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*Research article*

# Taylor coefficients condition and mean growth of random series in $Q_K$ spaces

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**Abstract:** Some properties for random power series $(\mathcal{R}f)(z) = \sum_{n=0}^{\infty} a_n \epsilon_n z^n$ in $Q_K$ spaces and $Q_{K,0}$ spaces were studied in this paper. It was proved that if the condition $\sum_{n=1}^{\infty} n|a_n|^2 K(\frac{1}{n}) < \infty$ holds, then $\mathcal{R}f \in Q_K$ a.s. Furthermore, this condition actually implied that $\mathcal{R}f \in Q_{K,0}$ a.s. As an application, the relationship between mean Lipschitz spaces and $Q_K$ spaces was shown.

**Keywords:** random power series; lacunary series; $Q_K$ space; Mean Lipschitz space; Carleson measure
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## 1. Introduction

Recall that an analytic function $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is called a lacunary series if

$$\lambda = \inf_k \frac{n_{k+1}}{n_k} > 1.$$

Such series are often used to construct examples of analytic functions in various function spaces. It is known that a lacunary series belongs to the Bloch space if, and only if, its Taylor coefficients are bounded [15], and belongs to BMOA, the space of analytic functions in the unit disk $\mathbb{D}$ of bounded mean oscillation (see [9]) if, and only if, $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ [5]. Also, if $0 \le p \le 1$, a lacunary series $f(z) = \sum_{k=1}^{\infty} a_k z^{n_k}$ is in $Q_p$ if, and only if, $\sum_{k=1}^{\infty} n_k^{1-p} |a_k|^2 < \infty$. Moreover, $f$ is actually in $Q_{p,0}$ (see [4]).

Let $\{\epsilon_n\}_{n=0}^{\infty}$ be a sequence of independent, identically distributed (i.i.d.) Rademacher random variables on a probability space $(\Omega, \mathcal{F}, P)$, that is, $P(\epsilon_n = 1) = P(\epsilon_n = -1) = \frac{1}{2}$ for all $n \ge 0$. For each $f(z)=\sum\limits_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$, we define the random power series

$$(\mathcal{R}f)(z)=\sum_{n=0}^{\infty} a_n \epsilon_n z^n,$$

where $H(\mathbb{D})$ is the space of all analytic functions on the unit disk $\mathbb{D}$.

Let $X$ be a certain analytic function space on $\mathbb{D}$. We say that $\mathcal{R}f$ almost surely belongs to $X$ (denoted by $\mathcal{R}f \in X\, a.s.$), if there exists $E \subset \Omega$ with $P(E)=1$ such that $(\mathcal{R}f)(z) \in X$ whenever $\epsilon_n \in E$.

Generally, random power series and lacunary series have many similar characteristics. For example, they are very well behaved if the coefficients are square-summable and very badly behaved if not (see [12, 14]). However, the condition $\sum\limits_{n=1}^{\infty} |a_n|^2 < \infty$ implies $\mathcal{R}f \in \bigcap\limits_{p<\infty} H^p$ a.s. ( [14]). Sledd [17] and Duren [5] have shown that $\sum\limits_{n=1}^{\infty} |a_n|^2 \log n < \infty$ implies that $\mathcal{R}f \in$ BMOA a.s. Furthermore, this condition actually gives the stronger conclusion that $\mathcal{R}f \in$ VMOA a.s. Aulaskari, Stegenga, and Zhao in [3] have shown that for $0<p<1$, $\sum\limits_{n=1}^{\infty} n^{1-p}|a_n|^2 < \infty$ implies $\mathcal{R}f \in Q_p$ a.s. Recently, we have proved that $\mathcal{R}f$ is actually in $Q_{p,0}$ a.s. in this condition [13]. However, compared with the known results for $Q_p$ spaces (see [3, 13]), the extension to the general $Q_K$ setting is not trivial because the weight function $K$ is not necessarily a power function.

