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Research article

Coupled sub-equations method for seeking analytical solutions of fractional partial differential equations

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Abstract: In this paper, we considered seeking interaction exact solutions with variable coefficients for fractional partial differential equations involving the conformable fractional derivative. Combining with the properties of the conformable fractional calculus, and using a new ansatz structure, we proposed a new coupled sub-equations method for seeking interaction exact solutions of nonlinear fractional partial differential equations, that is, the coupled extended $(\frac{R'}{R})$ equation-Riccati equation method. In the process of the method, based on conformable fractional calculus, unknown transformed functions of a fractional system were expressed in certain polynomials with respect to some variables, which satisfied two known coupled sub-equations, and the homogeneous balance principle was used to determine the degree of the polynomials. The method was applied to solve the time fractional two-dimensional Boussinesq equation. With the aid of mathematical software Maple, an abundance of interaction exact solutions with variable coefficients for the equation were obtained by the ansatz structure.

Keywords: coupled sub-equations method; fractional partial differential equations; exact solutions; variable coefficients; soliton
Mathematics Subject Classification: 35C07, 35Q51, 35Q53

1. Introduction

In recent decades, fractional differential equations, as an emerging field of modern mathematics, have gradually enriched and improved their theoretical foundations. The core concept is to extend the traditional integer order derivative to fractional order derivative, thereby making the description of differential equations more accurate and extensive. The theoretical research of fractional differential equations involves several aspects, including the definition of fractional derivatives, initial value problems, boundary value problems, numerical methods, exact solutions seeking, and qualitative properties of solutions such as stability, asymptotic behavior, oscillation, and so on.

In terms of applications, many practical problems can be modeled by fractional differential equations. Fractional differential equation models have advantages such as memory effect, genetic characteristics, and high accuracy in describing the dynamic behavior of systems. Therefore, research on the solving methods and the structure of solutions to fractional differential equations is currently a hot topic in the field of differential equations. Fractional calculus is widely used in anomalous diffusion, signal processing and control, fluid mechanics, image processing, soft matter research, seismic analysis, viscoelastic dampers, power fractal networks, fractional order sine oscillators, and fractal theory. In the field of physics, fractional differential equations are used to describe the dynamic behavior of various complex systems. For example, in quantum mechanics, fractional differential equations can describe the dynamic evolution process of particles in potential wells more accurately. In material science, the formation and fission process of materials under stress can be deeply understood through fractional differential equations. In the field of engineering, fractional differential equations provide new ideas for modeling and controlling complex systems. For example, in circuit design, fractional differential equations can describe the dynamic characteristics of circuits, providing theoretical support for circuit optimization design. In the field of biology, fractional differential equations are used to describe complex processes such as signal transduction, neural activity, population dynamics and so on.

Fractional differential equations are divided into linear fractional differential equations and nonlinear fractional differential equations. For linear fractional differential equations, due to their relatively simple structures, the properties of solutions are also relatively easy to analyze. Nonlinear fractional differential equations, due to their complex structures, have more complicated solving processes and analysis of properties. So far, there have been many effective methods for solving fractional differential equations, such as the $(\frac{G'}{G})$ method [1], the modified Kudryashov method [2–4], the fractional Nikiforov Uvarov method [5], the exp method [6–8], the exp(-$\varphi(\xi)$)-expansion method [9–11], the Riccati equation method [12–15], the Lie symmetry method [16,17], the project Riccati equation method [18,19], the Jacobi elliptic equation method [20–24], and so on. For other research on the theory and applications of fractional differential equations, we refer to [25–27].

Traditional methods for seeking exact solutions, such as the exp method, the $(\frac{G'}{G})$ method, the standard Riccati equation based method, and so on, all fix the independent variable as a linear traveling wave combination $\vartheta = x - ct$. Although this assumption can effectively obtain traveling wave solutions, it also brings about significant limitations: first, it cannot handle propagation problems in inhomogeneous media; second, it lacks degrees of freedom for regulating complex dynamical behaviors such as soliton-periodic wave coupling, resulting in a relatively simple structure of solutions; third, under a fixed traveling wave combination condition, it is difficult to construct non-traveling wave solutions for complex systems. To overcome these limitations, we propose an improved method that transfers the constraints of the Riccati equation from the solution function to the independent variable level. Specifically, we still adopt the $(\frac{G'}{G})$ expansion form, but require the independent variable ϑ itself to explicitly satisfy the Riccati equation, and the independent variable ζ of ϑ is an undetermined function of all independent variables in the original equation. This improvement allows for adjustable transformation rules of the independent variable, enabling the construction of interaction exact solutions and non-traveling wave solutions.

Motivated by the analysis above, in this paper, we mainly explore the methods of seeking exact solutions with interaction forms for nonlinear fractional partial differential equations (NFPDEs) involving the conformable fractional derivative [28,29]. We will use the combination of two types of sub-equations to construct a new ansatz structure, and furthermore construct interaction exact solutions, where the solutions of certain NFPDEs are expressed in polynomials in the variable $\left[\frac{R'_{\vartheta}(\vartheta(\zeta))}{R(\vartheta(\zeta))}\right]$, and the following two types of known sub-equations are satisfied

$$F_1(R, R'_{\vartheta}, R''_{\vartheta}, ...) = 0, \tag{1}$$

$$F_2(\vartheta, \vartheta'_{\zeta}, \vartheta''_{\zeta}, ...) = 0. \tag{2}$$

If $F_1,\ F_2$ are selected for different forms, then different coupled sub-equations methods can be constructed.

