AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 22052–22068.
DOI: 10.3934/math.2026892
Received: 25 March 2026
Revised: 04 May 2026
Accepted: 12 May 2026
Published: 23 July 2026

Research article

Asymptotic behavior of eigenvalues and eigenfunctions in a non-self-adjoint problem for the Schrödinger operator with involution

Akbope Beisebayeva and Abdizhahan M. Sarsenbi*

Department of Mathematics, M.Auezov South Kazakhstan University, Tauke Khan 5, Shymkent 160012, Kazakhstan

* **Correspondence:** Email: abdizhakhan.sarsenbi@auezov.edu.kz; Tel: +77058301361.

Abstract: In this paper, the spectral properties of non-self-adjoint boundary value problems for linear second-order differential equations with involution and variable complex-valued coefficients are investigated. The existence of linearly independent solutions is established, and the form of the general solution of linear second-order differential equations with involution and variable complex-valued coefficients is presented. Asymptotic formulas for linearly independent solutions of the equations under consideration are obtained. Using the asymptotics of linearly independent solutions, asymptotic formulas for eigenvalues and eigenfunctions of a non-self-adjoint spectral problem for a second-order differential equation with involution are proved. Estimates for the norms of derivatives of eigenfunctions are also established.

Keywords: differential equation with involution; non-self-adjoint boundary conditions; asymptotics of eigenvalues; asymptotics of eigenfunctions; asymptotics of solutions
Mathematics Subject Classification: 34K08, 35S16, 39B22

1. Introduction

The study of differential equations with involution has been the subject of numerous works. A substantial bibliography on the theory of differential equations with involution can be found in [1–3]. Applications of involution transformations in various fields of science are discussed, for example, in [1, 4–6], and in electrical engineering [7–9]. Apparently, one of the earliest physical problems that led to the concept of involution was the problem of mutual trajectories, considered by I. Bernoulli and L. Euler [10, 11]. In classical geometry, the application of involution is associated with the search for curves possessing symmetry properties [11]. The modeling of nonequilibrium systems in classical statistical mechanics [12], fluid flow through porous media [13], mathematical models of heat and mass transfer [14], and other areas of scientific research employ involution transformations.

Eigenvalue problems for differential equations with involution are of considerable interest. Such problems arise in solving certain types of partial differential equations with involution by the method of separation of variables [15, 16].

The use of eigenvalue problems for differential equations with involution requires knowledge of the asymptotics of eigenvalues and eigenfunctions of the problems under investigation. Among recent works devoted to the study of the asymptotics of eigenvalues and eigenfunctions of ordinary differential operators, one may note papers [17–21].

Let us recall [1] that a function $f(x)$, $f(x) \not\equiv x$, mapping a set $[-1, 1]$ into itself is called an involution if $f(f(x)) = x$.

Asymptotic formulas for the eigenvalues and eigenfunctions of various boundary value problems for an equation of the form

$$-u''(-x) = \lambda u(x), \ -1 < x < 1, \tag{1.1}$$

were studied in works [22, 23]. Here, the equation contains a linear involution transformation $f(x) = -x$. Problems for a more general equation with pure involution of the form

$$-u''(-x) + q(x)\, u(x) = \lambda u(x), \ -1 < x < 1, \tag{1.2}$$

with a complex-valued potential $q(x)$ were considered in [24, 25]. Equation (1.1) will be referred to as a second-order one-dimensional involutive differential equation.

In recent years, the study of differential equations with involution has advanced in various directions (see, for example, [26–28]).

Also worth noting is the work [29], devoted to the study of classical solutions of the multidimensional Schrödinger equation, as well as the work [30], devoted to the study of classical solutions for a class of coupled Euler–Poisson–Darboux–Tricomi equations.

In the present work, eigenvalue problems for second-order differential equations with involution (1.2) and non-self-adjoint boundary conditions are investigated. The results on the asymptotic formulas for eigenvalues and eigenfunctions of the non-self-adjoint spectral problem are presented.

The following new results have been obtained:

- An integral representation of solutions of the studied equations is derived;
- Asymptotic formulas for the solutions of the equation are obtained for large values of the spectral parameter;
- Asymptotic formulas for the eigenvalues are obtained for large values of the spectral parameter;
- Asymptotic formulas for the eigenfunctions and their derivatives are obtained for large values of the spectral parameter;
- Important estimates for the norms of the eigenfunctions are established.

All obtained results are new and can be applied in the theory of differential equations with involution.

It should be noted that, until now, no works have been devoted to studying the asymptotic properties of eigenvalues and eigenfunctions of boundary value problems for a differential equation with involution of the form (1.2).

Linearly independent solutions of Eq (1.1) are even and odd functions, which can be represented as the sum and difference of linearly independent solutions of ordinary differential equations (see Section 3). The structure of solutions to differential equations with involution does not allow one to directly extend known asymptotic results for solutions of second-order ordinary differential equations with variable coefficients - equations with involution of the form (1.2). The main features as described in Remark 1. It is well-known that non-self-adjoint spectral problems are difficult to study, as their systems of eigenfunctions are nonorthogonal. Therefore, one often has to consider the adjoint spectral problem (see Section 3), whereas in the self-adjoint case, the original and adjoint problems coincide. In our case, Eq (1.2) is non-self-adjoint due to a complex-valued this coefficient, and is the boundary conditions (4.1) are also non-self-adjoint.

This research is conducted with the aim of further applying the results to the study of solvability of problems for pseudoparabolic equations involving the studied operators, using the method of separation of variables.

The paper is organized as follows. In Section 2, the necessary preliminary information is presented, and lemmas on the asymptotic form of the solutions of the equation for large values of the spectral parameter are proved. Asymptotic formulas for the first derivative of the solutions are also given. In Section 3, asymptotic formulas for the eigenvalues and eigenfunctions of the studied problems are established. The obtained asymptotic formulas for eigenfunctions and their derivatives allow one to estimate the norms of the derivatives of the eigenfunctions in terms of the norms of the eigenfunctions themselves. Such estimates can be used when solving partial differential equation problems by the method of separation of variables.

2. Structure of solutions of a second-order differential equation with involution

In this section, we construct an integral representation of the solution of Eq (1.2). Then, we derive asymptotic formulas for the even and odd parts of the solutions of Eq (1.2). Using these formulas, the main results of the work are Eq obtained.

A function $u(x) \in C^2[-1, 1]$ that satisfies Eq (1.2) is called a regular solution of the equation.