For any $a \in \mathbb{D}$, let

$$\varphi_a(z)=\frac{a-z}{1-\bar{a}z}, \quad z \in \mathbb{D}$$

be the Möbius transformation of $\mathbb{D}$. Let $K:[0,\infty) \to [0,\infty)$ be a right-continuous and nondecreasing function. The space $Q_K$ consists of analytic functions $f$ on $\mathbb{D}$ with

$$\|f\|_{Q_K}^2=\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1-|\varphi_a(z)|^2)\,dA(z) < \infty.$$

The space $Q_{K,0}$ is the subspace of $Q_K$ with

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} |f'(z)|^2 K(1-|\varphi_a(z)|^2)\,dA(z)=0.$$

$Q_K$ is a Banach space with the norm $\|f\|=|f(0)|+\|f\|_{Q_K}$ and is Möbius invariant, that is, $\|f\circ\varphi_a\|_{Q_K}=\|f\|_{Q_K}$ for all $a \in \mathbb{D}$. It is also known that $Q_K$ is the $Q_p$ space and $Q_{K,0}$ is the little $Q_p$ space in the case of $K(t)=t^p, (0<p<\infty)$. Specifically, $Q_K$ and $Q_{K,0}$ coincide with BMOA and VMOA (the space of analytic functions in $\mathbb{D}$ of vanishing mean oscillation), respectively, if $K(t)=t$. For more materials about the $Q_p$ and BMOA theory, we refer to [9, 16, 20].

$Q_K$ spaces were introduced by Essén, Wulan, and Xiao et al. in [7, 8, 19], and it was proved that $Q_K$ is always contained in the Bloch space $\mathcal{B}$ and $Q_{K,0}$ is contained in $\mathcal{B}_0$ space, where the Bloch space $\mathcal{B}$ consists of all analytic functions $f$ on $\mathbb{D}$ satisfying

$$\|f\|_{\mathcal{B}}=\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty,$$

and the subspace $\mathcal{B}_0$ consists of all functions $f \in \mathcal{B}$ satisfying

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0.$$

Moreover, $Q_K$ space is nontrivial, i.e., $Q_K$ must contain nonconstant functions, if, and only if,

$$\int_0^{\frac{1}{e}} K\left(\log\frac{1}{r}\right) r dr < \infty.$$

For more results about the space $Q_K$, we refer to [19].

In this paper, we always assume that the weight function $K$ satisfies $K(0) = 0$, and

$$\int_1^{\infty} \varphi_K(s)\frac{ds}{s^{1+\sigma}} < \infty, \quad \sigma > 0 \tag{1.1}$$

where

$$\varphi_K(s) = \sup_{0<t\le 1}\frac{K(st)}{K(t)} < \infty, \quad 0 < s < \infty.$$

Note that $\varphi_K(s)$ is always well defined if it is allowed to take infinite values.

Suppose $f(z) = \sum\limits_{k=1}^{\infty} a_k z^{n_k}$ is a lacunary series and the weight function $K$ satisfies condition (1.1). The following equivalent relations hold in $Q_K$ spaces (see [18, 19])

$$\sum_{k=1}^{\infty} n_k|a_k|^2 K\left(\frac{1}{n_k}\right) < \infty \iff f \in Q_K \iff f \in Q_{K,0}. \tag{1.2}$$

The main purpose of this paper is to establish the sharp condition for the random series $\mathcal{R}f$ to belong to $Q_K$ (and $Q_{K,0}$) almost surely. This generalizes the previous criteria for $Q_p$ spaces (where $K(t) = t^p$) and also extends the lacunary series characterizations in [18, 19] to the random setting.

**Theorem 1.1.** *Suppose $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ and the weight function $K$ satisfies condition (1.1). If*

$$\sum_{n=1}^{\infty} n|a_n|^2 K\left(\frac{1}{n}\right) < \infty, \tag{1.3}$$

*then the random series*

$$(\mathcal{R}f)(z) = \sum_{n=0}^{\infty} a_n \epsilon_n z^n \in Q_K \; a.s.$$

Furthermore, this condition of Theorem 1.1 actually implies the stronger conclusion that $\mathcal{R}f \in Q_{K,0}$ a.s. We will obtain this result in Section 3. As an application, the relationship between mean Lipschitz spaces and $Q_K$ spaces is shown in Section 4.

In what follows, $C$ always denotes some positive constant which may differ at different situations. Moreover, $A \lesssim B$ means that there exists a positive constant $C$ such that $A \le CB$ and $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

## 2. Preliminaries and basic properties

A sequence of functions $\{\varphi_n(t)\}_{n=0}^{\infty}$ is called the Rademacher functions if

$$\varphi_1(t) = \begin{cases} 1, & 0 < t < \frac{1}{2}, \\ -1, & \frac{1}{2} < t < 1, \\ 0, & t = 0, \frac{1}{2}, 1, \end{cases}$$

and

$$\varphi_n(t) = \operatorname{sgn}\{\sin(2^n \pi t)\}, \quad 0 \le t \le 1.$$

In general, $\varphi_n(t)$ vanishes at all multiples of $2^{-n}$ and takes the values $\pm 1$ elsewhere. See [6, 10] for properties of these functions.

