Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 22069–22096.
DOI: 10.3934/math.2026893
Received: 04 May 2026
Revised: 16 July 2026
Accepted: 17 July 2026
Published: 23 July 2026

Research article

A stage-structured prey-predator mode with partially degenerate diffusion and free boundaries

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Abstract: This investigation focuses on a class of double free boundary problems pertaining to a stage-structured predator-prey model. The model features a predator population partitioned into immature and mature stages, where immature individuals are incapable of reproduction and diffusion. We start with the local existence and uniqueness, then prove global existence and uniqueness while obtaining corresponding estimates. Furthermore, we characterize the long-time behavior of solutions and demonstrate a spreading-vanishing dichotomy. Ultimately, sufficient criteria are provided to delineate the scenarios leading to spreading versus vanishing.

Keywords: stage-structured prey-predator model; partially degenerate diffusion; free boundary problem; long time behavior; spreading-vanishing dichotomy
Mathematics Subject Classification: 35B40, 35K57, 35R35, 92D25

1. Introduction

The life histories of plants, insects, and animals exhibit enormous diversity. Most species progress through multiple developmental stages throughout their life cycle, including immature and mature phases; some may also exhibit more specialized stages for dispersal or dormancy. Key demographic rates, such as survival, development, and reproduction, are almost invariably influenced by an individual's developmental stage, alongside a multitude of other factors. Such stage structures have been largely ignored in early population modeling, but have received much attention in recent years. Generally speaking, population growth models that include the stage structure predict more complex population dynamics than those that do not take this factor into account.

To discuss the impact of stage structure on species models, many prey-predator models with stage structure have been studied, including those with stage structure for predators [1–3] and those with stage structure for prey [4–6], and the references therein. The ODE version of the Lotka-Volterra stage-structured prey-predator model [1] is given by

$$\begin{cases} u' = bv - \theta_1 u - b_1 u^2, \\ v' = a_1 uw - \theta_2 v - bv, \\ w' = cw - b_2 w^2 - a_2 uw. \end{cases} \tag{1.1}$$

Here, u and v are the densities of mature and immature predators, respectively, while w denotes the prey density, $b, c, \theta_1, \theta_2, b_1, b_2, a_1, a_2$ are given positive constants in which θ_1 and θ_2 are the death rates of the mature and immature predators respectively, b denotes the maturation rate of immature predators into mature ones, b_1 and b_2 are the intra-specific competition rates of the mature predator and the prey respectively, c is the prey intrinsic growth rate, and a_1/a_2 is the conversion rate of prey into newborn immature predators. System (1.1) admits a trivial equilibrium $(0,0,0)$ and a semi-trivial equilibrium $(0,0,c/b_2)$. Moreover, if $a_1bc > b_2\theta_1(\theta_2+b)$, then there exists a unique positive equilibrium (u^*, v^*, w^*) for (1.1) where

$$u^* = \frac{a_1bc - b_2\theta_1(\theta_2+b)}{a_1a_2b + b_1b_2(\theta_2+b)}, \quad w^* = \frac{(b_1c + a_2\theta_1)(\theta_2+b)}{a_1a_2b + b_1b_2(\theta_2+b)}, \quad v^* = \frac{a_1u^*w^*}{\theta_2+b}. \tag{1.2}$$

It was shown in [1] that if $a_1bc < b_2\theta_1(\theta_2+b)$, then the equilibrium $(0,0,c/b_2)$ of (1.1) is globally stable. If $a_1bc > b_2\theta_1(\theta_2+b)$ and $a_1a_2b < b_1b_2(\theta_2+b)$, then (u^*, v^*, w^*) is globally asymptotically stable.

We know that, in the real world, the following phenomenon is quite natural: A kind of prey species initially occupies the entire spatial domain, and to achieve control of this prey species, a kind of predator species is introduced into a bounded spatial region. Considering that the predator species has two developmental stages (immature and mature), in its immature stage it has no reproductive predation ability and cannot migrate, and in its mature stage it can expand its habitat range beyond the boundary and undergo free diffusion. Thus, we establish double free boundary problems associated with a degenerate diffusive Lotka-Volterra predator-prey model as follows:

$$\begin{cases} u_t - d_1u_{xx} = bv - \theta_1 u - b_1u^2, & t>0, r(t)<x<s(t), \\ v_t = a_1uw - \theta_2 v - bv, & t>0, r(t)<x<s(t), \\ u = v = 0, & t>0, x \le r(t) \ or\ x \ge s(t), \\ w_t - d_2w_{xx} = cw - b_2w^2 - a_2uw, & t>0, -\infty<x<+\infty, \\ r'(t) = -\mu u_x(t, r(t)), s'(t) = -\beta u_x(t, s(t)), & t \ge 0, \\ u(0,x) = u_0(x), v(0,x) = v_0(x), & x \in [-s_0, s_0], \\ r(0) = -s_0, s(0) = s_0, w(0,x) = w_0(x), & -\infty<x<+\infty. \end{cases} \tag{1.3}$$

Here, $x = r(t)$ and $x = s(t)$ denote free boundaries to be determined; $d_1, d_2, s_0, \mu, \beta$ are given positive constants, $[-s_0, s_0]$ is the initial habitat of the predator, μ, β are, respectively, the ratios of expansion speed to population gradient at the left and right free expansion fronts, and initial functions satisfy

$$u_0 \in W_p^2([-s_0, s_0]), u_0 > 0\ x \in (-s_0, s_0),\ u_0(\pm s_0) = 0,\ u'(-s_0) > 0,\ u'(s_0) < 0,$$

$$v_0 \in C^{1-}([-s_0, s_0]), v_0 > 0\ x \in (-s_0, s_0),\ v_0(\pm s_0) = 0,$$

$$w_0 \in C^{2+\alpha}(\mathbb{R}), w_0 > 0, x \in \mathbb{R},$$

where $C^{1-}([-s_0, s_0])$ is the Lipschitz continuous function space in $[-s_0, s_0]$, and L_0 is the Lipschitz constant of v_0.

We mainly concern the dynamics of (1.3). First, We apply the contraction mapping principle to establish the local existence and uniqueness of solutions, derive regularity and uniform estimates, and further prove the global existence and uniqueness of solutions. Then, we construct appropriate upper and lower solutions and apply the comparison principle to derive a spreading-vanishing dichotomy, i.e., either:

(i) Spreading: the introduced predator successfully establishes itself in the new environment in the sense that

$$\lim_{t\to\infty} s(t) = -\lim_{t\to\infty} r(t) = \infty$$

and

$$\lim_{t\to\infty}(u(t,x), v(t,x), w(t,x)) = (u^*, v^*, w^*)$$

locally uniformly on $\mathbb{R}$ for the case $a_1a_2b < b_1b_2(\theta_2 + b)$, where (u^*, v^*, w^*) satisfies (1.2).

(ii) Vanishing: the predator spreads within a bounded area and vanishes eventually, i.e.,

$$-\infty < \lim_{t\to\infty} r(t) < \lim_{t\to\infty} s(t) < \infty$$

and

$$\lim\nolimits_{t\to\infty} \|u(t,\cdot)\|_{C([r(t),s(t)])} = \lim\nolimits_{t\to\infty} \|v(t,\cdot)\|_{C([r(t),s(t)])} = 0,$$
$$\lim\nolimits_{t\to\infty} w(t,x) = c/b_2 \text{ locally uniformly on } (-\infty, +\infty).$$

Moreover,

$$\lim_{t\to\infty} s(t) - \lim_{t\to\infty} r(t) \le \pi\sqrt{\frac{d_1b_2(\theta_2+b)}{a_1bc - b_2\theta_1(\theta_2+b)}} \overset{\text{def}}{=} \wedge,$$

if $a_1bc > b_2\theta_1(\theta_2 + b)$.

In this paper, the magnitude relationship between a_1bc and $b_2\theta_1(\theta_2 + b)$ plays a different role, comparing this with the corresponding ODE. When $a_1bc \le b_2\theta_1(\theta_2 + b)$, vanishing always happens. On the other hand, we have a criterion for spreading and vanishing as follows: If the initial occupying area $[-s_0, s_0]$ is beyond a critical size, namely $2s_0 \ge \wedge$, then spreading always happens regardless of moving parameters μ, β and the initial population density (u_0, v_0, w_0). On the other hand, if $2s_0 < \wedge$, then whether spreading or vanishing occurs is determined by moving parameters μ,β and the initial population density (u_0, v_0).

In the case in which we do not consider the stage structures, the Lotka-Volterra diffusive prey-predator model with free boundaries was studied by Wang and Zhao [7], and the corresponding higher-dimensional system was investigated by Zhao and Wang [8]. Following the pioneering work [9], researchers have increasingly adopted free boundary problems coupled with reaction-diffusion systems to describe the dynamical behaviors of species in recent years. For additional relevant references, apart from the studies cited earlier, interested readers may consult [10–13] regarding competition models and [14–17] with respect to predator-prey models.

In the absence of a native prey w, the problem (1.3) is reduced to partially degenerate reaction-diffusion systems with stage-structure and free boundaries

$$\begin{cases} u_t - d_1 u_{xx} = bv - \theta_1 u - b_1 u^2, & t > 0, r(t) < x < s(t), \\ v_t = au - \theta_2 v - bv, & t > 0, r(t) < x < s(t), \\ u = v = 0, & t > 0, x \le r(t) \text{ or } x \ge s(t), \\ r'(t) = -\mu u_x(t, r(t)), s'(t) = -\beta u_x(t, s(t)), & t \ge 0, \\ u(0, x) = u_0(x), v(0, x) = v_0(x), & x \in [-s_0, s_0], \\ r(0) = -s_0, s(0) = s_0, \end{cases}$$

which was studied in a concrete form by Liu and Wang [18]. The corresponding model with a fixed left boundary and a free right boundary was investigated by Wang and Cao [19], who proved global existence and uniqueness of solutions and analyzed their long-time behavior. Partially-degenerate diffusion complicates the analysis of corresponding problem, especially as the proof of existence and uniqueness of local solutions becomes relatively complex. However, the introduction of partially-degenerate diffusion is more consistent with practical situations.

The paper is organized as follows. In Section 2, we overcome the difficulties caused by partial degeneracy and prove the local and global existence, uniqueness, and uniform estimates of solutions to the coupled parabolic-ODE system. Section 3 first concerns some known results and a comparison principle, and is then devoted to the long-time behavior of the solution (u, v, w). In Section 4, we establish sharp sufficient criteria for spreading or vanishing in terms of the initial domain and ecological parameters. A short concluding discussion is presented in the final section.

2. Existence and uniqueness

For simplicity, we define

$$\begin{gathered} A_1 = \max_{[-s_0, s_0]} (u_0 + 1),\ A_2 = \max_{[-s_0, s_0]} (v_0 + 1),\ B = \max_{\mathbb{R}} (w_0 + 1), \\ \Lambda = \{b, c, \theta_1, \theta_2, b_1, b_2, a_1, a_2, d_1, d_2, s_0, \mu, \beta, A_1, A_2, B, \|u_0\|_{W_p^2((-s_0, s_0))}, u_0'(\pm s_0)\}, \\ \Delta_T = [0, T] \times [-1, 1],\ \Omega_T = [0, T] \times [-2s_0, 2s_0],\ D_T^{r,s} = \{0 < t \le T,\ r(t) < x < s(t)\}, \end{gathered}$$

and

$$C^{1,1-}(\Omega_T) = \left\{ \varphi \in C^{1,0}(\Omega_T) \,\middle|\, \begin{array}{l} \exists L(\varphi, T) \text{ s.t. } |\varphi(t, x_1) - \varphi(t, x_2)| \le L(\varphi, T)|x_1 - x_2| \\ (t, x_1), (t, x_2) \in \Omega_T \end{array} \right\},$$

$$\mathfrak{C}(D_T^{r,s}) = W_p^{1,2}(D_T^{r,s}) \cap C^{\frac{1+\alpha}{2}, 1+\alpha}(\overline{D}_T^{r,s}).$$

Theorem 2.1. *For any fixed $\alpha \in (0, 1)$ and $p > 3/(1 - \alpha)$, there exists a $T > 0$ such that (1.3) admits a unique solution $(u, v, w, r, s) \in \mathfrak{C}(D_T^{r,s}) \times C^{1,1-}(\overline{D}_T^{r,s}) \times C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T] \times \mathbb{R}) \times [C^{1+\frac{\alpha}{2}}([0, T])]^2$. Moreover*

$$u, v > 0 \text{ on } D_T^{r,s},\quad w > 0 \text{ on } [0, T] \times \mathbb{R},\quad r'(t) < 0,\ s'(t) > 0 \text{ on } [0, T].$$

Proof. **Step 1.** We denote $X_{v_0} = \{v \in C(\Omega_1) : v(0, x) = v_0(x), 0 \le v \le A_2\}$. For any fixed $v \in$

$X_{v_0} \bigcap C^{1,1-}(\Omega_1)$, we are concerned with the following problem:

$$
\begin{cases}
u_t - d_1 u_{xx} = bv - \theta_1 u - b_1 u^2, & 0 < t \le 1, r(t) < x < s(t), \\
u = 0, & x \notin (r(t), s(t)), \\
w_t - d_2 w_{xx} = cw - b_2 w^2 - a_2 uw, & 0 < t \le 1, -\infty < x < +\infty, \\
r'(t) = -\mu u_x(t, r(t)), s'(t) = -\beta u_x(t, s(t)), & t \ge 0, \\
u(0, x) = u_0(x), & x \in [-s_0, s_0], \\
r(0) = -s_0, s(0) = s_0, w(0, x) = w_0(x), & -\infty < x < +\infty.
\end{cases}
\tag{2.1}
$$

By employing similar argument as in [7], Theorem 2.1, one can prove that there exists a sufficiently small $T_1 > 0$ such that (2.1) has a unique solution $(u, w, r, s) \in \mathfrak{C}(D^{r,s}_{T_1}) \times C^{1+\frac{\alpha}{2}, 2+\alpha}([0, T_1] \times \mathbb{R}) \times \left[C^{1+\frac{\alpha}{2}}([0, T_1])\right]^2$ and

$$
\begin{cases}
\|u\|_{W^{1,2}_p(D^{r,s}_{T_1})},\ \|u, u_x\|_{C(\overline{D}^{r,s}_{T_1})},\ \|w\|_{C^{1+\frac{\alpha}{2},2+\alpha}([0,T_1]\times\mathbb{R})},\ \|r, s\|_{C^{1+\frac{\alpha}{2}}([0,T_1])} \le M, \\
\qquad -r'(t), s'(t) > 0,\ \ |r(t)|, |s(t)| \le 2s_0,\ t \in [0, T_1],
\end{cases}
\tag{2.2}
$$

where M depends only on Λ.