According to [28, Definition 2.1], the conformable fractional derivative of order γ for a differentiable function u is defined as

$$D_x^{\gamma} v(x) = \lim_{\delta \to 0} \frac{v(x + \delta x^{1-\gamma}) - v(x)}{\delta}.$$

By the definition above, one can verify that the following properties are satisfied:

$$\begin{cases} D_x^{\gamma} p[q(x)] = p'_q[q(x)] D_x^{\gamma} q(x), \\ D_x^{\gamma}(\frac{p}{q})(x) = \dfrac{q(x) D^{\gamma} p(x) - p(x) D^{\gamma} q(x)}{q^2(x)}, \\ D_x^{\gamma}[p(x) q(x)] = p(x) D^{\gamma} q(x) + q(x) D^{\gamma} p(x), \\ D_x^{\gamma} p(x) = p'(x) x^{1-\gamma}. \end{cases} \tag{3}$$

The remaining parts of this article will be arranged as follows. In Section II, we illustrate the proposed method, named the coupled extended $(\frac{R'}{R})$ equation-Riccati equation method, for seeking interaction exact solutions of NFPDEs. In Section III, we apply the method to seek various types of interaction exact solutions with variable coefficients for the time fractional two-dimensional Boussinesq equation. Finally, we provide some summary comments.

2. The coupled extended $(\frac{R'}{R})$ equation-Riccati equation method

In this section, we illustrate the coupled extended $(\frac{R'}{R})$ equation-Riccati equation method for seeking interaction exact solutions of NFPDEs.

Consider the following system of NFPDEs, in which $v_1, ...v_s$ are unknown differentiable functions involving variables $t,\ x_1, ..., x_i, ...$, and without loss of generality, suppose the fractional derivative acts on variables t and x_i. $\mathcal{A}_1,\ \mathcal{A}_2,\ ..., \mathcal{A}_m$ are polynomials with respect to each argument.

$$\begin{cases} \mathcal{A}_1(v_1, ...v_s,\ D_t^{\alpha} v_1, ..., D_t^{\alpha} v_s, (v_1)_{x_1}, ..., (v_s)_{x_1}, ..., D_{x_i}^{\beta} v_1, ..., D_{x_i}^{\beta} v_s, ...) = 0, \\ \mathcal{A}_2(v_1, ...v_s,\ D_t^{\alpha} v_1, ..., D_t^{\alpha} v_s, (v_1)_{x_1}, ..., (v_s)_{x_1}, ..., D_{x_i}^{\beta} v_1, ..., D_{x_i}^{\beta} v_s, ...) = 0, \\ ... \\ \mathcal{A}_m(v_1, ...v_s,\ D_t^{\alpha} v_1, ..., D_t^{\alpha} v_s, (v_1)_{x_1}, ..., (v_s)_{x_1}, ..., D_{x_i}^{\beta} v_1, ..., D_{x_i}^{\beta} v_s, ...) = 0. \end{cases} \tag{4}$$

Remark 1. The treatment of fractional derivatives in Eq (4) will follow the following approach. For example, for a differentiable function $v(x, y, z, t)$, if we set $X = k_1 \frac{x^\alpha}{\alpha}$, and $v(x, y, z, t) = V(X, y, z, t)$, then according to the first property of (3), one has $D_x^\alpha v = V'_X(X, y, z, t) D_x^\alpha X$. On the other hand, by the fourth property of (3), one can deduce that $D_x^\alpha X = k_1 \frac{\alpha x^{\alpha-1}}{\alpha} x^{1-\alpha} = k_1$. So it holds that $D_x^\alpha v = k_1 V'_X(X, y, z, t)$.

We give the main steps of the coupled extended $(\frac{R'}{R})$ equation-Riccati equation method for solving (4).

(1) Setting $T = \mu \frac{t^\alpha}{\alpha}$, $X_i = k_i \frac{x_i^\beta}{\beta}$, $v_i(t, x_1, x_2, ..., x_i, ...) = V_i(T, x_1, x_2, ..., X_i, ...)$, then (4) can be transformed to the following form:

$$\begin{cases} \widetilde{\mathcal{A}}_1(V_1, ...V_s,\ (V_1)_T, ..., (V_s)_T, (V_1)_{x_1}, ..., (V_s)_{x_1}, ..., (V_1)_{X_i}, ..., (V_s)_{X_i}, ...) = 0, \\ \widetilde{\mathcal{A}}_2(V_1, ...V_s,\ (V_1)_T, ..., (V_s)_T, (V_1)_{x_1}, ..., (V_s)_{x_1}, ..., (V_1)_{X_i}, ..., (V_s)_{X_i}, ...) = 0, \\ ... \\ \widetilde{\mathcal{A}}_m(V_1, ...V_s,\ (V_1)_T, ..., (V_s)_T, (V_1)_{x_1}, ..., (V_s)_{x_1}, ..., (V_1)_{X_i}, ..., (V_s)_{X_i}, ...) = 0. \end{cases} \tag{5}$$

