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Estimations for the Kumaraswamy distribution with a partially observed competing-risks model under progressive censoring data

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Abstract: This research examined the application of a two-competing-risks framework using the Kumaraswamy distribution within the context of progressive Type-II censoring. In this study, we assumed that the model follows the Kumaraswamy distribution and that the failure modes are independent and partially observed. The Type-II progressive censoring scheme, a cornerstone of modern reliability studies, was employed here as the primary data collection framework, particularly when evaluating highly reliable products where testing time and cost are significant factors. Maximum likelihood estimators (MLEs) were established. Also, the corresponding confidence intervals for the unknown parameters were computed, and the existence and uniqueness of these estimates were rigorously proven. Furthermore, approximate confidence intervals are constructed using the observed Fisher information matrix and the percentile bootstrap interval. In the Bayesian framework, Bayes estimates and their associated credible intervals were obtained utilizing Markov chain Monte Carlo (MCMC) sampling methods. Finally, a real-world application and simulation studies were presented to illustrate the practical utility of the proposed methodology.

Keywords: partially observed competing risks model; Bayesian estimation; Type-II progressive censoring scheme; maximum likelihood estimation; Kumaraswamy distribution; percentile bootstrap
Mathematics Subject Classification: 62N01, 62F10, 62F15

1. Introduction

This text provides a solid introduction to the importance of lifetime modeling and the mechanics of the progressive Type-II censoring (Type-II PC) scheme. The statistical modeling and analysis of lifetime data are pivotal across a broad spectrum of fields, including actuarial science, finance, engineering, and biomedical sciences. Consequently, a vast array of lifetime distributions has been developed to characterize these phenomena. Recently, modeling datasets have proven particularly useful for analyzing survival and failure rates. Statisticians have dedicated extensive efforts to understand the failure mechanisms of components, particularly within the complex systems of industrial and mechanical engineering. In these studies, operating units are observed until failure, their lifetimes are recorded, and statistical inference is applied to the resulting data. This process enables the estimation of reliability and hazard functions for the entire system. However, practical challenges arise when experimental units exhibit high reliability or involve high costs. In such scenarios, it becomes necessary to reduce both the number of units tested and the total experimental duration. To address these constraints, the Type-II PC scheme is employed. This framework allows for robust estimation while "saving" surviving units from failure for other uses. The Type-II PC scheme is implemented as follows: Initially, n independent and identical units are chosen for the test. Upon the occurrence of the first failure at time X_1, a set of R_1 units is randomly removed from the remaining $n - R_1 - 1$ survivors. At the second failure time X_2, a set of R_2 units are randomly withdrawn from the surviving $n - R_1 - R_2 - 2$. This procedure continues until the m-th failure (X_m) occurs, at which point all remaining $n - m - \sum_{i=1}^{m-1} R_i$ units are removed from the test. This experimental setup terminates at the m-th failure, as illustrated by Balakrishnan and Aggarwala [1]. The statistical literature extensively addresses inferential challenges associated with Type-II PC schemes. A significant portion of the literature focuses on the design of optimal censoring plans, for instance, Ng et al. [2] utilized expected Fisher information and asymptotic variance-covariance matrices to establish optimality for Weibull distributions. Other approaches incorporating random removal and applications to industrial and computer science data have been proposed by Fayomi et al [3]. Furthermore, researchers have extended these schemes to comparative lifetime experiments. Gao and Gui [4] developed statistical inference of the Burr-Hatke exponential distribution with a partially accelerated life test under a progressive Type-II scheme. Lee [5] developed an inference for parameters of the exponential distribution under a combined Type-II progressive hybrid censoring scheme. Mondal and Kundu [6] introduced a balanced two-sample scheme for Weibull populations. Novel complex models have also emerged, such as the six-parameter distribution developed by Sel et al. [7], which employs maximum likelihood, bootstrap, and Bayesian estimation. More recently, Goel and Krishna [8] proposed the progressive Type-II random censoring scheme for two Lindley distributions. The statistical inference and optimal censoring scheme of the competing-risks model under progressive Type-II censoring was investigated by Tian et al. [9]. Inferential techniques for the Burr XII lifetime model using progressive randomly censored samples were proposed by Goel et al. [10]. The review by Balakrishnan [11] remains the definitive resource. In practical applications involving competing-risks models, unit failures often result from several distinct causes. Measuring the specific risk of one failure cause relative to others has received significant attention in reliability literature. This area of study was first explored by Cox [12] using exponential distributions, with subsequent foundational properties established by Crowder [13] and Balakrishnan and Han [14]. Feizjavadian and Hashemi [15] and Ahmed et al. [16] proposed an inverse Lomax model with generalized Type-II hybrid censoring, offering practical applications in reliability studies. The model has been integrated into partially observed data contexts [17, 18]. Recent advancements in this field include the work of Nassar et al. [19] and Farghal et al. [20].

Let X be a random variable following the Kumaraswamy distribution (KuD) with positive shape parameters θ and η. Its cumulative distribution function (CDF) is defined as:

$$F(x) = 1 - (1 - x^{\eta})^{\theta}, \qquad 0 < x < 1, \quad \theta, \eta > 0. \tag{1.1}$$

This was originally proposed by Kumaraswamy [21] to model hydrological variables such as daily rainfall and streamflow. This model serves as a mathematically tractable alternative to the beta distribution. As noted by Nadarajah [22], the Kumaraswamy distribution is often preferred in hydrological sciences due to its superior flexibility and its explicit quantile function. Abou-Elheggag et al. [23] discussed estimation of $R = \mathrm{P}(Y < X)$ using k-upper record values. The distribution can take various shapes depending on its parameters: it is unimodal for $\theta > 1,\ \eta > 1$, uniantimodal for $\theta < 1,\ \eta < 1$, increasing for $\theta < 1,\ \eta \leq 1$, decreasing for $\theta \leq 1,\ \eta \leq 1$, and constant when $\theta = 1,\ \eta = 1$. Beyond hydrology, the distribution has been successfully applied to atmospheric temperatures and other natural phenomena. Its versatility allows it to approximate the uniform, triangular, and truncated normal distributions [24]. Motivated by its practical utility and analytical advantages, this paper investigates statistical inference for the Kumaraswamy distribution in the presence of competing risks. Specifically, we explore point and interval estimation under a Type-II PC scheme using both classical and Bayesian methodologies.

The novelty of this study lies in its integrated approach to handling partially observed competing risks within the framework of Type-II progressive censoring, specifically applied to the KuD, a configuration that remains largely unexplored in the reliability literature. While progressive censoring and competing-risks models have been widely studied using standard distributions such as the Weibull or exponential, most existing works assume that the causes of failure are fully known. In practice, however, diagnostic limitations or resource constraints often result in “masked” or partially observed failure causes. This research is motivated by the need for robust inferential frameworks capable of accommodating such data imperfections while preserving the flexibility required to model diverse failure time behaviors. The KuD is chosen for its unique combination of flexibility and mathematical simplicity. Unlike the beta distribution, which involves complex special functions (such as the incomplete beta function) in its cumulative and hazard functions, the KuD possesses closed-form expressions for both its CDF and probability density function (PDF). This property greatly enhances computational efficiency during MLE and MCMC simulation, making it particularly advantageous for inference in the presence of censored and partially observed data.

In this study, we assume that latent failure times follow a two-parameter KuD subject to two competing failure modes with partially observed causes. Our primary objective is to develop statistical inference for the KuD under a Type-II PC scheme. To this end, point estimation is explored using both maximum likelihood (ML) and Bayesian methods. Additionally, interval estimation is addressed via approximate ML-based intervals, the bootstrap technique, and Bayesian credible intervals. The proposed methodologies are evaluated through extensive numerical computations, including a Monte Carlo simulation study and the analysis of real-world data.

The subsequent sections of this paper are arranged as follows: Section 2 provides the model specification and core assumptions. Sections 3 and 4 detail the likelihood-based and parametric bootstrap techniques, respectively. Section 5 presents Bayesian estimation frameworks. In Section 6, the proposed methods are applied to two real-world datasets. A comparative analysis of these methodologies through comprehensive simulation studies is presented in Section 7. Finally, Section 8 provides concluding remarks for future work.

2. Model specification and basic assumptions

To formulate the model, assume n identical units are subjected to testing. The lifetimes X_i are independent and identically distributed (i.i.d.) random variables, where each failure is attributed to one of two competing risks. Thus, the observed failure times are $X_1, X_2, \ldots, X_n$. We denote the PDF, CDF, and survival function as $f(x)$, $F(x)$, and $S(x)$, respectively. The experimental design follows a Type-II PC framework, governed by a pre-specified monitoring point m and a predetermined censoring scheme $\mathbf{R}$, proposed to satisfy $n = m + \sum_{i=1}^{m} R_i$. During the experiment, at each observed failure time $X_{i;n}^{\mathbf{R}}$, the specific cause of failure ς_i is identified, and at the time of the i-th failure, R_i surviving units are randomly withdrawn from the experiment ($i = 1, 2, \ldots, m$). The resulting competing-risks Type-II PC sample is denoted as

$$\mathbf{x} = \left\{(x_{1;n}^{\mathbf{R}}, \varsigma_1), (x_{2;n}^{\mathbf{R}}, \varsigma_2), \ldots, (x_{m;n}^{\mathbf{R}}, \varsigma_m)\right\}.$$

Hence, under consideration are only two independent causes partially observed:

$$\varsigma_i = \{1, 2, 0\},\ i = 1, 2, \ldots, m,$$

where

$$\varsigma_i = \begin{cases} 1, \text{ means the failure occurs with respect to the first cause;} \\ 2, \text{ means the failure occurs with respect to the second cause;} \\ 0, \text{ means the failure occurs with respect to the unknown cause.} \end{cases}$$

The following points summarize the key characteristics and assumptions of the proposed model:

(1) Distributional choice and parameterization: We assume a common scale parameter and distinct shape parameters to ensure computational tractability, as the shape parameter exerts a more significant influence on the distribution's geometric profile. The CDF for the j-th failure cause is given by:

$$F_j(x) = 1 - (1 - x^{\eta})^{\theta_j}, \qquad 0 < x < 1, \quad \theta_j > 0,\ \eta > 0, \quad j = 1, 2. \tag{2.1}$$

(2) Latent failure times: For each unit $i = 1, 2, \ldots, m$, the observed lifetime is defined as $X_i = \min\{X_{i1}, X_{i2}\}$, where X_{ij} is the i-th failure time under cause j, the latent failure times associated with Cause 1 and 2, respectively.

(3) Total failure distribution: Given the independence of the competing risks, the observed failure time X_i also follows a Kumaraswamy distribution with the following survival function and PDF:

$$S(x) = \mathrm{P}(\min\{X_1, X_2\} > x) = \mathrm{P}(X_1 > x)\ \mathrm{P}(X_2 > x) = S_1(x) S_2(x),$$

$$\mathbf{S}(x) = (1 - x^{\eta})^{\theta_1+\theta_2}, \quad \mathbf{f}(x) = \eta(\theta_1 + \theta_2)x^{\eta-1}(1 - x^{\eta})^{\theta_1+\theta_2-1}, \quad 0 < x < 1,\ \theta_j, \eta > 0. \tag{2.2}$$

(4) Masked failure causes: Let m_0 denote the number of failures with masked (unknown) causes. The modeling of m_0 (the number of units with unknown or masked causes of failure) as the sum of independent Bernoulli random variables relies on the standard assumption that each failed unit's cause has a constant and independent probability of being recorded. We model m_0 as the sum of independent Bernoulli random variables, each with a success probability $P \in (0, 1)$.

(4) Distribution of observed counts: The observed number of failures attributed to the first and second causes (m_1, m_2, respectively) follows binomial distributions. Specifically, given $(m - m_0)$ non-masked failures, the distributions are:

(a) $m_1 \sim \text{Binomial}(m - m_0, \frac{\theta_1}{\theta_1+\theta_2})$;

(b) $m_2 \sim \text{Binomial}(m - m_0, \frac{\theta_2}{\theta_1+\theta_2})$.

The justification for using the binomial distribution is that the number of failures due to a particular cause may be modeled as a binomial random variable, conditioned on the number of observed failures and the probability of those causes. Such a model makes sense only under the assumption of independence and identical distribution of units, see Ahmed et al. [16] and Farghal et al. [20].