The following two important theorems have not yet been established in the theory of differential equations with involution [1–3]. Therefore, we present them here with Eq complete proofs. We prove the theorems on linearly independent solutions and on the general solution of Eq (1.2).

Theorem 1. The homogeneous linear second-order differential equation with involution (1.2) with a continuous coefficient $q(x) \in C[-1, 1]$ possesses two linearly independent solutions.

Proof. Consider the Cauchy problem at zero for Eq (1.2). Note that the linear second-order differential equation with involution (1.2) with Cauchy data $u(0) = a_1$, $u'(0) = a_2$ at the point $x = 0$ has a unique solution. Let us denote this solution by $u_1(x)$. Equation (1.2) with different initial data $u(0) = a_3$, $u'(0) = a_4$ also has a unique solution, which we denote by $u_2(x)$. These solutions are linearly independent if the determinant $\begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} \neq 0$ (see Theorem 2.1, p. 32 in [31]). By appropriately choosing the initial a_1, a_2, a_3, a_4, we obtain two linearly independent solutions. The theorem is proved.

A solution of the linear second-order differential equation with involution (1.2) containing two arbitrary constants will be called a general solution if any particular solution can be obtained from it by assigning specific values to the constants.

Theorem 2. The general solution of the linear second-order differential equation with involution (1.2) has the form

$$u(x) = C_1 u_1(x) + C_2 u_2(x),$$

where $u_1(x)$ and $u_2(x)$ are linearly independent solutions of Eq (1.2).

Proof. An arbitrary particular solution of Eq (1.2) is determined using the Cauchy problem at zero with $u(0) = u_0,\ u'(0) = u_{01}$. Then, at $x = 0$, we have

$$u(0) = C_1 u_1(0) + C_2 u_2(0).$$

Similarly, we obtain the corresponding expression for the derivative $u'(x)$ of the function $u(x)$.

$$u'(0) = C_1 u_1'(0) + C_2 u_2'(0).$$

Because $u(0) = u_0,\ u'(0) = u_{01}$, we obtain a system of two equations:

$$C_1 u_1(0) + C_2 u_2(0) = u_0,$$

$$C_1 u'_1(0) + C_2 u'_2(0) = u_{01}.$$

Because the determinant of the system is nonzero, the system has a unique solution, which corresponds to a particular solution of Eq (1.2). The theorem is proved.

Note that any function $u(x) \in L_2(-1,1)$ can be represented as the sum of an even and an odd function. Then, Eq (1.2) splits into two ordinary second-order differential equations, one corresponding to the even part and the other to the odd part. New representations of the solutions of the considered equations have been obtained.

3. Asymptotics of linearly independent solutions of Eq (1.2)

Consider Eq (1.2). Let us represent the function $u(x)$ in the form

$$u(x) = y(x) + z(x),\ u(-x) = y(x) - z(x),$$

where $y(x)$ is an even function, and $z(x)$ is an odd function. Substituting these functions into Eq (1.2) and assuming that $q(x)$ is an even function, we obtain two equations:

$$-y''(x) + q(x)\,y(x) = \lambda y(x),\ -1 < x < 1, \tag{3.1}$$

$$z''(x) + q(x)\,z(x) = \lambda z(x),\ -1 < x < 1, \tag{3.2}$$

where $y(x)$ is an even function, and $z(x)$ is an odd function. Let us denote $\rho^2 = \lambda$, $\lambda = |\lambda|\, e^{i\varphi}$, $\sqrt{\lambda} = \sqrt{|\lambda|}e^{\frac{i\varphi}{2}}$, $-\frac{\pi}{2} < \varphi < \frac{3\pi}{2}$. The linearly independent solutions of the homogeneous equation (3.1) $-y''(x) = \lambda y(x)$ are denoted by $y_1(x) = e^{i\rho x} + e^{-i\rho x}$, $y_2(x) = e^{i\rho x} - e^{-i\rho x}$. Similarly, the linearly independent solutions of the corresponding homogeneous equation (3.2) are denoted by $z_1(x) = e^{\rho x} + e^{-\rho x}$, $z_2(x) = e^{\rho x} - e^{-\rho x}$.

Thus, the linear second-order differential equation with involution (1.2) splits into two ordinary differential equations.

Lemma 1. Let $q(x) \in C[-1,1]$ be an even function. If $y(x)$ is the even part of a solution of Eq (1.2), that is, if the function $y(x)$ is a solution of the ordinary differential Eq (3.1), then the function $y(x)$ is also a solution of the integral equation

$$y(x) = \cos\rho x - \frac{1}{\rho}\int_0^x \sin\rho(x-\tau)\, q(\tau)\, y(\tau)\, d\tau. \tag{3.3}$$

If $z(x)$ is the odd part of a solution of Eq (1.2), that is, if the function $z(x)$ is a solution of the ordinary differential equation (3.2), then the function $z(x)$ is also a solution of the integral equation

$$z(x) = e^{\rho x} - e^{-\rho x} + \frac{1}{4\rho}\int_0^x [(e^{\rho x} + e^{-\rho x})(e^{\rho\tau} - e^{-\rho\tau}) - (e^{\rho\tau} + e^{-\rho\tau})(e^{\rho x} - e^{-\rho x})] q(\tau)\, z(\tau)\, d\tau, \tag{3.4}$$

where $z_1(x) = e^{\rho x} + e^{-\rho x},\ z_2(x) = e^{\rho x} - e^{-\rho x}$.

Proof. In the theory of differential equations [32], a more general form of Formula (3.3) is well-known. If the function under consideration is even, then from [32] we obtain Formula (3.3). Formula (3.4) is proved by the variation method. By simple calculations, we obtain Formula (3.4) for the odd function $z(x)$. The lemma is proved.

Let us write the asymptotic formulas for the solutions $y(x)$ of Eq (1.2) for large values of the parameter $\rho = \theta_1 + i\theta_2$. It is assumed that $q(x) \in C[-1,1]$.

As can be seen, the linearly independent solutions of the homogeneous equation with involution consist of even and odd functions. Therefore, in our considerations, even and odd solutions dominate. The asymptotic formulas for the solutions of the second-order linear differential equation with involution (1.2) are given in the following two lemmas.

