AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(7): 22034–22051.
DOI: 10.3934/math.2026891
Received: 04 May 2026
Revised: 10 July 2026
Accepted: 16 July 2026
Published: 23 July 2026

Research article

Shape-preserving properties of the local integro quartic spline

Yali Zhang and Yachao Wang*

School of Mathematics and Statistics, Yancheng Teachers University, Yancheng 224000, China

* **Correspondence:** Email: wyachao@foxmail.com.

Abstract: The purpose of this paper is to systematically investigate the shape-preserving properties of the local integro quartic spline. It aims to derive and establish sufficient theoretical conditions to guarantee the convexity, monotonicity, and positivity of the approximation results. Furthermore, several representative numerical examples are provided to intuitively verify and demonstrate the excellent shape-preserving performance and practical applicability of the constructed spline.

Keywords: local integro quartic spline; shape-preserving; convexity; monotonicity; positive; approximation
Mathematics Subject Classification: 65D05, 65D07

1. Introduction

Spline approximation is an important tool in numerical analysis and approximation theory because of its flexibility, smoothness, and good approximation performance. In recent years, many approximation methods based on splines and iterative techniques have been developed. For example, Guo and Liu [1] studied knot optimization in B-spline curve approximation. Liu et al. [2–4] investigated spline-based numerical schemes and progressive iterative approximation methods. Tan et al. [5] and Wu et al. [6] further developed accelerated iterative methods for approximation problems.

Classical spline interpolation is usually constructed from function values at knots. However, in many practical problems, the pointwise values of the target function are not directly available, while only the integral values or average values over successive subintervals can be measured. Such problems arise in mechanics, statistics, climatology, oceanography, electricity, and numerical analysis. This need is also consistent with the broader role of numerical approximation techniques in scientific and engineering problems, including fractional differential models and circuit analysis [7], where effective approximation tools are required when only limited information is available. Therefore, reconstructing an approximating function from integral values over successive intervals has become an important topic in spline approximation.

The study of integro splines was initiated by Behforooz [8], who constructed integro cubic splines by using integral values instead of function values at knots and analyzed approximation problems under several end conditions. Later, different types of integro spline interpolation methods were developed. Lang and Xu [9] studied integro quartic spline interpolation based on quartic B-splines, and further proposed a quintic B-spline method for integro interpolation in [10]. Zhanlav and Mijiddorj [11] investigated integro quintic splines and their approximation properties. Wu and Zhang [12] considered integro sextic spline interpolation and its superconvergence. Moreover, Wu et al. [13] analyzed even-degree spline interpolation from integral values of successive intervals and showed that the quartic spline has favorable smoothness properties. Boujraf et al. [14] proposed a simple method for constructing integro spline quasi-interpolants. These studies demonstrate that integro spline interpolation is an effective approach for reconstructing functions from integral data.

In addition to global integro spline constructions, local integro splines have attracted attention because they are easy to implement and usually avoid solving large systems of linear equations. Zhanlav and Mijiddorj [15] constructed local integro cubic splines and studied their approximation properties. They later proposed a local construction of integro quartic splines [16], deriving explicit formulae for the spline values and derivative values at the knots. Guo et al. [17] studied local integro splines with optimized knots, and Mijiddorj and Zhanlav [18] developed an algorithm for constructing local integro splines. More recently, integro cubic splines on non-uniform grids were investigated in [19]. These works provide useful theoretical and computational foundations for local integro spline approximation.

Besides approximation accuracy, shape preservation is also an important issue in spline approximation. In many applications, the reconstructed curve is expected to preserve qualitative properties such as positivity, monotonicity, and convexity. For classical spline approximation, shape-preserving problems have been widely studied; for example, Han [20] investigated convexity-preserving approximation by univariate cubic splines, and Zhu and Han [21] studied rational quartic interpolation splines with local shape-preserving properties. For integro splines, however, the corresponding shape-preserving theory is less developed. Zhanlav and Mijiddorj [22] discussed the convexity and monotonicity properties of the local integro cubic spline and provided sufficient conditions for shape preservation.

Compared with the local integro cubic spline, the local integro quartic spline has higher smoothness and better approximation performance [9–11]. Although the construction and approximation properties of local integro quartic splines were studied in [16], their shape-preserving properties have not been sufficiently investigated. In particular, sufficient conditions that guarantee convexity, monotonicity, and positivity preservation for the local integro quartic spline remain to be established. These factors inspire us to investigate the local integro quartic spline.

The main contributions of this paper are summarized as follows:

- The shape-preserving properties of the local integro quartic spline are investigated.
- Sufficient conditions for convexity and monotonicity preservation are established.
- A positivity-preserving condition is derived based on the B-spline coefficients.
- Numerical examples are provided to verify the theoretical results and illustrate the approximation performance.

The rest of this paper is organized as follows. Section 2 introduces preliminary results on the local integro quartic spline and its B-spline representation. Section 3 derives sufficient conditions for convexity, monotonicity, and positivity preservation. Section 4 presents numerical examples to verify the theoretical results and compare the approximation behavior of local integro cubic and quartic splines. Section 5 concludes the paper.

2. Preliminaries

Let $I = [a, b]$ be divided into n subintervals by the equidistant knots $x_i = a+ih$, where $h = (b-a)/n$. Let $S(x)$ be a local integro quartic spline belonging to $C^3[a, b]$. The spline $S(x)$ is constructed to satisfy the integral interpolation conditions

$$\int_{x_i}^{x_{i+1}} S(x)dx = I_i = \int_{x_i}^{x_{i+1}} y(x)dx,\ i = 0, 1, ..., n-1, \tag{2.1}$$

and

$$S(x_0) = y(x_0),\ S(x_1) = y(x_1),\ S(x_{n-1}) = y(x_{n-1}),\ S(x_n) = y(x_n).$$

Where I_i denotes the given integral value of $y(x)$ over the interval $[x_i, x_{i+1}]$.

The B-spline representation of $S(x)$ is given by [23]

$$S(x) = \sum_{j=-2}^{n+1} c_j B_j(x), \tag{2.2}$$

where $B_j(x)(j = -2, -1, ..., n+1)$ are linearly independent and form the basis for $S(I)$. For the sake of convenience in writing, we use the following notation: $y_i = y(x_i),\ S_i = S(x_i),\ m_i = S'(x_i),\ M_i = S''(x_i),\ T_i = S'''(x_i),\ i = 0, 1, ..., n.$