Let $\tilde{v}_0(x)$ be the zero extension of $v_0(x)$ to the interval $[r(T_1), s(T_1)]$. By $v_0 \in C^{1-}([-s_0, s_0])$, therefore $\tilde{v}_0 \in C^{1-}([r(T_1), s(T_1)])$. Obviously, the inverse functions of $x = r(t)$ and $x = s(t)$ both exist. Defining

$$
t^x = \begin{cases}
r^{-1}(x), & x \in [r(T_1), -s_0), \\
0, & |x| \le s_0, \\
s^{-1}(x), & x \in (s_0, s(T_1)],
\end{cases}
\tag{2.3}
$$

then $t^x \in C^{1-}([r(T_1), s(T_1)]$. For every $x \in (r(T_1), s(T_1))$, and the solution $(u(t, x), w(t, x))$ of (2.1), we consider the problem formulated below:

$$
\begin{cases}
\tilde{v}_t = a_1 u(t, x) w(t, x) - (\theta_2 + b)\tilde{v}(t, x), & t^x < t \le T_1, \\
\tilde{v}(t^x, x) = \tilde{v}_0(x).
\end{cases}
\tag{2.4}
$$

It follows from the standard theory of ordinary differential equations that there exists $T \in (0, T_1)$ with T depending only on Λ and M, such that $\tilde{v}(t, x)$ is well-defined on $[t^x, T]$ for all $x \in [r(T), s(T)]$. Consequently, $\tilde{v}$ is defined on the set $\overline{D}^{r,s}_T$. Furthermore, $0 \le \tilde{v} \le A_2$. We claim that $\tilde{v} \in C^{1,1-}(\overline{D}^{r,s}_T)$, whose Lipschitz constant will be determined in the subsequent step, and $\tilde{v}(t, r(t)) = \tilde{v}(t, s(t)) = 0$, $t \in [0, T]$. For each $x \in [r(T), s(T)]$, we extend $\tilde{v}$ by zero to the interval $[0, t^x]$. With this extension, $\tilde{v}$ belongs to $C^{1,1-}([0, T] \times [r(T), s(T)])$.

Next, we show that $\tilde{v}$ is Lipschitz continuous in x. Noting that $r'(t) < 0$ and $s'(t) > 0$ for all $t \in [0, T_1]$, we can see that there exists $\delta > 0$ satisfying

$$
|r'(t)|, |s'(t)| \ge \delta, \quad t \in [0, T_1]. \tag{2.5}
$$

For any $t' \in (t^x, T]$, we integrate the first equation of (2.4) over $[t^x, t']$. Then,

$$
\tilde{v}(t', x) = \tilde{v}(t^x, x) + \int_{t^x}^{t'} a_1 u(\tau, x) w(\tau, x) - (\theta_2 + b)\tilde{v}(\tau, x) d\tau.
$$

For fixed $(t,x'),(t,x'')$ in Ω_T, the argument can be split into several cases.

Case 1: Assume $(t,x'),(t,x'') \in \overline{D}_T^{r,s}$ and $x' < x'' \le s_0$. Then, it holds that $0 \le t^{x''} \le t^{x'} \le t$. Consequently, for any t' with $t^{x'} \le t' \le t$, it follows that

$$\begin{aligned}
& |\tilde{v}(t',x') - \tilde{v}(t',x'')| \\
\le\ & |\tilde{v}(t^{x'},x') - \tilde{v}(t^{x''},x'')| + \int_{t^{x''}}^{t^{x'}} |a_1 u(\tau,x'')w(\tau,x'') - (\theta_2+b)\tilde{v}(\tau,x'')|d\tau \\
& + \int_{t^{x'}}^{t'} a_1|w(\tau,x')|\cdot|u(\tau,x') - u(\tau,x'')| + a_1|u(\tau,x'')|\cdot|w(\tau,x') - w(\tau,x'')|d\tau \\
& + \int_{t^{x'}}^{t'} (\theta_2+b)|\tilde{v}(\tau,x') - \tilde{v}(\tau,x'')|d\tau.
\end{aligned}$$

Noticing the first inequality of (2.2), we obtain

$$\begin{aligned}
|\tilde{v}(t',x') - \tilde{v}(t',x'')| \le\ & |\tilde{v}(t^{x'},x') - \tilde{v}(t^{x''},x'')| + [a_1M^2 + (\theta_2+b)A_2]|t^{x'} - t^{x''}| \\
& + 2a_1M^2T|x'-x''| + (\theta_2+b)T\|\tilde{v}(\cdot,x') - \tilde{v}(\cdot,x'')\|_{C([t^{x'},t'])}.
\end{aligned} \tag{2.6}$$

If $x' < x'' < -s_0$, then $\tilde{v}(t^{x'},x') = \tilde{v}(t^{x''},x'') = 0$ and

$$|t^{x'} - t^{x''}| = |r^{-1}(x') - r^{-1}(x'')| \le \|(r^{-1})'\|_{C([0,T])}|x'-x''| \le \delta^{-1}|x'-x''|.$$

If $x' < -s_0 \le x''$, then $\tilde{v}(t^{x'},x') = 0$. Therefore,

$$|t^{x'} - t^{x''}| = |r^{-1}(x') - 0| = |r^{-1}(x') - r^{-1}(-s_0)| \le \delta^{-1}|x' - (-s_0)| \le \delta^{-1}|x'-x''|,$$

$$|\tilde{v}(t^{x'},x') - \tilde{v}(t^{x''},x'')| = |0 - v_0(x'')| = |v_0(-s_0) - v_0(x'')| \le L_0|-s_0-x''| \le L_0|x'-x''|,$$

where L_0 is the Lipschitz constant of v_0 in x.

If $-s_0 \le x' < x'' \le s_0$, then $t^{x'} = t^{x''} = 0$ and

$$|\tilde{v}(t^{x'},x') - \tilde{v}(t^{x''},x'')| = |v_0(x') - v_0(x'')| \le L_0|x'-x''|.$$

Inserting the above estimates into (2.6) and noticing that $T \le 1$, we can see that

$$|\tilde{v}(t',x') - \tilde{v}(t',x'')| \le (\theta_2+b)T\|(\tilde{v}(\cdot,x') - \tilde{v}(\cdot,x''))\|_{C([t^{x'},t'])} + \frac{K}{4}|x'-x''|,$$

where

$$K = 4\{[a_1M^2 + (\theta_2+b)A_2]\delta^{-1} + 2a_1M^2 + L_0\}. \tag{2.7}$$

Maximizing $|\tilde{v}(t',x') - \tilde{v}(t',x'')|$ over $[t^{x'},t]$ gives

$$\|\tilde{v}(\cdot,x') - \tilde{v}(\cdot,x'')\|_{C([t^{x'},t])} \le (\theta_2+b)T\|\tilde{v}(\cdot,x') - \tilde{v}(\cdot,x'')\|_{C([t^{x'},t])} + \frac{K}{4}|x'-x''|.$$

Let $T \le \min\{1, \frac{1}{2(\theta_2+b)}\}$. Then,

$$|\tilde{v}(t,x') - \tilde{v}(t,x'')| \le \|\tilde{v}(\cdot,x') - \tilde{v}(\cdot,x'')\|_{C([t^{x'},t])} \le \frac{K}{2}|x'-x''|. \tag{2.8}$$

Case 2: Assume $(t,x'),(t,x'') \in \overline{D}_T^{r,s}$ and $x' > x'' \ge -s_0$. By a similar argument, (2.8) holds.

Case 3: Assume $(t, x'), (t, x'') \in \overline{D}_T^{r,s}$ and $x'' < -s_0 < s_0 < x'$. We find

$$|\tilde{v}(t, x') - \tilde{v}(t, x'')| \leq |\tilde{v}(t, x') - \tilde{v}(t, s_0)| + |\tilde{v}(t, s_0) - \tilde{v}(t, x'')| \leq K|x' - x''|.$$

Case 4: Assume $(t, x'), (t, x'') \notin \overline{D}_T^{r,s}$. Obviously, $\tilde{v}(t, x') = \tilde{v}(t, x'') = 0$.

Case 5: Assume $(t, x') \notin \overline{D}_T^{r,s}$, $(t, x'') \in \overline{D}_T^{r,s}$. Without loss of generality, let $x' < r(t)$. Therefore, $\tilde{v}(t, x') = \tilde{v}(t, r(t)) = 0$, and

$$|\tilde{v}(t, x') - \tilde{v}(t, x'')| \leq |\tilde{v}(t, r(t)) - \tilde{v}(t, x'')| \leq K|r(t) - x''| \leq K|x' - x''|.$$

In conclusion, the estimate

$$|\tilde{v}(t, x') - \tilde{v}(t, x'')| \leq K|x' - x''|$$

is valid in all cases.

Step 2. Define

$$Y_{v_0}^T = \{\varphi \in C(\Omega_T) : \varphi(0, x) = v_0(x), 0 \leq \varphi \leq A_2, |\varphi(t, x') - \varphi(t, x'')| \leq K|x' - x''|\},$$

where K is the positive constant defined in (2.7). Evidently, $Y_{v_0}^T$ forms a closed subset of $C(\Omega_T)$. Since $C(\Omega_T)$ equipped with the uniform convergence metric is a complete metric space, the space $Y_{v_0}^T$ is complete with respect to the uniform norm. Each $v \in Y_{v_0}^T$ can be extended to $[T, 1] \times [-2s_0, 2s_0]$ via $v(t, x) = v(T, x)$. The extension satisfies $v \in X_{v_0} \bigcap C_x^{1,1-}(\Omega_1)$. We introduce a mapping $\mathfrak{F}$ via

$$\mathfrak{F}(v) = \tilde{v}, \ v \in Y_{v_0}^T.$$

From the preceding analysis, it follows that $\mathfrak{F}$ maps $Y_{v_0}^T$ into itself.

In what follows, we prove that $\mathfrak{F}$ constitutes a contraction mapping on $Y_{v_0}^T$ for sufficiently small $T > 0$. Denote (u_i, w_i, r_i, s_i) be the unique local solution to system (2.1) corresponding to $v = v_i, i = 1, 2$ and $t \in [0, T]$. We define t_i^x in accordance with (2.3), where (r, s) is replaced by (r_i, s_i). We denote by $\tilde{v}_i$ the unique solution of (2.4) corresponding to $t^x = t_i^x$, $u = u_i$, and $w = w_i$. We have

$$\tilde{v}_i(t, x) = \tilde{v}_i(t_i^x, x) + \int_{t_i^x}^t [a_1 u_i(\tau, x) w_i(\tau, x) - (\theta_2 + b)\tilde{v}_i(\tau, x)] d\tau, \quad x \in [r_i(T), s_i(T)].$$

Denote

$$V = v_2 - v_1, \tilde{V} = \tilde{v}_2 - \tilde{v}_1, s = s_2 - s_1, r = r_2 - r_1, \Pi_T = \overline{D_T^{r_1,s_1} \cup D_T^{r_2,s_2}}.$$

The ensuing reasoning is motivated by the approach developed in [20]. We extend u_i and $\tilde{v}_i$ by zero to $[0, T] \times \mathbb{R} \setminus \overline{D}_T^{r_i,s_i}$. For any fixed $(t, x) \in \Pi_T$, we derive a precise estimate for $|\tilde{V}(t, x)|$ by examining and treating each of the possible cases individually.

Case 1: $x \in (r_2(t), s_2(t)) \setminus (r_1(t), s_1(t))$. Either $x \in (r_2(t), r_1(t)]$ or $x \in [s_1(t), s_2(t))$, and $\tilde{v}_1(t, x) = \tilde{v}_2(t_2^x, x) = 0$. Therefore,

$$|\tilde{V}(t, x)| = |\tilde{v}_2(t, x)| = \left| \int_{t_2^x}^t [a_1 u_2(\tau, x) w_2(\tau, x) - (\theta_2 + b)\tilde{v}_2(\tau, x)] d\tau \right| \leq C_1 |t - t_2^x|,$$

where $C_1 = a_1M^2 + (\theta_2 + b)A_2$. If $x \in (r_2(t), r_1(t)]$, then $0 < t_2^x < t$ and $r_2(t) < r_2(t_2^x) = x \le r_1(t)$. Thus,

$$\begin{aligned} |\tilde{V}(t,x)| &\le C_1|t - t_2^x| = C_1|r_2^{-1}(r_2(t)) - r_2^{-1}(r_2(t_2^x))| \\ &\le C_1\|(r_2^{-1})'\|_{C([0,T])}|r_2(t) - r_2(t_2^x)| \\ &\le C_1\delta^{-1}|r_2(t) - r_2(t_2^x)| \\ &\le C_1\delta^{-1}|r_2(t) - r_1(t)| \\ &\le C_1\delta^{-1}\|r\|_{C([0,T])}, \end{aligned}$$

where $\delta > 0$ satisfies (2.5). If $s_1(t) < x \le s_2(t)$, we may similarly derive

$$|\tilde{V}(t,x)| = |\tilde{v}_2(t,x)| \le C_1\delta^{-1}\|s\|_{C([0,T])}.$$

Case 2: $x \in (r_1(t), s_1(t)) \setminus (r_2(t), s_2(t))$. Proceeding as in Case 1, it follows that

$$|\tilde{V}(t,x)| = |\tilde{v}_1(t,x)| \le C_1\delta^{-1}\|r, s\|_{C([0,T])}.$$

Case 3: $x \in (r_1(t), s_1(t)) \cap (r_2(t), s_2(t))$. When $x \in [-s_0, s_0]$, it follows that $t_1^x = t_2^x = 0$ and $\tilde{v}_1(t_1^x, x) = \tilde{v}_2(t_2^x, x) = \tilde{v}_0(x)$. Hence,

$$\begin{aligned} |\tilde{V}(t,x)| &\le \int_0^t |a_1(u_1(\tau,x)w_1(\tau,x) - u_2(\tau,x)w_2(\tau,x)) - (\theta_2 + b)(\tilde{v}_1(\tau,x) - \tilde{v}_2(\tau,x))|d\tau \\ &\le T(a_1M\|u_1 - u_2\|_{C(\Pi_T)} + a_1M\|w_1 - w_2\|_{C(\Pi_T)} + (\theta_2 + b)\|\tilde{V}\|_{C(\Pi_T)}). \end{aligned}$$