The joint likelihood function for the observed Type-II PC sample

$$\mathbf{x} = \{(x^{\mathbf{R}}_{1;n}, \varsigma_1), (x^{\mathbf{R}}_{2;n}, \varsigma_2), \ldots, (x^{\mathbf{R}}_{m;n}, \varsigma_m)\}$$

is formulated as follows:

$$L(\mathbf{x} \mid \Phi) \propto \prod_{i=1}^{m} \left\{ \left[f_1(x_i) S_2(x_i)\right]^{I(\varsigma_i=1)} \left[f_2(x_i) S_1(x_i)\right]^{I(\varsigma_i=2)} \right\} \left[\mathbf{f}(x_i)\right]^{I(\varsigma_i=0)} \left[\mathbf{S}(x_i)\right]^{R_i}, \tag{2.3}$$

where $\Phi = (\theta_1, \theta_2, \eta)$ denotes the parameter vector, $\mathbf{f}(\cdot)$ and $\mathbf{S}(\cdot)$ represent the PDF and the survival function of $X_i = \min(X_{i1}, X_{i2})$, respectively, given in Eq (2.2). Furthermore, the indicator function $\mathrm{I}(\varsigma_i = j)$ is defined as:

$$I(\varsigma_i = j) = \begin{cases} 1, & \varsigma_i = j, \\ 0, & \varsigma_i \neq j. \end{cases} \tag{2.4}$$

3. Classical inference

This section focuses on the point and interval estimation of the model parameters using the maximum likelihood estimation (MLE) method. We assume no specific order constraints on the parameters.

3.1. Maximum likelihood estimators

The MLEs of the KuD are derived by maximizing the likelihood function. For computational convenience, the likelihood function in Eq (2.3) is reformulated into the following compact expression:

$$L(\Phi \mid \mathbf{x}) = C\theta_1^{m_1}\theta_2^{m_2}\eta^m(\theta_1 + \theta_2)^{m_0} \prod_{i=1}^{m} x_i^{\eta-1}(1 - x_i^{\eta})^{(\theta_1+\theta_2)(R_i+1)-1}. \tag{3.1}$$

The log-likelihood function for the unknown parameters, denoted by $\ell(\Phi \mid \mathbf{x})$, is derived by taking the natural logarithm of Eq (3.1).

$$\begin{aligned} \ell(\Phi \mid \mathbf{x}) = {} & \log C + m_1 \log\theta_1 + m_2 \log\theta_2 + m \log\eta + m_0 \log(\theta_1 + \theta_2) \\ & + (\eta - 1)\sum_{i=1}^{m} \log x_i + (\theta_1 + \theta_2)\sum_{i=1}^{m}(R_i + 1)\log(1 - x_i^{\eta}) - \sum_{i=1}^{m} \log(1 - x_i^{\eta}). \end{aligned} \tag{3.2}$$

By equating the partial derivatives of $\ell(\Phi \mid \mathbf{x})$ with respect to the parameters and setting them to zero:

$$\frac{\partial \ell(\Phi \mid \mathbf{x})}{\partial \theta_j} = \frac{m_j}{\theta_j} + \frac{m_0}{\theta_1 + \theta_2} + \sum_{i=1}^{m} (R_i + 1) \log(1 - x_i^{\eta}) = 0 \tag{3.3}$$

and

$$\frac{\partial \ell(\Phi \mid \mathbf{x})}{\partial \eta} = \frac{m}{\eta} + \sum_{i=1}^{m} \log x_i - (\theta_1 + \theta_2) \sum_{i=1}^{m} \frac{(R_i + 1) x_i^{\eta} \log x_i}{1 - x_i^{\eta}} + \sum_{i=1}^{m} \frac{x_i^{\eta} \log x_i}{1 - x_i^{\eta}} = 0. \tag{3.4}$$

As shown in Eq (3.3), the ratio of the estimators θ_1 and θ_2 is given by:

$$\hat{\theta}_2 = \frac{m_2}{m_1} \times \hat{\theta}_1, \tag{3.5}$$

so that:

$$\hat{\theta}_j(\eta) = \frac{-m_j m}{(m_1 + m_2) \sum\limits_{i=1}^{m} (R_i + 1) \log(1 - x_i^{\eta})}, \qquad j = 1, 2. \tag{3.6}$$

Also, Eq (3.4) is reduced to

$$\hat{\eta} = \frac{m}{(\theta_1 + \theta_2) \sum\limits_{i=1}^{m} \frac{(R_i+1) x_i^{\eta} \log x_i}{1 - x_i^{\eta}} - \sum\limits_{i=1}^{m} \frac{x_i^{\eta} \log x_i}{1 - x_i^{\eta}} - \sum\limits_{i=1}^{m} \log x_i}. \tag{3.7}$$

Remark 3.1. The MLE of η, denoted as $\hat{\eta}$, is obtained via an iterative procedure. The process terminates when $|\eta^{(i+1)} - \eta^{(i)}| < \epsilon$, where ϵ is a pre-specified tolerance level ensuring convergence. Subsequently, the estimates for θ_1 and θ_2 (denoted as $\hat{\theta}_1$ and $\hat{\theta}_2$) are obtained by substituting $\hat{\eta}$ into Eq (3.6).

Remark 3.2.
(1) The standard competing-risks model is recovered when $m_0 = 0$, implying that all failure causes are perfectly observed.
(2) The model simplifies to a conventional single-cause lifetime distribution when $m_0 = 0$ and all failures are attributed to a single risk (i.e., $m_1 = m$ or $m_2 = m$).

Corollary 3.3. *To calculate the conditional estimator of η for a given Type-II PC sample under competing risks, we employ a fixed-point iteration approach, provided that the number of failures from each cause, m_j, is non-zero. The iterative process follows:*

$$\mathbf{x} = \{(x_{1;n}^{\mathbf{R}}, \delta_1), (x_{2;n}^{\mathbf{R}}, \delta_2), \ldots, (x_{m;n}^{\mathbf{R}}, \delta_m)\}$$

and with the conditional estimator of the parameter η given non-zero value m_j, $j = 1, 2$ is obtained with a fixed-point iteration with the iteration function given by

$$\eta^{(i+1)} = g(\eta^{(i)}), \tag{3.8}$$

with $g(\cdot)$ obtained from Eq (3.7). Given η, the conditional MLE for θ_j is then:

$$\hat{\theta}_j(\eta) = \frac{-m_j m}{(m_1 + m_2) \sum\limits_{i=1}^{m} (R_i + 1) \log(1 - x_i^{\eta})}, \quad j = 1, 2. \tag{3.9}$$

Proof. By differentiating the log-likelihood function $\ell(\Phi \mid \mathbf{x})$ in Eq (3.2) with respect to θ_j and setting the results to zero, we obtain:

$$\hat{\theta}_j(\eta) = \frac{-m_j m}{(m_1 + m_2)\sum_{i=1}^{m}(R_i + 1)\log(1 - x_i^{\eta})}, \quad j = 1, 2.$$

To confirm that $(\hat{\theta}_1, \hat{\theta}_2)$ maximizes $\ell(\Phi \mid \mathbf{x})$ for a given η, we consider the Hessian matrix $H(\theta_1, \theta_2)$. The elements of the Hessian are defined as:

$$H_{jj} = -\frac{m_j}{\theta_j^2} - \frac{m_0}{(\theta_1 + \theta_2)^2}, \qquad H_{12} = -\frac{m_0}{(\theta_1 + \theta_2)^2}.$$

The determinant at the critical point $(\hat{\theta}_1, \hat{\theta}_2)$ is

$$D(H(\hat{\theta}_1, \hat{\theta}_2)) = \frac{m_1 m_2}{\theta_1^2 \theta_2^2} + \frac{m_1 m_0}{\theta_1^2(\theta_1 + \theta_2)^2} + \frac{m_2 m_0}{\theta_2^2(\theta_1 + \theta_2)^2} \geq 0.$$

As the determinant is positive and $H_{jj} < 0$, the critical point is a local maximum. Since $\ell(\Phi \mid \mathbf{x})$ is a smooth function with a single critical point $(\hat{\theta}_1, \hat{\theta}_2)$, is provides the absolute maximum, completing the proof. Therefore, as given in Wang et al. [25], the proof is satisfied. □

Corollary 3.4 (Analytical results for the existence and uniqueness of MLEs)**.** *Having confirmed the existence and uniqueness of the MLE for θ_j, $j = 1, 2$, the estimates can be computed by solving the following expression:*

$$\frac{\partial \ell(\Phi \mid \mathbf{x})}{\partial \eta} = \frac{m}{\eta} + \sum_{i=1}^{m} \log x_i - (\theta_1 + \theta_2)\sum_{i=1}^{m} \frac{(R_i + 1)x_i^{\eta} \log x_i}{1 - x_i^{\eta}} + \sum_{i=1}^{m} \frac{x_i^{\eta} \log x_i}{1 - x_i^{\eta}} = 0. \tag{3.10}$$

Proof. This section establishes the necessary and sufficient conditions for the existence and uniqueness of the MLEs based on an arbitrary competing-risks Type-II PC sample.

Case 1: Shap parameter η is known.

When the shape parameter η is known and takes a fixed positive value, the shape parameter estimates $\hat{\theta}_j$ for $j = 1, 2$, as derived in Eq (3.6), is guaranteed to exist, be unique, and remain finite.

Case 2: The shape parameter θ_j is known.

When the shape parameter θ_j $(j = 1, 2)$ is fixed, the second derivative of the log-likelihood in Eq (3.2) is shown to be strictly less than zero. This implies that the log-likelihood function $\ell(\Phi \mid \mathbf{x})$ in Eq (3.2) is strictly concave with respect to η. Furthermore, the plot of the log-likelihood function $\ell(\Phi \mid \mathbf{x})$ (see Figure 1) clearly illustrates its concavity with respect to η. Consequently, if a solution for η exists, it must be unique. To verify existence, we examine the behavior of the score function $\frac{\partial \ell(\Phi|\mathbf{x})}{\partial \lambda}$ at the boundaries:

- As $\eta \to 0$,

$$\frac{\partial \ell(\Phi \mid \mathbf{x})}{\partial \eta} = \infty.$$

- As $\eta \to \infty$,

$$\frac{\partial \ell(\Phi \mid \mathbf{x})}{\partial \eta} = \sum_{i=1}^{m} \log x_i < 0.$$

For any sample size $m > 1$, the score function transitions from positive to negative values. By the intermediate value corollary, a finite and unique estimate for η exists.

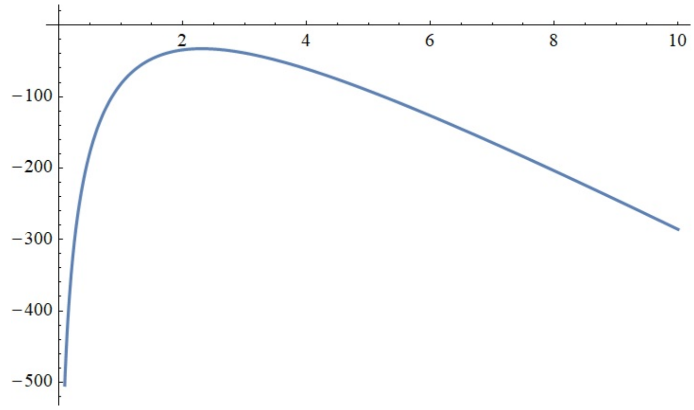

Figure 1. The profile log-likelihood function of η.

Case 3: All parameters are unknown.

In the case where all parameters are unknown, the MLE for η is obtained as the solution to the profile likelihood equation based on Eq (3.7). In this case, for any given η, the MLEs of θ_j ($j = 1, 2$) are determined by Eq (3.7). This inferential problem requires $m \geq 2$ and $m_1, m_2 > 1$. By substituting the expressions for $\theta_j(\eta)$ into the score equation $\frac{\partial \ell(\eta|\mathbf{x})}{\partial \eta}$, we observe the following limiting behavior on the interval $(0, \infty)$:

- $\lim\limits_{\eta \to 0} \dfrac{\partial \ell(\Phi \mid \mathbf{x})}{\partial \eta} = \infty$,
- $\lim\limits_{\eta \to \infty} \dfrac{\partial \ell(\Phi \mid \mathbf{x})}{\partial \eta} = \sum\limits_{i=1}^{m} \log x_i$.