(2) Suppose the solutions of (5) can be expressed in the following forms

$$V_i = \sum_{i=0}^{m} a_i(T, x_1, ..., X_i, ...) \left[\frac{R'_\vartheta(\vartheta(\zeta))}{R(\vartheta(\zeta))} \right]^i,\ i = 1, 2, ..., s, \tag{6}$$

where $\zeta = \zeta(T, x_1, ..., X_i, ...)$, $a_i,\ i = 0, 1, ..., m$ are unknown functions, and the following two equations are satisfied:

$$aRR''_\vartheta - bRR'_\vartheta - c(R'_\vartheta)^2 - dR^2 = 0, \tag{7}$$

$$\vartheta'_\zeta = \rho + \vartheta^2. \tag{8}$$

Eqs (7) and (8) are named as the extended $(\frac{R'}{R})$ equation and the Riccati equation, respectively. As one can see, the ansatz structure used here involves the coupling between the two sub-equations. So the structure of the solution is essentially different from those obtained by other existing methods, such as the $(\frac{G'}{G})$ method, the standard Riccati-based method, and so on. If we denote $\Delta_1 = b^2 + 4d(a-c)$, $\Delta_2 = d(a-c)$, then the solutions of (7) satisfy the following expressions

$$\left(\frac{R'_\vartheta(\vartheta)}{R(\vartheta)}\right) = \begin{cases} \frac{b}{2(a-c)} + \frac{\sqrt{\Delta_1}}{2(a-c)} [\frac{\delta_1 \sinh \frac{\sqrt{\Delta_1}\vartheta}{2(a-c)} + \delta_2 \cosh \frac{\sqrt{\Delta_1}\vartheta}{2(a-c)}}{\delta_1 \cosh \frac{\sqrt{\Delta_1}\vartheta}{2(a-c)} + \delta_2 \sinh \frac{\sqrt{\Delta_1}\vartheta}{2(a-c)}}],\ b \neq 0,\ \Delta_1 > 0, \\ \frac{b}{2(a-c)} + \frac{\sqrt{-\Delta_1}}{2(a-c)} [\frac{-\delta_1 \sin \frac{\sqrt{-\Delta_1}\vartheta}{2(a-c)} + \delta_2 \cos \frac{\sqrt{-\Delta_1}\vartheta}{2(a-c)}}{\delta_1 \cos \frac{\sqrt{-\Delta_1}\vartheta}{2(a-c)} + \delta_2 \sin \frac{\sqrt{-\Delta_1}\vartheta}{2(a-c)}}],\ b \neq 0,\ \Delta_1 < 0, \\ \frac{b}{2(a-c)} + \frac{\delta_2}{\delta_1 + \delta_2 \vartheta},\ b \neq 0,\ \Delta_1 = 0, \end{cases} \tag{9}$$

$$(\frac{R'_\vartheta(\vartheta)}{R(\vartheta)}) = \begin{cases} \frac{\sqrt{\Delta_2}}{(a-c)}[\frac{\delta_1 \sinh \frac{\sqrt{\Delta_2}\vartheta}{(a-c)} + \delta_2 \cosh \frac{\sqrt{\Delta_2}\vartheta}{(a-c)}}{\delta_1 \cosh \frac{\sqrt{\Delta_2}\vartheta}{(a-c)} + \delta_2 \sinh \frac{\sqrt{\Delta_2}\vartheta}{(a-c)}}],\ b = 0,\ \Delta_2 > 0, \\ \frac{\sqrt{-\Delta_2}}{(a-c)}[\frac{-\delta_1 \sin \frac{\sqrt{-\Delta_2}\vartheta}{(a-c)} + \delta_2 \cos \frac{\sqrt{-\Delta_2}\vartheta}{(a-c)}}{\delta_1 \cos \frac{\sqrt{-\Delta_2}\vartheta}{(a-c)} + \delta_2 \sin \frac{\sqrt{-\Delta_2}\vartheta}{(a-c)}}],\ b = 0,\ \Delta_2 < 0, \end{cases} \tag{10}$$

where $\delta_1,\ \delta_2$ are arbitrary constants.

As is known, the solutions of Eq (8) can be denoted as follows [30,31]:

$$\begin{cases} \vartheta_1(\zeta) = -\sqrt{-\rho}\tanh(\sqrt{-\rho}\zeta + C),\ \rho < 0, \\ \vartheta_2(\zeta) = -\sqrt{-\rho}\coth(\sqrt{-\rho}\zeta + C),\ \rho < 0, \\ \vartheta_3(\zeta) = \sqrt{\rho}\tan(\sqrt{\rho}\zeta + C),\ \rho > 0, \\ \vartheta_4(\zeta) = -\sqrt{\rho}\cot(\sqrt{\rho}\zeta + C),\ \rho > 0, \\ \vartheta_{5,\,6}(\zeta) = \sqrt{\rho}[\tan(2\sqrt{\rho}\zeta + C) \pm \sec(2\sqrt{\rho}\zeta + C)],\ \rho > 0, \\ \vartheta_7(\zeta) = -\frac{1}{\zeta + C},\ \rho = 0. \end{cases} \tag{11}$$

(3) By balancing the degrees of $(\frac{R'}{R})$ in (5), one can derive the value of m. After substituting (6) into (5), together with the help of Eqs (7) and (8), a system of equations with respect to $(\frac{R'}{R})$ can be obtained.

(4) After solving the system of equations, one can obtain the values of $\zeta,\ a_i$. Furthermore, by combining the results with (9)–(11), one can deduce the interaction exact solutions for Eq (4).

3. Application of the coupled extended $(\frac{R'}{R})$ equation-Riccati equation method

We consider the following time fractional two-dimensional Boussinesq equation:

$$D_t^\nu D_t^\nu u - u_{xx} - u_{yy} - (u^2)_{xx} - u_{xxxx} = 0,\ 0 < \nu \leq 1. \tag{12}$$

The fractional Boussinesq equation can be used in the analysis of long waves in shallow water, and also used in the analysis of the percolation of water in a porous subsurface of a horizontal layer of material. For the Boussinesq equation defined under conformable fractional derivative, its essential distinction from the classical integer order equation lies in the interpretation of the fractional order ν: it reflects the fractal dimension of the underlying spatial medium, rather than introducing memory effects. Consequently, the conformable fractional Boussinesq equation effectively maps the integer order equation from Euclidean space to a fractal space, thereby correcting the scaling relationship between dispersion and nonlinearity. In [32], Tasbozan etc. solved this equation by use of the Jacobi elliptic function expansion method.