If $x \in ((r_1(t), s_1(t)) \setminus [-s_0, s_0]) \cap ((r_2(t), s_2(t)) \setminus [-s_0, s_0])$, then $t_1^x > 0, t_2^x > 0, \tilde{v}_1(t_1^x, x) = \tilde{v}_2(t_2^x, x) = 0$. We may assume without loss of generality that $x < -s_0$ and $t_1^x > t_2^x > 0$, thus $r_1(t_2^x) > r_1(t_1^x) = x = r_2(t_2^x)$, $x \in (r_1(\tau), s_1(\tau)) \cap (r_2(\tau), s_2(\tau))$ for all $t_x^1 < \tau \le t$ and $x \in (r_2(t_1^x), s_2(t_1^x)) \setminus (r_1(t_1^x), s_1(t_1^x))$. From Case 1, we conclude that

$$|\tilde{V}(t_1^x, x)| = |\tilde{v}_2(t_1^x, x)| \le C_1\delta^{-1}\|r, s\|_{C([0,T])}.$$

Integrating both sides of the equation for $\tilde{v}_i$ over $[t_1^x, t]$, we arrive at

$$\tilde{v}_1(t,x) = \int_{t_1^x}^t [a_1u_1w_1 - (\theta_2 + b)\tilde{v}_1]d\tau,$$

$$\tilde{v}_2(t,x) = \tilde{v}_2(t_1^x, x) + \int_{t_1^x}^t [a_1u_2w_2 - (\theta_2 + b)\tilde{v}_2]d\tau.$$

A direct computation yields

$$\begin{aligned} &|\tilde{V}(t,x)| \\ \le\ & |\tilde{v}_2(t_1^x, x)| + \left|\int_{t_1^x}^t [a_1(u_1w_1 - u_2w_2) - (\theta_2 + b)(\tilde{v}_1 - \tilde{v}_2)]d\tau\right| \\ \le\ & \left|\tilde{V}(t_1^x, x)\right| + T(\theta_2 + b)\|\tilde{V}\|_{C(\Pi_T)} + Ta_1M\|u_1 - u_2\|_{C(\Pi_T)} + Ta_1M\|w_1 - w_2\|_{C(\Pi_T)} \\ \le\ & C_1\delta^{-1}\|r, s\|_{C([0,T])} + T(\theta_2 + b)\|\tilde{V}\|_{C(\Pi_T)} + Ta_1M\left(\|u_1 - u_2\|_{C(\Pi_T)} + \|w_1 - w_2\|_{C(\Pi_T)}\right). \end{aligned}$$

Combining Case 1, Case 2, and Case 3, we conclude that

$$\|\tilde{V}\|_{C(\Pi_T)} \le 2C_1\delta^{-1}\|r, s\|_{C([0,T])} + 2Ta_1M\left(\|u_1 - u_2\|_{C(\Pi_T)} + \|w_1 - w_2\|_{C(\Pi_T)}\right) \tag{2.9}$$

provided that $T(\theta_2 + b) \le 1/2$.

In Step 3 below, we prove that for all sufficiently small $T > 0$, there exists a positive constant C independent of T satisfying inequalities below:

$$\|r, s\|_{C^1([0,T])} \le C\|V\|_{C(\Pi_T)}, \quad \|u_1 - u_2\|_{C(\Pi_T)} \le C\|V\|_{C(\Pi_T)}, \quad \|w_1 - w_2\|_{C(\Pi_T)} \le C\|V\|_{C(\Pi_T)}. \tag{2.10}$$

Together with $r(0) = s(0) = 0$, we can see

$$\|r, s\|_{C([0,T])} \le T\|r, s\|_{C^1([0,T])} \le TC\|V\|_{C(\Pi_T)}. \tag{2.11}$$

Combining (2.9), (2.10), and (2.11), we arrive at

$$\|\tilde{V}\|_{C(\Pi_T)} \le \frac{1}{3}\|V\|_{C(\Pi_T)}$$

provided that T is sufficiently small. This confirms that $\mathfrak{F}$ is contractive in $Y^T_{v_0}$. Accordingly, $\mathfrak{F}$ admits a unique fixed point v in $Y^T_{v_0}$. Let (u, w, r, s) stand for the unique solution to system (2.1) corresponding to the above v. Then, (u, v, w, r, s) solves problem (1.3). In addition, we have $u \in W^{1,2}_p(D^{r,s}_T)$ and $v \in C^{1,1-}(\overline{D}^{r,s}_T)$. Since $p > 3$, it follows from the embedding theorem that $u \in C^{1+\alpha, \frac{1+\alpha}{2}}(D^{r,s}_T)$.

Step 3. We shall prove (2.10) in this step. Let

$$x_i(t, y) = \frac{1}{2}[s_i(t) - r_i(t)]y + \frac{1}{2}[s_i(t) + r_i(t)],$$

$$\eta_i(t) = \frac{2}{s_i(t) - r_i(t)}, \ \zeta_i(t, y) = \frac{s_i'(t) + r_i'(t)}{s_i(t) - r_i(t)} + \frac{s_i'(t) - r_i'(t)}{s_i(t) - r_i(t)}y,$$

$$\varphi_i(t, y) = u_i(t, x_i(t, y)), \ z_i(t, y) = v_i(t, x_i(t, y)), \ \psi_i(t, y) = w_i(t, x_i(t, y))$$

for $i = 1, 2$. Then,

$$\begin{cases} \varphi_{i,t} - d_1\eta_i^2\varphi_{i,yy} - \zeta_i\varphi_{i,y} = bz_i - \theta_1\varphi_i - b_1\varphi_i^2, & 0 < t \le 1, -1 < y < 1, \\ \varphi_i(t, y) = 0, & 0 < t \le 1, y \notin (-1, 1), \\ \varphi_i(0, y) = u_0(s_0 y), & y \in [-1, 1], \\ \psi_{i,t} - d_2\eta_i^2\psi_{i,yy} - \zeta_i\psi_{i,y} = c\psi_i - b_2\psi_i^2 - a_2\varphi_i\psi_i, & 0 < t \le 1, -\infty < y < +\infty, \\ \psi_i(0, y) = w_0(s_0 y), & -\infty < y < +\infty. \end{cases}$$

It is easy to verify that the above system is uniformly parabolic. The standard partial differential equation theory guarantees $\|\varphi_i\|_{W^{1,2}_p(\Delta_T)}, \|\psi_i\|_{C^{2+\alpha,1+\frac{\alpha}{2}}([0,T]\times\mathbb{R})} \le C_2$, where C_2 depends only on Λ. By the embedding theorem,

$$[\varphi_i]_{C^{\frac{\alpha}{2},\alpha}(\Delta_T)}, [\varphi_{i,y}]_{C^{\frac{\alpha}{2},\alpha}(\Delta_T)} \le C_3\|\varphi_i\|_{W^{1,2}_p(\Delta_T)}, \tag{2.12}$$

where C_3 is independent of T^{-1}, and we can see that

$$[\varphi_i]_{C^{\frac{\alpha}{2},\alpha}(\Delta_T)}, [\varphi_{i,y}]_{C^{\frac{\alpha}{2},\alpha}(\Delta_T)} \le C_4.$$

This implies

$$\|\varphi_{i,y}\|_{C(\Delta_T)} \le \|\varphi'_{i,0}(y)\|_{C[-1,1]} + C_4T^{\frac{\alpha}{2}} \le \|\varphi'_{i,0}(y)\|_{C[-1,1]} + C_4 = C_5. \tag{2.13}$$

Therefore,

$$\|u_{i,x}(t,x_i)\|_{C(\overline{D}_T^{r_i,s_i})} \le \|\varphi_{i,y}(t,y)\|_{C(\Delta_T)}\frac{2}{s_i(t)-r_i(t)} \le \frac{C_5}{s_0}. \tag{2.14}$$

Letting $\varphi=\varphi_1-\varphi_2,\ z=z_1-z_2$, and $\psi=\psi_1-\psi_2$, we can see that they satisfy

$$\begin{cases} \varphi_t - d_1\eta_1^2\varphi_{yy} - \zeta_1\varphi_y + (\theta_1 + b_1(\varphi_1+\varphi_2))\varphi \\ \quad = d_1(\eta_1^2-\eta_2^2)\varphi_{2,yy} + (\zeta_1-\zeta_2)\varphi_{2,y} + bz, \qquad t>0, -1<y<1, \\ \varphi(t,\pm1)=0,\ t\ge 0;\ \ \varphi(0,y)=0,\ -1\le y\le 1, \end{cases} \tag{2.15}$$

$$\begin{cases} \psi_t - d_2\eta_1^2\psi_{yy} - \zeta_1\psi_y + [b_2(\psi_1+\psi_2) - c + a_2\varphi_1]\psi \\ \quad = d_2(\eta_1^2-\eta_2^2)\psi_{2,yy} + (\zeta_1-\zeta_2)\psi_{2,y} - a_2\psi_2\varphi, \qquad t>0, -\infty<y<+\infty, \\ \psi(0,y)=0,\ \ -\infty<y<+\infty, \end{cases} \tag{2.16}$$

and $r(t), s(t)$ satisfy

$$\begin{cases} r'(t) = -\mu\eta_2(t)\varphi_y(t,-1) - \mu(\eta_2(t)-\eta_1(t))\varphi_{1,y}(t,-1), \quad 0<t\le T, \\ s'(t) = -\beta\eta_2(t)\varphi_y(t,1) - \beta(\eta_2(t)-\eta_1(t))\varphi_{1,y}(t,1), \quad 0<t\le T, \\ r(0)=s(0)=0. \end{cases}$$

Taking advantage of the standard theory and L_p theory, we can obtain

$$\|\varphi\|_{W_p^{1,2}(\Delta_T)} \le C_6[\|r,s\|_{C^1([0,T])} + \|z\|_{C(\Delta_T)}],$$

and

$$\|\psi\|_{C(\Delta_T)} \le C_7[\|r,s\|_{C^1([0,T])} + \|\varphi\|_{C(\Delta_T)}], \tag{2.17}$$

where C_6 and C_7 depend on M and Λ.

Making use of (2.13), we can obtain that

$$\begin{aligned} |r'(t)| &\le \mu\eta_2(t)|\varphi_y(t,-1)| + \mu|(\eta_2(t)-\eta_1(t))|\cdot|\varphi_{1,y}(t,-1)| \\ &\le \tfrac{\mu}{s_0}\|\varphi_y\|_{C(\Delta_T)} + \tfrac{\mu C_5}{4s_0^2}\|r,s\|_{C([0,T])} \\ &\le \tfrac{\mu}{s_0}\|\varphi_y\|_{C(\Delta_T)} + \tfrac{\mu C_5 T}{4s_0^2}\|r',s'\|_{C([0,T])} \end{aligned}$$

holds uniformly on $(0,T]$. Since $r'(0)=0$, it is easy to see that

$$\|r'\|_{C([0,T])} \le \frac{\mu}{s_0}\|\varphi_y\|_{C(\Delta_T)} + \frac{\mu C_5 T}{4s_0^2}\|r',s'\|_{C([0,T])}.$$

Similarly, we have

$$\|s'\|_{C([0,T])} \le \frac{\beta}{s_0}\|\varphi_y\|_{C(\Delta_T)} + \frac{\beta C_5 T}{4s_0^2}\|r',s'\|_{C([0,T])}.$$

Consequently, $\|r',s'\|_{C([0,T])} \le \frac{C_8}{2}\|\varphi_y\|_{C(\Delta_T)}$ provided T small enough. Noting that $|r'(t)| \le T\|r\|_{C([0,T])}$ and $|s'(t)| \le T\|s\|_{C([0,T])}$ uniformly on $[0,T]$, we can obtain

$$\|r,s\|_{C^1([0,T])} \le C_8\|\varphi_y\|_{C(\Delta_T)}.$$

Next, we estimate $\|z\|_{C(\Delta_T)}$. We can see that

$$\begin{aligned}|z_1(t,y)-z_2(t,y)| &\le |v_1(t,x_1(t,y))-v_1(t,x_2(t,y))|+|v_1(t,x_2(t,y))-v_2(t,x_2(t,y))|\\ &\le K|x_1(t,y)-x_2(t,y)|+\|V\|_{C(\Pi_T)}\\ &\le K\|r,s\|_{C([0,T])}+\|V\|_{C(\Pi_T)}\end{aligned}$$

holds uniformly on Δ_T. Then,

$$\|z\|_{C(\Delta_T)} \le K\|r,s\|_{C([0,T])}+\|V\|_{C(\Pi_T)}.$$

We easily have

$$\|\varphi\|_{W_p^{1,2}(\Delta_T)} \le C_9[\|r,s\|_{C^1([0,T])}+\|V\|_{C(\Pi_T)}], \tag{2.18}$$

where $C_9 = C_6(1+K)$.