Since $\frac{\partial \ell(\Phi|\mathbf{x})}{\partial \eta}$ is a continuous and strictly decreasing function that crosses zero, it possesses at least one finite positive root. Furthermore, because

$$\begin{aligned}\frac{\partial^2 \ell(\Phi \mid \mathbf{x})}{\partial \eta^2} &= -\frac{m}{\eta^2} - (\theta_1 + \theta_2) \sum_{i=1}^{m} \left[\frac{(R_i + 1) x_i^{\eta} \log^2 x_i}{1 - x_i^{\eta}} + \frac{(R_i + 1) x_i^{2\eta} \log^2 x_i}{(1 - x_i^{\eta})^2} \right] \\ &+ \sum_{i=1}^{m} \left[\frac{x_i^{\eta} \log^2 x_i}{1 - x_i^{\eta}} + \frac{x_i^{2\eta} \log^2 x_i}{(1 - x_i^{\eta})^2} \right] < 0,\end{aligned}$$

for $\eta > 0$, the root is unique. Therefore, the MLE of η is finite, unique, and exists if and only if $m \geq 2$. Similarly, θ_j exists and is unique provided that $m_1, m_2 > 1$, for more details see Dey and Dey [26]. □

3.2. Approximate confidence intervals based on MLEs

To construct interval estimates, we rely on the large-sample properties of the MLE $\hat{\Phi}$. The Fisher information matrix $M(\Phi)$ typically characterizes the asymptotic precision of these estimates. Since the exact expectation

$$-E\left(\frac{\partial^2 \ell(\Phi \mid \mathbf{x})}{\partial^2 \Phi}\right)$$

is often analytically complex, we utilize the observed Fisher information matrix,

$$M(\Phi) = [D_{ij}], \quad D_{ij} = -\frac{\partial^2 \ell(\Phi \mid \mathbf{x})}{\partial \Phi_i \partial \Phi_j}, \quad i, j \in \{1, 2, 3\},$$

as an approximation. This matrix consists of the second-order partial derivatives of the log-likelihood function evaluated at the MLEs. By inverting $M(\Phi)$, we obtain the asymptotic variance-covariance matrix, from which the $100(1-\alpha)\%$ confidence intervals (CIs) can be derived.

The construction of approximate confidence intervals (ACIs) for the unknown model parameters are based on the asymptotic properties of the MLEs. Under standard regularity conditions, it can be established that for a large sample size n, the estimators follow an asymptotic multivariate normal distribution:

$$\sqrt{n}\left(\hat{\Phi} - \Phi\right) \to N(0, M^{-1}(\Phi)),$$

where $M^{-1}(\Phi)$ represents the asymptotic variance-covariance matrix, which is approximated by the inverse of the observed Fisher information matrix evaluated at the MLEs, denoted as $M^{-1}(\hat{\Phi})$. This matrix is given by:

$$M^{-1}(\hat{\Phi}) = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}. \tag{3.11}$$

Applying Slutsky's corollary, the pivotal quantities

$$Z_{\Phi_i} = \frac{\hat{\Phi}_i - \Phi_i}{\sqrt{v_{ij}}}, \qquad i, j \in \{1, 2, 3\},$$

converge in distribution to a standard normal distribution, $N(0, 1)$. Consequently, the two-sided

$$100(1-\alpha)\%$$

ACIs for the parameters are calculated as:

$$\hat{\Phi}_i \mp Z_{\alpha/2} \sqrt{v_{ii}}, i, \quad i \in \{1, 2, 3\}, \quad \hat{\Phi}_i = \hat{\theta}_1,\ \hat{\theta}_2,\ \text{or } \hat{\eta}, \tag{3.12}$$

where v_{ii} represents the asymptotic variance of the i-th parameter and the percentile value, and $Z_{\alpha/2}$ is the upper $\alpha/2$-th percentile of the standard normal distribution. Based on Eq (3.2), the elements of the observed Fisher information matrix are determined by the following second-order partial derivatives:

$$\begin{aligned}
D_{11} &= -\frac{\partial^2 \ell(\Phi \mid \mathbf{x})}{\partial \theta_1^2} = \frac{m_1}{\theta_1^2} + \frac{m_0}{(\theta_1 + \theta_2)^2}, \\
D_{12} &= -\frac{\partial^2 \ell(\Phi \mid \mathbf{x})}{\partial \theta_1 \partial \theta_2} = \frac{m_0}{(\theta_1 + \theta_2)^2}, \\
D_{13} &= -\frac{\partial^2 \ell(\Phi \mid \mathbf{x})}{\partial \theta_1 \partial \eta} = \sum_{i=1}^{m} \frac{(R_i + 1) x_i^\eta \log x_i}{1 - x_i^\eta}, \\
D_{22} &= -\frac{\partial^2 \ell(\Phi \mid \mathbf{x})}{\partial \theta_2^2} = \frac{m_2}{\theta_2^2} + \frac{m_0}{(\theta_1 + \theta_2)^2},
\end{aligned}$$

$$D_{23} = -\frac{\partial^2 \ell(\Phi \mid \mathbf{x})}{\partial\theta_2 \partial\eta} = \sum_{i=1}^{m} \frac{(R_i + 1)x_i^{\eta} \log x_i}{1 - x_i^{\eta}},$$

$$D_{33} = -\frac{\partial^2 \ell(\Phi \mid \mathbf{x})}{\partial\eta^2} = \frac{m}{\eta^2} + (\theta_1 + \theta_2) \sum_{i=1}^{m} \left[\frac{(R_i + 1)x_i^{\eta} \log^2 x_i}{1 - x_i^{\eta}} + \frac{(R_i + 1)x_i^{2\eta} \log^2 x_i}{(1 - x_i^{\eta})^2} \right]$$

$$- \sum_{i=1}^{m} \frac{x_i^{\eta} \log^2 x_i}{1 - x_i^{\eta}} - \sum_{i=1}^{m} \frac{x_i^{2\eta} \log^2 x_i}{(1 - x_i^{\eta})^2}.$$

4. Parametric percentile bootstrap

In statistical literature, the bootstrap method is widely used for problems involving hypothesis testing and interval estimation. Here, the percentile parametric bootstrap-p technique is utilized to establish approximate confidence intervals for the parameters of interest. Detailed discussions on this approach can be found in Efron and Tibshirani [27]. The specific algorithm for obtaining these bootstrap estimates, based on the procedures, is presented as follows:

Algorithm 1.

Step 1: To generate a Type-II PC competing-risks sample from KuD (θ_1, η) and KuD (θ_2, η), the following procedure is implemented:

(1) Generate uniform samples: Generate m i.i.d. observations

$$w_1, w_2, \ldots, w_m$$

from a standard uniform $(0, 1)$ distribution.

(2) Apply power transformation: Transform the uniform variables into v_i, $i = 1, 2, \ldots, m$, using the following transformation based on the censoring scheme $(R_1, R_2, \ldots, R_m)$:

$$v_i = w_i^{1/(i + \sum_{j=m-i+1}^{m} R_j)}.$$

(3) Obtain uniform progressive samples: Calculate the Type-II PC uniform sample $\{u_1, u_2, \ldots, u_m\}$, where

$$u_i = 1 - \prod_{j=1}^{m-i+1} v_j.$$

(4) Inverse transform to the KuD: Using the shape and scale parameters θ_1, θ_2, and η, convert the uniform samples into KuD lifetimes:

$$x_i = \left[1 - (1 - u_i)^{(1/\theta_s)}\right]^{(1/\eta)}, \qquad s = 1, 2,\ i = 1, 2, \ldots, m.$$

(5) Assign competing risks: For each i, determine the observed failure time and cause indicator ς_i as follows:

$$x_{i1} \le x_{i2},\ \text{set } x_i = x_{i1},\ \varsigma_i = 1;$$
$$x_{i1} \ge x_{i2},\ \text{set } x_i = x_{i2},\ \varsigma_i = 2.$$

Using the Type-II PC sample

$$\mathbf{x} = \{(x_{1;n}^{\mathbf{R}}, \varsigma_1), (x_{2;n}^{\mathbf{R}}, \varsigma_2), \ldots, (x_{m;n}^{\mathbf{R}}, \varsigma_m)\},$$

we compute the MLEs for the parameter vector $\hat{\Phi} = \{\hat{\theta}_1, \hat{\theta}_2, \hat{\eta}\}$.
(6) Calculate failure counts: Compute the total number of failures for each cause, using the indicator function

$$m_s = \sum_{i=1}^{m} I(\varsigma_i = s), \qquad s = 1, 2.$$

Remark 4.1. Alternatively, the progressively censored failure times $\{x^{\mathbf{R}}_{1;n}, x^{\mathbf{R}}_{2;n}, ..., x^{\mathbf{R}}_{m;n}\}$ can be simulated directly from the KuD with two shape parameters $(\theta_1 + \theta_2)$ and η, followed by a random assignment of causes based on the probability

$$\frac{\theta_s}{(\theta_1 + \theta_2)}.$$

Step 2: To simulate the Type-II PC data under competing risks, we generate a sample

$$\mathbf{x}^* = \{x^*_1, x^*_2, \ldots, x^*_m\}$$

from the KuD $f(\cdot|\hat{\theta}_1, \hat{\theta}_2, \hat{\eta})$. This generation is based on the pre-specified parameter values $\Phi = (\theta_1, \theta_2, \eta)$ and the progressive censoring parameters n, m, and $\mathbf{R}$.
Step 3: To account for partially observed data, we introduce a masking probability p. Each failure is assigned a status via a Bernoulli trial, where 1 denotes a masked cause. The total number of masked cases, m_0, is therefore calculated as $\sum_{i=1}^{m} B_i$, where $B_i \sim \text{Bernoulli}(p)$, effectively making m_0 a binomial random variable with parameters m and p.
Step 4: Let $m - m_0$ represent the number of failures where the cause is observed. Given the independence of the competing risks, the relative probabilities of failure for each cause are

$$P_1 = \frac{\theta_1}{\theta_1 + \theta_2}, \quad P_2 = \frac{\theta_2}{\theta_1 + \theta_2}.$$

Accordingly, the observed counts m_1 and m_2 are sampled from binomial distributions with parameters $(m - m_0, P_1)$ and $(m - m_0, P_2)$ ensuring that

$$m_1 + m_2 = m - m_0.$$

Step 5: Iterate steps 2 through 4 N times to obtain the required sample size.
Step 6: Compute the bootstrap MLEs $\hat{\theta}^*_1, \hat{\theta}^*_2,$ and $\hat{\eta}^*$ using the generated sample $\mathbf{x}^*$.
Step 7: Generate N such bootstrap samples to obtain

$$(\hat{\Phi}^{*(1)}, \hat{\Phi}^{*(2)}, \ldots, \hat{\Phi}^{*(N)}).$$

Step 8: Arrange the bootstrap estimates in ascending order. The final point estimates are defined as the arithmetic mean of the bootstrap samples:

$$\hat{\Phi}^* = E(\hat{\Phi}^*) = \frac{1}{N} \sum_{i=1}^{N} \hat{\Phi}^{*(i)}. \tag{4.1}$$

The approximate $100(1 - \alpha)\%$ percentile bootstrap intervals for $\Phi = (\theta_1, \theta_2, \eta)$ are expressed as:

$$CI_{\Phi} = (\hat{\Phi}^*_{(L)}, \hat{\Phi}^*_{(U)}), \tag{4.2}$$

where $L = N\alpha/2$ and $U = N(1 - \alpha/2)$.

5. Bayesian inference

In this section, we describe the Bayesian estimation procedure for the model parameters $\Phi = (\theta_1, \theta_2, \eta)$. Since the joint posterior distribution is mathematically complex and does not admit a closed-form expression, we employ MCMC techniques, specifically the Metropolis-Hastings (MH) algorithm, to generate posterior samples and compute the required estimates. In this study, the unknown parameters $(\theta_1, \theta_2, \eta)$ are assumed to be independent random variables following gamma prior distributions with hyperparameters (a_i, b_i), $i = 1, 2, 3$, and thus we have:

$$P_i(\Phi_i \mid a_i, b_i) = \frac{b_i^{a_i}}{\Gamma(a_i)} \Phi_i^{a_i - 1} e^{-b_i \Phi_i}. \tag{5.1}$$

The gamma distribution has the maximum entropy of all distributions. This distribution is used in many fields, including business, science, and engineering, because of its positive skewness. Additionally, this distribution is quite flexible and can be used for modeling continuous random variables.