The following Khintchine's inequality (Appendix A of [6, 11]) for Rademacher functions will be used later.

**Lemma 2.1.** *(Khintchine's inequality) Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of complex numbers with* $\sum\limits_{n=1}^{\infty} |a_n|^2 < \infty$. *Then, the series* $\sum\limits_{n=1}^{\infty} a_n \varphi_n(t)$ *converges a.s. Furthermore, let* $0 < p < \infty$, *for all integers* $N \ge 1$,

$$A_p \left( \sum_{n=1}^{N} |a_n|^2 \right)^{1/2} \le \left( \mathbb{E} \left| \sum_{n=1}^{N} a_n \varphi_n(t) \right|^p \right)^{1/p} \le B_p \left( \sum_{n=1}^{N} |a_n|^2 \right)^{1/2} \tag{2.1}$$

*for some positive constants* $A_p$, $B_p$ *depending only on* $p$.

For $\alpha \ge 1$ and $\beta \ge 1$, the following inequality holds for the beta function $B(\alpha,\beta)$.

**Lemma 2.2.** *(Beta function inequality) Let* $\alpha \ge 1$ *and* $\beta \ge 1$. *Then, there exist two constants* $C_1$ *and* $C_2$ *depending only on* $\beta$ *such that*

$$C_1 \left( \frac{1}{\alpha} \right)^{\beta} \le B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt \le C_2 \left( \frac{1}{\alpha} \right)^{\beta}.$$

An important tool to study function spaces is Carleson type measure. Let $I$ be any arc on the unit circle $S^1$ and $|I|$ be the normalized length of $I$, i.e.,

$$|I| = \frac{1}{2\pi} \int_I |d\zeta|.$$

For any subarc $I$ of $S^1$, the Carleson sector $S(I)$, based on $I$, is defined by

$$S(I) = \{z = re^{i\theta} : 1 - |I| \le r < 1, e^{i\theta} \in I\}.$$

If $\mu$ is a positive Borel measure on the unit disk $\mathbb{D}$ and $K$ is a positive weight function, we say $\mu$ is a $K$-Carleson measure if

$$\|\mu\|_K = \sup_{I \subset S^1} \int_{S(I)} K\left( \frac{1 - |z|}{|I|} \right) d\mu(z) < \infty.$$

We say $\mu$ is a vanishing $K$-Carleson measure if

$$\lim_{|I|\to 0}\int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) d\mu(z) = 0.$$

If $K(t) = t^p, 0 < p < \infty$, then $\mu$ is a $K-$Carleson on $\mathbb{D}$ if, and only if, $(1-|z|^2)^p d\mu$ is a $p-$Carleson measure.

It is known in [8] that $f \in Q_K$ if, and only if, $d\mu(z) = |f'(z)|^2 dxdy$ is a $K$-Carleson measure on $\mathbb{D}$, and $f \in Q_{K,0}$ if, and only if, $d\mu(z) = |f'(z)|^2 dxdy$ is a vanishing $K$-Carleson measure, respectively.

If $0 < r < 1$ and $f \in H(\mathbb{D})$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, \quad 0 < p < \infty$$

and $M_\infty(r, f) = \sup_{|z|=r} |f(z)|$.

For $1 \le q \le \infty$, $0 \le \alpha \le 1$, the mean Lipschitz space $\Lambda(q, \alpha)$ consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{\Lambda(q,\alpha)} = |f(0)| + \sup_{0<r<1}(1-r)^{1-\alpha} M_q(r, f') < \infty.$$

Note that the size of $\Lambda(q, 1/q)$ increases with $q \in (1, \infty)$. It is clear that the space $\Lambda(\infty, 0)$ coincides with the Bloch space $\mathcal{B}$ and is contained in $\Lambda(q, 0)$ for every $q < \infty$.