Next we use the coupled extended $(\frac{R'}{R})$ equation-Riccati equation method to seek interaction exact solutions for Eq (12).

Let $T = \mu\frac{t^\nu}{\nu}$, and $u(x, y, t) = U(x, y, T)$. By Remark 1, Eq (12) is transformed into the equivalent form:

$$\mu^2 U_{TT} - U_{xx} - U_{yy} - (U^2)_{xx} - U_{xxxx} = 0. \tag{13}$$

We assume that the solutions of Eq (13) admit the following ansatz structure:

$$U(x,y,T) = \sum_{i=0}^{m} a_i(y,T)\left[\frac{R'_\vartheta(\vartheta(\zeta))}{R(\vartheta(\zeta))}\right]^i, \tag{14}$$

where $\zeta = \zeta(x,y,T)$, the coefficients $a_i,\ i = 0,1,...,m$ are unknown functions to be determined, and $R(\vartheta(\zeta)),\ \vartheta$ satisfy the coupled sub-equations (7) and (8). Substituting (14) into (13), along with Eqs (7) and (8), yields a polynomial in the variable $\left[\frac{R'_\vartheta(\vartheta(\zeta))}{R(\vartheta(\zeta))}\right]$.

We will determine the value of m using the homogeneous balance principle, which has been used by many authors [1,6–11,18,19]. The core of the homogeneous balance principle is to balance the degrees of the highest order linear partial derivative term and the dominant nonlinear term in the nonlinear equation, and then transform the problem of solving the original partial differential equation into solving a system of nonlinear algebraic equations.

As one can see from (14) and (7), the k-th partial derivative of U increases the degree of the ansatz by k compared with U itself. So, by balancing the degrees of $[\frac{R'_\vartheta(\vartheta(\zeta))}{R(\vartheta(\zeta))}]$ between the highest order linear partial derivatives U_{xxxx} and the nonlinear terms $(U^2)_{xx}$ in Eq (13), one can deduce $m+4 = 2m+2$, which implies $m = 2$. Therefore, the ansatz reduces to:

$$U(x,y,T) = a_0(y,T) + a_1(y,T)\left[\frac{R'_\vartheta(\vartheta(\zeta))}{R(\vartheta(\zeta))}\right] + a_2(y,T)\left[\frac{R'_\vartheta(\vartheta(\zeta))}{R(\vartheta(\zeta))}\right]^2. \tag{15}$$

Substituting (15), (7), and (8) into (13), and equating the coefficients of $\left[\frac{R'_\vartheta(\vartheta(\zeta))}{R(\vartheta(\zeta))}\right]^j$ to zero, we derive a system of under-determined partial differential equations for the unknown functions $\zeta(x,y,T),\ a_i(Y,T),\ i = 0,1,2$. After solving these equations with the aid of the mathematical software Maple, one can obtain the following three groups of results. Among these expressions, we mean $f_i,\ i = 1,2,...,7$ to denote arbitrary functions.

Group 1.

$$\zeta(x,y,T) = C_1x + C_2y + C_3T + C_4,\ a_2(y,T) = -\frac{6C_1^2(a-c)^2(\rho+\vartheta^2)^2}{a^2},$$

$$a_1(y,T) = \frac{6C_1^2 b(a-c)(\rho+\vartheta^2)^2}{a^2},\ a_0(y,T) = \frac{C_1^4[8(a-c)e-b^2](\rho+\vartheta^2)^2 - a^2C_1^2 + a^2(C_3^2\mu^2 - C_2^2)}{2C_1^2a^2}.$$

Group 2.

$$\zeta(x,y,T) = f_1(T\pm\mu y) + f_2(T\pm\mu y),\ a_2(y,T) = -\frac{6f_1^2(T\pm\mu y)(a-c)^2(\rho+\vartheta^2)^2}{a^2},$$

$$a_1(y,T) = \frac{6bf_1^2(T\pm\mu y)(a-c)(\rho+\vartheta^2)^2}{a^2},\ a_0(y,T) = \frac{[8(a-c)e-b^2](\rho+\vartheta^2)^2 f_1^2(T\pm\mu y) - a^2}{2a^2}.$$

Group 3.

$$\zeta(x,y,T) = f_1(T\pm\mu y),\ a_2(y,T) = f_2(T\pm\mu y),$$

$$a_1(y,T)=\frac{bf_2(T\pm\mu y)}{c-a},\ a_0(y,T)=f_3(-T-\mu y)+f_4(T-\mu y).$$

Combining with (9)–(11), (15), and $T=\mu\frac{t^\nu}{\nu}$, one can deduce an abundance of interaction exact solutions for the time fractional two-dimensional Boussinesq Eq (12). Next we only take Group1 for example.