By combining the individual prior densities, the joint prior density function is obtained as:

$$P(\Phi) = \theta_1^{a_1 - 1} \theta_2^{a_2 - 1} \eta^{a_3 - 1} e^{-(b_1\theta_1 + b_2\theta_2 + b_3\eta)}, \quad \theta_1, \theta_2, \eta > 0, \quad a_i, b_i > 0. \tag{5.2}$$

By applying Bayes' corollary, the likelihood function from Eq (3.1) is combined with the joint prior density in Eq (5.2) to derive the joint posterior distribution. The resulting posterior density function is expressed as follows:

$$\begin{aligned} P^*(\Phi \mid \mathbf{x}) \propto \exp\left\{ -b_1\theta_1 - b_2\theta_2 - b_3\eta + (\theta_1 + \theta_2) \sum_{i=1}^{m} (R_i + 1) \log(1 - x_i^{\eta}) - \sum_{i=1}^{m} \log(1 - x_i^{\eta}) + \eta \sum_{i=1}^{m} \log x_i \right\} \\ \times \theta_1^{m_1 + a_1 - 1} \theta_2^{m_2 + a_2 - 1} \eta^{m + a_3 - 1} (\theta_1 + \theta_2)^{m_0}. \end{aligned} \tag{5.3}$$

Considering both the symmetric squared error (SE) loss function and the asymmetric linear exponential (LINEX) loss function, the Bayes estimators of the parameters $(\theta_1, \theta_2, \eta)$ are obtained by minimizing the posterior risk. These estimators are given by:

$$\hat{\Phi}_{SE} = E(\Phi \mid \mathbf{x}) = \frac{\int_\Phi \Phi \, P^*(\Phi \mid \mathbf{x}) \, d\theta_1 \, d\theta_2 \, d\eta}{\int_\Phi P^*(\Phi \mid \mathbf{x}) \, d\theta_1 \, d\theta_2 \, d\eta}; \tag{5.4}$$

$$\hat{\Phi}_{LINEX} = -\frac{1}{c} \log[E(e^{-c\Phi} \mid \mathbf{x}) = -\frac{1}{c} \log\left(\frac{\int_\Phi e^{-c\Phi} \, P^*(\Phi \mid \mathbf{x}) \, d\theta_1 \, d\theta_2 \, d\eta}{\int_\Phi P^*(\Phi \mid \mathbf{x}) \, d\theta_1 \, d\theta_2 \, d\eta} \right). \tag{5.5}$$

As the Bayes estimators under the SE and LINEX loss functions lack explicit analytical solutions, they must be evaluated numerically. Specifically, a hybrid Metropolis-within-Gibbs sampling scheme is utilized to generate posterior samples for $(\theta_1, \theta_2, \eta)$. Following the methodologies of Ahmed and Mustafa [28], Fathi et al. [29], Mustafa et al. [30], and Ahmed et al. [31], an MCMC technique is implemented to simulate the empirical posterior distribution. This procedure is utilized to obtain both point estimates and Bayesian credible intervals for the parameters of interest. The full conditional posterior densities required for this algorithm are expressed as:

$$P^*_{\theta_1}(\theta_1 \mid \theta_2, \eta, \mathbf{x}) \propto \theta_1^{m_1+a_1-1}\, e^{-\theta_1\left[b_1-\sum_{i=1}^m (R_i+1)\log(1-x_i^\eta)\right]}$$
$$\sim \text{Gamma}\Big(m_1 + a_1,\ b_1 - \sum_{i=1}^{m}(R_i + 1)\log(1 - x_i^\eta)\Big), \tag{5.6}$$

$$P^*_{\theta_2}(\theta_2 \mid \theta_1, \eta, \mathbf{x}) \propto \theta_2^{m_2+a_2-1}\, e^{-\theta_2\left[b_2-\sum_{i=1}^m (R_i+1)\log(1-x_i^\eta)\right]}$$
$$\sim \text{Gamma}\Big(m_2 + a_2,\ b_2 - \sum_{i=1}^{m}(R_i + 1)\log(1 - x_i^\eta)\Big), \tag{5.7}$$

$$P^*_\eta(\eta \mid \theta_1, \theta_2, \mathbf{x}) \propto \frac{\eta^{m+a_3-1}\, e^{-\eta\left[b_3-\sum_{i=1}^m \log x_i\right]}}{\Big(b_1 - \sum_{i=1}^m (R_i + 1)\log(1 - x_i^\eta)\Big)^{m_1}\Big(b_1 - \sum_{i=1}^m (R_i + 1)\log(1 - x_i^\eta)\Big)^{a_2}}, \tag{5.8}$$

and

$$W(\theta_1, \theta_2, \eta \mid \mathbf{x}) = \frac{(\theta_1 + \theta_2)^{m_0}\, e^{-\log(1-x_i^\eta)}}{\Big(b_1 - \sum_{i=1}^m (R_i + 1)\log(1 - x_i^\eta)\Big)^{a_1}\Big(b_1 - \sum_{i=1}^m (R_i + 1)\log(1 - x_i^\eta)\Big)^{m_2}}. \tag{5.9}$$

The weighting function from Eq (5.9) uses these weights to calculate Bayes point estimates under both SE and LINEX loss functions (Eqs (5.10) and (5.11)). Specifically, it clarifies that since the hybrid sampling approach (MH and gamma) might not exactly follow the joint posterior, the weights act as a correction factor to ensure the final estimates are unbiased and accurate. Because the conditional posterior distribution in Eq (5.8) cannot be reduced to a known distribution for direct sampling, its graph (see Figure 2) indicates that it is semi-normal. So the MH algorithm is employed to obtain draws for the parameter η. The algorithm proceeds as follows.

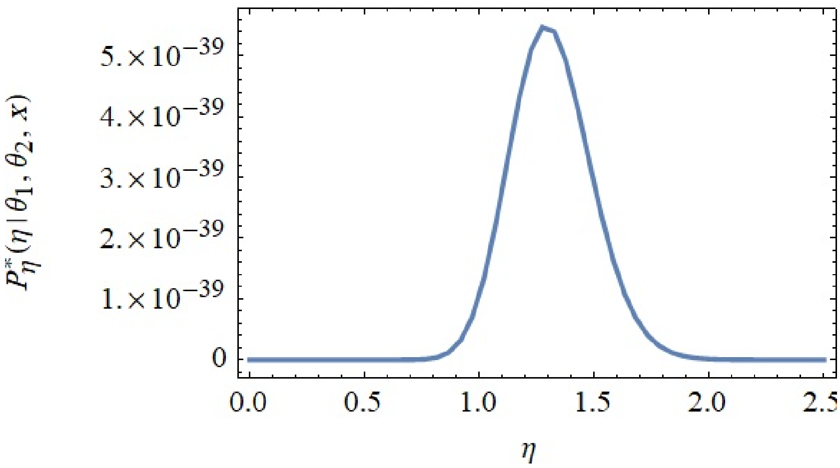

Figure 2. The conditional posterior $P^*_\eta(\eta \mid \theta_1, \theta_2, \mathbf{x})$ of η.

Algorithm 2. The importance sampling technique

Step 1: Initialization.

Set initial values for the parameters $(\theta_1^{(0)}, \theta_2^{(0)}, \eta^{(0)})$ and let $i = 1$. The MLEs are typically used as the starting point.

Step 2: Update η.

Generate a candidate $\eta^{(i)}$ from the conditional posterior $P_\eta^*(\eta \mid \theta_1, \theta_2, \mathbf{x})$ using the MH algorithm with a normal proposal distribution $N(\eta^{(i-1)}, v_{33})$.

• Calculate the acceptance probability:

$$q(\eta) = \min\left[1,\ \frac{P_\eta^*(\eta^* \mid \theta_1^{(i-1)}, \theta_2^{(i-1)}, \mathbf{x})}{P_\eta^*(\eta^{(i-1)} \mid \theta_1^{(i-1)}, \theta_2^{(i-1)}, \mathbf{x})}\right].$$

• Generate $U \sim \text{Uniform}(0,1)$. If $U \leq q(\eta)$, set $\eta^{(i)} = \eta^*$; otherwise, set $\eta^{(i)} = \eta^{(i-1)}$.

Step 3: Update θ_j.

Generate a candidate $\theta_j^{(i)}$, $j = 1, 2$, from the gamma conditional posterior

$$\theta_j^{(i)} \sim \Gamma\Big(m_j + a_j,\ b_j - \sum_{i=1}^{m}(R_i + 1)\log(1 - x_i^\eta)\Big).$$

Step 4: Iteration.

Set $i = i + 1$ and repeat Steps 2-3 N times to obtain the sequence of samples.

Step 5: Burn-in.

To ensure the chains have converged, discard the first M samples as burn-in. The remaining $N - M$ samples are used for inference.

Step 6: Point estimation.

The Bayes estimates under SE and LINEX loss functions for any parameter $\Phi = (\theta_1, \theta_2, \eta)$ are computed as:

$$\hat{\Phi}_{SE} = \frac{\sum_{i=M+1}^{N} \Phi^{(i)}\, W(\Phi^{(i)} \mid \mathbf{x})}{\sum_{i=M+1}^{N} W(\Phi^{(i)} \mid \mathbf{x})}, \tag{5.10}$$

$$\hat{\Phi}_{LINEX} = -\frac{1}{c}\log\left(\frac{\sum_{i=M+1}^{N} e^{-c\Phi^{(i)}}\, W(\Phi^{(i)} \mid \mathbf{x})}{\sum_{i=M+1}^{N} W(\Phi^{(i)} \mid \mathbf{x})}\right). \tag{5.11}$$

Step 7: Under the squared error loss function, the posterior variance for each parameter is computed as:

$$\hat{V}(h(\Phi)) = \frac{\sum_{J=M+1}^{N}\big(h(\Phi^{(J)}) - \hat{h}\big)^2\, W(\Phi^{(J)} \mid \mathbf{x})}{\sum_{J=M+1}^{N} W(\Phi^{(J)} \mid \mathbf{x})}. \tag{5.12}$$

Step 8: Symmetric Bayesian credible intervals.

We compute the $100(1-\alpha)\%$ symmetric credible intervals for the model parameters $\Phi = \{\theta_1, \theta_2, \eta\}$ and their functions $h(\Phi)$ using the ordered MCMC samples $\{\Phi^{(M+1)}, \ldots, \Phi^{(N)}\}$. As discussed in Chen and Shao [32] and Soliman et al. [33], these intervals are formulated by taking the lower and upper ascending-order $\alpha/2$-th percentiles of the generated samples. The procedure for any function $h(\Phi)$ follows the algorithm detailed below:

Algorithm 3. MCMC-based posterior interval construction
Step 1: Compute $h^{(i)}(\theta_1^{(i)}, \theta_2^{(i)}, \eta^{(i)})$ for each MCMC iteration and normalize the importance weights

$$w^{(i)} = \frac{W(\theta_1^{(i)}, \theta_2^{(i)}, \eta^{(i)} \mid \mathbf{x})}{\sum\limits_{i=M+1}^{N} W(\theta_1^{(i)}, \theta_2^{(i)}, \eta^{(i)} \mid \mathbf{x})}.$$

Step 2: Order the pairs $(h_{(i)}, w_{(i)})$ increasingly, which can be defined by

$$\hat{h}^{(\alpha)} = \begin{cases} h_{(1)}, & \text{if } \alpha = 0, \\ h_{(j)}, & \text{if } \sum_{i=1}^{j-1} w^{(i)} < \alpha < \sum_{i=1}^{j} w^{(i)}. \end{cases} \tag{5.13}$$

Step 3: For the $100(1-\alpha)\%$ symmetric (equal-tailed) credible interval for the parameter function $h(\theta_1, \theta_2, \eta)$, we utilize the $(\alpha/2)$, $(1-\alpha/2)$ empirical quantiles:

$$(\hat{h}^{(\alpha/2)},\ \hat{h}^{(1-\alpha/2)}). \tag{5.14}$$

6. Applications to real-world data

This section highlights the practical significance of our theoretical results through illustrative examples from the medical and industrial fields. The analysis of two empirical datasets is provided to support the validity of the proposed point and interval estimators. These applications confirm that the derived methodologies are well-suited for practical reliability analysis.

6.1. Application to male mice

In this application, the significance of the developed theoretical results is demonstrated through the analysis of radiation-exposed male mice data given by Hoel [34]. The data detailed in Table 1 attribute failure to two competing causes: reticulum cell sarcoma (Cause 1) or other causes (Cause 2). To validate the use of the KuD distribution, the data were first transformed via $Y = X/1000$. The appropriateness of the KuD for the transformed data was confirmed visually, by comparing the fitted and empirical survival functions, see Figure 3. The KuD is first fitted to the dataset. For comparison purposes, the Weibull distribution and generalized power unit half-logistic geometric distribution are also fitted. The competing models are evaluated using the following criteria:
(1) (-2 log)-likelihood:

$$-2\log L = -2\sum_{i=1}^{n} \log f(x_i \mid \hat{\theta}).$$

(2) Akaike information criterion (AIC):

$$AIC = -2\log L + 2k.$$

(3) Bayesian information criterion (BIC):

$$BIC = -2\log L + k\log n.$$

(4) Kolmogorov-Smirnov (K-S) statistic:

$$KS = \sup_x \left| F_n(x) - \hat{F}(x \mid \hat{\theta}) \right|,$$

with its corresponding p-value, where n is the sample size, k is the number of free parameters, $f(x \mid \hat{\theta})$ is the fitted density, and $\hat{F}(x \mid \hat{\theta})$ is the fitted CDF.

Table 1. Autopsy data for 61 mice describe the failure times of male mice that received a 300-R radiation dose at 5–6 weeks of age.