**Lemma 2.3.** *( [19]) Suppose $f \in H(\mathbb{D})$ and $2 < q \le \infty$. Let $K$ satisfy condition (1.1) for $\sigma = 1 - \frac{2}{q}$, namely,*

$$B_q = \int_1^\infty \frac{\varphi_K(s)}{s^{2-\frac{2}{q}}} ds < \infty.$$

*Then,*

$$\|f\|_{Q_K}^2 \le B_q \int_0^1 K(1-r) M_q^2(r, f') dr.$$

## 3. Proof of Theorem 1.1

If $K$ satisfies condition (1.1) for some $\sigma > 0$, define the auxiliary function

$$K^*(t) = \sigma t^\sigma \int_t^\infty K(s)\frac{ds}{s^{1+\sigma}}, \quad 0 < t < \infty. \tag{3.1}$$

We shall use the following standard properties of the auxiliary function $K^*$ , whose proof can be found in Lemma 3.1 of Chapter 3 of [19]. $K^*$ has the following properties:

(i) $K^*(t)$ is nondecreasing on $(0, \infty)$.
(ii) $K^*(t)/t^\sigma$ is decreasing on $(0, \infty)$.
(iii) $K^*(t) \ge K(t)$ for all $t \in (0, \infty)$.
(iv) $K^*(t) \le K(t)$ on $(0, 1]$, so $K^*(t) \approx K(t)$ on $t \in (0, 1]$.

In order to prove Theorem 1.1, we need the following inequality.

**Proposition 3.1.** *Suppose the weight function $K$ satisfies condition (1.1) and $\alpha \geq 1$. Then,*

$$\int_0^1 r^{\alpha-1} K(1-r)dr \lesssim \frac{1}{\alpha} K\left(\frac{1}{\alpha}\right). \tag{3.2}$$

*Proof.* Split the integral as

$$\int_0^1 r^{\alpha-1} K(1-r)dr = \int_0^{1-\frac{1}{\alpha}} r^{\alpha-1} K(1-r)dr + \int_{1-\frac{1}{\alpha}}^1 r^{\alpha-1} K(1-r)dr =: I_1 + I_2.$$

For $I_2$, since $r \leq 1$ and $K$ is nondecreasing,

$$I_2 \leq K\left(\frac{1}{\alpha}\right) \int_{1-\frac{1}{\alpha}}^1 r^{\alpha-1} dr \leq \frac{1}{\alpha} K\left(\frac{1}{\alpha}\right).$$

For $I_1$, substitute $1-t$ to obtain

$$I_1 = \int_{\frac{1}{\alpha}}^1 (1-t)^{\alpha-1} K(t)\, dt.$$

By properties (ii) and (iii) of $K^*$, we have $K(t) \leq K^*(t)$ and $K^*(t)/t^\sigma$ is decreasing for the $\sigma > 0$ in condition (1.1). Then, for $t \in [1/\alpha, 1]$,

$$\frac{K^*(t)}{t^\sigma} \leq \frac{K^*(\frac{1}{\alpha})}{(\frac{1}{\alpha})^\sigma}.$$

Hence,

$$I_1 \leq \int_{\frac{1}{\alpha}}^1 (1-t)^{\alpha-1} \frac{K^*(t)}{t^\sigma} t^\sigma dt \leq \frac{K^*\left(\frac{1}{\alpha}\right)}{\left(\frac{1}{\alpha}\right)^\sigma} \int_{\frac{1}{\alpha}}^1 t^\sigma (1-t)^{\alpha-1} dt.$$

The integral $\int_{\frac{1}{\alpha}}^1 t^\sigma (1-t)^{\alpha-1} dt$ is bounded by the Beta function $B(\sigma+1, \alpha) = \int_0^1 t^\sigma (1-t)^{\alpha-1} dt$. By Lemma 2.2,

$$I_1 \lesssim \frac{K^*\left(\frac{1}{\alpha}\right)}{\left(\frac{1}{\alpha}\right)^\sigma} \left(\frac{1}{\alpha}\right)^{\sigma+1} = \frac{1}{\alpha} K^*\left(\frac{1}{\alpha}\right).$$

Since $K^*\left(\frac{1}{\alpha}\right) \approx K\left(\frac{1}{\alpha}\right)$ on $(0,1]$ (property (iv)), we obtain $I_1 \lesssim \frac{1}{\alpha} K\left(\frac{1}{\alpha}\right)$. Combining with the estimate for $I_2$ gives the desired inequality.