By the same embedding inequality (2.12), we can see that

$$\|r,s\|_{C^1([0,T])} \le C_8 T^{\frac{\alpha}{2}}[\varphi_y]_{C^{\frac{\alpha}{2},\alpha}(\Delta_T)} \le C_3C_8T^{\frac{\alpha}{2}}\|\varphi\|_{W_p^{1,2}(\Delta_T)}, \tag{2.19}$$

and

$$\|\varphi\|_{C(\Delta_T)} \le T^{\frac{\alpha}{2}}[\varphi]_{C^{\frac{\alpha}{2},\alpha}(\Delta_T)} \le C_3T^{\frac{\alpha}{2}}\|\varphi\|_{W_p^{1,2}(\Delta_T)}. \tag{2.20}$$

Combining (2.18), (2.19), (2.20), and (2.17), we obtain

$$\|r,s\|_{C^1([0,T])} \le C_{10}\|V\|_{C(\Pi_T)},\ \ \|\varphi\|_{C(\Delta_T)} \le C_{10}\|V\|_{C(\Pi_T)},\ \ \|\psi\|_{C(\Delta_T)} \le C_{10}\|V\|_{C(\Pi_T)} \tag{2.21}$$

on the condition of $0 < T \ll 1$, where C_{10} is independent of T. Thus, the first inequality of (2.10) has been shown. We then show the last two inequalities of (2.10). For any given $(t,x) \in \Pi_T$, define

$$y_i = \frac{2x - r_i(t) - s_i(t)}{s_i(t) - r_i(t)}.$$

Next, we discuss the position of (t,x) in Π_T by cases.

Case 1: $x \in [r_1(t), s_1(t)] \cap [r_2(t), s_2(t)]$. It follows from (2.11), (2.13), and (2.21) that

$$\begin{aligned}|u_1(t,x)-u_2(t,x)| &= |\varphi_1(t,y_1)-\varphi_2(t,y_2)|\\ &\le |\varphi_1(t,y_1)-\varphi_1(t,y_2)|+|\varphi_1(t,y_2)-\varphi_2(t,y_2)|\\ &\le \|\varphi_{1,y}\|_{C(\Delta_T)}|y_1-y_2|+\|\varphi\|_{C(\Delta_T)}\\ &\le \tfrac{2}{s_0}\|\varphi_{1,y}\|_{C(\Delta_T)}\|r,s\|_{C([0,T])}+\|\varphi\|_{C(\Delta_T)}\\ &\le C_{11}\|V\|_{C(\Pi_T)}.\end{aligned} \tag{2.22}$$

Case 2: $x \in [r_2(t), s_2(t)]\backslash[r_1(t), s_1(t)]$. Then, $u_1(t,x) = 0$. We may assume without loss of generality that $x \in (r_2(t), r_1(t)]$, and by using of (2.11), (2.14), and (2.22), we can see that

$$\begin{aligned}|u_1(t,x)-u_2(t,x)| &= |u_1(t,r_1(t))-u_2(t,x)|\\ &\le |u_1(t,r_1(t))-u_2(t,r_1(t))|+|u_2(t,r_1(t))-u_2(t,x)|\\ &\le C_{11}\|V\|_{C(\Pi_T)}+\|u_{2,x}\|_{C(\bar{D}_T^{r_2,s_2})}|r_1(t)-r_2(t)|\\ &\le C_{12}\|V\|_{C(\Pi_T)}.\end{aligned}$$

Case 3: $x \in [r_1(t), s_1(t)] \backslash [r_2(t), s_2(t)]$. Proceeding as in Case 2, we derive

$$|u_1(t,x) - u_2(t,x)| \le C_{12} \|V\|_{C(\Pi_T)}.$$

Combining Case 1, Case 2, and Case 3, we can obtain $\|u_1 - u_2\|_{C(\Pi_T)} \le C_{12}\|V\|_{C(\Pi_T)}$, which is the second inequality in (2.10).

Similarly to the computational procedure of Case 1, we can obtain the third inequality of (2.10).

This proof is complete. □

Theorem 2.2. *For any fixed* $\alpha \in (0,1)$ *and* $p > 3/(1-\alpha)$, *problem* (1.3) *admits a unique global solution*

$$(u, v, w, r, s) \in \mathfrak{C}(D_\infty^{r,s}) \times C^{1,1-}(D_\infty^{r,s}) \times C^{1+\frac{\alpha}{2}, 2+\alpha}([0,\infty) \times \mathbb{R}) \times [C^{1+\alpha/2}([0,\infty))]^2$$

satisfying

$$0 < u(t,x), v(t,x) \le M_1 \text{ on } D_\infty^{r,s},\ 0 < w(t,x) \le M_1 \text{ on } [0,\infty) \times \mathbb{R},\ \ 0 < -r'(t), s'(t) \le M_2 \text{ on } [0,\infty), \tag{2.23}$$

and

$$\|u(t,\cdot)\|_{C^1([r(t),s(t)])} + \|r', s'\|_{C^{\alpha/2}([1,\infty))} \le M_3,\ \ t \ge 1, \tag{2.24}$$

where $M_1, M_2, M_3 > 0$ *depend on* α *and* Λ.

Proof. By using Theorem 2.1, we obtain that there exists $T > 0$ such that problem (1.3) admits a unique local solution (u, v, w, r, s) satisfying $u, v > 0$ on $D_T^{r,s}$, $w > 0$ on $[0,T] \times \mathbb{R}$, and $-r'(t), s'(t) > 0$ for all $t \in [0,T]$.

We can easily derive $0 < w \le G$ in $[0,T] \times \mathbb{R}$, where $G = \max\{\|w_0\|_\infty, \frac{c}{b_2}\}$. We recall that the comparison principle gives $(u(t,x), v(t,x)) \le (\bar{u}(t), \bar{v}(t))$ for $t \in [0,T]$, $x \in \mathbb{R}$, where $(\bar{u}(t), \bar{v}(t))$ is the solution of the problem

$$\begin{cases} \bar{u}_t = b\bar{v} - \theta_1\bar{u} - b_1\bar{u}^2, & t > 0, \\ \bar{v}_t = a_1 G\bar{u} - \theta_2\bar{v} - b\bar{v}, & t > 0, \\ \bar{u}(0) = \|u_0\|_\infty,\ \bar{v}(0) = \|v_0\|_\infty. \end{cases}$$

Letting $U(t) = \bar{u}(t) + \bar{v}(t)$, then U satisfies $U(0) = \|u_0\|_\infty + \|v_0\|_\infty$ and

$$U_t = a_1 G\bar{u} - \theta_1\bar{u} - b_1\bar{u}^2 - \theta_2\bar{v} \le -\theta_2 U + \frac{(a_1 G + \theta_2 - \theta_1)^2}{4b_1}.$$

By using of the comparison principle again, we have

$$\bar{u}(t) + \bar{v}(t) = U(t) \le M_1 := \max\left\{\|u_0\|_\infty + \|v_0\|_\infty,\ \frac{(a_1 G + \theta_2 - \theta_1)^2}{4b_1\theta_2},\ G\right\}.$$

Thus,

$$0 < u, v \le M_1 \text{ on } D_T^{r,s},\ \ 0 < w \le M_1 \text{ on } [0,T] \times \mathbb{R}.$$

Next we show that $-r'(t),\ s'(t) \le M_2$ for some positive constant M_2. To this end, let L be a positive constant, and define

$$\Omega_L^T := \{(t,r) : 0 \le t \le T,\ s(t) - 1/L < x < s(t)\}$$

and construct an auxiliary function

$$\bar{u}(t,x) = M_1[2L(s(t)-x) - L^2(s(t)-x)^2].$$

We will choose L so that $\bar{u}(t,x) \geq u(t,x)$ holds over Ω_L^T.

Direct calculations show that

$$\bar{u}_t - d_1\bar{u}_{xx} \geq 2d_1M_1L^2 \geq bM_1 \geq bv - \theta_1\bar{u} - b_1\bar{u}^2, \ \ (t,x) \in \Omega_L^T,$$

if $L^2 \geq \frac{b}{2d_1}$. On the other hand,

$$\bar{u}(t, s(t) - L^{-1}) = M_1 \geq u(t, s(t) - L^{-1}), \quad \bar{u}(t, s(t)) = 0 = u(t, s(t)).$$

If $M_1L \geq \|u_0'\|_{C([0,s_0])}$, then

$$u_0(x) = -\int_x^{s_0} u_0'(s)ds \leq (s_0 - x)\|u_0'\|_{C([0,s_0])} \leq M_1L(s_0 - x) \leq w(0,x) \ \text{ on } [s_0 - L^{-1}, s_0].$$

Let

$$L = \max\left\{\sqrt{\frac{b}{2d_1}}, \frac{\|u_0'\|_{C([0,s_0])}}{M_1}\right\}.$$

Applying the maximum principle to $\bar{u} - u$ over Ω_L^T, we can deduce that $u(t,x) \leq \bar{u}(t,x)$ for $(t,x) \in \Omega_L^T$. Thanks to $\bar{u}(t,s(t)) = 0 = u(t,s(t))$, it would then follow that

$$s'(t) = -\mu u_x(t,s(t)) \leq -\mu\bar{u}_x(t,s(t)) \leq 2\max\{\mu,\beta\}M_1L := M_2, \ 0 \leq t \leq T.$$

Similarly, $r'(t) \geq -M_2$ for $0 \leq t \leq T$.

Let T_0 be the maximal existence time of (u,v,w,r,s). It is sufficient to prove $T_0 = +\infty$. On the contrary, we assume $T_0 < \infty$. Fix $\delta \in [0, T_0)$. Following the proof of Theorem 2.1, we can see that $v(t,\cdot) \in C^{1-}([-2s_0, 2s_0])$, $w(t,\cdot) \in C^{2+\alpha}(\mathbb{R})$, and $\|w(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq M_1$ for $t \in [\delta, T_0)$, and can find a positive constant M independent of T_0 such that $\|u(t,\cdot)\|_{W_p^2([g(t),h(t)])} \leq M$ for $t \in [\delta, T_0)$. Similar to the proof of Theorem 2.1, there is a $\tau > 0$ independent of T_0 such that the solution of (1.3) with initial time $T_0 - \tau/2$ can be extended uniquely to the time $T_0 + \tau/2$. A contradiction is derived, and $T_0 = +\infty$. Therefore

$$(u,v,w,r,s) \in \mathfrak{C}(D_\infty^{r,s}) \times C^{1,1-}(D_\infty^{r,s}) \times C^{1+\frac{\alpha}{2},2+\alpha}([0,\infty)\times\mathbb{R}) \times [C^{1+\alpha/2}([0,\infty))]^2.$$

Inequalities (2.24) can be derived by a proof similar to the one for Theorem 2.3 in [20]. We omit the details.

The proof is complete.

Noting that $r'(t) < 0,\ s'(t) > 0$, and $\|r,s\|_{C^{1+\alpha/2}([1,\infty))} \leq M_3$, we can easily see that limits $\lim\limits_{t\to\infty} r(t)$ and $\lim\limits_{t\to\infty} s(t)$ exist. We denote

$$\lim_{t\to\infty} r(t) = r_\infty, \ \ \lim_{t\to\infty} s(t) = s_\infty.$$

3. Long-time behavior

To analyze the long-time asymptotic behavior of u, v, and w, we begin by presenting several auxiliary lemmas. Since the proofs of Lemmas 3.1, 3.2, and 3.4 are analogous to those of that of Proposition 3.1 in [15], Proposition 4.2, and Lemma 3.4 in [20] , respectively, we leave out the details.

Lemma 3.1. *Let D, η, σ, and h_0 be positive constants and C be any real number. Suppose $h \in C^1([0,+\infty)), z \in W_p^2(D_T)$ for any $T > 0$ and $p > 3$, where $D_T = \{(t,x) : 0 < t < T,\ x_0 < x < h(t)\}$. Assume that $h(t) \in C^1([0,\infty))$, $h(t) > x_0$ for $0 < t < \infty$, $\lim_{t\to\infty} h(t) = h_\infty < \infty$, $\lim_{t\to\infty} h'(t) = 0$; $z_0 \in W_p^2([x_0,h_0])$, $z_0(x) \geq 0$, $z_0(x) \not\equiv 0$ for $x \in [x_0,h_0]$; and $\|z(t,\cdot)\|_{C^1[0,h(t)]} \leq M$ for any $t \geq 1$ and some $M > 0$. If (z,h) satisfies*

$$\begin{cases} z_t - Dz_{xx} \geq Cz, & t > 0, \quad x_0 < x < h(t), \\ z \geq 0, & t > 0, \quad x = x_0, \\ z = 0, \ h'(t) \geq -\eta z_x, & t > 0, \quad x = h(t), \\ z(0,x) = z_0(x), \ h_0 = h(0), & x_0 \leq x \leq h_0, \end{cases}$$

then $\lim_{t\to\infty} \max_{x_0 \leq x \leq h(t)} z(t,x) = 0$.

