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Research article

Classification of certain n-Derivations of Hom-Lie superalgebras

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Abstract: Let χ be a Hom-Lie superalgebra and $n \geq 2$ be a fixed integer. In this paper, we study the Lie n-derivation algebra $hN\,\mathrm{Der}(\chi)$ as well as the generalized Lie n-derivation algebra $hGN\,\mathrm{Der}(\chi)$. We also analyze the structures of the n-centroids, the n-quasi centroids, and the central n-derivations, and examine how they relate to $hN\,\mathrm{Der}(\chi)$ and $hGN\,\mathrm{Der}(\chi)$. Moreover, we obtain a full classification of n-derivations of Hom-Lie superalgebras and demonstrate that, for simple Hom-Lie superalgebras, one has $hN\,\mathrm{Der}(\chi) = h\,\mathrm{Der}(\chi)$.

Keywords: n-derivation; Hom-Lie n-derivation; superalgebra; Hom-Lie superalgebra; centroid; quasi centroid
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1. Introduction

In 1973, Nambu generalized Hamiltonian mechanics [1], initiating the foundational study of n-ary algebras and shaping later theoretical developments. Further, in 1985, Filippov advanced the area by rigorously defining n-Lie algebras via a skew-symmetric bracket and by initiating their classification in the smallest nontrivial dimensions [2]. Subsequent work by Kasymov concentrated on the structural characteristics and representation theory of these algebras [3]. Since then, the field has progressively evolved, with numerous investigations deepening and refining the theoretical framework of n-ary algebras [4, 5]. Subsequently, the field has undergone substantial development of the theoretical framework of the same structure [6, 7], as well as [8–10]. Motivated by the work presented in [11, 12], we continue this line of study by introducing the notion of n-Hom-Lie superalgebras and examining their generalized derivations, which play a crucial role in advancing the theoretical understanding of the structure of n-Lie (super) algebras.

Hom-Lie algebraic structures were initially introduced by Makhlouf [13]. Later, Jonas [14] broadened this framework by examining particular deformations of the Virasoro and Witt algebras, along with graded quasi-Lie algebras [15–17] and quasi-Lie algebras [18]. Subsequent works, including [19, 20], extended these ideas to the context of Hom-Lie superalgebras in [21–23]. Sheng and Xiong [24] made important contributions to the study of representations of Hom-Lie algebras. The study of $\mathbb{Z}$-graded Hom-Lie superalgebras was then systematically developed in [25]. In addition, Armakan and Razavi [26] established criteria whereby a Hom-Lie superalgebra is complete, and this completeness under surjective twisting was further confirmed in [24, 27]. Conceptually, Hom-Lie superalgebras generalize Lie superalgebras by incorporating a twisted version of the super Jacobi identity. In the particular case where the twisting map is the identity, a Hom-Lie superalgebra reduces to an ordinary Lie superalgebra.

The study of derivations and generalized derivations plays a crucial role in understanding algebraic structures. Numerous authors, as reported in [28, 29], have explored derivations, generalized derivations seen in [30–32], centroids, and quasi-centroids from diverse viewpoints found in [33–35]. In particular, [36–38] highlight important properties of generalized derivations in the framework of Hom-Lie superalgebras, with a focus on their subalgebra structures and on how centroids and quasi-centroids interact with these derivation algebras. The notion of Lie triple derivations is likewise significantly obtained in [39–41]. Many studies, including those referenced in [42–44], have examined this concept in different settings, such as perfect Lie superalgebras, generalized Lie triple derivations of $gl(n, \mathbb{R})$, and their analogues in the algebras of strictly upper triangular matrices and in general Lie algebras. Moreover, important results concerning Lie triple derivations and biderivations in block algebras, incidence algebras, and triangular algebras are provided in [45–47].

This paper investigates the structure of Lie n-derivations, denoted by $hN\,\mathrm{Der}$, and generalized Lie n-derivations, denoted by $hGN\,\mathrm{Der}$, in the setting of Hom-Lie superalgebras χ. We study the associated subalgebras and identify several significant structural features. After introducing and discussing Hom-Lie n-derivations, their generalized counterparts, central n-derivations, and centroids, we prove that each of these objects forms a subalgebra of the general linear Hom-Lie superalgebra $hg(\chi)$. We also make their mutual relationships explicit. In particular, we show that the space $hN\,\mathrm{Der}(\chi)$ of all Hom-Lie n-derivations is stable under the usual super Lie bracket. Moreover, we establish that the sets of Hom-Lie n-centroids and n-quasi-centroids, $hNC(\chi)$ and $hNQC(\chi)$, are Hom-Lie subalgebras of $hg(\chi)$. Furthermore, we prove that the central n-derivations $hZN\,\mathrm{Der}(\chi)$ form a Hom-Lie superideal in both $hN\,\mathrm{Der}(\chi)$ and $hGN\,\mathrm{Der}(\chi)$. In the special case of centerless Hom-Lie superalgebras, we also show that $hNC(\chi)$ and $hNQC(\chi)$ commute. Our main theorem states that, under suitable hypotheses, the space of Hom-Lie n-derivations coincides with the space of Hom-Lie derivations, a result obtained via several supporting lemmas.

The structure of the paper is as follows. Section 2 recalls the key notations and foundational notions for Hom-Lie superalgebras. Section 3 introduces and studies the notions of n-centroids and n-quasi-centroids in this framework, and proves that these two structures commute whenever the underlying Hom-Lie superalgebra has a trivial center. Section 4 then shows that, for simple Hom-Lie superalgebras, Hom-Lie n-derivations do not bring any additional generality: They coincide exactly with ordinary Hom-Lie derivations.

Throughout this paper, we use the symbols $\mathbb{F}$, $\mathbb{Z}_2$, and $\mathbb{N}$: a field of characteristic 0, the ring of integers of modulo 2; and the set of non-negative integers; and the set $J = \{1, 2, \dots, n\}$. We also assume that $\tau : \chi \to \chi$ is a twisting map. We work with the common-twist case; that is, the family of twisting maps $(\tau_i)_{1 \leq i \leq n-1}$ is assumed to satisfy $\tau_1 = \tau_2 = \cdots = \tau_{n-1} = \tau$. Hence, all powers τ^k appearing later are understood to be the k-fold composition of this single map τ.