Lemma 2. Let $q(x) \in C[-1,1]$ be an even function. For sufficiently large $|\rho|$, $\rho = \theta_1 + i\theta_2$, and for all $x \in [-1,1]$, the even part $y(x)$ and the odd part $z(x)$ of a solution of Eq (1.2) have the following asymptotic forms:

$$y(x) = \left(e^{i\rho x} + e^{-i\rho x}\right) + O\left(\frac{1}{\rho}\right), \quad z(x) = (e^{\rho x} - e^{-\rho x}) + O\left(\frac{1}{\rho}\right).$$

Proof. Let us consider Formula (3.3), corresponding to an even solution. Let $y(x) = e^{|\theta_2|x}F(x)$. Then, from (3.3), we obtain the equality

$$F(x) = e^{-|\theta_2|x}\cos\rho x + \frac{1}{\rho}\int_0^x \sin\rho(x-\tau)\, q(\tau)\, e^{-|\theta_2|(x-\tau)}F(\tau)\, d\tau.$$

The last equality implies the inequality

$$|F(x)| \le |\cos\rho x| + \frac{1}{|\rho|}\int_0^x |q(\tau)|\, |F(\tau)|\, d\tau.$$

Let us denote $m = \max\limits_{-1\le x\le 1} |F(x)|$. Then, from the inequality

$$m \le |\cos\rho x| + \frac{m}{|\rho|}\int\limits_{-1}^{1} |q(\tau)|\, d\tau,$$

we obtain the estimate

$$m \le |\cos\rho x| \left[1 - \frac{1}{|\rho|}\int\limits_{-1}^{1} |q(\tau)|\, d\tau\right]^{-1}.$$

For sufficiently large $|\rho|$, $\rho = \theta_1 + i\theta_2$, the quantity $1 - \frac{1}{|\rho|}\int\limits_{-1}^{1} |q(\tau)|\, d\tau$ is positive. It follows that for sufficiently large $|\rho|$, $\rho = \theta_1 + i\theta_2$, there exists a number M_0 such that

$$m = \max_{-1\le x\le 1} |F(x)| \le M_0.$$

Using the last estimate, from Formula (3.3), we obtain the asymptotic formula

$$y(x) = \mathrm{O}\left(e^{|\theta_2||x|}\right). \tag{3.5}$$

Now, we substitute this value into the right-hand side of Formula (3.3). Then,

$$y(x) = \cos\rho x + \frac{1}{\rho}\int\limits_{0}^{x} \sin\rho(x-\tau)\, q(\tau)\, \mathrm{O}\left(e^{|\theta_2|\tau}\right) d\tau.$$

After the transformation, we have

$$y(x) = \left(e^{i\rho x} + e^{-i\rho x}\right) + \mathrm{O}\left(\frac{e^{2|\theta_2|}}{\rho}\right).$$

From the property of the function $y(x)$ being even on an interval $[-1, 1]$, we obtain the following. Because

$$y(x) = \left(e^{i\rho x} + e^{-i\rho x}\right) + \mathrm{O}\left(\frac{e^{|\theta_2|x}}{\rho}\right);$$

$$y(-x) = \left(e^{i\rho x} + e^{-i\rho x}\right) + \mathrm{O}\left(\frac{e^{-|\theta_2|x}}{\rho}\right),$$

evenness holds if

$$\mathrm{O}\left(\frac{e^{|\theta_2|x}}{\rho}\right) = \mathrm{O}\left(\frac{e^{-|\theta_2|x}}{\rho}\right),$$

that is, if $\theta_2 = 0$. This implies the assertion of the lemma.

Now let us consider Formula (3.4), corresponding to the odd solution. Because

$$z_2(x)\, z_1(\tau) = \left(e^{\rho x} - e^{-\rho x}\right)\left(e^{\rho\tau} + e^{-\rho\tau}\right) = e^{\rho(x+\tau)} + e^{\rho(x-\tau)} - e^{-\rho(x-\tau)} - e^{-\rho(x+\tau)},$$
$$z_2(\tau)\, z_1(x) = \left(e^{\rho\tau} - e^{-\rho\tau}\right)\left(e^{\rho x} + e^{-\rho x}\right) = e^{\rho(x+\tau)} + e^{-\rho(x-\tau)} - e^{\rho(x-\tau)} - e^{-\rho(x+\tau)},$$

$$z_2(\tau) z_1(x) - z_2(x) z_1(\tau) = 2\left(e^{-\rho(x-\tau)} - e^{\rho(x-\tau)}\right) \sim e^{|\theta_1|(x-\tau)},\ 0 \le \tau \le x,$$

from Formula (3.4), we obtain the equality

$$z(x) = z_2(x) + \frac{1}{4\rho} \int_0^x \left[e^{-\rho(x-\tau)} - e^{\rho(x-\tau)}\right] q(\tau) z(\tau) d\tau. \tag{3.6}$$

Let $z(x) = e^{|\theta_1|x} F(x)$. Then,

$$e^{|\theta_1|x} F(x) = z_2(x) + \frac{1}{4\rho} \int_0^x \left[e^{-\rho(x-\tau)} - e^{\rho(x-\tau)}\right] q(\tau) e^{|\theta_1|\tau} F(\tau) d\tau.$$

From the last equality, we obtain

$$F(x) = z_2(x) e^{-|\theta_1|x} + \frac{1}{4\rho} \int_0^x \left[e^{-\rho(x-\tau)} - e^{\rho(x-\tau)}\right] q(\tau) e^{-|\theta_1|x} e^{|\theta_1|\tau} F(\tau) d\tau.$$

By straightforward transformations, we obtain the inequality

$$|F(x)| \le |z_2(x)| e^{-|\theta_1|x} + \frac{1}{2\rho} \int_0^x e^{|\theta_1|(x-\tau)} q(\tau) e^{-|\theta_1|(x-\tau)} F(\tau) d\tau,$$

which can be written in the form

$$|F(x)| \le |z_2(x)| e^{-|\theta_1|x} + \frac{1}{2|\rho|} \int_{-x}^x |q(\tau)| |F(\tau)| d\tau.$$

Let us denote $m = \max\limits_{-1 \le x \le 1} |F(x)|$ again. Then,

$$m \le \left(1 - \frac{1}{2|\rho|} \int_{-1}^1 |q(\tau)| d\tau\right)^{-1}.$$

For sufficiently large $|\rho|$, $\rho = \theta_1 + i\theta_2$, there exists a number M_1 such that

$$m = \max_{-1 \le x \le 1} |F(x)| \le M_1.$$

Consequently,

$$z(x) = \mathrm{O}\left(e^{|\theta_1|x}\right). \tag{3.7}$$

We substitute the obtained value into the right-hand side of (3.4).