Lemma 3.2. *Let $k, m > 0$ be constants. If $\frac{a_1 mb}{\theta_2+b} > \theta_1$ and $k > \frac{\pi}{2}\sqrt{\frac{d_1(\theta_2+b)}{a_1mb-\theta_1(\theta_2+b)}}$, then for the problem*

$$\begin{cases} -d_1 u_{xx} = bv - \theta_1 u - b_1 u^2, & -k < x < k, \\ 0 = a_1 mu - \theta_2 v - bv, & -k < x < k, \\ u(-k) = u(k) = 0, \end{cases} \tag{3.1}$$

there exists a unique solution (u_k, v_k). Moreover,

$$\lim_{k\to\infty}(u_k, v_k) = (\hat{u}, \hat{v}) \ locally\ uniformly\ on\ (-\infty,+\infty),$$

where $(\hat{u},\hat{v})$ is the positive solution to

$$\begin{cases} bv - \theta_1 u - b_1 u^2 = 0, \\ a_1 mu - \theta_2 v - bv = 0. \end{cases} \tag{3.2}$$

Lemma 3.3. ([7], Proposition 8.1) *Let ϱ be a non-negative constant. For any given $\delta, N > 0$, there exist $T_\delta > 0$ and $l_\delta > \max\{N, \frac{\pi}{2}\sqrt{d/\lambda}\}$, such that when the continuous and non-negative function $z(t,x)$ satisfies*

$$\begin{cases} z_t - dz_{xx} \geq\ (\leq)\, z(\lambda - \theta z), & t > 0, \quad -l_\delta < x < l_\delta, \\ z(0,x) > 0, & -l_\delta < x < l_\delta, \end{cases}$$

and for $t > 0$, $z(t,\pm l_\delta) \geq\ (\leq)\, \varrho$ if $\varrho > 0$, while $z(t,\pm l_\delta) \geq\ (=)\, 0$ if $\varrho = 0$, then we have

$$z(t,x) > \lambda/\theta - \delta \ \ (z(t,x) < \lambda/\theta + \delta), \quad \forall\ t \geq T_\delta, \quad x \in [-N,N].$$

This implies

$$\liminf_{t\to\infty} z(t,x) \geq \lambda/\theta - \delta \ \left(\limsup_{t\to\infty} z(t,x) < \lambda/\theta + \delta\right) \ \ uniformly\ \ on\ \ [-N,N].$$

Lemma 3.4. (Comparison principle) *Assume* $\bar{u} \in C(\overline{D}_T^{\bar{r},\bar{s}}) \cap C^{1,2}(D_T^{\bar{r},\bar{s}})$, $\bar{v} \in C^{1,0}(\overline{D}_T^{\bar{r},\bar{s}})$, $\bar{w} \in C^{1,2}((0,T] \times \mathbb{R})$*, and* $\bar{r}, \bar{s} \in C^1([0,T])$*. If* $(\bar{u}, \bar{v}, \bar{w}, \bar{r}, \bar{s})$ *satisfies*

$$\begin{cases} \bar{u}_t - d_1\bar{u}_{xx} \geq b\bar{v} - \theta_1\bar{u} - b_1\bar{u}^2, & 0 < t \leq T, \;\; \bar{r}(t) < x < \bar{s}(t), \\ \bar{v}_t \geq a_1\bar{u}\bar{w} - \theta_2\bar{v} - b\bar{v}, & 0 < t \leq T, \;\; \bar{r}(t) < x < \bar{s}(t), \\ \bar{w}_t - d_2\bar{w}_{xx} \geq c\bar{w} - b_2\bar{w}^2, & 0 < t \leq T, \;\; x \in \mathbb{R}, \\ \bar{u}(t,x) = \bar{v}(t,x) = 0, & 0 < t \leq T, \;\; x \notin (\bar{r}(t), \; \bar{s}(t)), \\ \bar{r}'(t) \leq -\mu\bar{u}_x(t, \bar{r}(t)), \;\; \bar{s}'(t) \geq -\beta\bar{u}_x(t, \bar{s}(t)), & 0 < t \leq T, \\ \bar{u}(0,x) \geq 0, \; \bar{v}(0,x) \geq 0, & \bar{r}(0) \leq x \leq \bar{s}(0), \\ \bar{u}(0,x) \geq u_0(x), \; \bar{v}(0,x) \geq v_0(x), & -s_0 \leq x \leq \bar{s}_0, \\ \bar{r}(0) \leq -s_0, \; \bar{s}(0) \geq s_0, \; \bar{w}_0(x) \geq w(0,x), & x \in \mathbb{R}, \end{cases}$$

then the solution (u, v, w, r, s) *to problem* (1.3) *fulfills*

$$u \leq \bar{u}, \;\; v \leq \bar{v} \;\; \text{on } \overline{D}_T^{r,s}; \;\; w \leq \bar{w} \;\; \text{on } [0,T] \times \mathbb{R}; \;\; r \geq \bar{r}, \;\; s \leq \bar{s} \;\; \text{on } [0,T].$$

Theorem 3.1. *Assume that* (u, v, w, r, s) *solves problem* (1.3)*. If* $s_\infty - r_\infty < \infty$*, then*

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C([r(t),s(t)])} = \lim_{t\to\infty} \|v(t,\cdot)\|_{C([r(t),s(t)])} = 0,$$

$$\lim_{t\to\infty} w(t,x) = c/b_2 \; uniformly \; on \; any \; bounded \; subset \; of \; (-\infty, +\infty).$$

Proof. From (2.24), it follows that both r and s are uniformly continuous functions in $[1, \infty)$. Noting that $s_\infty - r_\infty < \infty$, we deduce $r'(t) \to 0$ and $s'(t) \to 0$ as $t \to \infty$. By using Theorem 2.2, we see that $u, v > 0$ for $t > 0, r(t) < x < s(t)$ and $\|u_x(t,\cdot)\|_{C([r(t),s(t)])} \leq M_3$ for $t \geq 1$. By applying Lemma 3.1 on $[0, s(t))$ and an analogous form of Lemma 3.1 on $(r(t), 0]$, we obtain $\lim_{t\to\infty} \|u(t,\cdot)\|_{C([r(t),s(t)])} = 0$.

For any $\varepsilon > 0$, there is $T > 0$ such that $a_1u(t,x)w(t,x) < \varepsilon$ for $t > T$ and $x \in [r(t), s(t)]$. Hence, v satisfies

$$\begin{cases} v_t \leq \varepsilon - (\theta_2 + b)v, & t > T, \;\; r(t) < x < s(t), \\ v(t, r(t)) = v(t, s(t)) = 0, & t \geq T, x \notin (r(t), s(t)), \\ v(T, x) > 0, & r(T) < x < s(T). \end{cases}$$

Applying the comparison principle, we can obtain $\lim_{t\to\infty} \max_{r(t)\leq x\leq s(t)} v(t,x) \leq \frac{\varepsilon}{\theta_2+b}$. Since ε is arbitrary, $\lim_{t\to\infty} \|v(t,\cdot)\|_{C([r(t),s(t)])} = 0$ holds.

Consider the solution $\bar{w}(t)$ of the following problem:

$$\bar{w}'(t) = \bar{w}(c - b_2\bar{w}), \;\; t > 0; \;\; \bar{w}(0) = \|w_0\|_\infty.$$

We can easily see that $w(t,x) \leq \bar{w}(t)$ for $t \in [0, \infty)$, $x \in \mathbb{R}$. Then,

$$\limsup_{t\to\infty} w(t,x) \leq \lim_{t\to\infty} \bar{w}(t) = c/b_2$$

uniformly on $\mathbb{R}$.

Notice that $u(t,x) \equiv 0$ for $t > 0$, $x \notin (r(t), s(t))$, and $\lim_{t\to\infty} \|u(t,\cdot)\|_{C([r(t),s(t)])} = 0$. For any fixed $\delta > 0$, one can find $T_\delta > 0$ such that $u(t,x) < \delta$ for $t > T_\delta$, $x \in \mathbb{R}$. Fix $0 < \varepsilon \ll 1$, $N \gg 1$, and define l_ε as in Lemma 3.3. Then, w satisfies

$$\begin{cases} w_t - d_2w_{xx} \geq w(c - b_2w - a_2\delta), & t > T_\delta, \;\; -l_\varepsilon < x < l_\varepsilon, \\ w(t, \pm l_\varepsilon) > 0, & t \geq T_\delta, \\ w(T_\sigma, x) > 0, & -l_\varepsilon \leq x \leq l_\varepsilon. \end{cases}$$

Thanks to Lemma 3.3, $\liminf_{t\to\infty} w(t,x) \geq \frac{c-a_2\sigma}{b_2} - \varepsilon$ uniformly on $[-N,N]$. Since ε, δ, and N are arbitrary, we derive that $\liminf_{t\to\infty} w(t,x) \geq c/b_2$ uniformly on a compact subset of $\mathbb{R}$.

This proof is complete. □

Next, we consider the spreading case $s_\infty - r_\infty = \infty$. For convenience, we denote

$$q = \frac{a_1 a_2 b}{b_1 b_2(\theta_2 + b)}.$$

Theorem 3.2. *Assume $s_\infty - r_\infty = \infty$. If $q < 1$, then we have*

$$\lim_{t\to\infty}(u(t,x), v(t,x), w(t,x)) = (u^*, v^*, w^*) \tag{3.3}$$

locally uniformly on $\mathbb{R}$, where (u^*, v^*, w^*) is given by (1.2).

Proof. The condition $s_\infty - r_\infty = \infty$ implies $a_1bc > b_2\theta_1(\theta_2 + b)$ (see Theorem 4.1) and $s_\infty = \infty, r_\infty = -\infty$ (see Theorem 4.3). Then, ODE system (1.1) has the positive equilibrium (u^*, v^*, w^*). In what follows, we construct an appropriate iterative sequence, whose conception is derived from the method adopted [7, 17].

Step 1. From the proof of Theorem 3.1, it follows that

$$\limsup_{t\to\infty} w(t,x) \leq \lim_{t\to\infty} \bar{w}(t) = \frac{c}{b_2} \overset{\text{def}}{=} \bar{w}_1 \text{ uniformly on } \mathbb{R}.$$

For any $\varepsilon > 0$, one can find $T > 0$ such that $w(t,x) \leq \bar{w}_1 + \varepsilon$ for $t \geq T$ and $x \in \mathbb{R}$. Thus, (u, v) fulfills

$$\begin{cases} u_t - d_1 u_{xx} = bv - \theta_1 u - b_1 u^2, & t > T, r(t) < x < s(t), \\ v_t \leq a_1 u(\bar{w}_1 + \varepsilon) - \theta_2 v - bv, & t > T, r(t) < x < s(t), \\ u(t,x) = v(t,x) = 0, & t > T,\ x \notin (r(t), s(t)), \\ u(T,x) \geq 0, v(T,x) \geq 0, & x \in [r(T), s(T)]. \end{cases}$$

Noting that $a_1bc > b_2\theta_1(\theta_2 + b)$, the unique positive equilibrium state $(\bar{u}_1^\varepsilon, \bar{v}_1^\varepsilon)$ corresponding to the following problem

$$\begin{cases} \bar{u}_t = b\bar{v} - \theta_1\bar{u} - b_1\bar{u}^2, & t > T, \\ \bar{v}_t = a_1\bar{u}(\bar{w}_1 + \varepsilon) - \theta_2\bar{v} - b\bar{v}, & t > T, \\ \bar{u}(T) = M_1,\ \bar{v}(T) = M_1, \end{cases}$$

is globally asymptotically stable. An application of the comparison principle yields

$$(u(t,x), v(t,x)) \leq (\bar{u}(t), \bar{v}(t))$$

for $t \geq T$ and $x \in \mathbb{R}$. Therefore, $\limsup_{t\to\infty}(u(t,x), v(t,x) \leq (\bar{u}_1^\varepsilon, \bar{v}_1^\varepsilon)$ uniformly on $\mathbb{R}$. Since ε is arbitrary, we obtain

$$\limsup_{t\to\infty} u(t,x) \leq \bar{u}_1,\ \limsup_{t\to\infty} v(t,x) \leq \bar{v}_1 \text{ uniformly on } \mathbb{R}, \tag{3.4}$$

where $(\bar{u}_1, \bar{v}_1)$ represents the unique positive solution to (3.2) when m is taken as $\bar{w}_1$.

Step 2. For any $0 < \varepsilon \ll 1$, it follows from (3.4) that there is $T > 0$ such that $u \leq \bar{u}_1 + \varepsilon$ holds in $[T, \infty) \times \mathbb{R}$. Thus, w fulfills

$$\begin{cases} w_t - d_2 w_{xx} \geq cw - b_2 w^2 - a_2 w(\bar{u}_1 + \varepsilon), & t \geq T,\ x \in \mathbb{R}, \\ w(T,x) > 0, & x \in \mathbb{R}. \end{cases}$$

We obtain $\liminf_{t\to\infty} w(t,x) \geq \frac{c-a_2(\bar{u}_1+\varepsilon)}{b_2}$ locally uniformly on $\mathbb{R}$. Since ε is arbitrary, we derive that

$$\liminf_{t\to\infty} w(t,x) \geq \frac{c-a_2\bar{u}_1}{b_2} \overset{\text{def}}{=} \underline{w}_1 \ \text{ locally uniformly on } \mathbb{R}. \tag{3.5}$$

On the condition of $q < 1$, we can easily see that $\underline{w}_1 > 0$.

Recalling Lemma 3.2, for any large k, we denote by $(\underline{u}_k(x), \underline{v}_k(x))$ and $(\underline{u}^\varepsilon, \underline{v}^\varepsilon)$ the unique positive solutions of (3.1) and (3.2) corresponding to $m = \underline{w}_1 - \varepsilon$, respectively. Therefore,

$$\lim_{k\to\infty}(\underline{u}_k(x), \underline{v}_k(x)) = (\underline{u}^\varepsilon, \underline{v}^\varepsilon)$$

locally uniformly on $(-\infty, \infty)$. For any fixed $L \gg 1$, we can find a sufficiently large $k > L$ such that

$$\underline{u}_k(x) > \underline{u}^\varepsilon - \varepsilon/2,\ \underline{v}_k(x) > \underline{v}^\varepsilon - \varepsilon/2,\ x \in [-L, L]. \tag{3.6}$$

For the above fixed $k > L$, consider the principal eigenpair (λ_1, ϕ) of problem

$$\begin{cases} -d_1\phi_{xx} = \lambda\phi, & x \in [-k,k], \\ \phi(-k) = \phi(k) = 0. \end{cases}$$

We have $\lambda_1 = d_1(\pi/2k)^2 < \frac{a_1(\underline{w}_1-\varepsilon)}{\theta_2+b} - \theta_1$. Direct computation gives that $(\delta\phi, \frac{a_1(\underline{w}_1-\varepsilon)\delta\phi}{\theta_2+b})$ is a positive lower solution for (3.1) with $m = \underline{w}_1 - \varepsilon$ when δ small enough. Furthermore, it is possible to select $T \gg 1$, $0 < \delta \ll 1$ such that $[-k,k] \subset (r(t), s(t))$ for $t \geq T$, $w(t,x) \geq \underline{w}_1 - \varepsilon$ for $(t,x) \in [T,\infty) \times [-k,k]$, and $(\delta\phi, \frac{a_1(\underline{w}_1-\varepsilon)\delta\phi}{\theta_2+b}) \leq (u(T,\cdot), v(T,\cdot))$ for $x \in [-k,k]$. Thus, (u,v) fulfills

$$\begin{cases} u_t - d_1u_{xx} = bv - \theta_1 u - b_1u^2, & t > T, -k < x < k, \\ v_t \geq a_1u(\underline{w}_1 - \varepsilon) - \theta_2 v - bv, & t > T, -k < x < k, \\ u(t, \pm k) > 0,\ v(t, \pm k) > 0, & t > T, \\ u(T,x) \geq \delta\phi(x),\ v(T,x) \geq \frac{a_1(\underline{w}_1-\varepsilon)\delta\phi(x)}{\theta_2+b}, & -k \leq x \leq k. \end{cases}$$