From [16], we obtained the explicit and approximate formulae as follows

$$c_i = \frac{13I_{i-3} - 39I_{i-2} - 94I_{i-1} + 746I_i - 159I_{i+1} + 13I_{i+2}}{480h},\ i = 3, 4, ..., n-3, \tag{2.3}$$

$$c_i = S_i + \frac{h}{2}m_i - \frac{h^2}{12}M_i - \frac{h^3}{12}T_i,\ i = 0, 1, .., n, \tag{2.4}$$

$$c_{i-2} + 26c_{i-1} + 66c_i + 26c_{i+1} + c_{i+2} = \frac{120I_i}{h}, i = 0, 1, ..., n-1, \tag{2.5}$$

in which we neglect the small term, and where

$$\begin{aligned}
&S_0 = (147I_0 - 213I_1 + 237I_2 - 163I_3 + 62I_4 - 10I_5)/(60h),\\
&S_1 = (10I_0 + 87I_1 - 63I_2 + 37I_3 - 13I_4 + 2I_5)/(60h),\\
&S_2 = (-2I_0 + 22I_1 + 57I_2 - 23I_3 + 7I_4 - I_5)/(60h),\\
&S_i = (I_{i-3} - 8I_{i-2} + 37I_{i-1} + 37I_i - 8I_{i+1} + I_{i+2})/(60h),\ i = 3, 4, ..., n-3,\\
&S_{n-2} = (-I_{n-6} + 7I_{n-5} - 23I_{n-4} + 57I_{n-3} + 22I_{n-2} - 2I_{n-1})/(60h),\\
&S_{n-1} = (2I_{n-6} - 13I_{n-5} + 37I_{n-4} - 63I_{n-3} + 87I_{n-2} + 10I_{n-1})/(60h),\\
&S_n = (-10I_{n-6} + 62I_{n-5} - 163I_{n-4} + 237I_{n-3} - 213I_{n-2} + 147I_{n-1})/(60h),
\end{aligned} \tag{2.6}$$

$$
\begin{aligned}
m_0 &= (-45I_0 + 109I_1 - 105I_2 + 51I_3 - 10I_4)/(12h^2),\\
m_1 &= (-10I_0 + 5I_1 + 9I_2 - 5I_3 + I_4)/(12h^2),\\
m_i &= (I_{i-2} - 15I_{i-1} + 15I_i - I_{i+1})/(12h^2),\ i = 2, 3, ..., n-2,\\
m_{n-1} &= (-I_{n-5} + 5I_{n-4} - 9I_{n-3} - 5I_{n-2} + 10I_{n-1})/(12h^2),\\
m_n &= (10I_{n-5} - 51I_{n-4} + 105I_{n-3} - 109I_{n-2} + 45I_{n-1})/(12h^2),
\end{aligned} \tag{2.7}
$$

$$
\begin{aligned}
M_0 &= (49I_0 - 183I_1 + 278I_2 - 218I_3 + 89I_4 - 15I_5)/(8h^3),\\
M_1 &= (15I_0 - 41I_1 + 42I_2 - 22I_3 + 7I_4 - I_5)/(8h^3),\\
M_2 &= (I_0 + 9I_1 - 26I_2 + 22I_3 - 7I_4 + I_5)/(8h^3),\\
M_i &= (-I_{i-3} + 7I_{i-2} - 6I_{i-1} - 6I_i + 7I_{i+1} - I_{i+2})/(8h^3),\ i = 3, 4, ..., n-3,\\
M_{n-2} &= (I_{n-6} - 7I_{n-5} + 22I_{n-4} - 26I_{n-3} + 9I_{n-2} + I_{n-1})/(8h^3),\\
M_{n-1} &= (-I_{n-6} + 7I_{n-5} - 22I_{n-4} + 42I_{n-3} - 41I_{n-2} + 15I_{n-1})/(8h^3),\\
M_n &= (-15I_{n-6} + 89I_{n-5} - 218I_{n-4} + 278I_{n-3} - 183I_{n-2} + 49I_{n-1})/(8h^3),
\end{aligned} \tag{2.8}
$$

and

$$
\begin{aligned}
T_0 &= (-3I_0 + 11I_1 - 15I_2 + 9I_3 - 2I_4)/h^4,\\
T_1 &= (-2I_0 + 7I_1 - 9I_2 + 5I_3 - I_4)/h^4,\\
T_i &= (-I_{i-2} + 3I_{i-1} - 3I_i + I_{i+1})/h^4,\ i = 2, 3, ..., n-2,\\
T_{n-1} &= (I_{n-5} - 5I_{n-4} + 9I_{n-3} - 7I_{n-2} + 2I_{n-1})/h^4,\\
T_n &= (2I_{n-5} - 9I_{n-4} + 15I_{n-3} - 11I_{n-2} + 3I_{n-1})/h^4.
\end{aligned} \tag{2.9}
$$

3. Shape-preserving properties of local integro quartic spline

3.1. Convexity-preserving and monotonicity-preserving conditions for the local integro quartic spline

Suppose that we have the data

$$
I_i,\ i = 0, 1, \dots, n-1, \tag{3.1}
$$

where I_i satisfy (2.1). The integral data sequence (3.1), or equivalently the corresponding cell-average sequence, is called monotonically increasing in the discrete sense if

$$
a_i = \frac{I_{i+1} - I_i}{h} > 0,\ i = 0, 1, \dots, n-2. \tag{3.2}
$$

It is called convex in the discrete sense if

$$
a_i \ge a_{i-1},\ i = 1, 2, \dots, n-2. \tag{3.3}
$$

It should be emphasized that the above definitions describe only the discrete behavior of the given integral data sequence. Since the partition is uniform, I_i/h can be regarded as the cell average of $y(x)$ on $[x_i, x_{i+1}]$. Thus, the quantity (3.2) represents the finite difference of the cell-average data. These definitions are not intended to characterize the pointwise monotonicity or convexity of the underlying function $y(x)$, since the integral values do not uniquely determine $y(x)$. In this paper, the shape-preserving results concern the constructed local integro quartic spline $S(x)$.

We first discuss sufficient conditions for convexity preservation. The following Lemma 3.1 provides algebraic conditions on the integral data that ensure the nonnegativity of the second derivatives of the spline at the knots in Theorem 3.1.

The following lemma gives a set of additional algebraic sufficient conditions on the integral data sequence to ensure $M_i \geq 0$ at all knots.

Lemma 3.1. *If data* (3.1) *satisfies the inequalities*

$$\begin{aligned}
&a_{i-3} - 6a_{i-2} + 6a_i - a_{i+1} \geq 0,\ i = 3, 4, ..., n-3,\\
&-49a_0 + 134a_1 - 144a_2 + 74a_3 - 15a_4 \geq 0,\\
&-15a_0 + 26a_1 - 16a_2 + 6a_3 - a_4 \geq 0,\\
&-a_0 - 10a_1 + 16a_2 - 6a_3 + a_4 \geq 0,\\
&-a_{n-6} + 6a_{n-5} - 16a_{n-4} + 10a_{n-3} + a_{n-2} \geq 0,\\
&a_{n-6} - 6a_{n-5} + 16a_{n-4} - 26a_{n-3} + 15a_{n-2} \geq 0,\\
&15a_{n-6} - 74a_{n-5} + 144a_{n-4} - 134a_{n-3} + 49a_{n-2} \geq 0,
\end{aligned} \tag{3.4}$$

then for $i = 0, 1, \ldots, n$, we have $M_i \geq 0$.