$$z(x) = z_2(x) + \frac{1}{4\rho}\int\limits_0^x \left[e^{-\rho(x-\tau)} - e^{\rho(x-\tau)}\right] \mathrm{O}\left(e^{|\theta_1|\tau}\right) q(\tau)\, d\tau \tag{3.8}$$

$$= z_2(x) + \frac{1}{4\rho}\int\limits_0^x \mathrm{O}\left(e^{|\theta|_1(x-\tau)}\right) \mathrm{O}\left(e^{|\theta_1|\tau}\right) q(\tau)\, d\tau. \tag{3.9}$$

After the transformation, we obtain

$$z(x) = \left(e^{\rho x} - e^{-\rho x}\right) + \mathrm{O}\left(\frac{e^{|\theta_1||x|}}{\rho}\right).$$

From the oddness property of the function $z(x)$, we obtain that the spectral parameter in the odd equation satisfies the condition $\theta_1 = 0$. Thus, the second statement of the lemma is proved. The lemma is then fully proved.

Remark 1. In the case of ordinary differential equations of order n, the asymptotic formulas for linearly independent solutions are well-known (see, for example, [33]). In the case of a second-order equation, the asymptotic formulas for the sum and difference of linearly independent solutions

$$y_1(x) = e^{i\rho x}\left[1 + \mathrm{O}\left(\frac{1}{\rho}\right)\right], \quad y_2(x) = e^{-i\rho x}\left[1 + \mathrm{O}\left(\frac{1}{\rho}\right)\right]$$

have the form

$$\frac{y_1(x) + y_2(x)}{2} = \cos\rho x + \mathrm{O}\left(\frac{1}{\rho}\right), \quad \frac{y_1(x) - y_2(x)}{2} = \sin\rho x + \mathrm{O}\left(\frac{1}{\rho}\right),$$

if the parameter ρ is real and positive [33]. Regarding the conditions of the parameter being real and positive, we note that the asymptotic formulas we obtained have the form

$$y(x) = \left(e^{i\rho x} + e^{-i\rho x}\right) + \mathrm{O}\left(\frac{e^{|\theta_2|x}}{\rho}\right);$$

$$y(-x) = \left(e^{i\rho x} + e^{-i\rho x}\right) + \mathrm{O}\left(\frac{e^{-|\theta_2|x}}{\rho}\right).$$

Therefore, parity holds if

$$\mathrm{O}\left(\frac{e^{|\theta_2|x}}{\rho}\right) = \mathrm{O}\left(\frac{e^{-|\theta_2|x}}{\rho}\right),$$

that is, if $\theta_2 = 0$. Recall that $\rho = \theta_1 + i\theta_2$. As Example 1 in the next section shows, the parameter $\rho = \theta_1 + i\theta_2$ can be complex, that is, not positive.

Let us proceed to deriving the asymptotic formulas for the derivatives of the solutions.

Lemma 3. Let $q(x) \in C[-1, 1]$ be an even function. For sufficiently large $|\rho|$, $\rho = \theta_1 + i\theta_2$, and for all $x \in [-1, 1]$, the derivative of the even part $y'(x)$ and the derivative of the odd part $z'(x)$ of the solution of Eq (1.2) respectively have the following asymptotic form:

$$y'(x) = \left[i\rho\left(e^{i\rho x} - e^{-i\rho x}\right) + \mathrm{O}(1)\right], z'(x) = \rho\left(e^{\rho x} + e^{-\rho x}\right) + \mathrm{O}(1). \tag{3.10}$$

Proof. Consider Formula (3.3), corresponding to the even part of the solution of Eq (1.2):

$$y(x) = \left(e^{i\rho x} + e^{-i\rho x}\right) + \frac{1}{\rho}\int_0^x \sin\rho(x-\tau)\, q(\tau)\, y(\tau)\, d\tau.$$

Let us compute its derivative,

$$y'(x) = i\rho\left(e^{i\rho x} - e^{-i\rho x}\right) + \int_0^x \cos\rho(x-\tau)\, q(\tau)\, y(\tau)\, d\tau.$$

In the right-hand side, instead of $y(\tau)$, we substitute the value (3.5) from Lemma 2.

$$y'(x) = i\rho\left(e^{i\rho x} - e^{-i\rho x}\right) + \int_0^x \cos\rho(x-\tau)\, q(\tau)\, O\left(e^{|\theta_2|\tau}\right) d\tau.$$

From this, we obtain

$$y'(x) = i\rho\left(e^{i\rho x} - e^{-i\rho x}\right) + \mathrm{O}(1).$$

The first formula from (3.10) is proved. To prove the second formula from (3.10), we compute the derivative of function (3.6),

$$z(x) = z_2(x) + \frac{1}{4\rho}\int_0^x \left[e^{-\rho(x-\tau)} - e^{\rho(x-\tau)}\right] q(\tau)\, z(\tau)\, d\tau.$$

As in the previous case, we substitute the value (3.7) from Lemma 2 for $z(\tau)$ in the right-hand side of the last equality. Then,

$$z'(x) = \rho\,\left(e^{\rho x} + e^{-\rho x}\right) - \frac{1}{4}\int_0^x \left[e^{-\rho(x-\tau)} + e^{\rho(x-\tau)}\right] q(\tau)\, \mathrm{O}\left(e^{|\theta_1||\tau|}\right) d\tau.$$

Because the function $z(x)$ is odd, this implies the equality

$$z'(x) = \rho\,\left(e^{\rho x} + e^{-\rho x}\right) + \mathrm{O}(1).$$

The second assertion of the lemma is proven, and the lemma is thus proved.

4. Asymptotic formulas for eigenvalues and eigenfunctions of the non-self-adjoint boundary value problem for a second-order differential equation with involution

Consider the linear second-order differential equation with involution (1.2), where λ is a certain parameter, and the second derivative of the unknown function $u(x)$ involves an involution transformation $S\,u(x) = u(-x)$. We will study Eq (1.2) with non-self-adjoint boundary conditions

$$u(-1) = 0, \quad u'(1) = u'(-1). \tag{4.1}$$

The function $u(x)$ will be called an eigenfunction of the boundary value problem (1.2), (4.1), corresponding to the eigenvalue λ, if it is absolutely continuous together with its first derivative on the interval $(-1,1)$ and satisfies Eq (1.2) and the boundary conditions (4.1).