2. Lie n-derivations on Hom-Lie superalgebras

In [48], the authors studied α^k-derivations (where k is a fixed positive integer) on general multiplicative n-Hom-Lie superalgebras endowed with an intrinsic n-ary operation. However, in the present paper, the notion of τ^k-n-derivation is formulated on a multiplicative Hom-Lie superalgebra. Consider χ is a Hom-Lie superalgebra. This section begins with several basic definitions and notational conventions that are standard in the literature on Hom-Lie superalgebras. We introduce the notion of Hom-Lie n-derivations, with particular attention to τ^k-n-derivations for $k \in \mathbb{N}$. Finally, we broaden the framework by introducing generalized n-derivations in this setting.

A vector space χ is called $\mathbb{Z}_2$-graded if it admits a decomposition $\chi = \chi_{\bar{0}} \oplus \chi_{\bar{1}}$. An element $P \in \chi_\lambda$ with $\lambda \in \mathbb{Z}_2$ is called homogeneous of degree λ, and we write $|P|$ for its degree. We define $|[\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]| = |\mathsf{x}_1| + |\mathsf{x}_2| + \cdots + |\mathsf{x}_n| \pmod 2$, and in particular we have $|\mathsf{x}_0| = 0$. Now, let χ and $\mathcal{R}$ be two $\mathbb{Z}_2$-graded vector spaces. A linear map $f : \chi \to \mathcal{R}$ is called homogeneous of degree $\mu \in \mathbb{Z}_2$ if, for every $s \in \chi_\lambda$, the image $f(s)$ lies in $\mathcal{R}_{\lambda+\mu}$. The collection of all such linear maps forms a graded vector space denoted by $\hom(\chi, \mathcal{R}) = \hom(\chi, \mathcal{R})_{\bar{0}} \oplus \hom(\chi, \mathcal{R})_{\bar{1}}$. Furthermore, if f satisfies $f(\chi_\lambda) \subseteq \mathcal{R}_{\lambda+\mu}$ for all $\lambda \in \mathbb{Z}_2$, then f is said to be even when $\mu = \bar{0}$.

Definition 2.1. *[48] Let $n \geq 2$ be a fixed integer. An n-Lie superalgebra is a pair $(\chi, [\cdot, \cdots, \cdot])$, where $\chi = \chi_{\bar{0}} \oplus \chi_{\bar{1}}$ is a $\mathbb{Z}_2$-graded vector space and $[\cdot, \cdots, \cdot] : \underbrace{\chi \times \chi \times \cdots \times \chi}_{n} \to \chi$ is an n-linear operation, subject to the following conditions:*

$$|[\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]| = |\mathsf{x}_1| + |\mathsf{x}_2| + \cdots + |\mathsf{x}_n| \quad mod\, 2.$$

Moreover, the bracket is graded skew-symmetric. Namely, for every $1 \leq i \leq n-1$,

$$[\mathsf{x}_1, \cdots, \mathsf{x}_i, \mathsf{x}_{i+1}, \cdots, \mathsf{x}_n] = -(-1)^{|\mathsf{x}_i||\mathsf{x}_{i+1}|} [\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_{i+1}, \mathsf{x}_i, \cdots, \mathsf{x}_n] .$$

Finally, the graded Filippov identity holds:

$$[\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_{n-1}, [\mathsf{y}_1, \dots, \mathsf{y}_n]] = \sum_{i=1}^{n} (-1)^{|\mathsf{x}_{n-1}||Z_{i-1}|} \Big[z_1, z_2, \dots, z_{i-1}, [\mathsf{x}_1, \mathsf{x}_2, \dots, \mathsf{x}_{n-1}, z_i], z_{i+1}, \dots, z_n \Big],$$

for any $\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n, \mathsf{y}_1, \mathsf{y}_2, \dots, \mathsf{y}_n, z_1, \dots, z_n \in hg(\chi)$, $i \in J$.

Where

$$|\mathsf{x}| = |\mathsf{x}_1| + \cdots + |\mathsf{x}_{n-1}|$$

and

$$|Z_{i-1}| = |z_1| + \cdots + |z_{i-1}|, \qquad |Z_0| = 0.$$

When $\chi_{\bar{1}} = \{0\}$*, then* χ *is actually an n-Lie algebra.*

Definition 2.2. *[48] An n-Hom-Lie superalgebra is* $(\chi, [\cdot, \cdots, \cdot], \tau)$ *consisting of a* $\mathbb{Z}_2$*-graded vector space* $\chi = \chi_{\bar{0}} \oplus \chi_{\bar{1}}$ *and a multilinear map* $[\cdot, \cdots, \cdot] : \underbrace{\chi \times \chi \times \cdots \times \chi}_{n} \to \chi$ *and a family* $(\tau)_{1 \le i \le n-1}$ *of even linear maps* $\tau : \chi \to \chi$*, satisfying*

$$|[\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]| = |\mathsf{x}_1| + |\mathsf{x}_2| + \cdots + |\mathsf{x}_n| \quad mod\, 2,$$
$$[\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_i, \mathsf{x}_{i+1}, \cdots, \mathsf{x}_n] = -(-1)^{|\mathsf{x}_i|\mathsf{x}_{i+1}|} [\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_{i+1}, \mathsf{x}_i, \cdots, \mathsf{x}_n],$$

and the Hom-Nambu identity is given by

$$\begin{aligned}&[\tau(\mathsf{x}_1), \ldots, \tau(\mathsf{x}_{n-1}), [z_1, \ldots, z_n]]\\ &= \sum_{i=1}^{n} (-1)^{|\mathsf{x}_{n-1}||Z_{i-1}|} [\tau(z_1), \ldots, \tau(z_{i-1}), [\mathsf{x}_1, \ldots, \mathsf{x}_{n-1}, z_i], \tau(z_{i+1}), \ldots, \tau(z_n)],\end{aligned}$$

for any $\mathsf{x}_i, z_i \in hg(\chi), i \in J$*. If* $\tau = id_\chi$*, then* χ *is just an n-Lie superalgebra.*

In particular, when τ preserves the bracket, that is,

$$\tau([\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \cdots, \mathsf{x}_n]) = [[\tau(\mathsf{x}_1), \tau(\mathsf{x}_2), \tau(\mathsf{x}_3), \cdots, \tau(\mathsf{x}_n)],$$

for all $\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \cdots, \mathsf{x}_n \in \mathcal{N}$, then we call $(\mathcal{N}, [\cdot, \cdots, \cdot], \tau)$, a multiplicative Hom-Lie superalgebra.