Proof. According to (2.8) and (3.2), the second-order derivatives of $S(x)$ at the equidistant knots are expressed as

$$M_i = \frac{a_{i-3} - 6a_{i-2} + 6a_i - a_{i+1}}{8h^2},\ i = 3, 4, ..., n-3.$$

Similarly, $M_0, M_1, M_2, M_{n-2}, M_{n-1}$, and M_n are expressed as

$$\begin{aligned}
M_0 &= \frac{-49a_0 + 134a_1 - 144a_2 + 74a_3 - 15a_4}{8h^2},\\
M_1 &= \frac{-15a_0 + 26a_1 - 16a_2 + 6a_3 - a_4}{8h^2},\\
M_2 &= \frac{-a_0 - 10a_1 + 16a_2 - 6a_3 + a_4}{8h^2},\\
M_{n-2} &= \frac{-a_{n-6} + 6a_{n-5} - 16a_{n-4} + 10a_{n-3} + a_{n-2}}{8h^2},\\
M_{n-1} &= \frac{a_{n-6} - 6a_{n-5} + 16a_{n-4} - 26a_{n-3} + 15a_{n-2}}{8h^2},\\
M_n &= \frac{15a_{n-6} - 74a_{n-5} + 144a_{n-4} - 134a_{n-3} + 49a_{n-2}}{8h^2}.
\end{aligned}$$

If condition (3.4) is satisfied, then for $i = 0, 1, \ldots, n$, we have $M_i \geq 0$. □

Theorem 3.1. *Assume that the integral data sequence satisfies the additional sufficient conditions in Lemma 3.1. If the following condition is also satisfied, then $S''(x) \geq 0$ on $[a, b]$, and hence $S(x)$ is convex on $[a, b]$.*

(i) $M_i \geq 0$.
(ii) *If, in addition, there exists interval* (x_i, x_{i+1}) *such that* $T_i < 0$ *and* $T_{i+1} > 0$*, then it is required that* $M_i + \frac{h}{2}T_i \geq 0$.

Then, $S''(x) \geq 0$ *on* $[a, b]$*, and* $S(x)$ *is convex on* $[a, b]$.

Proof. We represent the function $S''(x)$ in the form of a piecewise polynomial, where for $x \in [x_i, x_{i+1}],\ i = 0, 1, ..., n-1$, it can be represented as

$$S''(x) = M_i + T_i ht + \frac{T_{i+1} - T_i}{2} ht^2,\ t = \frac{x - x_i}{h}.$$

The critical point for $S''(x)$ is located at

$$t^* = \frac{T_i}{T_i - T_{i+1}}.$$

It is straightforward to show that $t^* \notin (0, 1)$ when $T_i T_{i+1} > 0$. Therefore, we have $0 \leq \min(M_i, M_{i+1}) < S''(x) \leq \max(M_i, M_{i+1})$.

Additionally, $t^* \in (0, 1)$ when $T_i T_{i+1} < 0$. The function $S''(x)$ reaches a local extremum at t^*. There are two possible cases:

(a) If $T_i > 0$ and $T_{i+1} < 0$, then $S''(x)$ has a local maximum at t^*. In this case, by virtue of (i), we have $S''(x) \geq \min(M_i, M_{i+1}) \geq 0$ on $[x_i, x_{i+1}]$.
(b) If $T_i < 0$ and $T_{i+1} > 0$, then $S''(x)$ has a local minimum at t^*. In this case, by virtue of (ii), we have

$$S''(t^*) = M_i + \frac{h}{2} T_i t^* > M_i + \frac{h}{2} T_i \geq 0.$$

Therefore, we have $S''(x) \geq S''(t^*) \geq 0$ on $[x_i, x_{i+1}]$. This completes the proof of the theorem. □

Next, we consider monotonicity preservation. The following Lemma 3.2 gives sufficient conditions under which the first derivatives of the local integro quartic spline at the knots are nonnegative in Theorem 3.2.

The following lemma provides additional sufficient conditions on the integral data sequence under which the first derivatives of the constructed spline at the knots satisfy $m_i \geq 0$.

Lemma 3.2. *If data* (3.1) *satisfies the inequalities*

$$\begin{aligned}
&-a_{i-2} + 14a_{i-1} - a_i \geq 0,\ i = 2, 3, ..., n-2,\\
&45a_0 - 64a_1 + 41a_2 - 10a_3 \geq 0,\\
&10a_0 + 5a_1 - 4a_2 + a_3 \geq 0,\\
&a_{n-5} - 4a_{n-4} + 5a_{n-3} + 10a_{n-2} \geq 0,\\
&-10a_{n-5} + 41a_{n-4} - 64a_{n-3} + 45a_{n-2} \geq 0,
\end{aligned} \tag{3.5}$$

then for $i = 0, 1, \ldots, n$*, we have* $m_i \geq 0$.

Proof. According to (2.7) and (3.2), the first-order derivatives of $S(x)$ are expressed as

$$m_i = \frac{-a_{i-2} + 14a_{i-1} - a_i}{12h}, \ i = 2, 3, ..., n-2.$$

Similarly, m_0, m_1, m_{n-1}, and m_n are expressed as

$$\begin{aligned} m_0 &= \frac{45a_0 - 64a_1 + 41a_2 - 10a_3}{12h}, \\ m_1 &= \frac{10a_0 + 5a_1 - 4a_2 + a_3}{12h}, \\ m_{n-1} &= \frac{a_{n-5} - 4a_{n-4} + 5a_{n-3} + 10a_{n-2}}{12h}, \\ m_n &= \frac{-10a_{n-5} + 41a_{n-4} - 64a_{n-3} + 45a_{n-2}}{12h}. \end{aligned}$$

If these satisfy (3.5), then for $i = 0, 1, \ldots, n$, we have $m_i \geq 0$. □

Theorem 3.2. *Assume that the integral data sequence satisfies the additional sufficient conditions in Lemma 3.2. If the following condition is also satisfied, then $S'(x) \geq 0$ on $[a, b]$, and hence $S(x)$ is monotonically increasing on $[a, b]$.*

(i) $m_i \geq 0$.

(ii) In addition, if there exist intervals $(x_i.x_{i+1})$ such that $T_i^2 h - 2(T_{i+1} - T_i)M_i \geq 0$, then, when any of the following four cases occur, (a)$T_{i+1} > 0$, $M_i < 0$, $(T_{i+1} + T_i)h + 2M_i > 0$, (b)$T_i > 0$, $M_i < 0$, $(T_{i+1} + T_i)h + 2M_i > 0$, (c)$T_i > 0 > T_{i+1}$, $M_i < 0$, $(T_{i+1} + T_i)h + 2M_i < 0$, (d)$T_i > 0$, $M_i < 0$, $(T_{i+1} + T_i)h + 2M_i > 0$, the following condition must be satisfied:

$$S'(x_i + ht_1) \geq 0,$$

where $t_1 = \frac{-T_i h + \sqrt{T_i^2 h^2 - 2h(T_{i+1} - T_i)M_i}}{(T_{i+1} - T_i)h}$.

Then, $S'(x) > 0$ on $[a, b]$, that is, $S(x)$ is monotone increasing on $[a, b]$.