□

**Proof of Theorem 1.1.** Assume that

$$\sum_{n=1}^{\infty} n|a_n|^2 K\left(\frac{1}{n}\right) < \infty$$

holds. By Fubini's theorem,

$$\mathbb{E}\left(\int_0^1 M_q^2(r, (\mathcal{R}f)')K(1-r)dr\right) = \int_0^1 \mathbb{E}\left(\int_0^{2\pi} \left|\sum_{n=1}^{\infty} n a_n \epsilon_n r^{(n-1)} e^{i(n-1)\theta}\right|^q \frac{d\theta}{2\pi}\right)^{\frac{2}{q}} K(1-r)dr. \tag{3.3}$$

For $q > 2$, the function $x \mapsto x^{\frac{2}{q}}$ is concave on $[0, \infty)$. Jensen's inequality yields

$$\mathbb{E}\left(\int_0^{2\pi}\left|\sum_{n=1}^{\infty} na_n\epsilon_n r^{(n-1)}e^{i(n-1)\theta}\right|^q \frac{d\theta}{2\pi}\right)^{\frac{2}{q}} \leq \left(\mathbb{E}\int_0^{2\pi}\left|\sum_{n=1}^{\infty} na_n\epsilon_n r^{(n-1)}e^{i(n-1)\theta}\right|^q \frac{d\theta}{2\pi}\right)^{\frac{2}{q}}$$
$$= \left(\int_0^{2\pi}\mathbb{E}\left|\sum_{n=1}^{\infty} na_n\epsilon_n r^{(n-1)}e^{i(n-1)\theta}\right|^q \frac{d\theta}{2\pi}\right)^{\frac{2}{q}}.$$

For each fixed $r$ and $\theta$, Khintchine's inequality gives

$$\mathbb{E}\left|\sum_{n=1}^{\infty} na_n\epsilon_n r^{(n-1)}e^{i(n-1)\theta}\right|^q \lesssim \left(\sum_{n=1}^{\infty} n^2|a_n|^2 r^{2n-2}\right)^{\frac{q}{2}}.$$

Since the righthand side is independent of $\theta$, substituting these into the previous formula (3.3) gives

$$\mathbb{E}\left(\int_0^1 M_q^2(r,(\mathcal{R}f)')K(1-r)dr\right) \lesssim \int_0^1 \sum_{n=1}^{\infty} n^2|a_n|^2 r^{2n-2}K(1-r)dr = \sum_{n=1}^{\infty} n^2|a_n|^2 \int_0^1 r^{2n-2}K(1-r)dr.$$

From Proposition 3.1,

$$\int_0^1 r^{2n-2}K(1-r)dr \lesssim \frac{1}{n}K(\frac{1}{n})$$

for $n \geq 1$. So we get

$$\mathbb{E}\left(\int_0^1 M_q^2(r,(\mathcal{R}f)')K(1-r)dr\right) \lesssim \sum_{n=1}^{\infty} n|a_n|^2 K(\frac{1}{n}) < \infty,$$

which implies that

$$\int_0^1 M_q^2(r,(\mathcal{R}f)')K(1-r)dr < \infty.$$

By Lemma 2.3, we have $\mathcal{R}f \in Q_K$ a.s. This theorem is proved.

□

The condition of Lemma 2.3 implies the stronger conclusion that $f$ actually belongs to the $Q_{K,0}$ space.

**Proposition 3.2.** *Suppose $f \in H(\mathbb{D})$ and $2 < q \leq \infty$. Let $K$ satisfy condition (1.1) for $\sigma = 1 - \frac{2}{q}$, namely,*

$$B_q = \int_1^{\infty} \frac{\varphi_K(s)}{s^{2-\frac{2}{q}}}ds < \infty.$$

*If*

$$\int_0^1 K(1-r)M_q^2(r,f')dr < \infty,$$

*then $f \in Q_{K,0}$.*

*Proof.* For any $I \subset S^1$, we first consider the case $q = \infty$. Since

$$\int_I |f'(re^{i\theta})|^2 d\theta \le M_\infty^2(r, f')\,|I|,$$

we have

$$\begin{aligned}\int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)|f'(z)|^2 dxdy &\le \int_{1-|I|}^1 |I| K\left(\frac{1-|z|}{|I|}\right) M_\infty^2(r, f')dr \\ &\le \int_{1-|I|}^1 |I|\varphi_K\left(\frac{1}{|I|}\right) K(1-r) M_\infty^2(r, f')dr \\ &\le \int_{1-|I|}^1 \left(\int_{\frac{1}{|I|}}^\infty \frac{\varphi_K(s)}{s^2} ds\right) K(1-r) M_\infty^2(r, f')dr \\ &\le B_\infty \int_{1-|I|}^1 K(1-r) M_\infty^2(r, f')dr.\end{aligned}$$