Equation (1.1) has linearly independent solutions $e^{i\rho x}+e^{-i\rho x}$, $e^{\rho x}-e^{-\rho x}$.

Example 1. Note that the non-self-adjoint spectral problem (1.1), (4.1) has two sequences of eigenvalues

$$\lambda_{k1}=-(k\pi)^2,\quad \lambda_{k2}=(k\pi)^2,\qquad k=1,\ 2,...$$

The system of nonorthogonal eigenfunctions of the spectral problem (1.2), (4.1) has the form

$$u_0(x)=x+1,\ u_{k1}(x)=\sin k\pi x,\ u_{k2}(x)=(-1)^k\frac{e^{k\pi x}-e^{-k\pi x}}{e^{k\pi}-e^{-k\pi}}+\cos k\pi x.$$

According to Theorem 2, the general solution of Eq (1.2) has the form

$$u(x)=C_1y(x)+C_2z(x),$$

where the even function $y(x)$ and the odd function $z(x)$ are linearly independent solutions of Eq (1.2), whose existence is established in Theorem 1. Note that the function $y(x)$ at the ends of the segment may not be even, and the function $z(x)$ at the ends of the segment may not be odd.

Theorem 3. Let $q(x)\in C[-1,1]$ be even functions. Then, the eigenvalues of the spectral problem (1.2), (4.1) form two infinite sequences:

$$\lambda_{k1}=-\left(k\pi+\mathrm{O}\left(\frac{1}{k}\right)\right)^2,\lambda_{k2}=\left(k\pi+\mathrm{O}\left(\frac{1}{k}\right)\right)^2.$$

Proof. Consider the general solution of Eq (1.2)

$$u(x)=C_1y(x)+C_2z(x),$$

where

$$y(x)=\left(e^{i\rho x}+e^{-i\rho x}\right)+\mathrm{O}\left(\frac{1}{\rho}\right),\quad z(x)=\left(e^{\rho x}-e^{-\rho x}\right)+\mathrm{O}\left(\frac{1}{\rho}\right).$$

We subject the function $u(x)$ to the boundary conditions (4.1) as follows:

$$C_1y(-1)+C_2z(-1)=0, \tag{4.2}$$

$$C_1y'(-1)+C_2z'(-1)=C_1y'(1)+C_2z'(1). \tag{4.3}$$

Considering that

$$y(x)=\left(e^{i\rho x}+e^{-i\rho x}\right)+\mathrm{O}\left(\frac{1}{\rho}\right),$$

$$z(x)=\left(e^{\rho x}-e^{-\rho x}\right)+\mathrm{O}\left(\frac{1}{\rho}\right),$$

$$y'(x)=\left[i\rho\left(e^{i\rho x}-e^{-i\rho x}\right)+\mathrm{O}(1)\right],$$

$$z'(x) = \rho\ (e^{\rho x} + e^{-\rho x}) + \mathrm{O}(1),$$

from (4.2) and (4.3), we obtain

$$C_1\left[\left(e^{i\rho} + e^{-i\rho}\right) + \mathrm{O}\left(\frac{1}{\rho}\right)\right] + C_2\left[-\left(e^{\rho} - e^{-\rho}\right) + \mathrm{O}\left(\frac{1}{\rho}\right)\right] = 0,$$

$$\begin{aligned} & C_1\left[-i\rho\left(e^{i\rho} - e^{-i\rho}\right) + \mathrm{O}(1)\right] + C_2\left[\rho\ \left(e^{\rho} + e^{-\rho}\right) + \mathrm{O}(1)\right] \\ = {} & C_1\left[i\rho\left(e^{i\rho} - e^{-i\rho}\right) + \mathrm{O}(1)\right] + C_2\left[\rho\ \left(e^{\rho} + e^{-\rho}\right) + \mathrm{O}(1)\right]. \end{aligned}$$

We obtain a system of algebraic equations,

$$\begin{aligned} C_1\left[\left(e^{i\rho} + e^{-i\rho}\right) + \mathrm{O}\left(\frac{1}{\rho}\right)\right] + C_2\left[-\left(e^{\rho} - e^{-\rho}\right) + \mathrm{O}\left(\frac{1}{\rho}\right)\right] &= 0, \\ C_1\left[-2i\rho\left(e^{i\rho} - e^{-i\rho}\right) + \mathrm{O}(1)\right] &= 0. \end{aligned}$$

The characteristic determinant is written in the form

$$\Delta = C_2\left[-\left(e^{\rho} - e^{-\rho}\right) + \mathrm{O}\left(\frac{1}{\rho}\right)\right] C_1\left[-2i\rho\left(e^{i\rho} - e^{-i\rho}\right) + \mathrm{O}(1)\right] = 0.$$

The equality

$$\left[-\left(e^{\rho} - e^{-\rho}\right) + \mathrm{O}\left(\frac{1}{\rho}\right)\right]\left[-2\left(e^{i\rho} - e^{-i\rho}\right) + \mathrm{O}\left(\frac{1}{\rho}\right)\right] = 0$$

implies the following relations:

$$\left(e^{\rho} - e^{-\rho}\right) = \mathrm{O}\left(\frac{1}{\rho}\right); \ \left(e^{i\rho} - e^{-i\rho}\right) = \mathrm{O}\left(\frac{1}{\rho}\right)$$

or

$$e^{-\rho}\left(e^{2\rho} - 1\right) = \mathrm{O}\left(\frac{1}{\rho}\right); e^{-i\rho}\left(e^{2i\rho} - 1\right) = \mathrm{O}\left(\frac{1}{\rho}\right).$$

We get two relations. The first relation

$$e^{2\rho} = 1 + \mathrm{O}\left(\frac{1}{\rho}\right) \Rightarrow 2\rho = 2k\pi i\left(1 + \mathrm{O}\left(\frac{1}{k}\right)\right); \ \rho_{k1} = k\pi i\left(1 + \mathrm{O}\left(\frac{1}{k}\right)\right),$$

gives us the asymptotics of the first series of eigenvalues $\rho_{k1} = k\pi i\left(1 + \mathrm{O}\left(\frac{1}{k}\right)\right)$. The second relation

$$e^{2i\rho} = 1 + \mathrm{O}\left(\frac{1}{\rho}\right)$$

gives us the asymptotics of the second series of eigenvalues $\rho_{k2} = k\pi\left(1 + \mathrm{O}\left(\frac{1}{k}\right)\right)$. The theorem is proved.