Definition 2.3. *[48] An n-Hom-Lie superalgebra* $(\chi, [\cdot, \cdots, \cdot], \tau)$ *is multiplicative, if* $(\tau)_{1 \le i \le n-1}$ *with* τ *and satisfying*

$$\begin{aligned}&\tau([\mathsf{x}_1, [\mathsf{x}_2, \cdots, \mathsf{x}_n]) = [\tau(\mathsf{x}_1), \tau(\mathsf{x}_2), \cdots, \tau(\mathsf{x}_n)],\\ &[\tau(\mathsf{x}_1), \tau(\mathsf{x}_2), \cdots, \tau(\mathsf{x}_{n-1}), [z_1, z_2, \cdots, z_n]]\\ &= \sum_{i=1}^{n} (-1)^{|\mathsf{x}_{n-1}||Z_{i-1}|} [\tau(z_1), \tau(z_2), \cdots, \tau(z_{i-1}), [\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_{n-1}, z_i], \tau(z_{i+1}), \cdots, \tau(z_n)],\end{aligned}$$

for any $\mathsf{x}_i, z_i \in hg(\chi), i \in J$.

Definition 2.4. *Let* $(\chi, [\cdot, \ldots, \cdot], \tau)$ *be a multiplicative n-Hom-Lie superalgebra, and a Hom-Lie linear map* $D_t : \chi \to \chi$ *is called a* τ^k*-n-derivation of* χ *for* $k \in \mathbb{N}$*, if*

$$D_t\tau = \tau D_t,$$
$$D_t([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]) = \sum_{i=1}^{n} (-1)^{D_t|\mathsf{x}_{i-1}|} \left[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2) \cdots, \tau^k(\mathsf{x}_{i-1}), D_t(\mathsf{x}_i), \tau^k(\mathsf{x}_{i+1}), \cdots, \tau^k(\mathsf{x}_n)\right]$$

$$\begin{aligned} D_t(\,[\mathsf{x}_1,\mathsf{x}_2,\cdots,\mathsf{x}_n]\,) &= \Big[D_t([\cdots[[\mathsf{x}_1,\mathsf{x}_2],\mathsf{x}_3],\cdots,\mathsf{x}_{n-1}]),\tau^k(\mathsf{x}_n)\Big] \\ &\quad +(-1)^{|\mathsf{x}_1||D_t|}\Big[\tau^k([\cdots[[\mathsf{x}_1,\mathsf{x}_2],\mathsf{x}_3],\cdots,\mathsf{x}_{n-1}]),D_t(\mathsf{x}_n)\Big] \\ &= \Big[\cdots[[D_t(\mathsf{x}_1),\tau^k(\mathsf{x}_2)],\tau^k(\mathsf{x}_3)],\cdots,\tau^k(\mathsf{x}_n)\Big] \\ &\quad +(-1)^{|D_t||\mathsf{x}_1|}\Big[\cdots[[\tau^k(\mathsf{x}_1),D_t(\mathsf{x}_2)],\tau^k(\mathsf{x}_3)],\cdots,\tau^k(\mathsf{x}_n)\Big] \\ &\quad +(-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^k(\mathsf{x}_1),\tau^k(\mathsf{x}_2)],D_t(\mathsf{x}_3)],\cdots,\tau^k(\mathsf{x}_n)\Big] \\ &\quad +\cdots \\ &\quad +(-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} \\ &\quad .\Big[\cdots[[\tau^k(\mathsf{x}_1),\tau^k(\mathsf{x}_2)],\tau^k(\mathsf{x}_3)],\cdots,D_t(\mathsf{x}_n)\Big] \end{aligned}$$

for any $\mathsf{x}_i \in hg(\chi), i \in J$, where we denotes $[\mathsf{x}_1,\mathsf{x}_2,\cdots,\mathsf{x}_n] = [\cdots[[\mathsf{x}_1,\mathsf{x}_2],\mathsf{x}_3],\cdots,\mathsf{x}_n]$ and τ^k denotes the k-fold composition of the single twisting map τ, with $\tau^0 = \mathrm{id}_\chi$. The collection of all τ^k-n-derivations of χ is denoted by

$$hN\,\mathrm{Der}(\chi) = \bigoplus_{k\geq 0} hN\,\mathrm{Der}_{\tau^k}(\chi).$$

Similarly, we define

$$\mathrm{Der}(\chi) = \bigoplus_{k\geq 0} \mathrm{Der}_{\tau^k}(\chi).$$

Equipped with the super-commutator and the even map

$$\widetilde{\tau} : \mathrm{Der}(\chi) \longrightarrow \mathrm{Der}(\chi), \qquad \widetilde{\tau}(D) = D\tau,$$

the space $\mathrm{Der}(\chi)$ forms a Hom-subalgebra of Ω and is called the derivation algebra of χ. (see [49]).

Definition 2.5. *[48] Consider the vector subspace $\Omega \subseteq \mathrm{End}(\chi)$, consisting of all linear operators on χ given by*

$$\Omega = \{u \in \mathrm{End}(\chi) \mid u\tau = \tau u\},$$

and

$$\widetilde{\tau} : \Omega \to \Omega; \widetilde{\tau}(u) = \tau u.$$

Then, $(\Omega, [\cdot,\cdot], \widetilde{\tau})$ is a Hom-Lie superalgebra over $\mathbb{F}$ and satisfies the following:

$$[D_{t_1}, D_{t_2}] = D_{t_1}D_{t_2} - (-1)^{|D_{t_1}||D_{t_2}|}D_{t_1}D_{t_2}$$

for any $D_{t_1}, D_{t_2} \in hg(\Omega)$.