Reticulum cell sarcoma	317	318	399	495	525	536	549	552
	554	552	558	571	586	594	596	605
	612	621	628	631	636	643	647	648
	649	661	663	666	670	695	697	700
	705	712	713	738	748	753		
Other causes as the second cause	40	42	51	62	163	179	206	222
	228	252	249	282	324	333	341	366
	385	407	420	431	441	461	462	482
	517	517	524	564	567	586	619	620
	621	622	647	651	686	761	763	

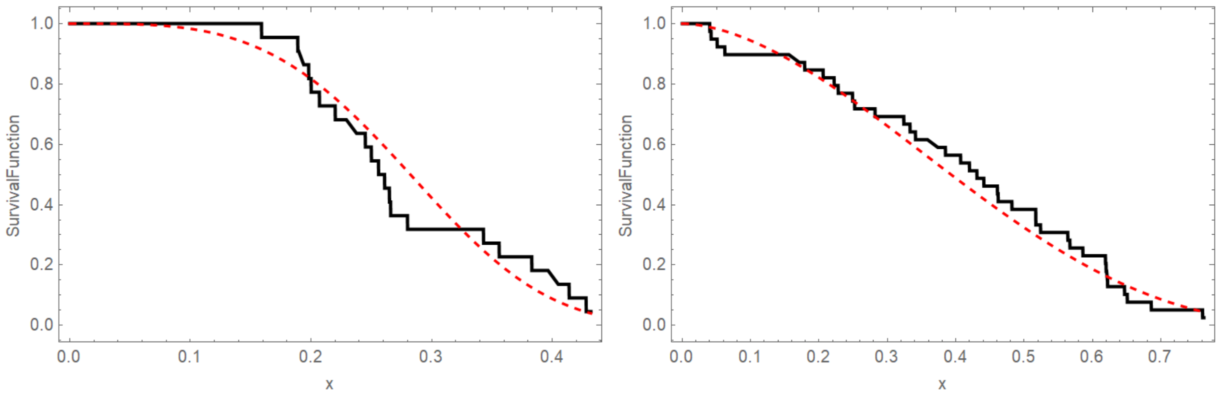

Figure 3. Empirical and fitted survival functions for Cause 1 and 2 of male mice data.

Model adequacy is indicated by smaller values of $-2\log L$, AIC, and BIC, together with a smaller K-S statistic and a larger associated p-value. The corresponding goodness of fit results are presented in Table 2. Table 2 indicates that the KuD provides the best fit among the considered models. It is clear that the KuD has the smallest $-2\log L$, AIC, BIC, and K-S statistic when $p = 0.15$, $m = 50$, and $R = (1^7, 0^8, 1^7, 0^8, 1^7, 0^7, 1^6)$. Using these data, we obtained the MLEs and Bayesian estimates for the parameters of the KuD, as shown in Table 3. We generated 10,000 MCMC samples using the previously described algorithm. The MLEs of the parameters were used as initial values for the unknown parameters in the MCMC sampling procedure. Table 3 also presents the two-sided asymptotic confidence intervals and credible intervals for the estimates, along with some optimality measures. This validated dataset is now used to demonstrate the application of our proposed Type-II PC sample. From this data:

(0.04,0)	(0.042,2)	(0.051,2)	(0.062,2)	(0.163,0)	(0.179,2)
(0.206,2)	(0.222,0)	(0.228,2)	(0.249,2)	(0.252,0)	(0.282,2)
(0.317,1)	(0.318,0)	(0.324,2)	(0.333,2)	(0.366,2)	(0.385,2)
(0.399,1)	(0.407,1)	(0.42,2)	(0.431,2)	(0.441,2)	(0.461,2)
(0.462,2)	(0.482,2)	(0.495,0)	(0.524,1)	(0.525,1)	(0.536,1)
(0.549,1)	(0.552,1)	(0.554,1)	(0.558,0)	(0.612,1)	(0.621,1)
(0.621,1)	(0.628,1)	(0.647,1)	(0.649,1)	(0.663,1)	(0.666,1)
(0.67,1)	(0.686,1)	(0.7,0)	(0.713,1)	(0.738,1)	(0.748,1)
(0.761,2)	(0.763,1).				

Table 2. Goodness-of-fit comparison under complete data.

Model	Data	-2 log L	AIC	BIC	p-value	K-S
Kumaraswamy	Reticulum cell sarcoma	-73.46	-69.46	-66.1849	0.9801	0.0762
	Other causes as the second cause	-16.7439	-12.7439	-9.41675	0.9233	0.088
Weibull	Reticulum cell sarcoma	-73.1522	-69.1522	-65.877	0.9866	0.0734
	Other causes as the second cause	-12.432	-8.43196	-5.10484	0.8109	0.1021
Generalized power unit	Reticulum cell sarcoma	-10.7832	-6.7832	-3.50803	0.7799	0.1067
half-logistic geometric	Other causes as the second cause	45.2134	49.2134	52.5405	0.5626	0.1263

Table 3. ML, bootstrap, and Bayes estimates, as well as 95% CIs for the model parameters using male mice data under different loss functions.

	θ_1	θ_2	η
MLE	1.4747	1.2182	2.1212
ACI	(0.6335, 2.3159)	(0.4867, 1.9497)	(1.5158, 2.7265)
boot	1.6060	1.3214	2.1962
boot CI	(0.8991, 2.8095)	(0.7006, 2.3856)	(1.6798, 2.9171)
Bayes SE	1.2590	1.4077	2.0092
Credibl interval	(0.6784, 1.8999)	(0.7829, 2.1220)	(1.5355, 2.3426)
Bayes LINEX ($c = -2$)	1.3660	1.5335	2.0556
Bayes LINEX ($c = 2$)	1.1589	1.2929	1.9564

In this framework, the indicator "1" denotes the first cause of failure, "2" represents the second cause, and "0" signifies an unobserved (masked) cause. Within this dataset, the observed frequencies are $m_0 = 8$, $m_1 = 23$, $m_2 = 19$, $m = 50$. Subsequently, the data are analyzed under the proposed competing-risks model considering these two primary causes of death.

Estimation and computational details:

(1) MLE: The MLEs for the model parameters $(\theta_1, \theta_2, \eta)$ were computed using an iterative numerical optimization method.

(2) Bayesian estimation: For the Bayesian analysis, we assigned non-informative gamma priors to the parameters, with the hyperparameters set to $a_i = b_i = 0.0001$ for $i = 1, 2, 3$. The posterior distributions were then sampled using an MCMC algorithm. We ran the MCMC chain for 11,000 iterations and discarded the first 1000 as a burn-in period to ensure convergence. To assess the convergence of

the empirical posterior distributions, trace plots and histograms of the generated MCMC samples for $(\theta_1, \theta_2, \eta)$ are provided in Figures 4 and 5.

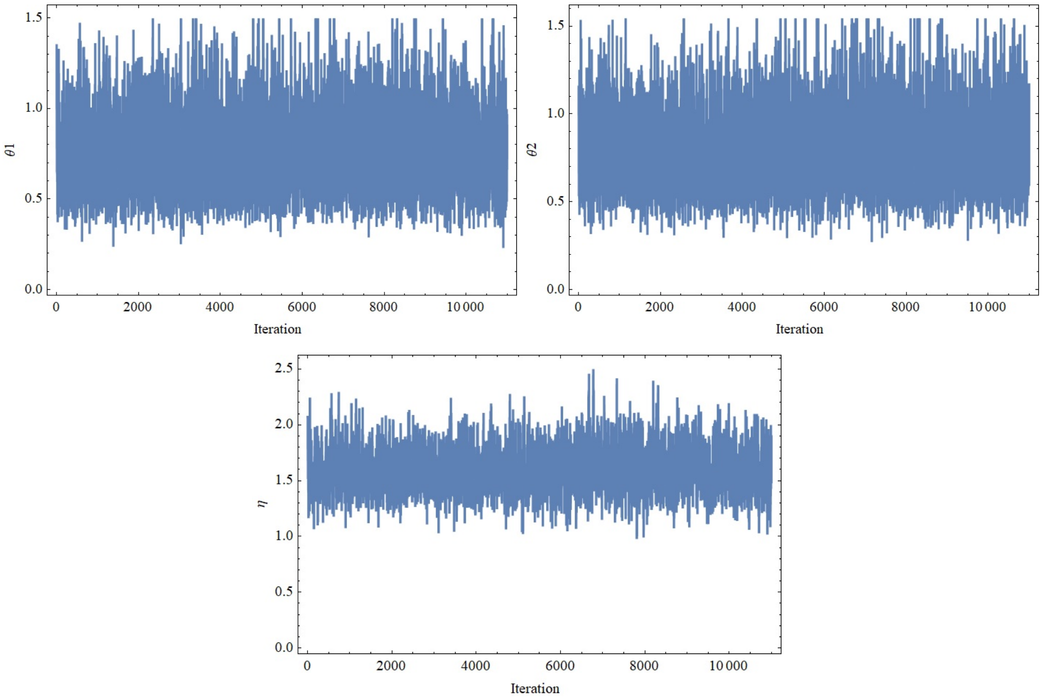

Figure 4. Male mice MCMC-generated trace plots of the parameters θ_1, θ_2, and η.

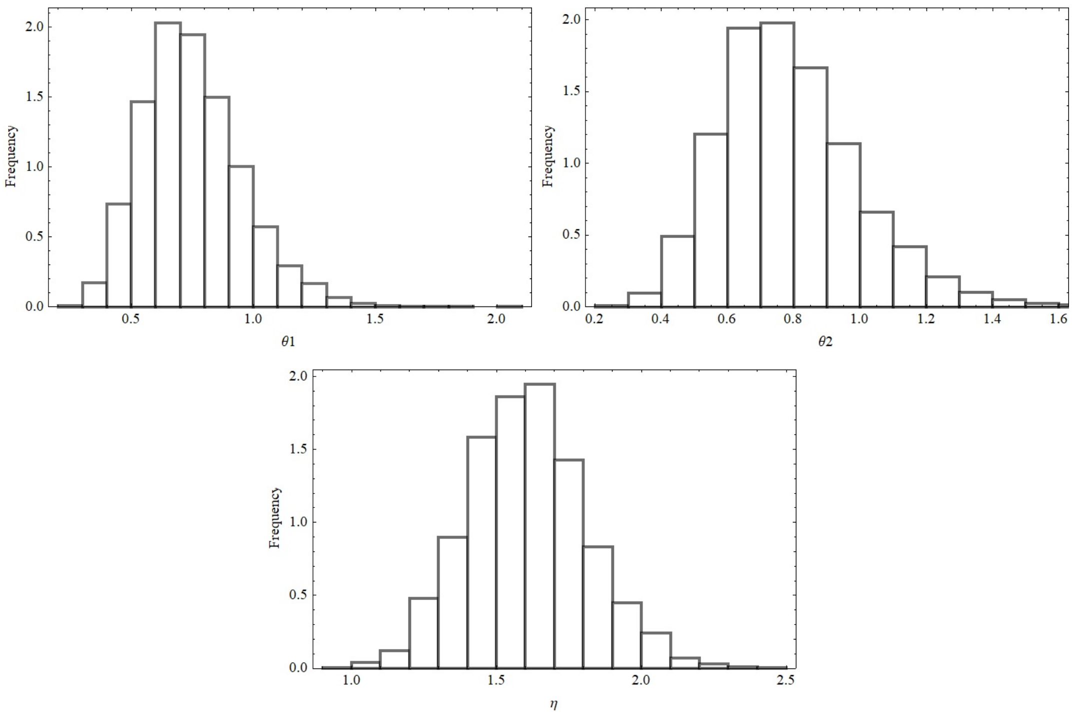

Figure 5. Male mice MCMC-generated histograms of the parameters θ_1, θ_2, and η.

Results.

Table 3 presents the point estimates and corresponding 95% intervals for the model parameters θ_1, θ_2, and η. The significance level for all confidence intervals was set at $\alpha = 0.05$. Table 3 provides a comprehensive comparison of all estimation methods: MLE, bootstrap, and the Bayesian approach under both SE and LINEX (with $c = -2, 2$) loss functions.

6.2. Application to the football data

In this section, we analyze a real-world dataset to illustrate the practical application of the proposed model. The data, originally reported by Meintanis [35], corresponds to football (soccer) matches in which at least one goal was scored by the home team and at least one goal resulted directly from a penalty kick. Wang [36] shows that the KuD provides a reasonable model for these data. In this analysis, the random variable X denotes the time (in minutes) of the first penalty kick goal scored by either team, while Y represents the time of the first goal of any type scored by the home team. The detailed observations are provided in Table 4. To ensure the data belong to the interval $(0, 1)$, each observation has been divided by 90. These transformed values represent the proportion of the match that elapsed before the event occurred. For this illustrative case, X and Y are treated as two independent competing-risks, Cause 1 and 2, respectively, and the complete competing risks dataset is summarized in Table 4 when

$$p = 0.15, \quad m = 50, \quad \text{and} \quad R = (5, 0^{10}, 5, 0^{10}, 5, 0^{10}, 5, 0^{15}, 4).$$

This validated dataset is now used to demonstrate the application of our proposed Type-II PC sample. We use the following data:

(0.0222,1)	(0.0222,2)	(0.0333,2)	(0.0333,2)	(0.0778,2)	(0.0889,1)
(0.0889,1)	(0.1,2)	(0.1222,0)	(0.1556,2)	(0.1667,2)	(0.1778,1)
(0.1778,1)	(0.1778,1)	(0.2,1)	(0.2,1)	(0.2,0)	(0.2111,1)
(0.2222,1)	(0.2444,1)	(0.2667,0)	(0.2889,1)	(0.3111,0)	(0.3111,1)
(0.3333,1)	(0.4,1)	(0.4333,1)	(0.4333,1)	(0.4333,1)	(0.4444,0)
(0.4667,1)	(0.4667,1)	(0.4889,0)	(0.5222,1)	(0.5333,1)	(0.5333,1)
(0.5444,1)	(0.5667,1)	(0.5778,1)	(0.6,1)	(0.6889,1)	(0.7111,1)
(0.7333,1)	(0.7333,1)	(0.7667,1)	(0.8444,1)	(0.9111,1)	(0.9444,2).