Using Theorem 3 and Lemma 2, one can find asymptotic expressions for the eigenfunctions.

Theorem 4. Let $q(x) \in C[-1, 1]$ be an even function. The asymptotic formulas for the eigenfunctions of the non-self-adjoint boundary value problem (1.2), (4.1) are given by

$$u_{k1}(x) = \sin k\pi x + \mathrm{O}\left(\frac{1}{k}\right), \quad u_{k2}(x) = \cos k\pi x + (-1)^k \frac{e^{k\pi x} - e^{-k\pi x}}{e^{k\pi} - e^{-k\pi}} + \mathrm{O}\left(\frac{1}{k}\right).$$

Proof. Let us consider the general solution of Eq (1.2)

$$u(x) = c_1 \left[\left(e^{i\rho x} + e^{-i\rho x}\right) + \mathrm{O}\left(\frac{1}{\rho}\right)\right] + c_2 \left[\left(e^{\rho x} - e^{-\rho x}\right) + \mathrm{O}\left(\frac{1}{\rho}\right)\right]. \tag{4.4}$$

We substitute the value $\rho_{k1} = \left(k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right) i$ from Theorem 3 into the asymptotic expression (4.4). Then, after some transformations, we obtain

$$u_{k1}(x) = c_1 \left(e^{-k\pi x} + e^{k\pi x} + \mathrm{O}\left(\frac{1}{k}\right)\right) + c_2 \left(2i \sin k\pi x + \mathrm{O}\left(\frac{1}{k}\right)\right).$$

Let us check that the boundary conditions are met:

$$u_{k1}(-1) = c_1 \left(e^{-k\pi x} + e^{k\pi x} + \mathrm{O}\left(\frac{1}{k}\right)\right) + c_2 \left(-2i \sin k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right) = 0.$$

From the equality, it follows that $c_1 = 0$. Consequently, we have obtained the asymptotics of the first series of eigenfunctions

$$u_{k1}(x) = \sin k\pi x + \mathrm{O}\left(\frac{1}{k}\right).$$

To derive the asymptotics of the second series of eigenfunctions in (4.4), we substitute $\rho_{k2} = \left(k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right)$. Then,

$$u_{k2}(x) = c_1 \left[2 \cos k\pi x + \mathrm{O}\left(\frac{1}{k}\right)\right] + c_2 \left[\left(e^{k\pi x} - e^{-k\pi x}\right) + \mathrm{O}\left(\frac{1}{k}\right)\right].$$

Satisfaction of the boundary conditions implies the equality

$$u_{k2}(-1) = c_1 \left[2 \cos k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right] - c_2 \left[\left(e^{k\pi} - e^{-k\pi}\right) + \mathrm{O}\left(\frac{1}{k}\right)\right] = 0.$$

We obtain

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{(-1)^k}{e^{k\pi} - e^{-k\pi}}.$$

Hence,

$$u_{k2}(x) = \cos k\pi x + (-1)^k \frac{e^{k\pi x} - e^{-k\pi x}}{e^{k\pi} - e^{-k\pi}} + \mathrm{O}\left(\frac{1}{k}\right).$$

We have obtained the asymptotics of the second series of eigenvalues. The theorem is proved.

In the spectral theory of differential operators, various estimates of eigenfunctions and their derivatives can be used (see, for example, [34]). Using the Formula (3.10), we can similarly obtain asymptotic formulas for the derivative of eigenfunctions.

Theorem 5. Let $q(x) \in C[-1, 1]$ be an even function.The asymptotic formulas for the derivatives of the eigenfunctions of the non-self-adjoint boundary value problem (1.2), (4.1) are given by

$$u_{k1}{}'(x) = [k\pi \cos k\pi x + \mathrm{O}(1)], \quad u'_{k2}(x) = k\pi \left[\sin k\pi x + \left(\frac{e^{kx} + e^{-k\pi x}}{e^{k\pi} + e^{-k\pi}}\right)\right] + \mathrm{O}(1)].$$

Proof. Using Theorem 2, we write the derivative of the general solution of Eq (1.2)

$$u'(x) = C_1 y'(x) + C_2 z'(x), \tag{4.5}$$

where

$$y'(x) = i\rho\left(e^{i\rho x} - e^{-i\rho x}\right) + \mathrm{O}(1),$$
$$z'(x) = \rho\left(e^{\rho x} + e^{-\rho x}\right) + \mathrm{O}(1).$$

Function (4.5) satisfies the second boundary condition. Therefore, for $\rho_{k1} = \left(k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right) i$, we have the equality

$$u'_{k1}(-1) = C_1\left[i\rho_{k1}\left(-e^{\left(k\pi+\mathrm{O}\left(\frac{1}{k}\right)\right)} + e^{-\left(k\pi+\mathrm{O}\left(\frac{1}{k}\right)\right)}\right) + \mathrm{O}(1)\right] + C_2\left[2\rho_{k1}\cos\left(k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right) + \mathrm{O}(1)\right]$$

$$= u'_{k1}(1) = C_1\left[i\rho_{k1}\left(e^{\left(k\pi+\mathrm{O}\left(\frac{1}{k}\right)\right)} - e^{-\left(k\pi+\mathrm{O}\left(\frac{1}{k}\right)\right)}\right) + \mathrm{O}(1)\right] + C_2\left[2\rho_{k1}\cos\left(k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right) + \mathrm{O}(1)\right].$$

The last equality is satisfied if $C_1 = 0$. Then, from (4.3), we obtain the asymptotics for

$$\rho = \rho_{k1} = \left(k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right) i;$$

$$u'_{k1}(x) = k\pi\cos k\pi x + \mathrm{O}(1).$$

At $\rho = \rho_{k2} = \left(k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right)$, the second boundary condition gives us equality

$$C_1\left[-\rho_{k1} 2i\sin\left(k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right) + \mathrm{O}(1)\right] + C_2\left[\rho_{k1}\left(e^{\left(k\pi+\mathrm{O}\left(\frac{1}{k}\right)\right)} + e^{-\left(k\pi+\mathrm{O}\left(\frac{1}{k}\right)\right)}\right) + \mathrm{O}(1)\right]$$

$$= C_1\left[\rho_{k1} 2i\sin\left(k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right) + \mathrm{O}(1)\right] + C_2\left[\rho_{k1}\left(e^{\left(k\pi+\mathrm{O}\left(\frac{1}{k}\right)\right)} + e^{-\left(k\pi+\mathrm{O}\left(\frac{1}{k}\right)\right)}\right) + \mathrm{O}(1)\right].$$