Proof. We represent the function $S'(x)$ in the form of a piecewise polynomial, where for $x \in [x_i, x_{i+1}]$, $i = 0, 1, ..., n-1$, it can be represented as

$$S'(x) = \frac{T_{i+1} - T_i}{6} h^2 t^3 + \frac{T_i}{2} h^2 t^2 + M_i ht + m_i, \ t = \frac{x - x_i}{h}.$$

Thus, we have

$$S''(x) = M_i + T_i ht + \frac{T_{i+1} - T_i}{2} ht^2.$$

If $T_i^2 h - 2(T_{i+1} - T_i)M_i < 0$, the inequality $S''(x) > 0$ (or $S''(x) < 0$) is established. Then, $S'(x)$ is monotonically increasing (or monotonically decreasing) on $[x_i, x_{i+1}]$, and the minimum occurs at $x = x_i$ (or $x = x_{i+1}$).

$$S'(x) \geq S'(x_i) = m_i \geq 0 \text{ (or } S'(x) \geq S'(x_{i+1}) = m_{i+1} \geq 0).$$

If $T_i^2 h - 2(T_{i+1} - T_i)M_i \geq 0$, letting $S''(x) = 0$, we obtain

$$t_1 = \frac{-T_i h + \sqrt{T_i^2 h^2 - 2h(T_{i+1} - T_i)M_i}}{(T_{i+1} - T_i)h},\ t_2 = \frac{-T_i h - \sqrt{T_i^2 h^2 - 2h(T_{i+1} - T_i)M_i}}{(T_{i+1} - T_i)h}.$$

When $T_{i+1} > T_i$, we have $t_2 < t_1$. We have the following possible cases.

- If $t_2 < t_1 < 0$ or $1 < t_2 < t_1$, then $S''(x) > 0$ for $t \in (0, 1)$. Therefore, $S'(x)$ is monotonically increasing on $[x_i, x_{i+1}]$, and the minimum occurs at $x = x_i$.

$$S'(x) \geq S'(x_i) = m_i \geq 0.$$

- If $t_2 < 0 < t_1 < 1$, i.e., $T_{i+1} > 0$, $M_i < 0$, $(T_{i+1} + T_i)h + 2M_i > 0$, then $S''(x) < 0$ for $t \in (0, t_1)$, and $S''(x) > 0$ for $t \in (t_1, 1)$. Therefore, $S'(x)$ is monotonically decreasing on $[x_i, x_i + ht_1]$, and the minimum occurs at $x = x_i + ht_1$. $S'(x)$ is monotonically increasing on $[x_i + ht_1, x_{i+1}]$, and the minimum occurs at $x = x_i + ht_1$. In this case, by virtue of (ii), we have

$$S'(x) \geq S'(x_i + ht_1) \geq 0.$$

- If $0 < t_2 < t_1 < 1$, i.e., $T_i < 0$, $T_{i+1} > 0$, $M_i > 0$, $(T_{i+1} + T_i)h + 2M_i > 0$, then $S''(x) > 0$ for $t \in (0, t_2)$, $S''(x) < 0$ for $t \in (t_2, t_1)$, and $S''(x) > 0$ for $t \in (t_1, 1)$. Therefore, $S'(x)$ is monotonically increasing on $[x_i, x_i + ht_2]$, and the minimum occurs at $x = x_i$; $S'(x)$ is monotonically decreasing on $[x_i + ht_2, x_i + ht_1]$, and the minimum occurs at $x = x_{i+1}$; and $S'(x)$ is monotonically increasing on $[x_i + ht_1, 1]$, and the minimum occurs at $x = x_i + ht_1$. In this case, by virtue of (ii), we have

$$S'(x) \geq \min(S'(x_i), S'(x_{i+1}), S'(x_i + ht_1)) = \min(m_i, m_{i+1}, S'(x_i + ht_1)) \geq 0.$$

- If $t_2 < 0 < 1 < t_1$, then $S''(x) < 0$ for $t \in (0, 1)$. Therefore, $S'(x)$ is monotonically decreasing on $[x_i, x_{i+1}]$, and the minimum occurs at $x = x_{i+1}$.

$$S'(x) \geq S'(x_{i+1}) = m_{i+1} \geq 0.$$

- If $0 < t_2 < 1 < t_1$, then $S''(x) > 0$ for $t \in (0, t_2)$, and $S''(x) < 0$ for $t \in (t_2, 1)$. Therefore, $S'(x)$ is monotonically increasing on $[x_i, x_i + ht_2]$, and the minimum occurs at $x = x_i$. $S'(x)$ is monotonically decreasing on $[x_i + ht_2, x_{i+1}]$, and the minimum occurs at $x = x_{i+1}$.

$$S'(x) \geq \min(S'(x_i), S'(x_{i+1})) = \min(m_i, m_{i+1}) \geq 0.$$

When $T_{i+1} < T_i$, we have $t_1 < t_2$. There are the following possible cases.

- If $t_1 < t_2 < 0$ or $1 < t_1 < t_2$, then $S''(x) < 0$ for $t \in (0, 1)$. Therefore, $S'(x)$ is monotonically decreasing on $[x_i, x_{i+1}]$, and the minimum occurs at $x = x_{i+1}$.

$$S'(x) \geq S'(x_{i+1}) = m_{i+1} \geq 0.$$

- If $t_1 < 0 < t_2 < 1$, then $S''(x) > 0$ for $t \in (0, t_2)$, and $S''(x) < 0$ for $t \in (t_2, 1)$. Therefore, $S'(x)$ is monotonically increasing on $[x_i, x_i + ht_2]$, and the minimum occurs at $x = x_i$, and $S'(x)$ is monotonically decreasing on $[x_i + ht_2, x_{i+1}]$, and the minimum occurs at $x = x_{i+1}$.

$$S'(x) \geq \min(S'(x_i), S'(x_{i+1})) = \min(m_i, m_{i+1}) \geq 0.$$

- If $t_1 < 0 < 1 < t_2$, then $S''(x) > 0$ for $t \in (0, 1)$. Therefore, $S'(x)$ is monotonically increasing on $[x_i, x_{i+1}]$, and the minimum occurs at $x = x_i$.

$$S'(x) \geq S'(x_i) = m_i > 0.$$

- If $0 < t_1 < t_2 < 1$, i.e., $T_i > 0 > T_{i+1}$, $M_i < 0$, $(T_{i+1} + T_i)h + 2M_i < 0$, then $S''(x) < 0$ for $t \in (0, t_1)$, $S''(x) > 0$ for $t \in (t_1, t_2)$, and $S''(x) < 0$ for $t \in (t_2, 1)$. Therefore, $S'(x)$ is monotonically decreasing on $[x_i, x_i + ht_1]$, and the minimum occurs at $x = x_i + ht_1$; $S'(x)$ is monotonically increasing on $[x_i + ht_1, x_i + ht_2]$, and the minimum occurs at $x = x_i + ht_1$; and $S'(x)$ is monotonically decreasing on $[x_i + ht_2, x_{i+1}]$, and the minimum occurs at $x = x_{i+1}$. In this case, by virtue of (ii), we have

$$S'(x) \geq \min(S'(x_{i+1}), S'(x_i + ht_1)) = \min(m_{i+1}, S'(x_i + ht_1)) \geq 0.$$

- If $0 < t_1 < 1 < t_2$, i.e., $T_i > 0$, $M_i < 0$, $(T_{i+1} + T_i)h + 2M_i > 0$, then $S''(x) < 0$ for $t \in (0, t_1)$, and $S''(x) > 0$ for $t \in (t_1, 1)$. Therefore, $S^{'}(x)$ is monotonically decreasing on $[x_i, x_i + ht_1]$, and the minimum occurs at $x = x_i + ht_1$; and $S'(x)$ is monotonically increasing on $[x_i + ht_1, x_{i+1}]$, and the minimum occurs at $x = x_i + ht_1$. In this case, by virtue of (ii), we have

$$S'(x) \geq S'(x_i + ht_1) \geq 0.$$

Thereby, we have $S'(x) \geq 0$ on $[x_i, x_{i+1}]$. This completes the proof of the theorem. □

3.2. Positivity-preserving conditions for the local integro quartic spline

The following theorem provides sufficient conditions for the positivity-preserving property of the local integro quartic spline.