Definition 2.6. *[48] Let $(L, [\cdot,\cdot], \tau)$ be a Hom-Lie superalgebra. A graded subspace $M \subseteq L$ is called a Hom-subalgebra of L if it is stable under the twisting map and closed under the bracket, that is,*

$$\tau(M) \subseteq M \quad \text{and} \quad [D_{t_1}, D_{t_2}] \in M \text{ for all } D_{t_1}, D_{t_2} \in M.$$

A graded subspace $I \subseteq L$ is said to be a Hom-ideal of $(L, [\cdot,\cdot], \tau)$ if $\tau(I) \subseteq I$ and I is stable under the bracket operation $[\cdot,\cdot]$. In other words, for any $D_{t_1} \in L$ and $D_{t_2} \in I$, one has $[D_{t_1}, D_{t_2}] \in I$.

Proposition 2.1. *Prove that the set of n-derivations, $hN\,\mathrm{Der}(\chi)$, is always closed with respect to the standard super Lie bracket, for any Hom-Lie superalgebra χ.*

Proof. Let $D_{t1} \in hN\,\mathrm{Der}_{\tau^s}(\chi)$ and $D_{t2} \in hN\,\mathrm{Der}_{\tau^k}(\chi)$ be given. The standard super Lie bracket is defined as follows:

$$[D_{t_1}, D_{t_2}] = D_{t_1}D_{t_2} - (-1)^{|D_{t_1}||D_{t_2}|}D_{t_2}D_{t_1}. \tag{1}$$

Consider,

$$\begin{aligned}
&D_{t_1}D_{t_2}([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]) \\
&= D_{t_1}\Big(\Big[\cdots[[D_{t_2}(\mathsf{x}_1), lign\tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_2}||\mathsf{x}_1|}\Big[\cdots[[\tau^k(\mathsf{x}_1), D_{t_2}(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], D_{t_2}(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} \\
&\cdot\Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, D_{t_2}(\mathsf{x}_n)\Big]\Big) \\
&= \Big[\cdots[[D_{t_1}D_{t_2}(\mathsf{x}_1), \tau^s\tau^k(\mathsf{x}_2)], \tau^s\tau^k(\mathsf{x}_3)], \cdots, \tau^s\tau^k(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}||\mathsf{x}_1|}\Big[\cdots[[\tau^s D_{t_2}(\mathsf{x}_1), D_{t_1}\tau^k(\mathsf{x}_2)], \tau^s\tau^k(\mathsf{x}_3)], \cdots, \tau^s\tau^k(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^s D_{t_2}(\mathsf{x}_1), \tau^s\tau^k(\mathsf{x}_2)], D_{t_1}\tau^k(\mathsf{x}_3)], \cdots, \tau^s\tau^k(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[\tau^s D_{t_2}(\mathsf{x}_1), \tau^s\tau^k(\mathsf{x}_2)], \tau^s\tau^k(\mathsf{x}_3)], \cdots, D_{t_1}\tau^k(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_2}||\mathsf{x}_1|}\Big[\cdots[[D_{t_1}\tau^k(\mathsf{x}_1), \tau^s D_{t_2}(\mathsf{x}_2)], \tau^s\tau^k(\mathsf{x}_3)], \cdots, \tau^s\tau^k(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}||\mathsf{x}_1|}\Big[\cdots[[\tau^s\tau^k(\mathsf{x}_1), D_{t_1}D_{t_2}(\mathsf{x}_2)], \tau^s\tau^k(\mathsf{x}_3)], \cdots, \tau^s\tau^k(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^s\tau^k(\mathsf{x}_1), \tau^s D_{t_2}(\mathsf{x}_2)], D_{t_1}\tau^k(\mathsf{x}_3)], \cdots, \tau^s\tau^k(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[\tau^s\tau^k(\mathsf{x}_1), \tau^s D_{t_2}(\mathsf{x}_2)], \tau^s\tau^k(\mathsf{x}_3)], \cdots, D_{t_1}\tau^k(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[D_{t_1}\tau^k(\mathsf{x}_1), \tau^s\tau^k(\mathsf{x}_2)], \tau^s D_{t_2}(\mathsf{x}_3)], \cdots, \tau^s\tau^k(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}||\mathsf{x}_1|}\Big[\cdots[[\tau^s\tau^k(\mathsf{x}_1), D_{t_1}\tau^k(\mathsf{x}_2)], \tau^s D_{t_2}(\mathsf{x}_3)], \cdots, \tau^s\tau^k(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^s\tau^k(\mathsf{x}_1), \tau^s\tau^k(\mathsf{x}_2)], D_{t_1}D_{t_2}(\mathsf{x}_3)], \cdots, \tau^s\tau^k(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[\tau^s\tau^k(\mathsf{x}_1), \tau^s\tau^k(\mathsf{x}_2)], D_{t_2}\tau^k(\mathsf{x}_3)], \cdots, D_{t_1}\tau^k(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[D_{t_1}\tau^k(\mathsf{x}_1), \tau^s\tau^k(\mathsf{x}_2)], \tau^s\tau^k(\mathsf{x}_3)], \cdots, \tau^s D_{t_2}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}||\mathsf{x}_1|}\Big[\cdots[[\tau^s\tau^k(\mathsf{x}_1), D_{t_1}\tau^k(\mathsf{x}_2)], \tau^s\tau^k(\mathsf{x}_3)], \cdots, \tau^s D_{t_2}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^s\tau^k(\mathsf{x}_1), \tau^s\tau^k(\mathsf{x}_2)], D_{t_1}\tau^k(\mathsf{x}_3)], \cdots, \tau^s D_{t_2}(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[\tau^s\tau^k(\mathsf{x}_1), \tau^s\tau^k(\mathsf{x}_2)], \tau^s\tau^k(\mathsf{x}_3)], \cdots, D_{t_1}D_{t_2}(\mathsf{x}_n)\Big]
\end{aligned} \tag{2}$$