Table 4. The proportion of the time of a football game.

Time in minutes of the first kick goal	0.0222	0.4556	0.4667	0.6000	0.0889	0.2778	0.6111	0.4889
	0.2444	0.2778	0.2889	0.7111	0.1778	0.1778	0.7000	0.2000
	0.2111	0.2667	0.2889	0.3000	0.3111	0.5667	0.4889	0.3778
	0.4000	0.4333	0.5889	0.4444	0.4667	0.9111	0.5444	0.5444
	0.7333	0.8444	0.7333	0.7667	0.8000			
The first goal of any type scored	0.0222	0.0333	0.0333	0.0778	0.0889	0.1	0.1222	0.1444
	0.1556	0.1556	0.1667	0.1778	0.8333	0.2	0.2	0.2111
	0.2222	0.2667	0.5333	0.5222	0.3111	0.3111	0.3333	0.3778
	0.5778	0.4333	0.4333	0.4444	0.4667	0.5333	0.5444	0.5444
	0.6889	0.7111	0.9444	0.7889	0.8			

In this framework, the indicator 1 denotes the first cause of failure, 2 represents the second cause, and 0 signifies an unobserved (masked) cause. Within this dataset, the observed frequencies are $m_0 = 6$, $m_1 = 36$, $m_2 = 8$, and $m = 50$. Subsequently, the data are analyzed under the proposed competing-risks model considering these two primary times of a football game.

Estimation and computational details:

(1) MLE: The MLEs for the model parameters $(\theta_1, \theta_2, \eta)$ were computed using an iterative numerical optimization method.

(2) Bayesian estimation: For the Bayesian analysis, we assigned non-informative gamma priors to the parameters, with hyperparameters set to $a_i = b_i = 0.0001$ for $i = 1, 2, 3$. The posterior distributions were then sampled using an MCMC algorithm. We ran the MCMC chain for 11,000 iterations and discarded the first 1000 as a burn-in period to ensure convergence. To assess the convergence of the empirical posterior distributions, trace plots and histograms of the generated MCMC samples for $(\theta_1, \theta_2, \eta)$ are provided in Figures 6 and 7. Using these data, we obtained the MLEs and Bayesian estimates for the parameters of the KuD, as shown in Table 5. We generated 10,000 MCMC samples using the previously described algorithm. The MLEs of the parameters were used as initial values for the unknown parameters in the MCMC sampling procedure. Table 5 also presents the two-sided asymptotic confidence intervals and credible intervals for the estimates, along with some optimality measures.

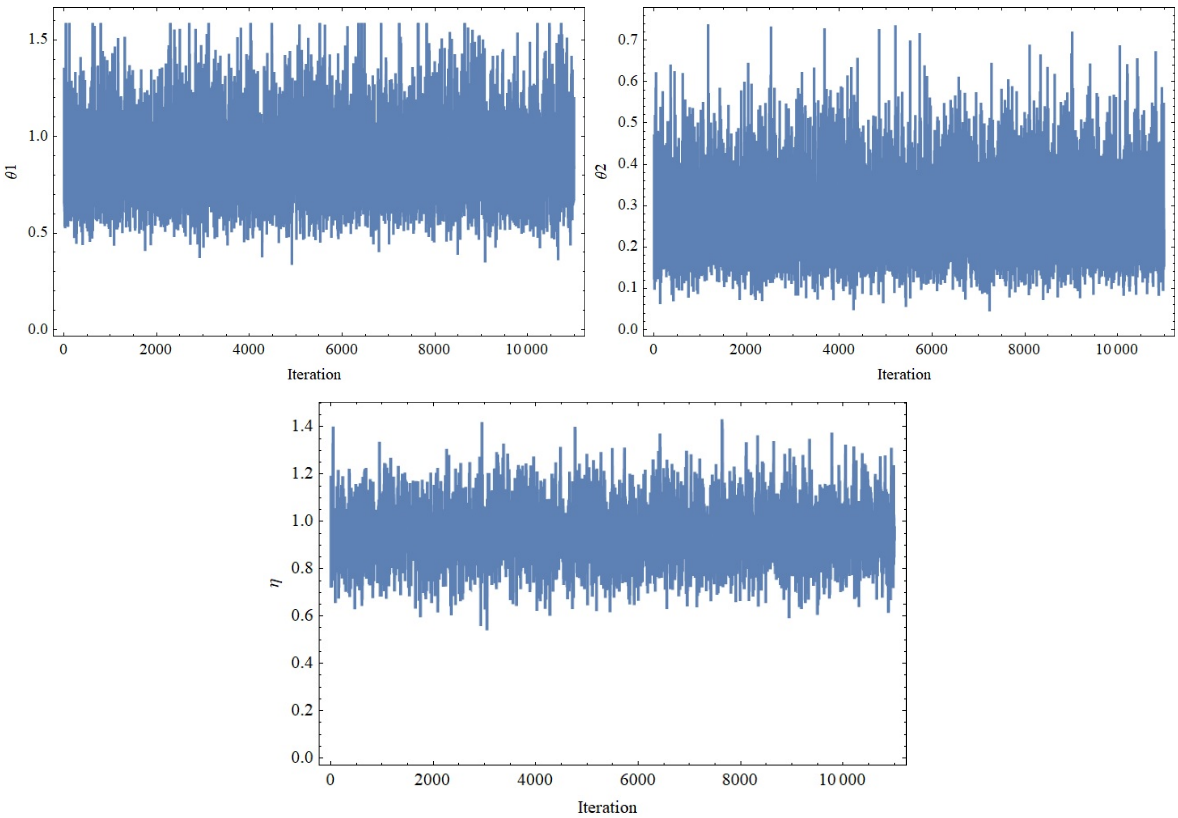

Figure 6. Football data MCMC-generated trace plots of the parameters θ_1, θ_2, and η.

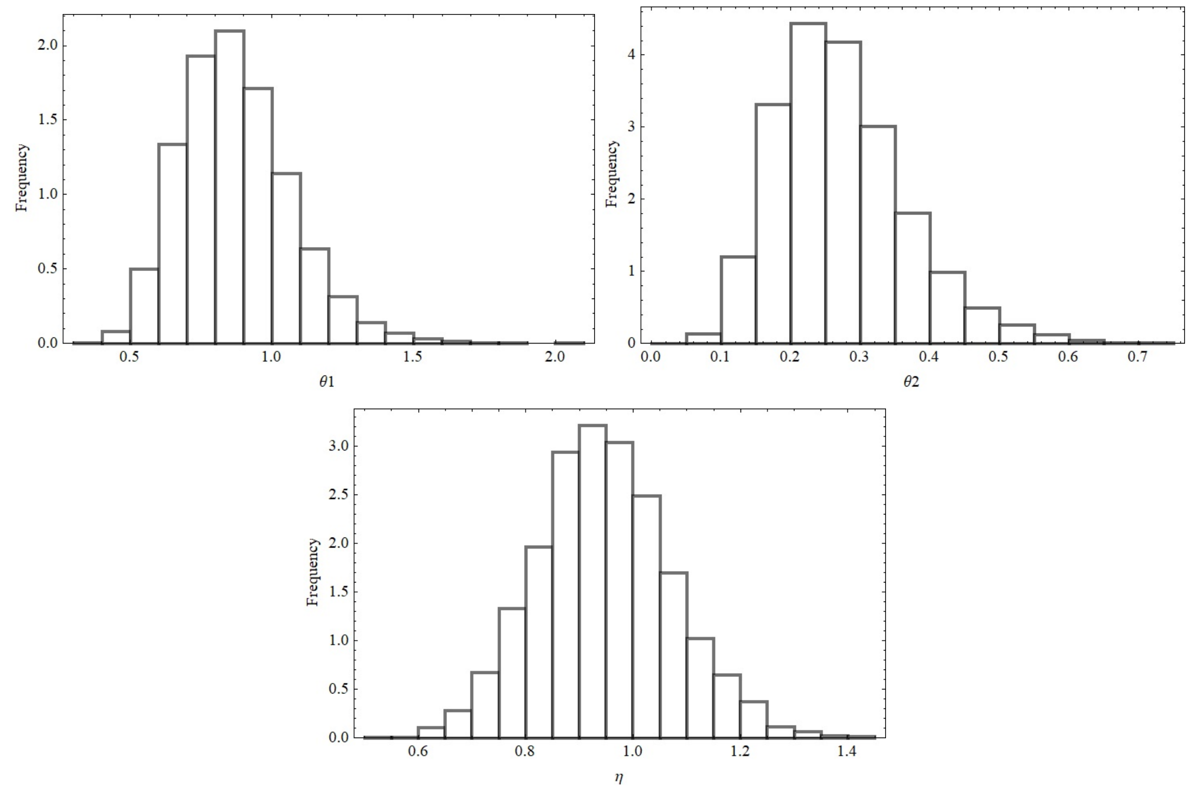

Figure 7. Football data MCMC-generated histograms of the parameters θ_1, θ_2, and η.

Table 5. ML, bootstrap, and Bayes estimates, as well as 95% CIs, for the model parameters using male mice data under different loss functions.

	θ_1	θ_2	η
MLE	1.5573	0.2946	1.2435
ACI	(0.8838, 2.2307)	(0.0604, 0.5288)	(0.8961, 1.5908)
Boot	1.6607	0.3163	1.2833
Boot CI	(1.0467, 2.5709)	(0.0970, 0.6279)	(0.9570, 1.6764)
Bayes SE	1.5718	0.4981	1.2169
Credibl interval	(1.0263, 2.1130)	(0.2619, 0.6480)	(0.9746, 1.3579)
Bayes LINEX ($c = -1$)	1.6337	0.5063	1.2244
Bayes LINEX ($c = 1$)	1.5119	0.4895	1.2088

Results.

Table 5 presents the point estimates and corresponding 95% intervals for the model parameters θ_1, θ_2, and η. The significance level for all confidence intervals was set at $\alpha = 0.05$. The table provides a comprehensive comparison of all estimation methods: MLE, bootstrap, and the Bayesian approach under both SE and LINEX (with $c = -1, 1$) loss functions.

7. Monte-Carlo simulation

To compare the efficiency of the proposed estimation methods, we implemented a Monte Carlo simulation. Specifically, the MLE and bootstrap estimate were evaluated and compared against Bayesian estimates, considering scenarios where parameters are estimated both without prior information (frequentist approach) and with the inclusion of prior distributions (Bayesian approach).

7.1. Simulation design

The following procedure was utilized to generate the censored samples from the KuD model for the simulation study:

Algorithm 4.

Step 1: Total sample size (n) and number of replications: The total sample sizes considered are $n = 40$ and 60, and estimator performance was assessed via 1000 Monte Carlo samples from the Ku distribution.

Step 2: Observed failure count (m): The number of observed failures, m, is selected such that $m < n$ and the progressive censoring condition

$$n = m + \sum_{i=1}^{m} R_i$$

is satisfied. Specifically, we set $m = 20, 30, 40, 50$ for $n = 40, 60$.

Step 3: True parameter values: In the simulation studies, we select parameters based on the optimal shape of the density function. The true values for the parameters $(\theta_1, \theta_2, \eta)$ are specified as $(0.5, 0.63, 2)$.

Step 4: Binomial removal probability (p): The probability parameter for the binomial removal scheme is varied across two levels: $p = 0.1, 0.15$.

Step 5: The performance of the proposed estimates is investigated under three different progressive censoring schemes. These schemes are chosen to reflect scenarios where units are withdrawn early, midway, or at the end of the life test. These schemes are formulated as:

- Scheme I:

$$\mathbf{R}_1 = (n - m, 0, 0, \ldots, 0).$$

- Scheme II:

$$\mathbf{R}_2 = (0^{m-\frac{n}{2}}, 1^{n-m}, 0^{m-\frac{n}{2}}).$$

- Scheme III:

$$\mathbf{R}_3 = (0, 0, \ldots, n - m).$$

Step 6: To specify the informative prior, the hyperparameters a_i and b_i are determined by equating the prior mean to the estimated parameter values, i.e.,

$$\theta_i \cong \frac{a_i}{b_i}, \quad i = 1, 2, 3.$$

Step 7: Specific values are assigned to the hyperparameters for each of the three pre-defined parameter sets.