Theorem 3.3. *Let $S(x)$ be the local integro quartic spline defined by* (2.2)*, with coefficients determined by* (2.3), (2.4)*, and* (2.5). *If the data in* (3.1) *satisfy the inequalities*

$$\begin{aligned}
&13I_{i-3} - 39I_{i-2} - 94I_{i-1} + 746I_i - 159I_{i+1} + 13I_{i+2} \geq 0,\ i = 3(1)n-3,\\
&151I_0 + 951I_1 - 994I_2 + 446I_3 - 69I_4 - 5I_5 \geq 0,\\
&-115I_0 + 721I_1 - 174I_2 + 106I_3 - 79I_4 + 21I_5 \geq 0,\\
&39I_0 - 289I_1 + 1006I_2 - 354I_3 + 91I_4 - 13I_5 \geq 0,\\
&-13I_{n-6} + 91I_{n-5} - 234I_{n-4} + 166I_{n-3} + 551I_{n-2} - 81I_{n-1} \geq 0,\\
&21I_{n-6} - 199I_{n-5} + 706I_{n-4} - 1254I_{n-3} + 1081I_{n-2} + 125I_{n-1} \geq 0,\\
&-5I_{n-6} + 171I_{n-5} - 874I_{n-4} + 2006I_{n-3} - 2529I_{n-2} + 1711I_{n-1} \geq 0,\\
&2637I_0 - 7159I_1 + 11266I_2 - 10134I_3 + 4801I_4 - 931I_5 \geq 0,\\
&-17977I_0 + 104911I_1 - 223794I_2 + 231646I_3 - 118309I_4 + 24003I_5 \geq 0,\\
&-931I_{n-6} + 6361I_{n-5} - 17707I_{n-4} + 25546I_{n-3} - 19759I_{n-2} + 6970I_{n-1} \geq 0,
\end{aligned} \tag{3.6}$$

then $S(x) > 0$ on $[a, b]$.

Proof. According to (2.3), it is clear that $c_i \geq 0$, for $i = 3, 4, ..., n-3$, under conditions (3.6). From (2.4), (2.6), (2.7), (2.8), and (2.9), we obtain

$$c_0 = \frac{151I_0 + 951I_1 - 994I_2 + 446I_3 - 69I_4 - 5I_5}{480h},$$

$$c_1 = \frac{-115I_0 + 721I_1 - 174I_2 + 106I_3 - 79I_4 + 21I_5}{480h},$$

$$c_2 = \frac{39I_0 - 289I_1 + 1006I_2 - 354I_3 + 91I_4 - 13I_5}{480h},$$

$$c_{n-2} = \frac{-13I_{n-6} + 91I_{n-5} - 234I_{n-4} + 166I_{n-3} + 551I_{n-2} - 81I_{n-1}}{480h},$$

$$c_{n-1} = \frac{21I_{n-6} - 199I_{n-5} + 706I_{n-4} - 1254I_{n-3} + 1081I_{n-2} + 125I_{n-1}}{480h},$$

$$c_n = \frac{-5I_{n-6} + 171I_{n-5} - 874I_{n-4} + 2006I_{n-3} - 2529I_{n-2} + 1711I_{n-1}}{480h}.$$

From (2.5) and the above c_i, $i = 0, 1, 2, n-2, n-1, n$, we have

$$c_{-1} = \frac{2637I_0 - 7159I_1 + 11266I_2 - 10134I_3 + 4801I_4 - 931I_5}{480h},$$

$$c_{-2} = \frac{-17977I_0 + 104911I_1 - 223794I_2 + 231646I_3 - 118309I_4 + 24003I_5}{480h},$$

$$c_{n+1} = \frac{-931I_{n-6} + 6361I_{n-5} - 17707I_{n-4} + 25546I_{n-3} - 19759I_{n-2} + 6970I_{n-1}}{480h}.$$

By virtue of (3.6), we have $c_i \geq 0$, $i = -2, -1, 0, 1, 2, n-2, n-1, n, n+1$. □

4. Numerical examples

In this section, numerical examples are presented to verify the shape-preserving properties of the local integro quartic spline. Different from a simple comparison of approximation errors, we check whether the sufficient conditions in Theorems 3.1–3.3 are satisfied for each data set. We also compute the minimum values of $S(x)$, $S'(x)$, and $S''(x)$ on a fine grid to verify positivity, monotonicity, and convexity preservation, respectively.

The error between the target function $f(x)$ and the spline approximation $S(x)$ on $[a, b]$ is measured by

$$d = \|S(x) - f(x)\|_\infty, \quad x \in [a, b].$$

Example 4.1. Let

$$f(x) = e^x, \quad x \in [0, 2].$$

The function $f(x) = e^x$ is positive, monotonically increasing, and convex on $[0, 2]$. The integral data are generated by

$$I_i = \int_{x_i}^{x_{i+1}} e^x \, dx = e^{x_{i+1}} - e^{x_i}, \quad i = 0, 1, \ldots, n-1.$$

In this example, we take $n = 20$, and construct both the local integro cubic spline and the local integro quartic spline.

For this data set, the sufficient conditions in Theorems 3.1–3.3 are satisfied. Therefore, according to Theorems 3.1–3.3, the constructed local integro quartic spline should preserve convexity, monotonicity, and positivity. To verify these shape-preserving properties, we compute the minimum values of $S(x)$, $S'(x)$, and $S''(x)$ on a fine grid of $[0, 2]$. The numerical results are listed in Table 1. They show that

$$\min_{x \in [0,2]} S(x) > 0, \quad \min_{x \in [0,2]} S'(x) > 0, \quad \min_{x \in [0,2]} S''(x) > 0.$$

Hence, the local integro quartic spline is positive, monotonically increasing, and convex on $[0, 2]$. This verifies the conclusions of Theorems 3.1–3.3 and shows that the proposed spline preserves the shape of the data when the sufficient conditions are satisfied (see Table 1).

In addition to the verification of shape-preserving properties, we compare the approximation errors of the local integro cubic spline and the local integro quartic spline. As shown in Figure 1, both splines approximate e^x well, but the local integro quartic spline gives a smaller approximation error. This indicates that the local integro quartic spline not only preserves the desired shape properties, but also improves the approximation accuracy compared with the local integro cubic spline.