If $b\neq 0,\ \Delta_1>0,\ \rho<0$, then from the first equality of (9) and the first two equalities of (11) we obtain the following two interaction soliton-soliton solutions:

$$\begin{aligned}
u_1(x,y,t)=&\frac{C_1^4[8(a-c)e-b^2]\rho^2 sech^4(\sqrt{-\rho}\zeta+C)-a^2C_1^2+a^2(C_3^2\mu^2-C_2^2)}{2C_1^2a^2}\\
&+\frac{6C_1^2b(a-c)\rho^2 sech^4(\sqrt{-\rho}\zeta+C)}{a^2}\{\widetilde{\sigma}_1+\widetilde{\sigma}_2\\
&[\frac{\delta_1\sinh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\tanh(\sqrt{-\rho}\zeta+C)}{2(c-a)}+\delta_2\cosh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\tanh(\sqrt{-\rho}\zeta+C)}{2(c-a)}}{\delta_1\cosh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\tanh(\sqrt{-\rho}\zeta+C)}{2(c-a)}+\delta_2\sinh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\tanh(\sqrt{-\rho}\zeta+C)}{2(c-a)}}]\}\\
&-\frac{6C_1^2(a-c)^2\rho^2 sech^4(\sqrt{-\rho}\zeta+C)}{a^2}\{\widetilde{\sigma}_1+\widetilde{\sigma}_2\\
&[\frac{\delta_1\sinh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\tanh(\sqrt{-\rho}\zeta+C)}{2(c-a)}+\delta_2\cosh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\tanh(\sqrt{-\rho}\zeta+C)}{2(c-a)}}{\delta_1\cosh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\tanh(\sqrt{-\rho}\zeta+C)}{2(c-a)}+\delta_2\sinh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\tanh(\sqrt{-\rho}\zeta+C)}{2(c-a)}}]\}^2,
\end{aligned}\tag{16}$$

$$\begin{aligned}
u_2(x,y,t)=&\frac{C_1^4[8(a-c)e-b^2]\rho^2 csch^4(\sqrt{-\rho}\zeta+C)-a^2C_1^2+a^2(C_3^2\mu^2-C_2^2)}{2C_1^2a^2}\\
&+\frac{6C_1^2b(a-c)\rho^2 csch^4(\sqrt{-\rho}\zeta+C)}{a^2}\{\widetilde{\sigma}_1+\widetilde{\sigma}_2\\
&[\frac{\delta_1\sinh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\coth(\sqrt{-\rho}\zeta+C)}{2(c-a)}+\delta_2\cosh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\coth(\sqrt{-\rho}\zeta+C)}{2(c-a)}}{\delta_1\cosh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\coth(\sqrt{-\rho}\zeta+C)}{2(c-a)}+\delta_2\sinh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\coth(\sqrt{-\rho}\zeta+C)}{2(c-a)}}]\}\\
&-\frac{6C_1^2(a-c)^2\rho^2 csch^4(\sqrt{-\rho}\zeta+C)}{a^2}\{\widetilde{\sigma}_1+\widetilde{\sigma}_2\\
&[\frac{\delta_1\sinh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\coth(\sqrt{-\rho}\zeta+C)}{2(c-a)}+\delta_2\cosh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\coth(\sqrt{-\rho}\zeta+C)}{2(c-a)}}{\delta_1\cosh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\coth(\sqrt{-\rho}\zeta+C)}{2(c-a)}+\delta_2\sinh\frac{\sqrt{\Delta_1}\sqrt{-\rho}\coth(\sqrt{-\rho}\zeta+C)}{2(c-a)}}]\}^2,
\end{aligned}\tag{17}$$

where $\zeta=C_1x+C_2y+C_3\mu\frac{t^\nu}{\nu}+C_4,\ \widetilde{\sigma}_1=\frac{b}{2(a-c)},\ \widetilde{\sigma}_2=\frac{\sqrt{\Delta_1}}{2(a-c)}$.

In Figure 1, we illustrate the interaction soliton-soliton solution $u_1(x,y,t)$ under some certain parameters.

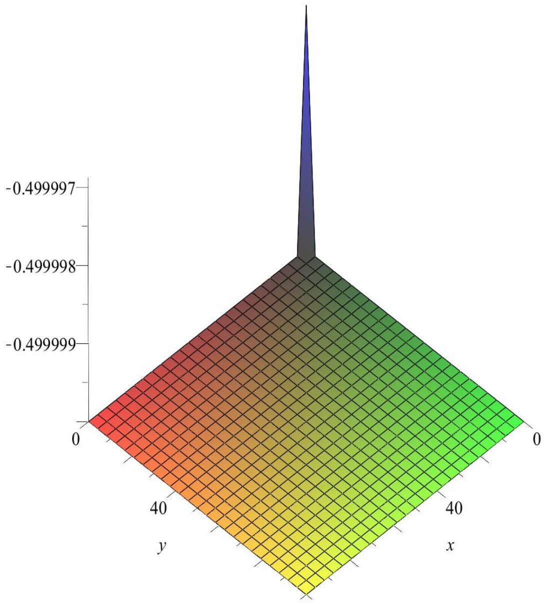

Figure 1. The interaction soliton-soliton solution u_1 under $a = d = 2,\ b = c = e = \mu = C_1 = C_2 = C_3 = C_4 = 1,\ \rho = -1,\ C = 0,\ \delta_1 = 1,\ \delta_2 = 2,\ \nu = 0.5,\ t = 2.$

As can be seen from Figure 1, the peak intensity values are relatively stable. This indicates that a stable amplitude-locked bound state has been formed between the two soliton components, with no energy exchange or periodic oscillation, exhibiting pure soliton molecule behavior and a pure soliton bound state. This dynamic characteristic can be utilized to simulate the stable bundled transmission of optical soliton pairs in nonlinear optical fibers or spin solitons in Bose-Einstein condensates [33,34].