The integral

$$\int_0^1 K(1-r) M_\infty^2(r, f')dr < \infty$$

implies that for any $\epsilon > 0$, there exists a $\delta, 0 < \delta < 1$, such that

$$\int_{1-\delta}^1 K(1-r) M_\infty^2(r, f')dr < \epsilon.$$

Thus

$$\int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)|f'(z)|^2 dxdy \le B_\infty \epsilon \tag{3.4}$$

holds for any subarc $I$ on $S^1$ with $|I| < \delta$.

For the case $2 < q < \infty$, by Hölder's inequality, we have

$$\int_I |f'(re^{i\theta})|^2 d\theta \le \left(\int_I |f'(re^{i\theta})|^q d\theta\right)^{\frac{2}{q}} \left(\int_I d\theta\right)^{\frac{q-2}{q}} \le M_q^2(r, f')|I|^{1-\frac{2}{q}}.$$

It follows that

$$\begin{aligned}\int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)|f'(z)|^2 dxdy &\le \int_{1-|I|}^1 \int_I \varphi_K\left(\frac{1}{|I|}\right) K(1-r)|f'(re^{i\theta})|^2 drd\theta \\ &\le \int_{1-|I|}^1 \varphi_K\left(\frac{1}{|I|}\right) K(1-r) M_q^2(r, f')|I|^{1-\frac{2}{q}} dr.\end{aligned}$$

By condition (1.1),

$$\varphi_K\left(\frac{1}{|I|}\right)|I|^{1-\frac{2}{q}} = \left(1-\frac{2}{q}\right)\varphi_K\left(\frac{1}{|I|}\right)\int_{\frac{1}{|I|}}^{\infty}\frac{ds}{s^{2-\frac{2}{q}}}$$
$$\leq \left(1-\frac{2}{q}\right)\int_{\frac{1}{|I|}}^{\infty}\frac{\varphi_K(s)}{s^{2-\frac{2}{q}}}ds$$
$$\leq B_q.$$

The integral

$$\int_0^1 K(1-r)M_q^2(r,f')dr < \infty$$

implies that for any $\epsilon > 0$, there exists a $\delta, 0<\delta<1$, such that

$$\int_{1-\delta}^1 K(1-r)M_q^2(r,f')dr < \epsilon.$$

Thus, we have

$$\int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)|f'(z)|^2dxdy \leq B_q\int_{1-|I|}^1 K(1-r)M_q^2(r,f')dr < B_q\epsilon \tag{3.5}$$

holds for any subarc $I$ on $S^1$ with $|I| < \delta$. (3.4) and (3.5) show that the measure $d\mu(z) = |f'(z)|^2dxdy$ is a vanishing $K-$Carleson measure, hence $f \in Q_{K,0}$. □

The following result characterizing random power series in the $Q_{K,0}$ space can be obtained immediately from Theorem 1.1 and Proposition 3.2.

**Theorem 3.3.** *Suppose* $f(z) = \sum_{n=0}^{\infty} a_nz^n \in H(\mathbb{D})$ *and the weight function K satisfies condition (1.1). If*

$$\sum_{n=1}^{\infty} n|a_n|^2K\left(\frac{1}{n}\right) < \infty,$$

*then the random series*

$$(\mathcal{R}f)(z) = \sum_{n=0}^{\infty} a_n\epsilon_nz^n \in Q_{K,0}\ a.s.$$

## 4. Mean Lipschitz spaces and $Q_K$

Let $q \geq 1$ and $0 < \alpha \leq 1$. The mean Lipschitz spaces $\Lambda(q,\alpha)$ can be depicted by (see [6])

$$M_q(r,f') = O\left((1-r)^{a-1}\right), \quad r \in (0,1).$$

Furthermore, if

$$M_q(r,f') = o\left((1-r)^{a-1}\right), \quad as\ \ r \to 1,$$

we say that $f \in \lambda(q,\alpha)$.