Table 1. Shape-preserving verification for $f(x) = e^x$ on $[0, 2]$.

n	Conditions satisfied	$\min S(x)$	$\min S'(x)$	$\min S''(x)$
20	Theorems 3.1–3.3	0.9999998	0.9999061	0.9997513

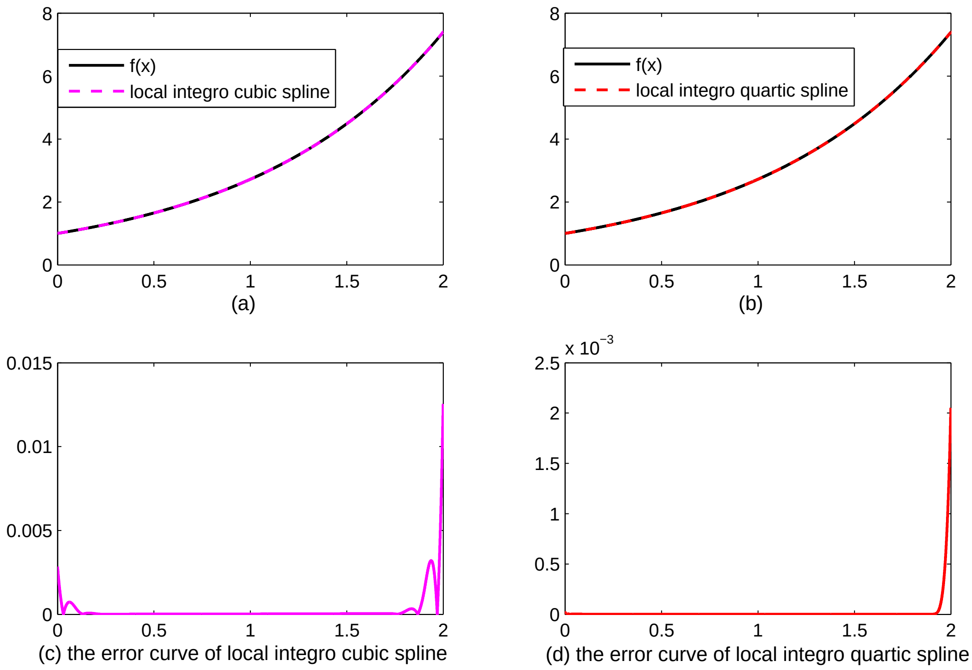

Figure 1. Comparison of local integro cubic and quartic splines for $f(x) = e^x$ on $[0, 2]$.

Example 4.2. Let

$$g(x) = \sin x, \quad x \in [0, 2.5].$$

The integral data are generated by

$$I_i = \int_{x_i}^{x_{i+1}} \sin x \, dx = \cos x_i - \cos x_{i+1}, \quad i = 0, 1, \ldots, n-1.$$

We take $n = 32$ and construct both the local integro cubic spline and the local integro quartic spline.

For this data set, the positivity condition in Theorem 3.3 is satisfied, while the sufficient conditions for convexity and monotonicity in Theorems 3.1 and 3.2 are not satisfied on the whole interval. Therefore, the local integro quartic spline is expected to preserve positivity, whereas global monotonicity and convexity are not guaranteed. To verify this, we compute the minimum values of $S(x)$, $S'(x)$, and $S''(x)$ on a fine grid of $[0, 2.5]$. The numerical results show that

$$\min_{x \in [0,2.5]} S(x) = 6.6046 \times 10^{-9} > 0,$$

whereas

$$\min_{x \in [0,2.5]} S'(x) = -0.8011269 < 0, \quad \min_{x \in [0,2.5]} S''(x) = -0.9999978 < 0.$$

Moreover, $S'(x)$ and $S''(x)$ change sign on the interval. This confirms that the constructed local integro quartic spline preserves positivity, but is not globally monotone or convex, which is consistent with the theoretical results (see Table 2).

The approximation errors are also shown in Figure 2. The error curves indicate that the local integro quartic spline provides better approximation accuracy than the local integro cubic spline.

Table 2. Shape-preserving verification for $g(x) = \sin x$ on $[0, 2.5]$.

n	Conditions satisfied	$\min S(x)$	Sign of $S'(x)$	Sign of $S''(x)$
32	Theorem 3.3	6.6046×10^{-9}	sign-changing	sign-changing

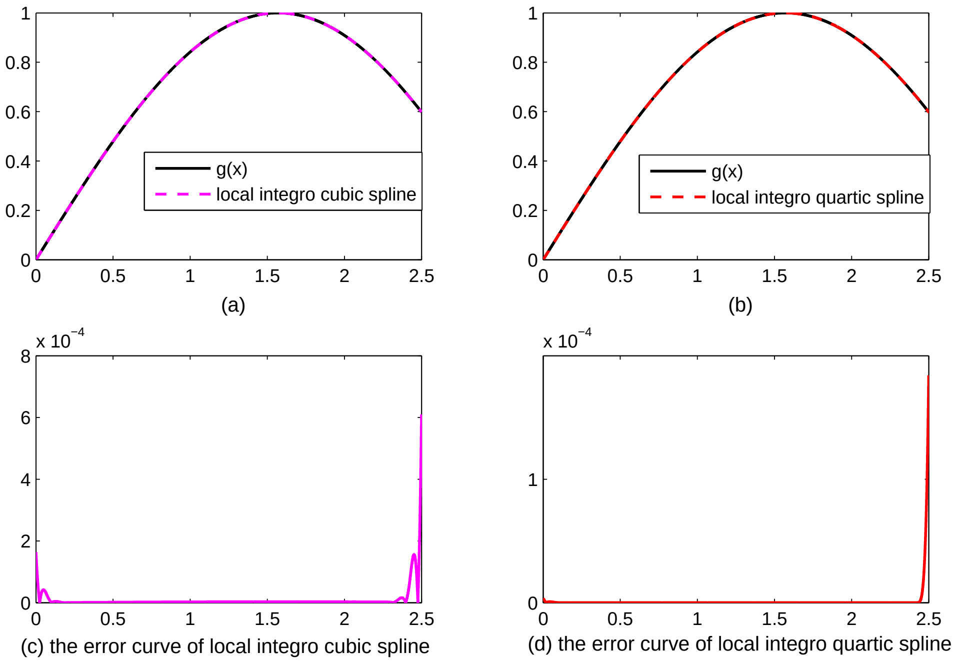

Figure 2. Comparison of local integro cubic and quartic splines for $g(x) = \sin x$ on $[0, 2.5]$.

Example 4.3. Let

$$p(x) = \sin(4x), \quad x \in [0, 2\pi].$$

The integral data are generated by

$$I_i = \int_{x_i}^{x_{i+1}} \sin(4x)\,dx = \frac{\cos(4x_i) - \cos(4x_{i+1})}{4}, \quad i = 0, 1, \ldots, n-1.$$

We take $n = 64$ and construct both the local integro cubic spline and the local integro quartic spline.