$$
\begin{aligned}
&= \Big[\cdots[[D_{t_1}D_{t_2}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|D_{t_2}|)}\Big[\cdots[[\tau^{s}D_{t_2}(\mathsf{x}_1), D_{t_1}\tau^{k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)]\cdots\tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|D_{t_2}|)}\Big[\cdots[[\tau^{s}D_{t_2}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], D_{t_1}\tau^{k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|+|D_{t_2}|)}\Big[\cdots[[\tau^{s}D_{t_2}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, D_{t_1}\tau^{k}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_2}||\mathsf{x}_1|}\Big[\cdots[[D_{t_1}\tau^{k}(\mathsf{x}_1), \tau^{s}D_{t_2}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}||\mathsf{x}_1|}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), D_{t_1}D_{t_2}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|D_{t_2}|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s}D_{t_2}(\mathsf{x}_2)], D_{t_1}\tau^{k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|+|D_{t_2}|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s}D_{t_2}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, D_{t_1}\tau^{k}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[D_{t_1}\tau^{k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s}D_{t_2}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}||\mathsf{x}_1|}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), D_{t_1}\tau^{k}(\mathsf{x}_2)], \tau^{s}D_{t_2}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], D_{t_1}D_{t_2}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|+|D_{t_2}|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s}D_{t_2}(\mathsf{x}_3)], \cdots, D_{t_1}\tau^{k}(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[D_{t_1}\tau^{k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s}D_{t_2}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}||\mathsf{x}_1|}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), D_{t_1}\tau^{k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s}D_{t_2}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], D_{t_1}\tau^{k}(\mathsf{x}_3)], \cdots, \tau^{s}D_{t_2}(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, D_{t_1}D_{t_2}(\mathsf{x}_n)\Big] \\
&= \Big[\cdots[[D_{t_1}D_{t_2}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}|(|D_{t_2}|+|\mathsf{x}_1|)}\Big[\cdots[[\tau^{s}D_{t_2}(\mathsf{x}_1), D_{t_1}\tau^{k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_1}|(|D_{t_2}|+|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^{s}D_{t_2}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], D_{t_1}\tau^{k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_{t_1}|\left(|D_{t_2}|+|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|\right)}\Big[\cdots[[\tau^{s}D_{t_2}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, D_{t_1}\tau^{k}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_2}||\mathsf{x}_1|}\Big[\cdots[[D_{t_1}\tau^{k}(\mathsf{x}_1), \tau^{s}D_{t_2}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ (-1)^{(|D_{t_1}|+|D_{t_2}|)|\mathsf{x}_1|}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), D_{t_1}D_{t_2}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_{t_2}||\mathsf{x}_1|}(-1)^{|D_{t_1}|(|D_{t_2}|+|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s}D_{t_2}(\mathsf{x}_2)], D_{t_1}\tau^{k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_{t_2}||\mathsf{x}_1|}(-1)^{|D_{t_1}|\left(|D_{t_2}|+|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|\right)}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1),\tau^{s}D_{t_2}(\mathsf{x}_2)],\tau^{s+k}(\mathsf{x}_3)],\cdots,D_{t_1}\tau^{k}(\mathsf{x}_n)\Big]\\
&+(-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[D_{t_1}\tau^{k}(\mathsf{x}_1),\tau^{s+k}(\mathsf{x}_2)],\tau^{s}D_{t_2}(\mathsf{x}_3)],\cdots,\tau^{s+k}(\mathsf{x}_n)\Big]\\
&+(-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}(-1)^{|D_{t_1}|(|\mathsf{x}_1|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1),D_{t_1}\tau^{k}(\mathsf{x}_2)],\tau^{s}D_{t_2}(\mathsf{x}_3)],\cdots,\tau^{s+k}(\mathsf{x}_n)\Big]\\
&+(-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}(-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1),\tau^{s+k}(\mathsf{x}_2)],D_{t_1}D_{t_2}(\mathsf{x}_3)],\cdots,\tau^{s+k}(\mathsf{x}_n)\Big]\\
&+\cdots\\
&+(-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}(-1)^{|D_{t_1}|\left(|D_{t_2}|+|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|\right)}\\
&\cdot\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1),\tau^{s+k}(\mathsf{x}_2)],\tau^{s}D_{t_2}(\mathsf{x}_3)],\cdots,D_{t_1}\tau^{k}(\mathsf{x}_n)\Big]\\
&+\cdots\\
&+(-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[D_{t_1}\tau^{k}(\mathsf{x}_1),\tau^{s+k}(\mathsf{x}_2)],\tau^{s+k}(\mathsf{x}_3)],\cdots,\tau^{s}D_{t_2}(\mathsf{x}_n)\Big]\\
&+(-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}(-1)^{|D_{t_1}||\mathsf{x}_1|}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1),D_{t_1}\tau^{k}(\mathsf{x}_2)],\tau^{s+k}(\mathsf{x}_3)],\cdots,\tau^{s}D_{t_2}(\mathsf{x}_n)\Big]\\
&+(-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}(-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\\
&\cdot\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1),\tau^{s+k}(\mathsf{x}_2)],D_{t_1}\tau^{k}(\mathsf{x}_3)],\cdots,\tau^{s}D_{t_2}(\mathsf{x}_n)\Big]\\
&+\cdots\\
&+(-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}(-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\\
&\cdot\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1),\tau^{s+k}(\mathsf{x}_2)],\tau^{s+k}(\mathsf{x}_3)],\cdots,D_{t_1}D_{t_2}(\mathsf{x}_n)\Big].
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&D_{t_2}D_{t_1}([\mathsf{x}_1,\mathsf{x}_2,\cdots,\mathsf{x}_n])\\
&=D_{t_2}\Big(\Big[\cdots[[D_{t_1}(\mathsf{x}_1),\tau^{k}(\mathsf{x}_2)],\tau^{k}(\mathsf{x}_3)],\cdots,\tau^{k}(\mathsf{x}_n)\Big]\\
&+(-1)^{|D_{t_1}||\mathsf{x}_1|}\Big[\cdots[[\tau^{k}(\mathsf{x}_1),D_{t_1}(\mathsf{x}_2)],\tau^{k}(\mathsf{x}_3)],\cdots,\tau^{k}(\mathsf{x}_n)\Big]\\
&+(-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^{k}(\mathsf{x}_1),\tau^{k}(\mathsf{x}_2)],D_{t_1}(\mathsf{x}_3)],\cdots,\tau^{k}(\mathsf{x}_n)\Big]\\
&+\cdots\\
&+(-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\\
&\cdot\Big[\cdots[[\tau^{k}(\mathsf{x}_1),\tau^{k}(\mathsf{x}_2)],\tau^{k}(\mathsf{x}_3)],\cdots,D_{t_1}(\mathsf{x}_n)\Big]\Big)\\
&=\Big[\cdots[[D_{t_2}D_{t_1}(\mathsf{x}_1),\tau^{s+k}(\mathsf{x}_2)],\tau^{s+k}(\mathsf{x}_3)],\cdots,\tau^{s+k}(\mathsf{x}_n)\Big]\\
&+(-1)^{|D_{t_2}|(|D_{t_1}|+|\mathsf{x}_1|)}\Big[\cdots[[\tau^{s}D_{t_1}(\mathsf{x}_1),D_{t_2}\tau^{k}(\mathsf{x}_2)],\tau^{s+k}(\mathsf{x}_3)],\cdots,\tau^{s+k}(\mathsf{x}_n)\Big]\\
&+(-1)^{|D_{t_2}|(|D_{t_1}|+|\mathsf{x}_1|+|\mathsf{x}_2|)}\\
&\cdot\Big[\cdots[[\tau^{s}D_{t_1}(\mathsf{x}_1),\tau^{s+k}(\mathsf{x}_2)],D_{t_2}\tau^{k}(\mathsf{x}_3)],\cdots,\tau^{s+k}(\mathsf{x}_n)\Big]\\
&+\cdots\\
&+(-1)^{|D_{t_2}|\left(|D_{t_1}|+|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|\right)}\Big[\cdots[[\tau^{s}D_{t_1}(\mathsf{x}_1),\tau^{s+k}(\mathsf{x}_2)],\tau^{s+k}(\mathsf{x}_3)],\cdots,D_{t_2}\tau^{k}(\mathsf{x}_n)\Big]\\
&+(-1)^{|D_{t_1}||\mathsf{x}_1|}\Big[\cdots[[D_{t_2}\tau^{k}(\mathsf{x}_1),\tau^{s}D_{t_1}(\mathsf{x}_2)],\tau^{s+k}\mathsf{x}_3],\cdots,\tau^{s+k}(\mathsf{x}_n)\Big]\\
&+(-1)^{|D_{t_1}||\mathsf{x}_1|}(-1)^{|D_{t_2}||\mathsf{x}_1|}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1),D_{t_2}D_{t_1}(\mathsf{x}_2)],\tau^{s+k}(\mathsf{x}_3)],\cdots,\tau^{s+k}(\mathsf{x}_n)\Big]
\end{aligned}
\tag{3}
$$