The last equality is satisfied for any C_1, C_2. Therefore, from (4.3), we obtain the asymptotic formula

$$u'_{k2}(x) = -2C_1\left(k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right)\sin\left(k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right)x \tag{4.6}$$

$$+ C_2\left(k\pi + \mathrm{O}\left(\frac{1}{k}\right)\right)\left(e^{\left(k\pi+\mathrm{O}\left(\frac{1}{k}\right)\right)x} + e^{-\left(k\pi+\mathrm{O}\left(\frac{1}{k}\right)\right)x}\right) + \mathrm{O}(1). \tag{4.7}$$

By choosing constants C_1, C_2, after simple transformations, we obtain an asymptotic formula for the derivative of an eigenfunction

$$u'_{k2}(x) = k\pi\left[\sin k\pi x + \left(\frac{e^{k\pi x} + e^{-k\pi x}}{e^{k\pi} + e^{-k\pi}}\right)\right] + \mathrm{O}(1)].$$

The theorem is proved.

The proven theorem shows that the asymptotic behavior of the derivative $u'_{k2}(x)$ does not coincide with the derivative of the asymptotic behavior of the eigenfunctions $u_{k2}(x)$.

From Theorems 4 and 5 follow estimates of the norms of the derivative of the eigenfunctions.

Theorem 6. Let $q(x) \in C[-1,1]$ be an even function. For the derivatives of the eigenfunctions of the non-self-adjoint boundary value problem (1.2), (4.1), the following estimates hold:

$$\|u_{k1}{}'(x)\|_{C[-1,1]} \le C_3\sqrt{|\lambda_{k1}|}\|u_{k1}(x)\|_{C[-1,1]},\ \ \|u_{k2}{}'(x)\|_{C[-1,1]} \le C_4\sqrt{|\lambda_{k2}|}\|u_{k2}(x)\|_{C[-1,1]}$$

for all indices k,where the constants C_3, C_4 do not depend on the index.

Proof. By Theorem 5, we have

$$u'_{k1}(x) = k\pi\cos k\pi x + \mathrm{O}(1).$$

Hence, it follows that

$$\max_{-1\le x\le 1}|u'_{k1}(x)| = k\pi + \mathrm{O}(1) = k\pi\left(1 + \mathrm{O}\left(\frac{1}{k}\right)\right).$$

Because

$$u_{k1}(x) = \sin k\pi x + \mathrm{O}\left(\frac{1}{k}\right),$$

it follows from Theorem 4 that

$$\max_{-1\le x\le 1}|u_{k1}(x)| = 1 + \mathrm{O}\left(\frac{1}{k}\right).$$

Consequently,

$$\max_{-1\le x\le 1}|u'_{k1}(x)| = k\pi\left(\max_{-1\le x\le 1}|u_{k1}(x)| + \mathrm{O}\left(\frac{1}{k}\right)\right).$$

This means that there exists a number C_3 such that the first assertion of the theorem holds. Analogously, the second inequality is proved. The theorem is proved.

The following theorem provides estimates for various norms of the eigenfunctions.

Theorem 7. Let $q(x) \in C[-1,1]$ be an even function. The norms of the eigenfunctions of the non-self-adjoint boundary value problem (1.2), (4.1) satisfy the following estimates

$$\|u_{k1}(x)\|_{C[-1,1]} \le C_5\|u_{k1}(x)\|_{L_2(-1,1)},\ \ \|u_{k2}(x)\|_{C[-1,1]} \le C_6\|u_{k2}(x)\|_{L_2(-1,1)}$$

for all indices k, where the constants C_5, C_6 do not depend on the index.

Proof. Let us compute the $L_2(-1,1)$-norm of the eigenfunction $u_{k1}(x)$:

$$\int_{-1}^{1}|u_{k1}(x)|^2dx = \int_{-1}^{1}\left|\cos k\pi x + \mathrm{O}\left(\tfrac{1}{k}\right)\right|^2dx = \int_{-1}^{1}\left(\cos^2 k\pi x + \mathrm{O}\left(\tfrac{1}{k}\right) + \mathrm{O}\left(\tfrac{1}{k^2}\right)\right)dx$$
$$= \int_{-1}^{1}\left(\tfrac{1+\cos 2k\pi x}{2} + \mathrm{O}\left(\tfrac{1}{k}\right) + \mathrm{O}\left(\tfrac{1}{k^2}\right)\right)dx = 1 + \mathrm{O}\left(\tfrac{1}{k}\right) + \mathrm{O}\left(\tfrac{1}{k^2}\right).$$

The $L_2(-1,1)$-norm of the eigenfunction $u_{k1}(x)$ is equal to

$$\|u_{k1}(x)\|_{L_2(-1,1)} = \sqrt{1 + \mathrm{O}\left(\frac{1}{\rho_{k1}}\right)}.$$

Similarly, we compute the $L_2(-1,1)$-norm of the eigenfunction $u_{k2}(x)$. Comparing with Theorem 5, we obtain the statement of the theorem. The theorem is proved.

5. Conclusions

In this work, asymptotic formulas for the eigenvalues and eigenfunctions of a non-self-adjoint boundary value problem for a second-order involutive differential equation with a complex-valued potential have been obtained.

The results of this study can be applied to the analysis of problems for pseudoparabolic equations with involution. The developed approaches can also be extended to investigate the asymptotics of eigenvalues and eigenfunctions of other boundary value problems.

The obtained estimates of the norms of the derivatives of eigenfunctions, as well as the relationships between different norms of eigenfunctions, may have important applications.

Author contributions

Akbope Beisebayeva: Conceptualization; software; validation; formal analysis; investigation; resources; data curation; writing—original draft preparation; writing—review and editing; supervision.

Abdizhahan M. Sarsenbi: Methodology; software; validation; formal analysis; investigation; resources; data curation; writing—original draft preparation; writing—review and editing; supervision.

All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The researchers would like to thank the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan for the financial support provided (Grant No. AP32716173).

Conflict of interest

The authors do not have any conflicts of interest.

References

1. J. Wiener, *Generalized solutions of functional differential equations*, World Scientific, 1993.
2. A. Cabada, F. A. F. Tojo, *Differential equations with involutions*, New York: Atlantis Press, 2015.
3. D. Przeworska-Rolewicz, *Equations with transformed argument*, Elsevier-PWN, Amsterdam, Warszawa, 1973.
4. A. Signorini, Sul moto di un punto soggelto a resistenza idraulica e forza di richiamo, *Atti R. Ist. Veneto di Sc. Lett. Ed. Arti.*, **73** (1913), 803–858.
5. J. P. Van Zandt, Oscillating systems damped by resistance proportional to the square of the velocity, *Phys. Rev.*, **10** (1917), 415–431.