If $b \neq 0,\ \Delta_1 > 0,\ \rho > 0$, then from the first equality of (9) and the results of (11) we obtain the following interaction soliton-periodic solutions:

$$u_3(x,y,t) = \frac{C_1^4[8(a-c)e - b^2]\rho^2 \sec^4(\sqrt{\rho}\zeta + C) - a^2C_1^2 + a^2(C_3^2\mu^2 - C_2^2)}{2C_1^2a^2}$$

$$+\frac{6C_1^2b(a-c)\rho^2 \sec^4(\sqrt{\rho}\zeta + C)}{a^2}\{\widetilde{\sigma}_1 + \widetilde{\sigma}_2$$

$$[\frac{\delta_1 \sinh\dfrac{\sqrt{\Delta_1}\sqrt{\rho}\tan(\sqrt{\rho}\zeta + C)}{2(a-c)} + \delta_2 \cosh\dfrac{\sqrt{\Delta_1}\sqrt{\rho}\tan(\sqrt{\rho}\zeta + C)}{2(a-c)}}{\delta_1 \cosh\dfrac{\sqrt{\Delta_1}\sqrt{\rho}\tan(\sqrt{\rho}\zeta + C)}{2(a-c)} + \delta_2 \sinh\dfrac{\sqrt{\Delta_1}\sqrt{\rho}\tan(\sqrt{\rho}\zeta + C)}{2(a-c)}}]\}$$

$$-\frac{6C_1^2(a-c)^2\rho^2 \sec^4(\sqrt{\rho}\zeta + C)}{a^2}\{\widetilde{\sigma}_1 + \widetilde{\sigma}_2$$

$$[\frac{\delta_1 \sinh \frac{\sqrt{\Delta_1}\sqrt{\rho}\tan(\sqrt{\rho}\zeta + C)}{2(a-c)} + \delta_2 \cosh \frac{\sqrt{\Delta_1}\sqrt{\rho}\tan(\sqrt{\rho}\zeta + C)}{2(a-c)}}{\delta_1 \cosh \frac{\sqrt{\Delta_1}\sqrt{\rho}\tan(\sqrt{\rho}\zeta + C)}{2(a-c)} + \delta_2 \sinh \frac{\sqrt{\Delta_1}\sqrt{\rho}\tan(\sqrt{\rho}\zeta + C)}{2(a-c)}}]\}^2, \tag{18}$$

$$u_4(x,y,t) = \frac{C_1^4[8(a-c)e - b^2]\rho^2 \csc^4(\sqrt{\rho}\zeta + C) - a^2C_1^2 + a^2(C_3^2\mu^2 - C_2^2)}{2C_1^2a^2}$$

$$+\frac{6C_1^2 b(a-c)\rho^2 \csc^4(\sqrt{\rho}\zeta + C)}{a^2}\{\widetilde{\sigma}_1 + \widetilde{\sigma}_2$$

$$[\frac{\delta_1 \sinh \frac{\sqrt{\Delta_1}\sqrt{\rho}\cot(\sqrt{\rho}\zeta + C)}{2(c-a)} + \delta_2 \cosh \frac{\sqrt{\Delta_1}\sqrt{\rho}\cot(\sqrt{\rho}\zeta + C)}{2(c-a)}}{\delta_1 \cosh \frac{\sqrt{\Delta_1}\sqrt{\rho}\cot(\sqrt{\rho}\zeta + C)}{2(c-a)} + \delta_2 \sinh \frac{\sqrt{\Delta_1}\sqrt{\rho}\cot(\sqrt{\rho}\zeta + C)}{2(c-a)}}]\}$$

$$-\frac{6C_1^2(a-c)^2\rho^2 \csc^4(\sqrt{\rho}\zeta + C)}{a^2}\{\widetilde{\sigma}_1 + \widetilde{\sigma}_2$$

$$[\frac{\delta_1 \sinh \frac{\sqrt{\Delta_1}\sqrt{\rho}\cot(\sqrt{\rho}\zeta + C)}{2(c-a)} + \delta_2 \cosh \frac{\sqrt{\Delta_1}\sqrt{\rho}\cot(\sqrt{\rho}\zeta + C)}{2(c-a)}}{\delta_1 \cosh \frac{\sqrt{\Delta_1}\sqrt{\rho}\cot(\sqrt{\rho}\zeta + C)}{2(c-a)} + \delta_2 \sinh \frac{\sqrt{\Delta_1}\sqrt{\rho}\cot(\sqrt{\rho}\zeta + C)}{2(c-a)}}]\}^2, \tag{19}$$

$$u_{5,6}(x,y,t) = \frac{C_1^4[8(a-c)e - b^2]\rho^2(1+\widetilde{\zeta}^2)^2 - a^2C_1^2 + a^2(C_3^2\mu^2 - C_2^2)}{2C_1^2a^2} + \frac{6C_1^2 b(a-c)\rho^2(1+\widetilde{\zeta}^2)^2}{a^2}$$

$$\{\widetilde{\sigma}_1 + \widetilde{\sigma}_2[\frac{\delta_1 \sinh \frac{\sqrt{\Delta_1}\sqrt{\rho}\widetilde{\zeta}}{2(a-c)} + \delta_2 \cosh \frac{\sqrt{\Delta_1}\sqrt{\rho}\widetilde{\zeta}}{2(a-c)}}{\delta_1 \cosh \frac{\sqrt{\Delta_1}\sqrt{\rho}\widetilde{\zeta}}{2(a-c)} + \delta_2 \sinh \frac{\sqrt{\Delta_1}\sqrt{\rho}\widetilde{\zeta}}{2(a-c)}}]\}$$

