes of non-normal operators, and applying these findings to problems in physics, network theory, and numerical analysis.

Author contributions

All authors contributed equally to this work. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest. Prof. Takao Komatsu is an editorial board member for AIMS Mathematics, and was not involved in the editorial review and the decision to publish this article.

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A. Computational implementation, verification, and visualization of results

In this section, we present the MATLAB code implementations that underpin the computational analysis and numerical experiments discussed in the main body of the paper. These scripts illustrate the construction of the special structured matrices T_n, the computation of their characteristic polynomials, and the LU decomposition without pivoting. By including these codes, we aim to provide transparency, reproducibility, and practical insight into the numerical methods employed, as well as to facilitate further research and experimentation based on our results.

Table A1. LU Decomposition of matrix T_n and determinant comparison with Leonardo numbers.

```
function demo_lu_Tn()

% DEMO_LU_TN - LU decomposition of special matrices T_n without pivoting.
% Computes L and U such that A = LU, checks the result,
% computes det(T_n), and compares it to the n-th Leonardo number.

ns = [4,5,6];
for ii = 1:length(ns)
n = ns(ii);
A = zeros(n); if n>=1, A(1,1)=1; end
if n>=2, A(1,2)=1; end
if n>=3, A(1,3)=1; end
for i=2:n, A(i,i)=2; A(i,i-1)=1; end
for i=2:n-2, A(i,i+2)=-1;  end
fprintf('\nT_%d =\n', n); disp(A)
L = eye(n); U = zeros(n);
for i=1:n
for j=i:n, U(i,j)=A(i,j)-L(i,1:i-1)*U(1:i-1,j); end
for j=i+1:n, L(j,i)=(A(j,i)-L(j,1:i-1)*U(1:i-1,i))/U(i,i); end
end
fprintf('L =\n'); disp(L)
fprintf('U =\n'); disp(U)
fprintf('LU = A? %d\n', norm(L*U-A,'fro')<1e-9)
fprintf('det(T_n) = %g\n', prod(diag(U)))
Ls=[0 1 1]; for k=3:n, Ls(end+1)=Ls(end)+Ls(end-1)+1; end
fprintf('Leonardo L_%d = %d\n', n, Ls(n))
end
end
```

Code listing 1: The following MATLAB code computes the LU decomposition (without pivoting) of a special structured matrix T_n, verifies the decomposition, calculates the determinant from the upper triangular matrix U, and compares it to the corresponding Leonardo number L_n. The matrix T_n has a specific sparsity pattern and is constructed recursively.

Table A2. MATLAB code that plots the characteristic polynomial over a range of λ values for n = 4, 5, and 6.

```
function characteristic_polynomial_matlab()
lambda_vals = linspace(0,5,500); % lambda range and n values
n_values = [4,5,6];
figure; hold on; grid on;
colors = ['r','g','b'];
for idx = 1:length(n_values)
n = n_values(idx);
p_vals = zeros(size(lambda_vals));
for k = 1:length(lambda_vals)
p_vals(k) = characteristic_poly(n, lambda_vals(k));
end
plot(lambda_vals, p_vals, colors(idx), 'DisplayName', ['n=', num2str(n)]);
end
yline(0,'k--');
xlabel('\lambda');
ylabel('p_n(\lambda)');
title('Karakteristik Polinom p_n(\lambda)');
legend show;
hold off;
end
function p = characteristic_poly(n, lam)
a = lam - 2;
r = cardano_roots(a);
alphas = alpha_coefficients(r, a);
total = 0;
for j = 1:3
total = total + alphas(j) * r(j)^(n-3) * ((a+1)*r(j)^2 - r(j) + 1);
end
p = real(total); % ignore the imaginary part
end
function r = cardano_roots(a)
% Cubic equation: r^3 - a r^2 + 1 = 0
p = -a^2 / 3;
q = 1 - 2*a^3 / 27;
delta = (q/2)^2 + (p/3)^3;
if delta >= 0
sqrt_delta = sqrt(delta);
u = nthroot(-q/2 + sqrt_delta, 3);
v = nthroot(-q/2 - sqrt_delta, 3);
y1 = u + v;
omega = (-1 + 1i*sqrt(3))/2;
y2 = omega*u + conj(omega)*v;
y3 = conj(omega)*u + omega*v;
else
rho = sqrt(-p^3 / 27);
theta = acos(-q / (2*rho));
rho_cbrt = nthroot(-p/3, 0.5);
y1 = 2 * rho_cbrt * cos(theta / 3);
y2 = 2 * rho_cbrt * cos((theta + 2*pi) / 3);
y3 = 2 * rho_cbrt * cos((theta + 4*pi) / 3);
end
r = [y1, y2, y3] + a/3;
end
function alphas = alpha_coefficients(r, a)
A = [1 1 1; r; r.^2];
b = [1; a; a^2];
alphas = A \ b;
end
```

Code listing 2: The following MATLAB code calculates and visualizes the characteristic polynomial over a specified range of λ values for matrices of dimensions 4, 5, and 6. The roots of the associated cubic equation are determined using Cardano's formula, enabling precise evaluation of the characteristic polynomial. This approach facilitates detailed analysis of the polynomial's behavior and spectral properties of the corresponding matrices.