$$+(-1)^{|D_{t_1}||x_1|}(-1)^{|D_{t_2}|(|D_{t_1}|+|x_1|+|x_2|)}\Big[\cdots[[\tau^{s+k}(x_1),\tau^s D_{t_1}(x_2)],D_{t_2}\tau^k(x_3)],\cdots,\tau^{s+k}(x_n)\Big]$$
$$+\cdots$$
$$+(-1)^{|D_{t_1}||x_1|}(-1)^{|D_{t_2}|\left(|D_{t_1}|+|x_1|+|x_2|+|x_3|+\cdots+|x_{n-1}|\right)}$$
$$\cdot\Big[\cdots[[\tau^{s+k}(x_1),\tau^s D_{t_1}(x_2)],\tau^{s+k}(x_3)],\cdots,D_{t_2}\tau^k(x_n)\Big]$$
$$+(-1)^{|D_{t_1}|(|x_1|+|x_2|)}\Big[\cdots[[D_{t_2}\tau^k(x_1),\tau^{s+k}(x_2)],\tau^s D_{t_1}(x_3)],\cdots,\tau^{s+k}(x_n)\Big]$$
$$+(-1)^{|D_{t_1}|(|x_1|+|x_2|)}(-1)^{|D_{t_2}|(|x_1|)}\Big[\cdots[[\tau^{s+k}(x_1),D_{t_2}\tau^k(x_2)],\tau^s D_{t_1}(x_3)],\cdots,\tau^{s+k}(x_n)\Big]$$
$$+(-1)^{|D_{t_1}|(|x_1|+|x_2|)}(-1)^{|D_{t_2}|(|x_1|+|x_2|)}\Big[\cdots[[\tau^{s+k}(x_1),\tau^{s+k}(x_2)],D_{t_2}D_{t_1}(x_3)],\cdots,\tau^{s+k}(x_n)\Big]$$
$$+\cdots$$
$$+(-1)^{|D_{t_1}|(|x_1|+|x_2|)}(-1)^{|D_{t_2}|\left(|D_{t_1}|+|x_1|+|x_2|+|x_3|+\cdots+|x_{n-1}|\right)}$$
$$\cdot\Big[\cdots[[\tau^{s+k}(x_1),\tau^{s+k}(x_2)],\tau^s D_{t_1}(x_3)],\cdots,D_{t_2}\tau^k(x_n)\Big]$$
$$+\cdots$$
$$+(-1)^{|D_{t_1}|(|x_1|+|x_2|+|x_3|+\cdots+|x_{n-1}|)}\Big[\cdots[[D_{t_2}\tau^k(x_1),\tau^{s+k}(x_2)],\tau^{s+k}(x_3)],\cdots,\tau^s D_{t_1}(x_n)\Big]$$
$$+(-1)^{|D_{t_1}|(|x_1|+|x_2|+|x_3|+\cdots+|x_{n-1}|)}(-1)^{|D_{t_2}||x_1|}\Big[\cdots[[\tau^{s+k}(x_1),D_{t_2}\tau^k(x_2)],\tau^{s+k}(x_3)],\cdots,\tau^s D_{t_1}(x_n)\Big]$$
$$+(-1)^{|D_{t_1}|(|x_1|+|x_2|+|x_3|+\cdots+|x_{n-1}|)}(-1)^{|D_{t_2}|(|x_1|+|x_2|)}$$
$$\cdot\Big[\cdots[[\tau^{s+k}(x_1),\tau^{s+k}(x_2)],D_{t_2}\tau^k(x_3)],\cdots,\tau^s D_{t_1}(x_n)\Big]$$
$$+\cdots$$
$$+(-1)^{|D_{t_1}|(|x_1|+|x_2|+|x_3|+\cdots+|x_{n-1}|)}(-1)^{|D_{t_2}|(|x_1|+|x_2|+|x_3|+\cdots+|x_{n-1}|)}$$
$$\cdot\Big[\cdots[[\tau^{s+k}(x_1),\tau^{s+k}(x_2)],\tau^{s+k}(x_3)],\cdots,D_{t_2}D_{t_1}(x_n)\Big].$$