It follows from the comparison principle that $(u(t,x), v(t,x)) \geq (\underline{u}(t,x), \underline{v}(t,x))$ for all $(t,x) \in [T,\infty) \times [-k,k]$, where $(\underline{u}, \underline{v})$ is a positive solution to the following problem:

$$\begin{cases} \underline{u}_t - d_1\underline{u}_{xx} = b\underline{v} - \theta_1\underline{u} - b_1\underline{u}^2, & t > T, -k < x < k, \\ \underline{v}_t = a_1\underline{u}(\underline{w}_1 - \varepsilon) - \theta_2\underline{v} - b\underline{v}, & t > T, -k < x < k, \\ \underline{u}(t, \pm k) = 0,\ \underline{v}(t, \pm k) = 0, & t > T, \\ \underline{u}(T,x) = \delta\phi(x),\ \underline{v}(T,x) = \frac{a_1(\underline{w}_1-\varepsilon)\delta\phi(x)}{\theta_2+b}, & -k \leq x \leq k. \end{cases}$$

Obviously, $\underline{u}$ and $\underline{v}$ are nondecreasing with respect to t. Applying the standard parabolic regularity, we obtain

$$\lim_{t\to\infty}(\underline{u}(t,x), \underline{v}(t,x)) = (\underline{u}_k(x), \underline{v}_k(x))$$

uniformly on $[-k,k]$. So, we can find $T_1 > T$ such that $u(t,x) > \underline{u}_k(x) - \varepsilon/2$ and $v(t,x) > \underline{v}_k(x) - \varepsilon/2$ for $(t,x) \in [T_1,\infty) \times [-k,k]$. Combining with (3.6) yields $u(t,x) > \underline{u}^\varepsilon - \varepsilon$ and $v(t,x) > \underline{v}^\varepsilon - \varepsilon$ on $[T_1,\infty) \times [-L,L]$. Since ε and L are arbitrary, we have

$$\liminf_{t\to\infty} u(t,x) \geq \underline{u}_1,\ \liminf_{t\to\infty} v(t,x) \geq \underline{v}_1 \text{ locally uniformly on } \mathbb{R}, \tag{3.7}$$

where $(\underline{u}_1, \underline{v}_1)$ is the positive solution of (3.2) with $m = \underline{w}_1$.

Step 3. Fix $N \gg 1$ and $0 < \delta, \varepsilon \ll 1$. The quantity l_δ is determined by Lemma 3.3 with $d = d_2$, $\lambda = c - a_2\left(\underline{u}_1 - \varepsilon\right)$, and $\theta = b_2$, $\rho = 0$. According to (3.7), there is $T > 0$ such that $u \geq \underline{u}_1 - \varepsilon$ on $[T, \infty) \times [-l_\delta, l_\delta]$. Consequently, w fulfills

$$\begin{cases} w_t - d_2 w_{xx} \leq cw - b_2 w^2 - a_2 w(\underline{u}_1 - \varepsilon), & t \geq T,\ x \in [-l_\delta, l_\delta], \\ w(t, \pm l_\delta) \geq 0, & t \geq T. \end{cases}$$

As $w(T, x) > 0$ in $(-l_\delta, l_\delta)$, we obtain $\limsup_{t\to\infty} w(t, x) \leq \frac{c - a_2(\underline{u}_1 - \varepsilon)}{b_2}$ uniformly on $[-N, N]$ by Lemma 3.3. Since N, δ, and ε are arbitrary, we derive that

$$\limsup_{t\to\infty} w(t, x) \leq \frac{c - a_2\underline{u}_1}{b_2} \stackrel{\text{def}}{=} \bar{w}_2 \text{ locally uniformly on } \mathbb{R}. \tag{3.8}$$

Direct computation gives $\frac{a_1 b \bar{w}_2}{\theta_2 + b} - \theta_1 > 0$. For any fixed $0 < \varepsilon \ll 1$ and large enough k, let $\bar{u}_k(x)$ denote the unique positive solution to the boundary value problem

$$\begin{cases} -d_1 u_{xx} = \frac{a_1 b(\bar{w}_2 + \varepsilon)}{\theta_2 + b} u - \theta_1 u - b_1 u^2,\ x \in [-k, k], \\ u(-k) = u(k) = \bar{M}, \end{cases} \tag{3.9}$$

where $\bar{M} = \max\{M_1, \frac{1}{b_1}[\frac{a_1 b(\bar{w}_2 + \varepsilon)}{\theta_2 + b} - \theta_1]\}$. In addition, $0 < \bar{u}_k(x) < \bar{M}$. Let $(\bar{u}_2^\varepsilon, \bar{v}_2^\varepsilon)$ be the unique positive root of (3.2) with $m = \bar{w}_2 + \varepsilon$. By using the comparison principle, we can see that $\bar{u}_k(x)$ is decreasing in k and $\lim_{k\to\infty} \bar{u}_k(x) = \bar{u}_2^\varepsilon$ locally uniformly on $\mathbb{R}$. Taking $\bar{v}_k(x) = \frac{a(\bar{w}_2 + \varepsilon) u_k(x)}{\theta_2 + b}$, then $\lim_{k\to\infty} \bar{v}_k(x) = \bar{v}_2^\varepsilon$ locally uniformly on $\mathbb{R}$.

For any fixed $L \gg 1$, we can find a sufficiently large $k > L$ such that $\bar{u}_k(x) < \bar{u}_2^\varepsilon + \varepsilon/2$ and $\bar{v}_k(x) < \bar{v}_2^\varepsilon + \varepsilon/2$ on $[-L, L]$. Noting that (3.8), we can find $T > 0$ such that $w(t, x) \leq \bar{w}_2 + \varepsilon$ on $[T, \infty) \times [-k, k]$ and $(-k, k) \subset [r(t), s(t)]$. By virtue of the equation of v, it is possible to find $T_1 > T$ such that $v(t, x) \leq \frac{a_1(\bar{w}_2 + \varepsilon)\bar{M}}{\theta_2 + b}$ on $[T_1, \infty) \times [-k, k]$. Hence, (u, v) fulfills

$$\begin{cases} u_t - d_1 u_{xx} = bv - \theta_1 u - b_1 u^2, & t > T_1, -k < x < k, \\ v_t \leq a_1 u(\bar{w}_2 + \varepsilon) - \theta_2 v - bv, & t > T_1, -k < x < k, \\ u(t, \pm k) \leq \bar{M},\ v(t, \pm k) \leq \frac{a_1(\bar{w}_2 + \varepsilon)\bar{M}}{\theta_2 + b}, & t > T_1, \\ u(T_1, x) \leq \bar{M},\ v(T_1, x) \leq \frac{a_1(\bar{w}_2 + \varepsilon)\bar{M}}{\theta_2 + b}, & -k \leq x \leq k. \end{cases}$$

Let $(\bar{u}, \bar{v})$ denote the positive solution corresponding to the following problem:

$$\begin{cases} \bar{u}_t - d_1 \bar{u}_{xx} = b\bar{v} - \theta_1 \bar{u} - b_1 \bar{u}^2, & t > T_1, -k < x < k, \\ \bar{v}_t = a_1 \bar{u}(\bar{w}_2 + \varepsilon) - \theta_2 \bar{v} - b\bar{v}, & t > T_1, -k < x < k, \\ \bar{u}(t, \pm k) = \bar{M},\ \bar{v}(t, \pm k) = \frac{a(\bar{w}_2 + \varepsilon)\bar{M}}{\theta_2 + b}, & t > T_1, \\ \bar{u}(T_1, x) = \bar{M},\ \bar{v}(T_1, x) = \frac{a_1(\bar{w}_2 + \varepsilon)\bar{M}}{\theta_2 + b}, & -k \leq x \leq k. \end{cases}$$

Then, we have

$$\lim_{t\to\infty} (\bar{u}(t, x), \bar{v}(t, x)) = (\bar{u}_k(x), \bar{v}_k(x))$$

uniformly on $[-k,k]$. So, we can find $T_2 > T_1$ such that $u(t,x) \le \bar{u}_k(x) + \varepsilon/2$ and $v(t,x) \le \bar{v}_k(x) + \varepsilon/2$ for $t > T_2$ and $x \in [-k,k]$. Therefore, it follows that $u(t,x) \le \bar{u}_2^\varepsilon + \varepsilon$ and $v(t,x) \le v_2^\varepsilon + \varepsilon$ for $t > T_2$ and $x \in [-L,L]$. Since ε and L are arbitrary, we obtain

$$\limsup_{t\to\infty} u(t,x) \le \bar{u}_2,\ \limsup_{t\to\infty} v(t,x) \le \bar{v}_2 \text{ locally uniformly on } \mathbb{R},$$

where $(\bar{u}_2, \bar{v}_2)$ is the positive root of (3.2) with $m = \bar{w}_2$.

By arguing in a manner analogous to Step 2, we obtain the conclusion that

$$\liminf_{t\to\infty} w(t,x) \ge \underline{w}_2,\ \liminf_{t\to\infty} u(t,x) \ge \underline{u}_2,\ \liminf_{t\to\infty} v(t,x) \ge \underline{v}_2$$

uniformly on the compact subset of $\mathbb{R}$, where $\underline{w}_2 = \frac{c_2 - a_2\bar{u}_2}{b}$, $(\underline{u}_2, \underline{v}_2)$ is the unique positive root of (3.2) with $m = \underline{w}_2$.

Step 4. From the above steps and careful calculations, we have

$$\bar{w}_1 = \frac{c}{b_2},\ \ \bar{u}_1 = \frac{c}{a_2}q - \frac{\theta_1}{b_1},\ \ \bar{v}_1 = \frac{a_1\bar{w}_1\bar{u}_1}{\theta_2 + b},$$

$$\underline{w}_1 = \frac{c}{b_2}(1-q) + \frac{a_2\theta_1}{b_1b_2},\ \ \underline{u}_1 = \left(\frac{c}{a_2}q - \frac{\theta_1}{b_1}\right)(1-q),\ \ \underline{v}_1 = \frac{a_1\underline{w}_1\underline{u}_1}{\theta_2 + b},$$

$$\bar{w}_2 = \frac{c}{b_2}(1-q+q^2) + \frac{a_2\theta_1}{b_1b_2}(1-q),\ \ \bar{u}_2 = \left(\frac{c}{a_2}q - \frac{\theta_1}{b_1}\right)(1-q+q^2),\ \ \bar{v}_2 = \frac{a_1\bar{w}_2\bar{u}_2}{\theta_2 + b},$$

$$\underline{w}_2 = \frac{c}{b_2}(1-q+q^2-q^3) + \frac{a_2\theta_1}{b_1b_2}(1-q+q^2),\ \ \underline{u}_2 = \left(\frac{c}{a_2}q - \frac{\theta_1}{b_1}\right)(1-q+q^2-q^3),\ \ \underline{v}_2 = \frac{a_1\underline{w}_2\underline{u}_2}{\theta_2 + b}.$$

So, we know that

$$\underline{u}_1 < \underline{u}_2 < \overline{u}_2 < \overline{u}_1,\ \ \underline{v}_1 < \underline{v}_2 < \overline{v}_2 < \overline{v}_1,\ \ \underline{w}_1 < \underline{w}_2 < \overline{w}_2 < \overline{w}_1.$$

By following the above procedures repeatedly, we can obtain three monotonically increasing sequences $\{\underline{u}_n\}$, $\{\underline{v}_n\}$, and $\{\underline{w}_n\}$ and three monotonically decreasing sequences $\{\bar{u}_n\}$, $\{\bar{v}_n\}$, and $\{\bar{w}_n\}$, which satisfy

$$\begin{aligned}
&\underline{u}_1 < \cdots < \underline{u}_n < \cdots < \overline{u}_n < \cdots < \overline{u}_1,\\
&\underline{v}_1 < \cdots < \underline{v}_n < \cdots < \overline{v}_n < \cdots < \overline{v}_1,\\
&\underline{w}_1 < \cdots < \underline{w}_n < \cdots < \overline{w}_n < \cdots < \overline{w}_1.
\end{aligned}$$

Moreover, the solution (u,v,w) of (1.3) satisfies

$$\begin{aligned}
&\underline{u}_n \le \liminf_{t\to\infty}\ u(t,x) \le \limsup_{t\to\infty}\ u(t,x) \le \overline{u}_n,\\
&\underline{v}_n \le \liminf_{t\to\infty}\ v(t,x) \le \limsup_{t\to\infty}\ v(t,x) \le \overline{v}_n,\\
&\underline{w}_n \le \liminf_{t\to\infty}\ w(t,x) \le \limsup_{t\to\infty}\ w(t,x) \le \overline{w}_n,
\end{aligned}$$

locally uniformly on $\mathbb{R}$. We obtain the following iteration format:

$$
\begin{aligned}
&\bar{w}_1 = \frac{c}{b_2}, && \bar{u}_i = \frac{a_1 b}{b_1(\theta_2+b)}\bar{w}_i - \frac{\theta_1}{b_1}, && \bar{v}_i = \frac{a_1\bar{w}_i\bar{u}_i}{\theta_2+b},\\
&\underline{w}_i = \frac{c}{b_2} - \frac{a_2}{b_2}\bar{u}_i, && \underline{u}_i = \frac{a_1 b}{b_1(\theta_2+b)}\underline{w}_i - \frac{\theta_1}{b_1}, && \underline{v}_i = \frac{a_1\underline{w}_i\underline{u}_i}{\theta_2+b},\\
&\overline{w}_{i+1} = \frac{c}{b_2} - \frac{a_2}{b_2}\underline{u}_i, && i = 1, 2, ...
\end{aligned}
$$

When $i \geq 2$, we can see that

$$
\begin{aligned}
\bar{w}_i &= \frac{c}{b_2}(1 - q + q^2 - \cdots + q^{2i-2}) + \frac{a_2\theta_1}{b_1 b_2}(1 - q + \cdots - q^{2i-3}),\\
\bar{u}_i &= \left(\frac{c}{a_2}q - \frac{\theta_1}{b_1}\right)(1 - q + q^2 - \cdots + q^{2i-2}),\\
\bar{v}_i &= \frac{a_1}{\theta_2+b}\bar{w}_i\bar{u}_i,\\
\underline{w}_i &= \frac{c}{b_2}(1 - q + q^2 - \cdots - q^{2i-1}) + \frac{a_2\theta_1}{b_1 b_2}(1 - q + q^2 - \cdots + q^{2i-2}),\\
\underline{u}_i &= \left(\frac{c}{a_2}q - \frac{\theta_1}{b_1}\right)(1 - q + q^2 - \cdots - q^{2i-1}),\\
\underline{v}_i &= \frac{a_1}{\theta_2+b}\underline{w}_i\underline{u}_i.
\end{aligned}
$$

Note that $q < 1$, so all six sequences mentioned above possess well-defined limits, and

$$
\begin{aligned}
&\lim_{i\to\infty}\overline{w}_i = \lim_{i\to\infty}\underline{w}_i = \frac{b_1 c + a_2\theta_1}{b_1 b_2(1+q)} = \frac{(b_1 c + a_2\theta_1)(\theta_2+b)}{a_1 a_2 b + b_1 b_2(\theta_2+b)} = w^*,\\
&\lim_{i\to\infty}\overline{u}_i = \lim_{i\to\infty}\underline{u}_i = \frac{b_1 c q - a_2\theta_1}{b_1 a_2(1+q)} = \frac{a_1 b c - b_2\theta_1(\theta_2+b)}{a_1 a_2 b + b_1 b_2(\theta_2+b)} = u^*,\\
&\lim_{i\to\infty}\overline{v}_i = \lim_{i\to\infty}\underline{v}_i = \frac{a_1}{\theta_2+b}u^* w^* = v^*.
\end{aligned}
$$

Thus, (3.3) holds. The proof is complete.