For this oscillatory data set, the sufficient conditions in Theorems 3.1–3.3 are not satisfied on the whole interval. Therefore, positivity, monotonicity, and convexity are not guaranteed. To verify this, we compute the minimum and maximum values of $S(x)$, $S'(x)$, and $S''(x)$ on a fine grid of $[0, 2\pi]$. The numerical results show that

$$\min_{x\in[0,2\pi]} S(x) = -1.0001322 < 0, \quad \max_{x\in[0,2\pi]} S(x) = 1.0001321 > 0,$$

$$\min_{x\in[0,2\pi]} S'(x) = -4.0095224 < 0, \quad \max_{x\in[0,2\pi]} S'(x) = 4.0095224 > 0,$$

and

$$\min_{x\in[0,2\pi]} S''(x) = -15.9784255 < 0, \quad \max_{x\in[0,2\pi]} S''(x) = 15.9784255 > 0.$$

Thus, $S(x)$, $S'(x)$, and $S''(x)$ all change sign on the interval. This confirms that when the sufficient conditions in Theorems 3.1–3.3 are violated, the corresponding shape-preserving properties may fail, which is consistent with the theoretical results (see Table 3).

The approximation errors are also shown in Figure 3. The error curves indicate that the local integro quartic spline provides better approximation accuracy than the local integro cubic spline.

Table 3. Shape-preserving verification for $p(x) = \sin(4x)$ on $[0, 2\pi]$.

n	Conditions satisfied	Sign of $S(x)$	Sign of $S'(x)$	Sign of $S''(x)$
64	Not satisfied	sign-changing	sign-changing	sign-changing

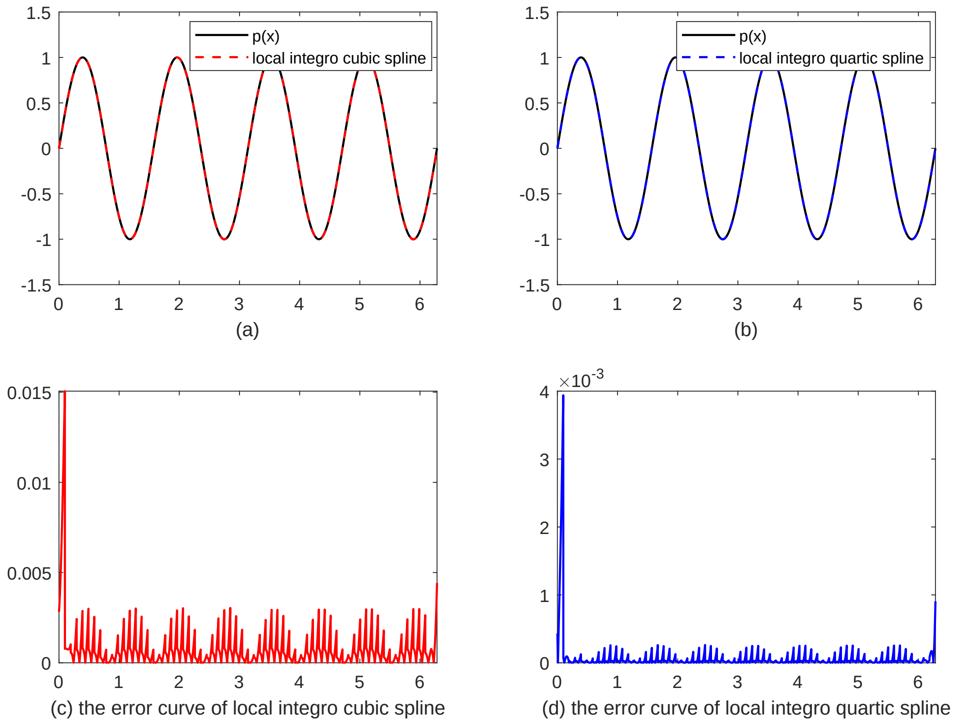

Figure 3. Comparison of local integro cubic and quartic splines for $p(x) = \sin(4x)$ on $[0, 2\pi]$.

Example 4.4. To investigate the behavior of the local integro quartic spline for discontinuous functions, we consider the piecewise function

$$q(x) = \begin{cases} 1, & 0 \le x < 1, \\ 2, & 1 \le x \le 2. \end{cases}$$

The integral data are generated by

$$I_i = \int_{x_i}^{x_{i+1}} q(x)\,dx, \quad i = 0, 1, \ldots, n-1.$$

We take $n = 20$ and construct the local integro quartic spline from these integral values on a uniform partition of $[0, 2]$.

This example is not intended to verify the convexity- or monotonicity-preserving theorems, because the target function is discontinuous and the constructed local integro quartic spline belongs to $C^3[a, b]$. Instead, it is used to illustrate the behavior of the method for discontinuous integral data.

The numerical results show that the spline remains positive on $[0, 2]$:

$$\min_{x\in[0,2]} S(x) = 0.8788954 > 0.$$

At the discontinuity point, the spline gives

$$S(1) = 1.5000,$$

which lies between the left and right limits of the discontinuous function. This indicates that the jump discontinuity is smoothed out. Moreover, small undershoot and overshoot appear near the jump point, for example,

$$S(0.9) = 0.8833 < 1, \quad S(1.1) = 2.1167 > 2.$$

Therefore, the approximation is stable away from the discontinuity, but larger errors occur near the jump point. This behavior is natural for smooth spline approximations of $q(x)$ (see Table 4).

Table 4. Behavior of the local integro quartic spline for the discontinuous function.

n	$\min S(x)$	$S(1)$	Shape conclusion
20	0.8788954	1.5000	positivity preserved; jump not preserved

The approximation behavior is also illustrated in Figure 4. The upper two subfigures compare the discontinuous function with the local integro cubic and quartic splines, respectively. The lower two subfigures show the corresponding approximation error curves. It can be observed that both splines give smooth approximations and cannot reproduce the jump exactly. The error curves show that the largest errors occur near the discontinuity point $x = 1$, while the approximation is stable away from the jump.

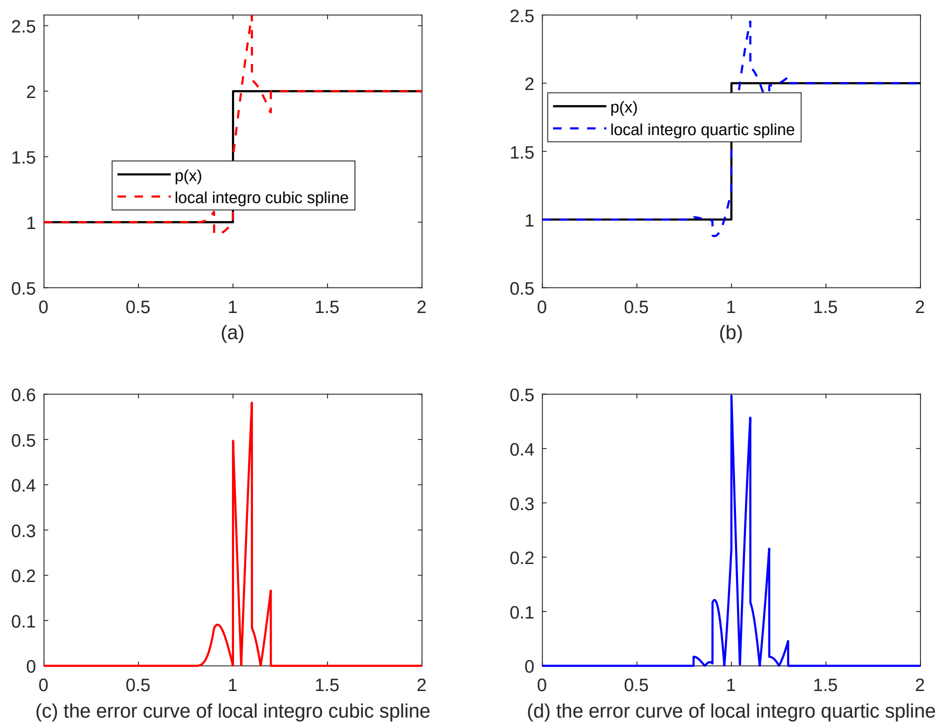

Figure 4. Comparison of local integro cubic and quartic splines for $q(x)$ on $[0, 2]$.