From Eqs (1)–(3), it is not hard to show $h\,\mathrm{NDer}(\chi)$ is always closed under the usual super Lie bracket. □

Definition 2.7. *Let χ be a Hom-Lie superalgebra. The super Lie bracket on* $\mathrm{End}(\chi)$ *is defined as*

$$(P)=D_{t_1}D_{t_2}(P)-(-1)^{|D_{t_1}||D_{t_2}|}D_{t_2}D_{t_1}(P),\quad \textit{for all} P\in\chi.$$

This structure is referred to as the general Hom-Lie superalgebra of χ, and is represented by $hg(\chi)$.

Proposition 2.2. *Prove that $hN\,\mathrm{Der}(\chi)$ is Hom-Lie subalgebra of $hg(\chi)$.*

Proof. To show $hN\,\mathrm{Der}(\chi)\subseteq hg(\chi)$. For any $D_{t_1}\in hN\,\mathrm{Der}_{\tau^s}(\chi)$ and $D_{t_2}\in hNDer_{\tau^k}(\chi)$, we must satisfy $[D_{t_1},D_{t_2}]\in hN\,\mathrm{Der}_{\tau^s+\tau^k}(\chi)$. Consequently, by invoking Proposition 2.1, the argument is concluded, and the proof is thereby complete. □

Definition 2.8. *[48] An endomorphism $D\in\mathrm{End}(\chi)$ is a homogeneous generalized α^k-derivation of χ for n if the endomorphisms $D_t\in\mathrm{End}(\chi)$ exist such that $D_t\alpha=\alpha D_t$, and*

$$D_t([x_1,x_2,\ldots,x_n])=[D_t(x_1),\alpha^k(x_2),\ldots,\alpha^k(x_n)]$$
$$+\sum_{i=2}^{n}(-1)^{|D_t||x_{i-1}|}[\alpha^k(x_1),\ldots,\alpha^k(x_{i-1}),D_t^{(i-1)}(x_i),\alpha^k(x_{i+1}),\ldots,\alpha^k(x_n)],$$

for any $x_i\in hg(\chi),i\in J$.

Definition 2.9. *[48] A multiplicative Hom-Lie superalgebra χ having $D_t : \chi \to \chi$ is termed as τ^k -generalized n-derivation of χ, if a τ^k-n-derivation map $E_t : \chi \to \chi$ such that, $D_t\tau = \tau D_t, E_t\tau = \tau E_t$,*

$$\begin{aligned} D_t([\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n]) &= [D_t(\mathsf{x}_1), \tau^k(\mathsf{x}_2), \ldots, \tau^k(\mathsf{x}_n)] \\ &\quad + \sum_{i=2}^{n} (-1)^{|D_t||X_{i-1}|}[\tau^k(\mathsf{x}_1), \ldots, \tau^k(\mathsf{x}_{i-1}), E_t(\mathsf{x}_i), \tau^k(\mathsf{x}_{i+1}), \ldots, \tau^k(\mathsf{x}_n)], \end{aligned}$$

for any $\mathsf{x}_i \in hg(\chi), i \in J$.

$$\begin{aligned} D_t(\,[\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]\,) &= \Big[D_t([\cdots[[\mathsf{x}_1, \mathsf{x}_2], \mathsf{x}_3], \cdots, \mathsf{x}_{n-1}]), \tau^k(\mathsf{x}_n)\Big] \\ &\quad +(-1)^{|\mathsf{x}_1||D_t|}\Big[\tau^k([\cdots[[\mathsf{x}_1, \mathsf{x}_2], \mathsf{x}_3], \cdots, \mathsf{x}_{n-1}), E_t(\mathsf{x}_n)\Big] \\ &= \Big[\cdots[[D_t(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\ &\quad +(-1)^{|D_t||\mathsf{x}_1|}\Big[\cdots[[\tau^k(\mathsf{x}_1), E_t(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\ &\quad +(-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], E_t(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\ &\quad +\cdots \\ &\quad +(-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} \\ &\quad \cdot\Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)]\cdots E_t(\mathsf{x}_n)\Big]. \end{aligned}$$

Observe that $|D_t| = |E_t|$. We represent all n-derivations on a multiplicative Hom-Lie superalgebra as $hGNDer_{\tau^k}(\chi)$. Indeed, if $D_t = E_t$, then $hN\,\mathrm{Der}(\chi)$ is contained in $hGN\,\mathrm{Der}(\chi)$, although the reverse implication does not hold in all cases.

Proposition 2.3. *Let χ be a Hom-Lie superalgebra. Prove that the set of generalized Hom-Lie n-derivations $hGNDer(\chi)$ of χ is a Hom-Lie subalgebra of $hg(\chi)$.*

Proof. To prove $hGN\,\mathrm{Der}(\chi) \subseteq hg(\chi)$. for any $D_{t_1} \in hGNDer_{\tau^s}(\chi)$ and $D_{t_2} \in hGNDer_{\tau^k}(\chi)$, we must satisfy $[D_{t_1}, D_{t_2}] \in hGN\,\mathrm{Der}_{\tau^{k+s}}(\chi)$. Since