$$-\frac{6C_1^2(a-c)^2\rho^2(1+\widetilde{\zeta}^2)^2}{a^2}\{\widetilde{\sigma}_1 + \widetilde{\sigma}_2[\frac{\delta_1 \sinh \frac{\sqrt{\Delta_1}\sqrt{\rho}\widetilde{\zeta}}{2(a-c)} + \delta_2 \cosh \frac{\sqrt{\Delta_1}\sqrt{\rho}\widetilde{\zeta}}{2(a-c)}}{\delta_1 \cosh \frac{\sqrt{\Delta_1}\sqrt{\rho}\widetilde{\zeta}}{2(a-c)} + \delta_2 \sinh \frac{\sqrt{\Delta_1}\sqrt{\rho}\widetilde{\zeta}}{2(a-c)}}]\}^2, \tag{20}$$

where $\widetilde{\zeta} = \tan(2\sqrt{\rho}\zeta + C) \pm \sec(2\sqrt{\rho}\zeta + C)$, $\zeta = C_1x + C_2y + C_3\mu\frac{t^\nu}{\nu} + C_4$.

In Figures 2 and 3, we illustrate the interaction soliton-periodic solutions $u_3(x,y,t)$ and $u_4(x,y,t)$ under some certain parameters.

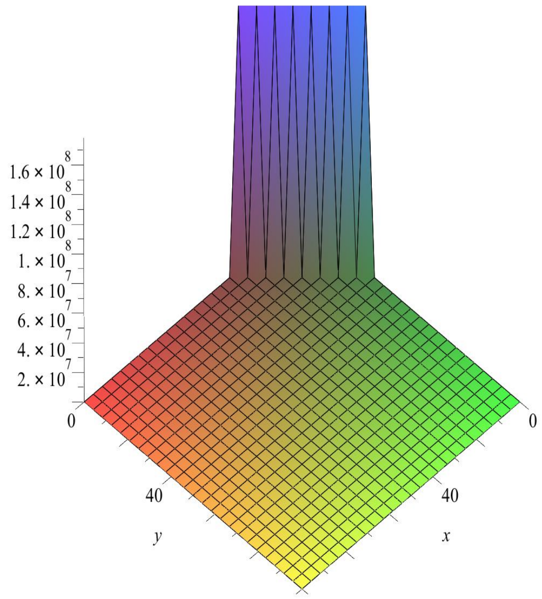

Figure 2. The interaction soliton-soliton solution u_3 under $a = d = 2,\ b = c = e = \mu = C_1 = C_2 = C_3 = C_4 = 1,\ \rho = -1,\ C = 0,\ \delta_1 = 1,\ \delta_2 = 2,\ \nu = 0.5,\ t = 2.$

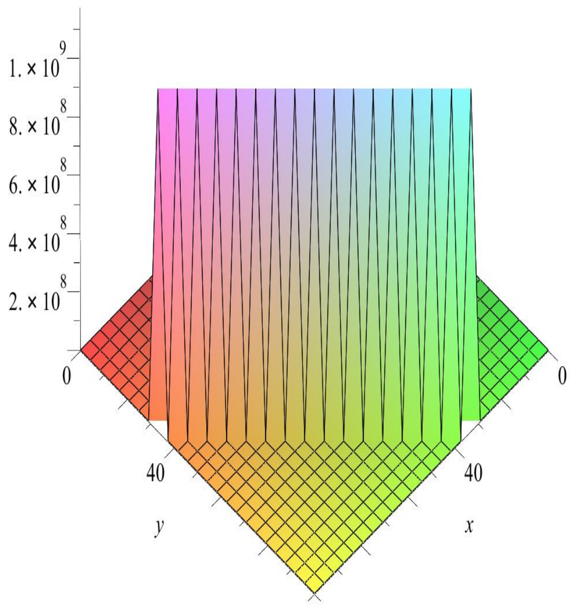

Figure 3. The interaction soliton-soliton solution u_4 under $a = d = 2,\ b = c = e = \mu = C_1 = C_2 = C_3 = C_4 = 1,\ \rho = -1,\ C = 0,\ \delta_1 = 1,\ \delta_2 = 2,\ \nu = 0.7,\ t = 2.$

Figures 2 and 3 correspond to $\nu = 0.5$ and 0.7, respectively. A comparison reveals that the waveform in Figure 2 exhibits compact soliton peaks and a weakly periodic background, with stronger localization, while the soliton peaks in Figure 3 become wider, and the periodic modulation amplitude is significantly enhanced, making the overall waveform more dominated by periodic waves. This difference stems from the regulation of the fractional order ν on the system. A smaller ν strengthens energy localization, with soliton components dominating, while a larger ν weakens non-localization, releasing periodic components. Simultaneously, introducing different singularity distributions through the Riccati equation further affects the localization-periodicity competition. The traditional $(\frac{G'}{G})$ method and Riccati equation method, which fix the independent variable as a linear traveling wave, cannot characterize the properties of such soliton-periodic wave coupled solutions. However, the method proposed in this paper elevates the Riccati equation constraint to the independent variable level. By selecting different Riccati parameters and fractional orders, it can successfully capture and regulate the continuous evolution of the soliton-periodic wave coupled morphology with varying fractional orders.