7.2. Simulation analysis

Parameters for the simulated data are estimated following the methodologies outlined in Sections 3–5. The ML package is utilized to implement the iterative algorithm for computing the MLEs. Interval estimation includes ACIs derived from normal approximations, as well as bootstrap intervals generated via Boot-p techniques. For the Bayesian framework, gamma priors are assumed, and point and interval estimates are generated using the MH-within-Gibbs MCMC approach.

7.3. Simulation outcome

The performance of the estimated model parameters is evaluated using the bias, mean squared error (MSE), and the average length (AL) of the CIs, MLEs, bootstrap, and Bayes estimates of the unknown parameters. The Bayesian estimators are obtained using the MH-within-Gibbs MCMC approach. The Bayes estimates are obtained under SE and LINEX loss functions for various parameter values for comparison's sake. Tables 6–9 present the point estimation results under various censoring schemes, while Tables 10 and 11 summarize the corresponding interval estimation results. The quality of the interval estimates is further examined using coverage probabilities (CPs). The findings in Tables 6–9 reveal several significant trends. Specifically, the accuracy of the estimates improves as the sample size n increases, confirming their asymptotic unbiasedness. Furthermore, the consistent decrease in MSE with larger sample sizes across all cases demonstrates the consistency of the proposed estimators. Regarding interval estimation, the ALs for parameters $(\theta_1, \theta_2, \eta)$ under the Bayesian strategy are superior (narrower) to those produced by the likelihood function approach. Based on the simulation study results presented in Tables 6–11, the following observations can be made:
(1) With increasing sample sizes, the MSEs and ALs decrease, and the coverage probabilities align more closely with the nominal 0.95 level.
(2) The Bayes estimation under SE and LINEX loss functions based on the MCMC method shows minimum MSEs compared to the MLE method and the bootstrap method.
(3) In the LINEX loss function, when c has increased, MSE decreases.
(4) Bayesian estimation based on a gamma prior generally outperforms classical methods. Specifically, Bayes estimates with informative priors exhibit the lowest bias and MSE, except in a few cases, which may be due to fluctuations in the data.
(5) Finally, the censoring scheme $\mathbf{R}_1 = (n - m, 0, 0, \ldots, 0)$ is most efficient for all choices; it usually provides the smallest MSE for all estimators.

Table 6. Average values and MSEs for the ML, bootstrap, and Bayes estimates with ($\theta_1 = 0.5, \theta_2 = 0.63, \eta = 2$) and ($P = 0.1, n = 40$).

n	m	CS	Method	θ_1		θ_2		η	
				Mean	MSE	Mean	MSE	Mean	MSE
40	20	I	ML	0.5738	0.0680	0.7551	0.1298	2.1920	0.3206
			Bootstrap	0.6663	0.1209	0.8778	0.2363	2.3627	0.4499
			Bayes SE	0.6006	0.0529	0.7492	0.0726	2.2847	0.3130
			Bayes LINEX ($c = -0.5$)	0.6107	0.0576	0.7620	0.0798	2.2926	0.3210
			Bayes LINEX ($c = 1$)	0.5817	0.0451	0.7255	0.0606	2.2682	0.3171
40	20	II	ML	0.6259	0.1856	0.7876	0.2194	2.2286	0.5421
			Bootstrap	0.8040	0.4311	1.0084	0.5424	2.4481	0.7688
			Bayes SE	0.6020	0.1051	0.7315	0.1160	2.1619	0.4002
			Bayes LINEX ($c = -0.5$)	0.6130	0.1139	0.7444	0.1259	2.1694	0.4068
			Bayes LINEX ($c = 1$)	0.5817	0.0903	0.7072	0.0992	2.1462	0.3865
40	20	III	ML	0.6611	0.1831	0.8546	0.2516	2.3075	0.4574
			Bootstrap	0.9321	0.6064	1.2040	0.8893	2.5827	0.7713
			Bayes SE	0.6249	0.0992	0.7652	0.1291	2.1497	0.3402
			Bayes LINEX ($c = -0.5$)	0.6365	0.1083	0.7793	0.1400	2.1602	0.3474
			Bayes LINEX ($c = 1$)	0.6034	0.0839	0.7391	0.1109	2.1263	0.3239
40	30	I	ML	0.5613	0.0369	0.6833	0.0556	2.1786	0.2900
			Bootstrap	0.6204	0.0362	0.7555	0.0847	2.3243	0.3002
			Bayes SE	0.6115	0.0277	0.7219	0.0396	2.3413	0.2655
			Bayes LINEX ($c = -0.5$)	0.6183	0.0286	0.7299	0.0424	2.3470	0.2674
			Bayes LINEX ($c = 1$)	0.5986	0.0255	0.7067	0.0346	2.3294	0.2602
40	30	II	ML	0.5532	0.0375	0.7342	0.0701	2.1963	0.3005
			Bootstrap	0.6227	0.0617	0.8266	0.1192	2.3488	0.3776
			Bayes SE	0.5698	0.0295	0.7223	0.0426	2.2202	0.2704
			Bayes LINEX ($c = -0.5$)	0.5761	0.0312	0.7302	0.0456	2.2254	0.2742
			Bayes LINEX ($c = 1$)	0.5578	0.0262	0.7070	0.0375	2.2096	0.2628
40	30	III	ML	0.5553	0.0438	0.7430	0.0720	2.1583	0.3720
			Bootstrap	0.6393	0.0788	0.8559	0.1398	2.3311	0.3893
			Bayes SE	0.5731	0.0410	0.7322	0.0551	2.1746	0.2747
			Bayes LINEX ($c = -0.5$)	0.5797	0.0436	0.7407	0.0590	2.1801	0.2784
			Bayes LINEX ($c = 1$)	0.5606	0.0365	0.7160	0.0483	2.1629	0.2671

Table 7. Average values and MSEs for the ML, bootstrap, and Bayes estimates with ($\theta_1 = 0.5, \theta_2 = 0.63, \eta = 2$) and ($P = 0.15$, $n = 40$).

n	m	CS	Method	θ_1		θ_2		η	
				Mean	MSE	Mean	MSE	Mean	MSE
40	20	I	ML	0.5669	0.0615	0.7439	0.0750	2.1766	0.2877
			Bootstrap	0.6572	0.1072	0.8610	0.1440	2.3472	0.3105
			Bayes SE	0.6083	0.0529	0.7551	0.0512	2.3247	0.2796
			Bayes LINEX ($c = -0.5$)	0.6187	0.0574	0.7676	0.0567	2.3323	0.2875
			Bayes LINEX ($c = 1$)	0.5891	0.0450	0.7316	0.0421	2.3086	0.2633
40	20	II	ML	0.6567	0.1499	0.8508	0.2123	2.3533	0.5118
			Bootstrap	0.8409	0.3645	1.0898	0.5373	2.5818	0.7862
			Bayes SE	0.6418	0.0949	0.7844	0.1243	2.2735	0.3902
			Bayes LINEX ($c = -0.5$)	0.6538	0.1037	0.7988	0.1354	2.2823	0.3993
			Bayes LINEX ($c = 1$)	0.6196	0.0801	0.7575	0.1059	2.2545	0.3707
40	20	III	ML	0.6503	0.1946	0.7818	0.2072	2.2649	0.4486
			Bootstrap	0.9059	0.6137	1.0859	0.6947	2.5361	0.73614
			Bayes SE	0.6168	0.0945	0.7135	0.0905	2.1640	0.3473
			Bayes LINEX ($c = -0.5$)	0.6277	0.1027	0.7256	0.0974	2.1729	0.3537
			Bayes LINEX ($c = 1$)	0.5967	0.0807	0.6907	0.0790	2.1446	0.3331
40	30	I	ML	0.5667	0.0545	0.6908	0.0415	2.1577	0.2690
			Bootstrap	0.6268	0.0800	0.7638	0.0673	2.3022	0.3623
			Bayes SE	0.6307	0.0464	0.7431	0.0409	2.3909	0.2307
			Bayes LINEX ($c = -0.5$)	0.6379	0.0498	0.7515	0.0441	2.3966	0.2381
			Bayes LINEX ($c = 1$)	0.6170	0.0404	0.7270	0.0354	2.3788	0.2149
40	30	II	ML	0.5637	0.0548	0.7087	0.0658	2.0678	0.1929
			Bootstrap	0.6357	0.0869	0.7984	0.1094	2.2151	0.2563
			Bayes SE	0.5932	0.0465	0.7146	0.0436	2.1441	0.1959
			Bayes LINEX ($c = -0.5$)	0.5998	0.0492	0.7227	0.0466	2.1496	0.1991
			Bayes LINEX ($c = 1$)	0.5805	0.0417	0.6993	0.0384	2.1325	0.1894
40	30	III	ML	0.5961	0.0571	0.7520	0.0707	2.2154	0.2226
			Bootstrap	0.6871	0.1043	0.8670	0.1398	2.3907	0.3511
			Bayes SE	0.6171	0.0510	0.7455	0.0499	2.2628	0.3068
			Bayes LINEX ($c = -0.5$)	0.6244	0.0543	0.7543	0.0536	2.2682	0.3122
			Bayes LINEX ($c = 1$)	0.6032	0.0453	0.7289	0.0435	2.2513	0.2951

Table 8. Average values and MSEs for the ML, bootstrap, and Bayes estimates with ($\theta_1 = 0.5, \theta_2 = 0.63, \eta = 2$), ($P = 0.10, n = 60$).

n	m	CS	Method	θ_1		θ_2		η	
				Mean	MSE	Mean	MSE	Mean	MSE
60	40	I	ML	0.5545	0.0359	0.7000	0.0441	2.1312	0.1476
			Bootstrap	0.5947	0.0361	0.7509	0.0612	2.2297	0.1547
			Bayes SE	0.6078	0.0333	0.7386	0.0351	2.2849	0.1434
			Bayes LINEX ($c = -0.5$)	0.6127	0.0349	0.7447	0.0372	2.2889	0.1504
			Bayes LINEX ($c = 1$)	0.5982	0.0303	0.7268	0.0313	2.2766	0.1309
60	40	II	ML	0.5317	0.0290	0.6760	0.0453	2.0247	0.1612
			Bootstrap	0.5811	0.0423	0.7387	0.0666	2.1277	0.1576
			Bayes SE	0.5528	0.0292	0.6792	0.0382	2.0445	0.1530
			Bayes LINEX ($c = -0.5$)	0.5573	0.0306	0.6847	0.0399	2.0481	0.1510
			Bayes LINEX ($c = 1$)	0.5442	0.0269	0.6686	0.0352	2.0370	0.14093
60	40	III	ML	0.5832	0.0380	0.7012	0.0646	2.1220	0.1818
			Bootstrap	0.6510	0.0646	0.7829	0.1037	2.2457	0.2425
			Bayes SE	0.6034	0.038015	0.7064	0.0627	2.1399	0.1612
			Bayes LINEX ($c = -0.5$)	0.6086	0.0399	0.7126	0.0656	2.1443	0.1631
			Bayes LINEX ($c = 1$)	0.5935	0.0345	0.6944	0.0576	2.1307	0.1573
60	50	I	ML	0.5402	0.0196	0.6210	0.0239	2.0559	0.1256
			Bootstrap	0.5714	0.0255	0.6572	0.0283	2.1435	0.1293
			Bayes SE	0.6079	0.0181	0.6845	0.0203	2.2671	0.1150
			Bayes LINEX ($c = -0.5$)	0.6117	0.0190	0.6889	0.0212	2.2704	0.1227
			Bayes LINEX ($c = 1$)	0.6005	0.0183	0.6760	0.0185	2.2602	0.1146
60	50	II	ML	0.5421	0.0266	0.6699	0.0324	2.0525	0.1146
			Bootstrap	0.5783	0.0353	0.7148	0.0436	2.1417	0.13926
			Bayes SE	0.5888	0.0277	0.7067	0.0252	2.1935	0.1332
			Bayes LINEX ($c = -0.5$)	0.5927	0.0289	0.7113	0.0264	2.1968	0.1352
			Bayes LINEX ($c = 1$)	0.5811	0.0256	0.6976	0.0230	2.1866	0.1291
60	50	III	ML	0.5351	0.0223	0.6978	0.0297	2.1398	0.1641
			Bootstrap	0.5744	0.0301	0.7492	0.0442	2.2390	0.2119
			Bayes SE	0.5647	0.0236	0.7127	0.0238	2.2117	0.1877
			Bayes LINEX ($c = -0.5$)	0.5684	0.0245	0.7174	0.0250	2.2155	0.1900
			Bayes LINEX ($c = 1$)	0.5574	0.0219	0.7035	0.0215	2.2040	0.1831