Under certain conditions, the mean Lipschitz spaces can be embedded in $Q_p$ spaces and $Q_K$ spaces. It is proved in [1, 2] that $\Lambda(q, \frac{1}{q}) \subsetneq Q_p$ if $0 < p < 1, 2 < q < \infty$, and $1 - \frac{2}{q} < p$. It is known in [19] that the condition

$$A_q = \int_0^1 \frac{\varphi_K(s)}{s^{2-\frac{2}{q}}} < \infty$$

means $\Lambda(q, \frac{1}{q}) \subsetneq Q_K$ for $2 \leq q < \infty$.

For the random series, we can get the converse result.

**Theorem 4.1.** *Let* $f(z) = \sum\limits_{n=1}^{\infty} a_n z^n \in H(\mathbb{D})$*. If the weight function $K$ satisfies condition (1.1) and the random series*

$$(\mathcal{R}f)(z) = \sum_{n=0}^{\infty} a_n \epsilon_n z^n \in Q_K \ a.s.$$

*then* $\mathcal{R}f \in \Lambda(q, \frac{1}{q})$ *a.s. where* $q > 2$*. Furthermore,* $\mathcal{R}f \in \lambda(q, \frac{1}{q})$ *a.s.*

*Proof.* From the proof of Theorem 1.1 and Proposition 3.2 , the condition $\mathcal{R}f(z) \in Q_K$ *a.s.* implies

$$\int_0^1 M_q^2(r, (\mathcal{R}f)')K(1 - r)dr < \infty.$$

Furthermore, for any $\epsilon > 0$, there exists a $\delta, 0 < \delta < 1$, such that

$$\int_{1-\delta}^1 M_q^2(r, (\mathcal{R}f)')K(1 - r)dr < \epsilon.$$

Since $K$ satisfies (1.1) with $\sigma = 1 - 2/q$, the associated function $K^*$ has the property that $K^*(t)/t^\sigma$ is decreasing. Consequently, for $0 < t \leq 1$,

$$\frac{K^*(t)}{t^\sigma} \geq K^*(1) \quad \text{and} \quad K(t) \approx K^*(t).$$

Hence, we have

$$\begin{aligned}
\int_{1-\delta}^1 M_q^2(r, (\mathcal{R}f)')K(1 - r)dr &= \int_{1-\delta}^1 M_q^2(r, (\mathcal{R}f)')(1 - r)^{1-\frac{2}{q}} \frac{K(1 - r)}{(1 - r)^{1-\frac{2}{q}}} dr \\
&\approx \int_{1-\delta}^1 M_q^2(r, (\mathcal{R}f)')(1 - r)^{1-\frac{2}{q}} \frac{K^*(1 - r)}{(1 - r)^{1-\frac{2}{q}}} dr \\
&\geq K^*(1) \int_{1-\delta}^1 M_q^2(r, (\mathcal{R}f)')(1 - r)^{1-\frac{2}{q}} dr
\end{aligned}$$

which implies that

$$M_q(r, (\mathcal{R}f)') = o\left(\frac{1}{(1 - r)^{1-\frac{1}{q}}}\right), \quad as \ \ r \to 1.$$

So $\mathcal{R}f$ actually belongs to $\lambda(q, \frac{1}{q})$ a.s., and this proof is complete. □

## 5. Conclusions

In this paper, we have studied the almost sure behavior of random power series $\mathcal{R}f$ in the general $Q_K$ and $Q_{K,0}$ spaces. Under a mild regularity condition on the weight function $K$, we proved that the coefficient condition

$$\sum_{n=1}^{\infty} n|a_n|^2 K\left(\frac{1}{n}\right) < \infty$$

is sufficient for $\mathcal{R}f$ to belong to $Q_K$ almost surely. Moreover, this condition actually implies the stronger conclusion $\mathcal{R}f \in Q_{K,0}$ almost surely. These results extend the classical characterizations for lacunary series and for random series in $Q_p$ spaces to the more general $Q_K$ spaces. As an application, we showed that if $\mathcal{R}f \in Q_K$ a.s., then it belongs to the mean Lipschitz space $\Lambda(q, 1/q)$ a.s., and even to its little counterpart $\lambda(q, 1/q)$ a.s. for $q > 2$.

## Author contributions

RuiLan ZhuGe: Conceptualization, Formal analysis, Writing-original draft; Xinxin Zhao: Validation, Writing-review & editing; Yan Wu: Supervision, Project administration, Funding acquisition, Writing-review & editing.

All authors have read and approved the final manuscript.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

All authors declare no conflicts of interest in this paper.

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