If $b \neq 0,\ \Delta_1 > 0,\ \rho = 0$, then from the first equality of (9) and the last equality of (11) we obtain the following interaction soliton-rational solutions:

$$\begin{aligned} u_7(x,y,t) = &\frac{C_1^4[8(a-c)e-b^2](\rho+\vartheta^2)^2 - a^2C_1^2 + a^2(C_3^2\mu^2 - C_2^2)}{2C_1^2a^2} \\ &+\frac{6C_1^2b(a-c)(\rho+\vartheta^2)^2}{a^2}\{\widetilde{\sigma}_1+\widetilde{\sigma}_2[\frac{\delta_1\sinh\frac{\sqrt{\Delta_1}\vartheta}{2(c-a)}+\delta_2\cosh\frac{\sqrt{\Delta_1}\vartheta}{2(c-a)}}{\delta_1\cosh\frac{\sqrt{\Delta_1}\vartheta}{2(c-a)}+\delta_2\sinh\frac{\sqrt{\Delta_1}\vartheta}{2(c-a)}}]\} \\ &-\frac{6C_1^2(a-c)^2(\rho+\vartheta^2)^2}{a^2}\{\widetilde{\sigma}_1+\widetilde{\sigma}_2[\frac{\delta_1\sinh\frac{\sqrt{\Delta_1}\vartheta}{2(c-a)}+\delta_2\cosh\frac{\sqrt{\Delta_1}\vartheta}{2(c-a)}}{\delta_1\cosh\frac{\sqrt{\Delta_1}\vartheta}{2(c-a)}+\delta_2\sinh\frac{\sqrt{\Delta_1}\vartheta}{2(c-a)}}]\}^2, \end{aligned} \tag{21}$$

where $\vartheta = -\dfrac{1}{C_1x + C_2y + C_3\mu\frac{t^\nu}{\nu} + C_4 + C}$.

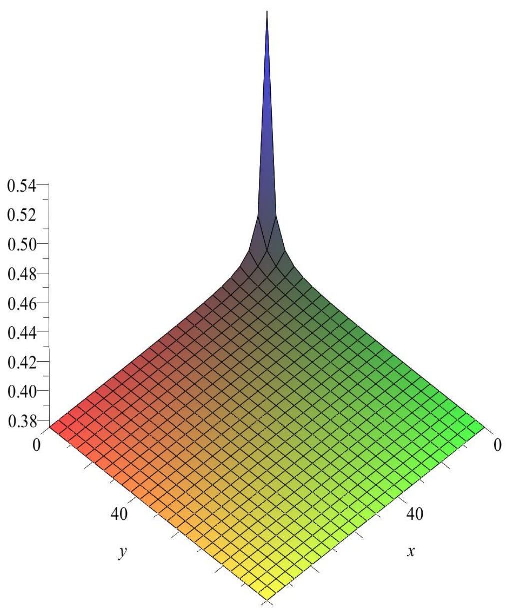

Figure 4. The interaction soliton-soliton solution u_7 under $a = d = 2,\ b = c = e = \mu = C_1 = C_2 = C_3 = C_4 = 1,\ \rho = -1,\ C = 0,\ \delta_1 = 1,\ \delta_2 = 2,\ \nu = 0.7,\ t = 2.$

In Figure 4, we illustrate the interaction soliton-rational solution $u_7(x, y, t)$ under some certain parameters.

As can be seen from Figure 4, the overall waveform exhibits a main soliton peak, but on both sides or in the background, there are uplifts or plateau structures caused by algebraic decay due to rational function components, rather than periodic oscillations. Unlike the previous soliton-soliton and soliton-periodic scenarios in Figures 1–3, the rational function components here do not produce repeating wave packets, but instead form localized, non-oscillatory modulation distortions. Dynamically, the main soliton maintains its localization and amplitude stability, but the rational component causes the wave to retain a slowly decaying tail over a longer distance. This type of coupling can be used to describe soliton excitation in non-uniform backgrounds, or localized states coupled with the continuum spectrum in Bose-Einstein condensates.

According to to Group 1, one can also obtain other interaction solutions for $b \neq 0,\ \Delta_1 \leq 0$; $b = 0,\ \Delta_2 > 0$; or $b = 0,\ \Delta_2 < 0$ by use of (9)–(11). Similarly, by a combination of the results in Groups 2,3 and (9)–(11), (15), one can obtain a series of other interaction exact solutions for the time fractional two-dimensional Boussinesq Eq (12).

Remark 2. All the solutions obtained above have been verified by the mathematical software Maple. After substituting the solutions (16)–(21) to Eq (12), one can symbolically verify the equation satisfies explicitly.

4. Conclusions

We have introduced a new ansatz structure, and used it to construct a new coupled sub-equations method for seeking interaction exact solutions with variable coefficients for nonlinear fractional partial differential equations. The homogeneous balance principle is used in the solving process. By the method, with the aid of mathematical software Maple, some new interaction solutions with variable coefficients for the time fractional two-dimensional Boussinesq equation have been found. This method transfers the constraints of the Riccati equation from the solution function to the independent variable ϑ itself, and allows the independent variable ζ of ϑ to be an undetermined function of the original equation's independent variables. Therefore, it has further application value in multiple fields. For example, it can be used in variable-coefficient nonlinear optical fiber systems, fractional or stochastic wave equations, soliton control in inhomogeneous media, and localized state design in Bose-Einstein condensates, especially in complex scenarios that require dynamic adjustment of soliton-period coupling patterns. However, this method also has some significant shortcomings. First, the assumption that ϑ satisfies the Riccati equation is strong, and not all physical systems can naturally satisfy or be precisely solved. Second, the introduction of the undetermined function $\zeta(x, y, t)$ increases the complexity of the algebraic equation system, often requiring the assistance of symbolic computation or numerical methods to complete. Finally, this method is currently mainly applicable to models with dimensions below (2+1), and when extended to higher dimensions or coupled equation systems, the handling of compatibility conditions becomes very difficult. Therefore, while enhancing flexibility, the new method sacrifices some universality and computational convenience.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interest in this paper.

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