4. Criteria for spreading and vanishing

This section focuses on discussing the criteria for spreading($s_\infty - r_\infty = \infty$) and vanishing ($s_\infty - r_\infty < \infty$), which are divided into two cases: $q \leq \frac{a_2\theta_1}{b_1 c}$ and $q > \frac{a_2\theta_1}{b_1 c}$, namely $a_1 bc \leq b_2\theta_1(\theta_2+b)$ and $a_1 bc > b_2\theta_1(\theta_2+b)$.

4.1. The case $a_1 bc \leq b_2\theta_1(\theta_2+b)$

Theorem 4.1. *Denote (u, v, w, r, s) as the solution to (1.3). If $a_1 bc \leq b_2\theta_1(\theta_2+b)$, then $s_\infty - r_\infty < \infty$.*

Proof. We denote by $\hat{w}$ the solution of the problem

$$
\hat{w}_t = c\hat{w} - b_2\hat{w}^2, \qquad \hat{w}(0) = \max\{\|w_0\|_{L_\infty}, \tfrac{c}{b_2}\}. \tag{4.1}
$$

Then, we have

$$\hat{w}(t) = \frac{c}{b_2} + \frac{b_2 c\hat{w}(0) - c^2}{b_2^2\hat{w}(0)\mathrm{e}^{ct} - b_2^2\hat{w}(0) + b_2 c}.$$

It follows from the comparison principle that $\hat{w}(t) \geq w(t,x)$ on $[0,\infty)\times\mathbb{R}$ and v satisfies

$$\begin{cases} v_t \leq a_1 u\hat{w} - (\theta_2 + b)v, & t > 0, r(t) < x < s(t), \\ v(t, r(t)) = v(t, s(t)) = 0, & t > 0, \\ v(0,x) = v_0(x), & -s_0 \leq x \leq s_0. \end{cases}$$

Denote $\rho = \max\{\mu, \beta\}$ and

$$f(t) = \int_{r(t)}^{s(t)} [(\theta_2 + b)u + bv]dx, \quad h(t) = s(t) - r(t), \quad \varphi(t) = \frac{ba_1(b_2 c\hat{w}(0) - c^2)M_1}{b_2^2\hat{w}(0)\mathrm{e}^{ct} - b_2^2\hat{w}(0) + b_2 c}.$$

We can easily see that $\int_0^t \varphi(\tau)d\tau$ is uniformly bounded with respect to t on $[0,+\infty)$. Noting that $a_1 bc \leq b_2\theta_1(\theta_2 + b)$, by simple calculations, we can obtain

$$\begin{aligned} f'(t) &= \int_{r(t)}^{s(t)} [(\theta_2 + b)u_t + bv_t]dx \\ &\leq \int_{r(t)}^{s(t)} \left[(\theta_2 + b)d_1 u_{xx} + \frac{ba_1(b_2 c\hat{w}(0) - c^2)u}{b_2^2\hat{w}(0)\mathrm{e}^{ct} - b_2^2\hat{w}(0) + b_2 c} + \left(\frac{a_1 bc}{b_2} - (\theta_2 + b)\theta_1\right)u\right]dx \\ &\leq -\tfrac{d_1(\theta_2+b)}{\rho}(s'(t) - r'(t)) + \frac{ba_1(b_2 c\hat{w}(0) - c^2)M_1}{b_2^2\hat{w}(0)\mathrm{e}^{ct} - b_2^2\hat{w}(0) + b_2 c}(s(t) - r(t)) \\ &= -\tfrac{d_1(\theta_2+b)}{\rho}h'(t) + \varphi(t)h(t). \end{aligned}$$

Then,

$$h'(t) \leq -\frac{\rho}{d_1(\theta_2 + b)}f'(t) + \frac{\rho}{d_1(\theta_2 + b)}\varphi(t)h(t).$$

Integration of the above inequality over the interval $[0,t]$ leads to

$$h(t) \leq h(0) + \frac{\rho}{d_1(\theta_2 + b)}f(0) + \frac{\rho}{d_1(\theta_2 + b)}\int_0^t \varphi(\tau)h(\tau)d\tau.$$

By using the Gronwall inequality, we have

$$h(t) \leq \left[h(0) + \frac{\rho}{d_1(\theta_2 + b)}f(0)\right]\exp\left\{\frac{\rho}{d_1(\theta_2 + b)}\int_0^t \varphi(\tau)d\tau\right\} < \infty.$$

Hence, $s_\infty - r_\infty < \infty$. The proof is complete.

In light of Theorem 3.1, the introduced predator faces ultimate extinction when the condition $a_1 bc \leq b_2\theta_1(\theta_2 + b)$ holds true.

4.2. The case $a_1 bc > b_2\theta_1(\theta_2 + b)$

In this subsection, we always assume $a_1 bc > b_2\theta_1(\theta_2 + b)$.

Theorem 4.2. *Denote (u,v,w,r,s) as the solution of (1.3). If $s_\infty - r_\infty < \infty$, then*

$$s_\infty - r_\infty \le \pi\sqrt{\frac{d_1 b_2(\theta_2 + b)}{a_1 bc - b_2\theta_1(\theta_2 + b)}} \overset{\text{def}}{=} \wedge. \tag{4.2}$$

Thus, if $s_0 \ge \frac{\wedge}{2}$, noticing $r'(t) < 0, s'(t) > 0$, then we obtain $s_\infty - r_\infty = \infty$. Equivalently, successful invasion of the predator occurs provided that its initial habitat domain exceeds the critical threshold range.

Proof. Taking advantage of ($s_\infty - r_\infty < \infty$) and Theorem 3.1, we have

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C([r(t),s(t)])} = \lim_{t\to\infty} \|v(t,\cdot)\|_{C([r(t),s(t)])} = 0,$$
$$\lim_{t\to\infty} w(t,x) = c/b_2 \text{ locally uniformly on } \mathbb{R}.$$

By contradiction, we suppose that $s_\infty - r_\infty > \wedge$. Therefore, for any $\varepsilon > 0$, there exists $T \ge 1$ such that

$$s(t) - r(t) > \max\left\{2s_0, \pi\sqrt{\frac{d_1 b_2(\theta_2 + b)}{ba_1(c-\varepsilon) - b_2\theta_1(\theta_2 + b)}}\right\},$$

and $w(t,x) \ge \dfrac{c-\varepsilon}{b_2}$ for all $(t,x) \in [T,\infty) \times [r(T), s(T)]$. Letting $r(T) = k_1, s(T) = k_2$, then $k_2 - k_1 > \pi\sqrt{\frac{d_1 b_2(\theta_2+b)}{ba_1(c-\varepsilon)-b_2\theta_1(\theta_2+b)}}$, and u, v satisfy

$$\begin{cases} u_t - d_1 u_{xx} = bv - \theta_1 u - b_1 u^2, & t > T, k_1 < x < k_2, \\ v_t \ge \frac{a_1(c-\varepsilon)}{b_2} u - (\theta_2 + b)v, & t > T, k_1 < x < k_2, \\ u(t,x) > 0,\ v(t,x) > 0, & t > T, x = k_i, i = 1,2, \\ u(T,x) > 0, v(T,x) > 0, & k_1 \le x \le k_2. \end{cases} \tag{4.3}$$

We claim that $(\delta\phi, \frac{a_1(c-\varepsilon)\delta\phi}{b_2(\theta_2+b)})$ is a lower solution of the following problem:

$$\begin{cases} \underline{u}_t - d_1 \underline{u}_{xx} = b\underline{v} - \theta_1 \underline{u} - b_1 \underline{u}^2, & t > T, k_1 < x < k_2, \\ \underline{v}_t = \frac{a_1(c-\varepsilon)}{b_2} \underline{u} - (\theta_2 + b)\underline{v}, & t > T, k_1 < x < k_2, \\ \underline{u}(t,x) = u(t,x),\ \underline{v}(t,x) = v(t,x), & t > T, x = k_i, i = 1,2, \\ \underline{u}(T,x) = u(T,x),\ \underline{v}(T,x) = v(T,x), & k_1 \le x \le k_2, \end{cases}$$

when δ is sufficiently small, where ϕ is the principle eigen-function of

$$\begin{cases} -d_1\phi_{xx} = \lambda\phi, & \|\phi\|_{L_\infty} = 1, \\ \phi(k_1) = \phi(k_2) = 0. \end{cases}$$

Denoting λ_1 as the principle eigenvalue, then $\lambda_1 = d_1\frac{\pi^2}{(k_2-k_1)^2} < \frac{ba_1(c-\varepsilon)}{b_2(\theta_2+b)} - \theta_1$. When δ is sufficiently small, we have

$$\begin{aligned} -d_1(\delta\phi)_{xx} &= \delta\lambda_1\phi \\ &= \frac{ba_1(c-\varepsilon)\delta\phi}{b_2(\theta_2+b)} - \theta_1\delta\phi - b_1\delta^2\phi^2 + \delta\phi\left(\lambda_1 - \frac{ba_1(c-\varepsilon)}{b_2(\theta_2+b)} + \theta_1 + b_1\delta\phi\right) \\ &< \frac{ba_1(c-\varepsilon)\delta\phi}{b_2(\theta_2+b)} - \theta_1\delta\phi - b_1\delta^2\phi^2, \end{aligned}$$

and

$$0 = \frac{a_1(c-\varepsilon)}{b_2}\delta\phi - (\theta_2 + b)\frac{a_1(c-\varepsilon)\delta\phi}{b_2(\theta_2 + b)}.$$

It follows from the comparison principle that

$$u(t,x) \geq \delta\phi,\ v(t,x) \geq \frac{a_1(c-\varepsilon)\delta\phi}{b_2(\theta_2+b)}, \quad t > T,\ k_1 < x < k_2, \tag{4.4}$$

which contradicts with $\lim_{t\to\infty} \|u(t,\cdot)\|_{C([r(t),s(t)])} = \lim_{t\to\infty} \|v(t,\cdot)\|_{C([r(t),s(t)])} = 0$. Therefore, (4.2) holds.

Theorem 4.3. *If $s_\infty - r_\infty = \infty$, then $s_\infty = \infty$ and $r_\infty = -\infty$.*

Proof. We proceed by contradiction. Without loss of generality, we assume that $s_\infty < \infty$ and $r_\infty = -\infty$. Let $k > \wedge + 2$ with $\wedge$ given in (4.2). We can find $T_1 > 0$ such that $r(T_1) < -k$ and u fulfills

$$\begin{cases} u_t - d_1 u_{xx} = bv - \theta_1 u - b_1 u^2, & t > T_1, -k < x < s(t), \\ u(t,-k) > 0, u(t,s(t)) = 0, & t > T_1, \\ s'(t) = -\beta u_x(t,s(t)), & t > T_1, \\ u(T_1,x) \geq 0, & -k \leq x \leq s(T_1). \end{cases}$$

It follows from $s_\infty < \infty$ that $\lim_{t\to\infty} s'(t) = 0$. Applying Lemma 3.1, we can easily see that

$$\lim_{t\to\infty} \|u(t,\cdot)\|_{C([-k,s(t)])} = 0. \tag{4.5}$$

Then, following a similar argument as in Theorem 3.1 with slight modifications, we derive that

$$\lim_{t\to\infty} \|v(t,\cdot)\|_{C([-k,s(t)])} = 0,\ \lim_{t\to\infty} w(t,x) = \frac{c}{b_2} \text{ uniformly on } [1-k, s(t)].$$

One can find $T_2 > T_1$ such that $w(t,x) \geq \frac{c-\varepsilon}{b_2}$ on $[T_2,\infty) \times [1-k, s(T_1)]$. Let $\varepsilon > 0$ be sufficiently small and take an interval $[k_1,k_2] \subset (1-k, s(T_1))$ such that

$$k_2 - k_1 > \pi\sqrt{\frac{d_1 b_2(\theta_2+b)}{ba_1(c-\varepsilon) - b_2\theta_1(\theta_2+b)}}.$$

We can easily see that (u,v) satisfies (4.3) with $T = T_2$. Following a similar argument as in the proof of Theorem 4.2, we can derive (4.4), which contradicts (4.5).

The proof is complete.

Theorem 4.2 and Theorem 4.3 show that if $s_0 \geq \frac{\wedge}{2}$, then $r_\infty = -\infty$ and $s_\infty = \infty$, equivalently, the introduced predator achieves full spatial spread over the entire domain as time tends to infinity.

The following result addresses the case where $s_0 < \frac{\wedge}{2}$.