Table 9. Average values and MSEs for the ML, bootstrap, and Bayes estimates with (θ_1 = 0.5, θ_2 = 0.63, η = 2), (P = 0.15, n = 60).

n	m	CS	Method	θ_1		θ_2		η	
				Mean	MSE	Mean	MSE	Mean	MSE
60	40	I	ML	0.5422	0.0410	0.7111	0.0582	2.1356	0.1974
			Bootstrap	0.5824	0.0401	0.7639	0.0790	2.2353	0.1952
			Bayes SE	0.6055	0.0381	0.7568	0.0524	2.3267	0.1553
			Bayes LINEX (c = −0.5)	0.6107	0.0400	0.7633	0.0553	2.3300	0.1574
			Bayes LINEX (c = 1)	0.5955	0.0346	0.7441	0.0471	2.3196	0.1437
60	40	II	ML	0.5585	0.0546	0.7079	0.0475	2.1488	0.2285
			Bootstrap	0.6117	0.0774	0.7743	0.0730	2.2571	0.2552
			Bayes SE	0.5899	0.0481	0.7201	0.0385	2.2175	0.2173
			Bayes LINEX (c = −0.5)	0.5949	0.0504	0.7263	0.0406	2.2211	0.2202
			Bayes LINEX (c = 1)	0.5800	0.0439	0.7083	0.0346	2.2099	0.2111
60	40	III	ML	0.5351	0.0563	0.6416	0.0469	2.0674	0.2036
			Bootstrap	0.5945	0.0417	0.7129	0.0678	2.1905	0.1984
			Bayes SE	0.5678	0.0467	0.6596	0.0414	2.1315	0.1693
			Bayes LINEX (c = −0.5)	0.5723	0.0479	0.6649	0.0429	2.1354	0.1707
			Bayes LINEX (c = 1)	0.5591	0.0445	0.6494	0.0387	2.1234	0.1566
60	50	I	ML	0.5326	0.0155	0.6424	0.0234	2.1209	0.1781
			Bootstrap	0.5638	0.0194	0.6801	0.0290	2.2110	0.2185
			Bayes SE	0.6028	0.0131	0.7066	0.0220	2.3461	0.1422
			Bayes LINEX (c = −0.5)	0.6066	0.0142	0.7110	0.0222	2.3488	0.1449
			Bayes LINEX (c = 1)	0.5954	0.0121	0.6979	0.0219	2.3402	0.13661
60	50	II	ML	0.5309	0.0253	0.6673	0.0386	2.0700	0.1969
			Bootstrap	0.5657	0.0255	0.7113	0.0386	2.1586	0.1947
			Bayes SE	0.5852	0.0243	0.7107	0.0307	2.2190	0.1856
			Bayes LINEX (c = −0.5)	0.5889	0.0253	0.7153	0.0320	2.2215	0.1869
			Bayes LINEX (c = 1)	0.5778	0.0224	0.7017	0.0263	2.2138	0.1827
60	50	III	ML	0.5228	0.0226	0.6472	0.0247	2.0773	0.1992
			Bootstrap	0.5606	0.0302	0.6936	0.0330	2.1756	0.1839
			Bayes SE	0.5687	0.0243	0.6808	0.0304	2.2036	0.1705
			Bayes LINEX (c = −0.5)	0.5722	0.0252	0.6851	0.0313	2.2061	0.1721
			Bayes LINEX (c = 1)	0.5618	0.0226	0.6723	0.0268	2.1984	0.1672

Table 10. The estimates of AIs and CPs of 95% interval estimators when ($\theta_1 = 0.5, \theta_2 = 0.63, \eta = 2$) and $P = 0.1$.

(n, m)	CS	Par.	MLE	Bootstrap	MCMC
(40, 20)	I	θ_1	0.9134(0.964)	0.9507(0.926)	0.7149(0.944)
		θ_2	1.0887(0.954)	1.2569(0.938)	0.8135(0.938)
		η	2.0804(0.970)	2.2778(0.956)	1.5893(0.968)
	II	θ_1	1.1261(0.942)	1.5500(0.944)	0.7166(0.914)
		θ_2	1.3184(0.936)	1.9194(0.918)	0.8003(0.942)
		η	2.0656(0.900)	2.3788(0.934)	1.6569(0.900)
	III	θ_1	1.3278(0.960)	1.2215(0.934)	0.7451(0.964)
		θ_2	1.6143(0.964)	1.8444(0.948)	0.8355(0.938)
		η	2.307(0.946)	2.7243(0.954)	1.7763(0.918)
(40, 30)	I	θ_1	0.7187(0.924)	0.6786(0.914)	0.6027(0.900)
		θ_2	0.8121(0.950)	0.8285(0.928)	0.6597(0.954)
		η	1.8398(0.934)	2.0081(0.956)	1.5705(0.954)
	II	θ_1	0.7494(0.972)	0.7420(0.946)	0.5532(0.952)
		θ_2	0.9039(0.956)	0.9826(0.948)	0.6613(0.942)
		η	1.7852(0.944)	1.9723(0.918)	1.5625(0.936)
	III	θ_1	0.7877(0.954)	0.8336(0.968)	0.5913(0.928)
		θ_2	0.9629(0.954)	1.1187(0.912)	0.6764(0.980)
		η	1.8533(0.962)	2.0691(0.920)	1.5774(0.948)
(60, 40)	I	θ_1	0.6165(0.984)	0.5424(0.984)	0.5253(0.932)
		θ_2	0.7138(0.952)	0.6854(0.936)	0.5858(0.958)
		η	1.5360(0.964)	1.6298(0.924)	1.4887(0.920)
	II	θ_1	0.6214(0.990)	0.5805(0.920)	0.5327(0.942)
		θ_2	0.7297(0.970)	0.7382(0.972)	0.5497(0.964)
		η	1.5116(0.964)	1.5123(0.980)	1.5067(0.940)
	III	θ_1	0.7213(0.962)	0.7407(0.914)	0.5329(0.966)
		θ_2	0.8251(0.954)	0.8939(0.966)	0.5815(0.928)
		η	1.5639(0.950)	1.6950(0.956)	1.5047(0.942)
(60, 50)	I	θ_1	0.5235(0.966)	0.4620(0.900)	0.4609(0.950)
		θ_2	0.5716(0.912)	0.5207(0.966)	0.4956(0.942)
		η	1.4024(0.968)	1.4893(0.962)	1.3380(0.968)
	II	θ_1	0.5505(0.944)	0.4852(0.942)	0.4689(0.970)
		θ_2	0.6307(0.956)	0.5997(0.982)	0.5140(0.928)
		η	1.3505(0.950)	1.4396(0.946)	1.3442(0.942)
	III	θ_1	0.5630(0.954)	0.5059(0.958)	0.4684(0.890)
		θ_2	0.6713(0.970)	0.6587(0.982)	0.5193(0.958)
		η	1.4498(0.974)	1.5416(0.914)	1.4567(0.942)

Table 11. The estimates of Als and CPs of 95% interval estimators when ($\theta_1 = 0.5, \theta_2 = 0.63, \eta = 2$) and $P = 0.15$.

(n, m),	CS	Par.	MLE	Bootstrap	MCMC
		θ_1	0.9082(0.926)	0.9235(0.958)	0.7295(0.924)
	I	θ_2	1.0728(0.984)	1.2016(0.970)	0.8185(0.984)
		η	2.0435(0.990)	2.2635(0.928)	1.6653(0.946)
		θ_1	1.1974(0.924)	1.6013(0.974)	0.7660(0.980)
(40, 20)	II	θ_2	1.4394(0.946)	2.0670(0.982)	0.8514(0.978)
		η	2.1559(0.944)	2.4791(0.936)	1.7050(0.914)
		θ_1	1.2913(0.942)	1.1098(0.914)	0.7836(0.966)
	III	θ_2	1.4758(0.936)	1.4989(0.954)	0.9238(0.970)
		η	2.2794(0.962)	2.6686(0.934)	1.7315(0.916)
		θ_1	0.7353(0.966)	0.6872(0.964)	0.6218(0.968)
	I	θ_2	0.8269(0.964)	0.8344(0.966)	0.6798(0.958)
		η	1.8170(0.954)	1.9939(0.900)	1.5494(0.972)
		θ_1	0.7661(0.944)	0.7553(0.970)	0.5939(0.968)
(40, 30)	II	θ_2	0.8862(0.972)	0.9437(0.966)	0.6587(0.934)
		η	1.6972(0.944)	1.8748(0.942)	1.5585(0.956)
		θ_1	0.8490(0.972)	0.9023(0.934)	0.6218(0.900)
	III	θ_2	0.9929(0.982)	1.1363(0.956)	0.6885(0.948)
		η	1.8881(1.000)	2.0941(0.954)	1.5772(0.946)
		θ_1	0.6148(0.932)	0.6153(0.954)	0.5299(0.966)
	I	θ_2	0.7280(0.938)	0.6923(0.980)	0.6534(0.974)
		η	1.5468(0.942)	1.6333(0.900)	1.4668(0.946)
		θ_1	0.6669(0.924)	0.6263(0.932)	0.5192(0.976)
(60, 40)	II	θ_2	0.7748(0.980)	0.7882(0.946)	0.5785(0.928)
		η	1.4822(0.958)	1.5909(0.970)	1.4698(0.956)
		θ_1	0.6557(0.946)	0.6582(0.982)	0.5053(0.924)
	III	θ_2	0.7450(0.912)	0.7917(0.900)	0.5444(0.942)
		η	1.5431(0.946)	1.6781(0.958)	1.5115(0.946)
		θ_1	0.5286(0.974)	0.4470(0.966)	0.4622(0.934)
	I	θ_2	0.5932(0.972)	0.5396(0.934)	0.5002(0.900)
		η	1.4352(0.942)	1.5088(0.918)	1.4169(0.914)
		θ_1	0.5472(0.954)	0.4706(0.966)	0.4575(0.928)
(60, 50)	II	θ_2	0.6324(0.964)	0.5922(0.954)	0.5052(0.932)
		η	1.4692(0.974)	1.4531(0.952)	1.4250(0.968)
		θ_1	0.5481(0.914)	0.4910(0.918)	0.4414(0.966)
	III	θ_2	0.6286(0.954)	0.6054(0.938)	0.4906(0.952)
		η	1.4639(0.980)	1.5153(0.968)	1.4505(0.962)

8. Conclusions

In this paper, partially observed competing-risks models were discussed. In this study, the lifetimes of the units were modeled by the Ku distribution subject to a Type-II PC framework. We discussed the ML, bootstrap, and Bayes estimates of the unknown parameters. The Bayesian estimators were obtained using MH-within-Gibbs MCMC approach. The Bayes estimates are obtained under SE and LINEX loss functions for various parameter values for comparison's sake. To evaluate the performance of the suggested approaches, we proposed a Monte Carlo simulation study and analyzed two real data sets. This research provided a robust inferential framework for the KuD model subjected to competing risks and Type-II CP. By evaluating point and interval estimators under classical and Bayesian paradigms, we confirmed that both procedures demonstrate desirable statistical properties, with Bayesian results proving more efficient in most simulated scenarios. We computed parameter estimates by minimizing the expected loss under both SE and LINEX specifications, considering various parameter configurations. A Monte Carlo simulation study confirmed that Bayesian point and interval estimation based on gamma priors provides superior performance and is highly recommended for practical use. While we specifically addressed two independent causes of failure, the results are generalizable to larger systems and alternative censoring mechanisms. Moving forward, of particular interest is the investigation of optimality criteria for censoring designs and sampling plans, which will be addressed in future studies.

9. Future work

We intend to investigate the application of the expectation-maximization (EM) algorithm to estimate parameters within the proposed censoring scheme and compare its effectiveness with the maximum-likelihood and Bayesian estimators described in this paper. Future research directions can be categorized into modeling complexity, advanced censoring schemes, experimental design optimization, and computational paradigms. In addition, a comprehensive prior sensitivity analysis could be performed to assess the robustness of the Bayesian estimates.

Author contributions

Manal H. Alloqmani: Conceptualization, methodology, software, formal analysis, writing—original draft preparation; Faten S. Alamri: Methodology, software, formal analysis, writing—review; Gamal M. Ismail: Funding acquisition, software, formal analysis, writing—original draft preparation; Samah M. Ahmed: Conceptualization, methodology, software, formal analysis, writing—review and editing; Raga Hassan Ali Shiekh: Conceptualization, formal analysis and editing. Al-Wageh A. Farghal: Conceptualization, methodology, software, formal analysis, resources and supervision. All authors have read and approved the final version of the manuscript for publication.

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Use of Generative-AI tools declaration

The authors declare that no Artificial Intelligence (AI) tools were used in the creation of this article.

Conflict of interest

The authors declare no conflicts of interest.

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