6. E. Cartan, H. Cartan, Note sur la generation des oscillations entretenues, *Ann. des PTT*, **14** (1925), 1196–1207.

7. F. Tricomi, Integrazione di un'equazione differenziale presentatasi in Elettrotecnica, *Ann. Sc. Norm. Sup. di Pisa*, **2** (1933), 1–20.

8. G. Duffing, *Erzwungene schwingungen bei veranderlicher eigenfrequenz und ihre technische bedeutung*, Berlin, 1918.

9. L. H. Thomas, The calculation of atomic fields, *Mathematical Proceedings of the Cambridge Philosophical Society*, **23** (1927), 542–548. https://doi.org/10.1017/S0305004100011683

10. S. F. Lacroix, *Traite du calcul differential et du calcul integral*, Paris, 1797.

11. K. L. Cooke, *Differential-difference equations*, New York: Academic Press, 1963.

12. D. N. Zubarev, *Statistical mechanics of nonequilibrium processes (in Russian)*, Fizmatlit, Moscow, 2002.

13. N. M. Gersevanov, *Iterative calculus and its applications (in Russian)*, Moscow, 1950.

14. M. Kirane, M. A. Sadybekov, A. A. Sarsenbi, On an inverse problem of reconstructing a subdiffusion process from nonlocal data, *Math. Methods Appl. Sci.*, **42** (2019), 2043–2052. https://doi.org/10.1002/mma.5498

15. E. Mussirepova, A. A. Sarsenbi, A. M. Sarsenbi, Solvability of mixed problems for the wave equation with reflection of the argument, *Math. Methods Appl. Sci.*, **45** (2022), 11262–11271. https://doi.org/10.1002/mma.8448

16. E. Mussirepova, A. Sarsenbi, A. Sarsenbi, The inverse problem for the heat equation with reflection of the argument and with a complex coefficient, *Bound. Value Probl.*, **2022** (2022), 99. https://doi.org/10.1186/s13661-022-01675-1

17. A. Štikonas, E. Şen, Asymptotic analysis of Sturm-Liouville problem with Dirichlet and nonlocal two-point boundary conditions, *Math. Model. Anal.*, **28** (2023), 308–329. https://doi.org/10.3846/mma.2023.17617

18. E. Şen, A. Štikonas, Asymptotic distribution of eigenvalues and eigenfunctions of a nonlocal boundary value problem, *Math. Model. Anal.*, **26** (2021), 253–266. https://doi.org/10.3846/mma.2021.13056

19. N. S. Imanbaev, Distribution of eigenvalues of a third-order differential operator with strongly regular boundary conditions, *AIP Conf. Proc.*, **1997** (2021), 020027. https://doi.org/10.1063/1.5049021

20. N. S. Imanbaev, M. A. Sadybekov, Characteristic determinant of the spectral problem for the ordinary differential operator with the boundary load, *AIP Conf. Proc.*, **1611** (2014), 261–265. https://doi.org/10.1063/1.4893844

21. N. S. Imanbaev, On stability of basis property of root vectors system of the Sturm-Liouville operator with an integral perturbation of conditions in nonstrongly regular Samarskii-Ionkin type problems, *Int. J. Differ. Equ.*, **2015** (2015), 641481. https://doi.org/10.1155/2015/641481

22. A. M. Sarsenbi, Unconditional bases related to a nonclassical second-order differential operator, *Diff. Equat.*, **46** (2010), 509–514. https://doi.org/10.1134/S0012266110040051

23. M. A. Sadybekov, A. M. Sarsenbi, Criterion for the basis property of the eigenfunction system of a multiple differentiation operator with an involution, *Diff. Equat.*, **48** (2012), 1112–1118. https://doi.org/10.1134/S001226611208006X

24. L. V. Kritskov, A. M. Sarsenbi, Equiconvergence property for spectral expansions related to perturbations of the operator $-u''$ (–x) with initial data, *Filomat*, **32** (2018), 1069–1078.

25. A. A. Sarsenbi, B. K. Turmetov, Basis property of a system of eigenfunctions of a second-order differential operator with involution, *Vestn. Udmurtsk. Univ. Mat. Mekh. Komp. Nauki*, **29** (2019), 183–196. https://doi.org/10.20537/vm190204

26. B. Turmetov, V. Karachik, On solvability of some inverse problems for a nonlocal fourth-order parabolic equation with multiple involution, *AIMS Math.*, **9** (2024), 6832–6849. https://doi.org/10.3934/math.2024333

27. F. Dekhkonov, B. Turmetov, On one time-optimal control problem for a parabolic equation with involution in a bounded domain, *AIMS Math.*, **10** (2025), 20531–20549. https://doi.org/10.3934/math.2025916

28. A. Sarsenbi, A. Sarsenbi, Boundary value problems for a second-order differential equation with involution in the second derivative and their solvability, *AIMS Math.*, **8** (2023), 26275–26289. https://doi.org/10.3934/math.20231340

29. S. G. Georgiev, A. Boukarou, K. Zennir, K. Bouhali, Classical solutions with new topological properties for n-dimensional physical models: Schrödinger and wave equations, *J. Nonlinear Convex Anal.*, **26** (2025), 2723 –2740.

30. S. G. Georgiev, S. M. Mirgani, K. Zennir, K. Bouhali, On nonnegative classical solutions of coupled Euler–Poisson–Darboux–Tricomi equations with nonlinear sources, *AIMS Math.*, **11** (2026), 5557–5574. https://doi.org/10.3934/math.2026229

31. S. B. Norkin, *Differential equations of the second order with retarded argument: some problems of the theory of vibrations of systems with retardation*, 1972.

32. G. Sansone, *Ordinary differential equations (in Russian)* , Vol. 1, Moscow: IL, 1953.

33. M. A. Naimark, *Linear differential operators*, Ungar: New York, NY, USA, 1968.

34. V. A. Il'in, L. V. Kritskov, Properties of spectral expansions corresponding to non-self-adjoint differential operators, *J. Math. Sci.*, **116** (2003), 3489–3550. https://doi.org/10.1023/A:1024180807502

AIMS Press