Lemma 4.1. *If $s_0 < \frac{\wedge}{2}$, then one can find $\bar{\mu} > 0 (\bar{\beta} > 0)$ such that $s_\infty - r_\infty = \infty$ whenever $\mu \geq \bar{\mu} (\beta \geq \bar{\beta})$.*

The proof of Lemma 4.1 is omitted here, since its procedure and idea are largely identical to those of Lemma 3.2 in reference [10], with only minor modifications needed. From Lemma 4.1, we see $s_\infty - r_\infty = \infty$ if $\rho = \max\{\mu,\beta\} \geq \max\{\bar{\mu},\bar{\beta}\}$. Intuitively, we expect vanishing to hold under the conditions $a_1 bc \geq b_2\theta_1(\theta_2 + b)$, $s_0 < \wedge/2$, and small μ, β. Our intuition is confirmed by the lemma presented below.

Lemma 4.2. *If $s_0 < \frac{\wedge}{2}$, then one can find $\underline{\rho} > 0$ such that $s_\infty - r_\infty < \infty$ whenever $\rho = \max\{\mu, \beta\} \le \underline{\rho}$.*

Proof. For any given $k \in (s_0, \frac{\wedge}{2})$, we introduce the eigenvalue problem below:

$$\begin{cases} d_1\phi_{xx} - \theta_1\phi + b\psi = \lambda\phi, & -k < x < k, \\ \frac{a_1 c}{b_2}\phi - (\theta_2 + b)\psi = \lambda\psi, & -k < x < k, \\ \phi(-k) = \phi(k) = 0. \end{cases}$$

By Corollary 3.2 in [20] and $2k < \wedge$, we obtain that the principle eigenvalue $\lambda_1 < 0$ with a corresponding positive eigenvector (ϕ_1, ψ_1). Therefore,

$$\begin{cases} d_1\phi_{1,xx} - \theta_1\phi_1 + b\psi_1 = \lambda_1\phi_1, & -k < x < k, \\ \frac{a_1 c}{b_2}\phi_1 - (\theta_2 + b)\psi_1 = \lambda_1\psi_1, & -k < x < k. \end{cases} \tag{4.6}$$

Denote (η_1, φ_1) as the eigenpair of the problem

$$-\varphi_{xx} = \eta\varphi, \;\; -k < x < k, \;\; \varphi(-k) = \varphi(k) = 0.$$

Obviously, $\eta_1 = \frac{\pi^2}{4k^2}$. Multiplying (4.6) by φ_1 and integrating over $[-k, k]$, by direct computation, we obtain that there exists a positive constant ξ such that

$$\begin{cases} \frac{a_1 c}{b_2}\xi - (\theta_2 + b) = \lambda_1, \\ -d_1\frac{\pi^2}{4k^2}\xi - \theta_1\xi + b = \lambda_1\xi. \end{cases} \tag{4.7}$$

Noting that $\hat{w}$ satisfies (4.1), we have $\hat{w} \ge \frac{c}{b_2}, \lim\limits_{t\to\infty} \hat{w}(t) = \frac{c}{b_2}$, and $\hat{w}(t) \ge w(t, x)$ in $[0, \infty) \times \mathbb{R}$. Defining $\rho = \max\{\mu, \beta\}$,

$$g(t) = Kexp\left\{\int_0^t [a_1\xi(\hat{w}(\tau) - \tfrac{c}{b_2}) + \lambda_1]d\tau\right\}, \;\; h(t) = \left(s_0^2 + \rho\pi\xi\int_0^t g(\tau)d\tau\right)^{\frac{1}{2}},$$

$$\hat{u}(t, x) = \xi g(t)\cos\frac{\pi x}{2h(t)}, \;\; \hat{v}(t, x) = g(t)\cos\frac{\pi x}{2h(t)}, \;\; (t, x) \in [0, \infty) \times [-h(t), h(t)],$$

and, obviously, $h'(t) > 0$ for $t \ge 0$. Choose a sufficiently large constant K satisfying

$$u_0(x) \le K\xi\cos\frac{\pi x}{2s_0} = \hat{u}(0, x), \quad v_0(x) \le K\cos\frac{\pi x}{2s_0} = \hat{v}(0, x), \;\; x \in [-s_0, s_0]. \tag{4.8}$$

As $\lim\limits_{t\to\infty} \hat{w}(t) = \frac{c}{b_2}$ and $\lambda_1 < 0$, $\int_0^{+\infty} g(\tau)d\tau < \infty$ is bounded. Let

$$\underline{\rho} = \frac{k^2 - s_0^2}{\pi\xi\int_0^\infty g(\tau)d\tau}.$$

If $0 < \rho \le \underline{\rho}$, then $h(t) < k$ for $t \ge 0$. Combining with (4.8) and $\hat{w}(t) \ge \frac{c}{b_2}$, it follows from straightforward calculations that

$$\begin{aligned} & \hat{u}_t - d_1\hat{u}_{xx} - b\hat{v} + \theta_1\hat{u} + b_1\hat{u}^2 \\ = \;& \xi g(t)\cos\tfrac{\pi x}{2h(t)}\left[a_1\theta\left(\hat{w}(t) - \tfrac{c}{b_2}\right) + \lambda_1 + d_1\left(\tfrac{\pi}{2h(t)}\right)^2 - \tfrac{b}{\xi} + \theta_1 + b_1\right] + g(t)\sin\tfrac{\pi x}{2h(t)} \cdot \tfrac{\pi x h'(t)}{2h^2(t)} \\ \ge \;& \xi g(t)\cos\tfrac{\pi x}{2h(t)}\left[d_1\left(\tfrac{\pi}{2h(t)}\right)^2 - d_1\left(\tfrac{\pi}{2k}\right)^2\right] \\ \ge \;& 0 \end{aligned}$$

and

$$\begin{aligned}
& \hat{v}_t - a_1\hat{u}\hat{w} + (\theta_2 + b)\hat{v} \\
\geq\ & g(t)\cos\frac{\pi x}{2h(t)}\left[a_1\xi\left(\hat{w}(t) - \frac{c}{b_2}\right) + \lambda_1 - a_1\hat{w}\xi + \theta_2 + b\right] \\
\geq\ & 0
\end{aligned}$$

for $(t, x) \in [0, \infty) \times (-h(t), h(t))$. In addition,

$$-h'(t) = -\rho\hat{u}_x(t, -h(t)), \qquad h'(t) = -\rho\hat{u}_x(t, h(t)).$$

Therefore, if $0 < \rho \leq \underline{\rho}$, then $(\hat{u}, \hat{v}, \hat{w}, -h(t), h(t))$ fulfills

$$\begin{cases}
\hat{u}_t - d_1\hat{u}_{xx} \geq b\hat{v} - \theta_1\hat{u} - b_1\hat{u}^2, & t > 0, -h(t) < x < h(t), \\
\hat{v}_t \geq a_1\hat{u}\hat{w} - (\theta_2 + b)\hat{v}, & t > 0, -h(t) < x < h(t), \\
\hat{u} = \hat{v} = 0, & t > 0, x \leq -h(t) \text{ or } x \geq h(t), \\
\hat{w}_t - d_2\hat{w}_{xx} \geq c\hat{w} - b_2\hat{w}^2, & t > 0, -\infty < x < +\infty, \\
\hat{u}(t, \pm h(t)) = \hat{v}(t, \pm h(t)) = 0, & t > 0, \\
-h'(t) \leq -\mu\hat{u}_x(t, -h(t)), h'(t) \geq -\beta\hat{u}_x(t, h(t)), & t \geq 0, \\
\hat{u}(0, x) \geq u_0(x), \hat{v}(0, x) \geq v_0(x), & -s_0 \leq x \leq s_0, \\
h(0) = s_0, \ \ \hat{w}(0) \geq w_0(x), & -\infty < x < +\infty.
\end{cases}$$

By virtue of Lemma 3.4, $-h(t) \leq r(t)$, and $s(t) \leq h(t)$ holds for all $t \geq 0$. It follows that

$$r_\infty \geq -\lim_{t\to\infty} h(t) \geq -k, \quad s_\infty \leq \lim_{t\to\infty} h(t) \leq k,$$

and hence $s_\infty - r_\infty < \infty$. The proof is complete.

Obviously, it follows from the proof above that $\underline{\rho}$ does not depend on u_0, v_0, and is strictly decreasing with K. Therefore, if μ and β are fixed, then we can find a small enough $K > 0$ such that $\rho \leq \underline{\rho}$. At the same time, if u_0 and v_0 are sufficiently small such that (4.8) holds, then by a simple argument, we can still obtain $s_\infty - r_\infty < \infty$.

Remark 4.1. *If $s_0 < \frac{\wedge}{2}$ and (u_0, v_0) are small enough, then $s_\infty - r_\infty < \infty$.*

Theorem 4.4. *If $s_0 < \frac{\wedge}{2}$, then we can find $0 < \rho_* < \rho^*$ such that $s_\infty - r_\infty < \infty$ provided that $\rho \leq \rho_*$ or $\rho = \rho^*$; moreover, $s_\infty - r_\infty = \infty$ provided that $\rho > \rho^*$.*

Proof. This proof is similar to that of Theorem 5.2 in [7]. For the convenience of the readers we shall give the details. Write $(u_\rho, v_\rho, w_\rho, r_\rho, s_\rho)$ in place of (u, v, w, r, s) to clarify the dependence of the solution of (1.3) on ρ. Define

$$\Sigma^* = \left\{\rho > 0 : \ s_{\rho,\infty} - r_{\rho,\infty} \leq \wedge\right\}.$$

In consideration of Lemma 4.2 and Theorem 4.2, we have $(0, \underline{\rho}] \subset \Sigma^*$. According to Lemma 4.1, $\Sigma^* \cap [\bar{\mu}, \infty) = \emptyset$. Therefore, $\rho^* \overset{\text{def}}{=} \sup \Sigma^* \in [\underline{\rho}, \bar{\mu}]$. By this definition and Theorem 4.2, it follows that $r_{\rho,\infty} = -\infty$, $s_{\rho,\infty} = \infty$ for $\rho > \rho^*$. Hence, $\Sigma^* \subset (0, \rho^*]$.

We will show that $\rho^* \in \Sigma^*$. Otherwise, $r_{\rho^*,\infty} = -\infty$ and $s_{\rho^*,\infty} = \infty$. There exists $T > 0$ such that $s_{\rho^*}(T) - r_{\rho^*}(T) > \wedge$. Utilizing the continuous dependence of $(u_\rho, v_\rho, w_\rho, r_\rho, s_\rho)$ on ρ, we can find $\varepsilon > 0$ such that $s_\rho(T) - r_\rho(T) > \wedge$ for $\rho \in (\rho^* - \varepsilon, \rho^* + \varepsilon)$. It follows that

$$\lim_{t\to\infty} [s_\rho(t) - r_\rho(t)] \geq s_\rho(T) - r_\rho(T) > \wedge, \quad \forall\, \rho \in (\rho^* - \varepsilon, \rho^* + \varepsilon).$$

Therefore, $(\rho^* - \varepsilon, \rho^* + \varepsilon) \cap \Sigma^* = \emptyset$, and $\sup \Sigma^* \le \rho^* - \varepsilon$. This contradicts the definition of ρ^*.

Define

$$\Sigma_* = \left\{ \nu : \ \nu \ge \underline{\rho} \text{ such that } s_{\rho,\infty} - r_{\rho,\infty} \le \wedge \text{ for all } \rho \le \nu \right\},$$

where $\underline{\rho}$ is given by Lemma 4.2. Then, $\rho_* \overset{\text{def}}{=} \sup \Sigma_* \le \rho^*$ and $(0, \rho_*) \subset \Sigma_*$. Similarly to the above, it can be obtained that $\rho_* \in \Sigma_*$. The proof is complete.

Theorem 4.4 gives that there exist threshold values for boundary moving parameters μ and β, and gives sufficient conditions for predator spreading or vanishing.

5. Conclusions and discussion

In the present work, we investigate a prey-predator model with stage structure and free boundaries, which is a degenerate diffusion system consisting of one ordinary differential equation and two partial differential equations. Considering this new mathematical framework, we overcome the difficulties caused by partial degeneracy and prove the local and global existence, uniqueness, and uniform estimates of solutions to the coupled parabolic-ODE system. We also characterize the long-time behavior of solutions and provide sharp criteria for spreading and vanishing.

For the corresponding ordinary differential systems, it has been shown that if $a_1bc < b_2\theta_1(\theta_2 + b)$, then the equilibrium $(0, 0, c/b_2)$ is globally stable; if $a_1bc > b_2\theta_1(\theta_2 + b)$ and $a_1a_2b < b_1b_2(\theta_2 + b)$, then the positive equilibrium (u^*, v^*, w^*) is globally asymptotically stable.

Within the framework of the present model, it is shown that when $a_1bc \le b_2\theta_1(\theta_2+b)$, the introduced predator is bound to vanish; when $a_1bc > b_2\theta_1(\theta_2 + b)$, the successful invasion and spread of the introduced predator are determined by the initial data and variable parameters. Provided that the initial interval $[-s_0, s_0]$ is greater than a critical threshold, namely

$$2s_0 \ge \pi \sqrt{\frac{d_1b_2(\theta_2 + b)}{a_1bc - b_2\theta_1(\theta_2 + b)}},$$

then spreading always occurs. On the other hand, if

$$2s_0 < \pi \sqrt{\frac{d_1b_2(\theta_2 + b)}{a_1bc - b_2\theta_1(\theta_2 + b)}},$$

the dichotomy between spreading and vanishing depends on the parameters μ, β and the initial distributions (u_0, v_0).

Biologically, the proposed model and corresponding theoretical results better reflect realistic ecological processes. Mathematically, this system exhibits richer and more refined dynamical behaviors.

Author contributions

Hongtao Zhang: Conceptualization, methodology, writing-original draft; Jingfu Zhao: Formal analysis, investigation; Jina Li: writing-review & editing. All authors contributed to the study conception and design. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by Natural Science Foundation of Henan (No. 252300421500). The authors are grateful to the anonymous referees for their valuable comments and constructive suggestions.

Conflict of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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