5. Conclusions

In this paper, we systematically investigated the shape-preserving properties of the local integro quartic spline. Sufficient theoretical conditions were derived to guarantee the preservation of convexity, monotonicity, and positivity of the constructed spline. These results provide a theoretical basis for applying local integro quartic splines to approximation problems in which only integral data are available.

Several numerical examples were presented to support the theoretical analysis. The results show that when the proposed sufficient conditions are satisfied, the local integro quartic spline preserves the corresponding shape properties. When these conditions are not satisfied, the expected shape-preserving properties may fail, which further illustrates the role and effectiveness of the derived conditions. The numerical comparisons also indicate that the local integro quartic spline has good approximation accuracy and practical applicability.

Future work may include extending the present analysis to nonuniform partitions, higher-degree local integro splines, and other types of shape-preserving constraints.

Author contributions

Yali Zhang: Conceptualization, methodology, writing-original draft, and formal analysis; Yachao Wang: Conceptualization, methodology, writing-original draft, and formal analysis. All authors agree to take responsibility for the content and conclusions of the manuscript.

Use of Generative-AI tools declaration

An AI assistant was used to check the language. All scientific content was reviewed and approved by the authors.

Acknowledgments

The authors would like to thank the anonymous reviewers for their valuable comments and constructive suggestions, which helped improve the quality of this manuscript. This work was supported by the National Natural Science Foundation of China (No.12501173), the Natural Science Foundation of Jiangsu Province (BK20241087, BK20251096).

Conflict of interest

The authors declare that they have no conflict of interest.

References

1. X. Guo, C. Liu, Two-step optimization of knots in B-spline curve approximation, *Math. Comput. Simul.*, **245** (2026), 53–64. https://doi.org/10.1016/j.matcom.2026.01.004
2. C. Liu, J. Li, L. Hu, Anderson accelerated preconditioning iterative method for RBF interpolation, *Eng. Anal. Boundary Elem.*, **169** (2024), 105970. https://doi.org/10.1016/j.enganabound.2024.105970
3. C. Liu, X. Han, J. Li, A class of spline functions for solving 2-order linear differential equations with boundary conditions, *Algorithms*, **13** (2020), 231. https://doi.org/10.3390/a13090231
4. C. Liu, X. Han, J. Li, Preconditioned progressive iterative approximation for triangular Bézier patches and its application, *J. Comput. Appl. Math.*, **366** (2020), 112389. https://doi.org/10.1016/j.cam.2019.112389
5. A. Tan, N. Wu, C. Liu, Two accelerated residual-based iterative methods and their convergence analyses, *Appl. Math. Lett.*, **171** (2025), 109657. https://doi.org/10.1016/j.aml.2025.109657
6. N. Wu, H. Cao, C. Liu, Momentum-accelerated randomized geometric iterative methods for curve and surface approximation, *Comput.-Aided Des.*, **192** (2026), 104011. https://doi.org/10.1016/j.cad.2025.104011
7. M. Abd El-Hady, M. El-Gamel, H. Emadifar, A. El-shenawy, Analysis of RL electric circuits modeled by fractional Riccati IVP via Jacobi-Broyden Newton algorithm, *PLoS One*, **20** (2025), e0316348. https://doi.org/10.1371/journal.pone.0316348
8. H. Behforooz, Approximation by integro cubic splines, *Appl. Math. Comput.*, **175** (2006), 8–15. https://doi.org/10.1016/j.amc.2005.07.066
9. F. Lang, X. Xu, On integro quartic spline interpolation, *J. Comput. Appl. Math.*, **236** (2012), 4214–4226. https://doi.org/10.1016/j.cam.2012.05.017

10. F. Lang, X. Xu, Quintic B-spline method for integro interpolation, *Appl. Math. Comput.*, **263** (2015), 353–360. https://doi.org/10.1016/j.amc.2015.04.076
11. T. Zhanlav, R. Mijiddorj, Integro quintic splines and their approximation properties, *Appl. Math. Comput.*, **231** (2014), 536–543. https://doi.org/10.1016/j.amc.2014.01.043
12. J. Wu, X. Zhang, Integro sextic spline interpolation and its super convergence, *Appl. Math. Comput.*, **219** (2013), 6431–6436. https://doi.org/10.1016/j.amc.2012.12.062
13. J. Wu, Y. Zhang, X. Zhang, C. Zhu, Even degree spline interpolation from the integral values of successive interval, *J. Syst. Sci. Math. Sci.*, **37** (2017), 2085–2094. https://doi.org/10.12341/jssms13265
14. A. Boujraf, D. Sbibih, M. Tahrichi, A. Tijini, A simple method for constructing integro spline quasi-interpolants, *Math. Comput. Simulat.*, **111** (2015), 36–47. https://doi.org/10.1016/j.matcom.2014.11.019
15. T. Zhanlav, R. Mijiddorj, The local integro cubic splines and their approximation properties, *Appl. Math. Comput.*, **216** (2010), 2215–2219. https://doi.org/10.1016/j.amc.2010.03.037
16. T. Zhanlav, R. Mijiddorj, On local integro quartic splines, *Appl. Math. Comput.*, **269** (2015), 301–307. https://doi.org/10.1016/j.amc.2015.07.077
17. X. Guo, X. Han, Y. Zhang, The local integro splines with optimized knots, *Comput. Appl. Math.*, **38** (2019), 156. https://doi.org/10.1007/s40314-019-0960-z
18. R. Mijiddorj, T. Zhanlav, Algorithm to construct integro splines, *ANZIAM J.*, **63** (2021), 359–375. https://doi.org/10.21914/anziamj.v63.15855
19. T. Zhanlav, R. Mijiddorj, Integro cubic splines on non-uniform grids and their properties, *East Asian J. Appl. Math.*, **11** (2021), 406–420. https://doi.org/10.4208/eajam.030920.251220
20. X. Han, Convexity-preserving approximation by univariate cubic splines, *J. Comput. Appl. Math.*, **287** (2015), 196–206. https://doi.org/10.1016/j.cam.2015.03.051
21. Y. Zhu, X. Han, C^2 rational quartic interpolation spline with local shape preserving property, *Appl. Math. Lett.*, **46** (2015), 57–63. https://doi.org/10.1016/j.aml.2015.02.005
22. T. Zhanlav, R. Mijiddorj, Convexity and monotonicity properties of the local integro cubic spline, *Appl. Math. Comput.*, **293** (2017), 131–137. https://doi.org/10.1016/j.amc.2016.08.017
23. C. De Boor, *A practical guide to splines*, New York: Springer, 1978.