$$\begin{aligned} D_{t_1}(\,[\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]\,) &= [D_{t_1}([\cdots[[\mathsf{x}_1, \mathsf{x}_2], \mathsf{x}_3], \cdots, \mathsf{x}_{n-1}]), \tau^s(\mathsf{x}_n)] \\ &\quad +(-1)^{|\mathsf{x}_1||D_{t_1}|}[\tau^s([\cdots[[\mathsf{x}_1, \mathsf{x}_2], \mathsf{x}_3], \cdots, \mathsf{x}_{n-1}]), E_{t_1}(\mathsf{x}_n)] \\ &= [\cdots[[D_{t_1}(\mathsf{x}_1), \tau^s(\mathsf{x}_2)], \tau^s(\mathsf{x}_3)], \cdots, \tau^s(\mathsf{x}_n)] \\ &\quad +(-1)^{|D_{t_1}||\mathsf{x}_1|}[\cdots[[\tau^s(\mathsf{x}_1), E_{t_1}(\mathsf{x}_2)], \tau^s(\mathsf{x}_3)], \cdots, \tau^s(\mathsf{x}_n)] \\ &\quad +(-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}[\cdots[[\tau^s(\mathsf{x}_1), \tau^s(\mathsf{x}_2)], E_{t_1}(\mathsf{x}_3)], \cdots, \tau^s(\mathsf{x}_n)] \\ &\quad +\cdots \\ &\quad +(-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} \\ &\quad \cdot[\cdots[[\tau^s(\mathsf{x}_1), \tau^s(\mathsf{x}_2)], \tau^s(\mathsf{x}_3)], \cdots, E_{t_1}(\mathsf{x}_n)]. \end{aligned}$$

$$\begin{aligned} D_{t_2}(\,[\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]\,) &= \Big[D_{t_2}([\cdots[[\mathsf{x}_1, \mathsf{x}_2], \mathsf{x}_3], \cdots, \mathsf{x}_{n-1}]), \tau^k(\mathsf{x}_n)\Big] \\ &\quad +(-1)^{|\mathsf{x}_1||D_{t_2}|}\Big[\tau^k([\cdots[[\mathsf{x}_1, \mathsf{x}_2], \mathsf{x}_3], \cdots, \mathsf{x}_{n-1}]), E_{t_2}(\mathsf{x}_n)\Big] \\ &= \Big[\cdots[[D_{t_2}(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\ &\quad +(-1)^{|D_{t_2}||\mathsf{x}_1|}\Big[\cdots[[\tau^k(\mathsf{x}_1), E_{t_2}(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\ &\quad +(-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], E_{t_2}(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\ &\quad +\cdots \\ &\quad +(-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} \\ &\quad \cdot\Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, E_{t_2}(\mathsf{x}_n)\Big]. \end{aligned}$$

For any $\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \cdots, \mathsf{x}_{n-1} \in hg(\chi), \mathsf{x}_n \in \chi$, Furthermore, we can obtain,

$$
\begin{aligned}
& D_{t_1}D_{t_2}([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]) \\
& = D_{t_1}(\Big[\cdots[[D_{t_2}(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
& + (-1)^{|D_{t_2}||\mathsf{x}_1|}\Big[\cdots[[\tau^k(\mathsf{x}_1), E_{t_2}(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
& + (-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], E_{t_2}(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
& + \cdots \\
& + (-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} \\
& \cdot \Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, E_{t_2}(\mathsf{x}_n)\Big]) \\
= & D_{t_1}(\Big[\cdots[[D_{t_2}(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big]) \\
& + (-1)^{|D_{t_2}||\mathsf{x}_1|}D_{t_1}(\Big[\cdots[[\tau^k(\mathsf{x}_1), E_{t_2}(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big]) \\
& + (-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}D_{t_1}(\Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], E_{t_2}(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big]) \\
& + \cdots \\
& + (-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} \\
& \cdot D_{t_1}(\Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, E_{t_2}(\mathsf{x}_n)\Big]) \\
& = \Big[\cdots[[D_{t_1}D_{t_2}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
& + (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|D_{t_2}|)}\Big[\cdots[[\tau^s D_{t_2}(\mathsf{x}_1), E_{t_1}\tau^k(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
& + (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|D_{t_2}|)}\Big[\cdots[[\tau^s D_{t_2}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], E_{t_1}\tau^k(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
& + \cdots \\
& + (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|+|D_{t_2}|)}\Big[\cdots[[\tau^s D_{t_2}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, E_{t_1}\tau^k(\mathsf{x}_n)\Big] \\
& + (-1)^{|D_{t_2}||\mathsf{x}_1|}\Big[\cdots[[D_{t_1}\tau^k(\mathsf{x}_1), \tau^s E_{t_2}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
& + (-1)^{|D_{t_1}||\mathsf{x}_1|}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), E_{t_1}E_{t_2}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
& + (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|D_{t_2}|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^s E_{t_2}(\mathsf{x}_2)], E_{t_1}\tau^k(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
& + \cdots \\
& + (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|+|D_{t_2}|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^s E_{t_2}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, E_{t_1}\tau^k(\mathsf{x}_n)\Big] \\
& + (-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[D_{t_1}\tau^k(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^s E_{t_2}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
& + (-1)^{|D_{t_1}||\mathsf{x}_1|}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), E_{t_1}\tau^k(\mathsf{x}_2)], \tau^s E_{t_2}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
& + (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], E_{t_1}E_{t_2}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big] \\
& + \cdots \\
& + (-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|+|D_{t_2}|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^s E_{t_2}(\mathsf{x}_3)], \cdots, E_{t_1}\tau^k(\mathsf{x}_n)\Big] \\
& + \cdots \\
& + (-1)^{|D_{t_2}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[D_{t_1}\tau^k(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^s E_{t_2}(\mathsf{x}_n)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+(-1)^{|D_{t_1}||\mathsf{x}_1|}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), E_{t_1}\tau^{k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s}E_{t_2}(\mathsf{x}_n)\Big]\\
&+(-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], E_{t_1}\tau^{k}(\mathsf{x}_3)], \cdots, \tau^{s}E_{t_2}(\mathsf{x}_n)\Big]\\
&+\cdots\\
&+(-1)^{|D_{t_1}|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, E_{t_1}E_{t_2}(\mathsf{x}_n)\Big].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&D_{t_1}D_{t_2}([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n])\\
=&\Big[\cdots[[D_{t_1}D_{t_2}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big]\\
&+(-1)^{(|D_{t_2}|+|D_{t_1}|)|\mathsf{x}_1|}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), E_{t_1}E_{t_2}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big]\\
&+(-1)^{(|D_{t_1}|+|D_{t_2}|)(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], E_{t_1}E_{t_2}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big]\\
&+\cdots\\
&+(-1)^{(|D_{t_1}|+|D_{t_2}|)(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)]\cdots E_{t_1}E_{t_2}(\mathsf{x}_n)\Big].
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&D_{t_2}D_{t_1}([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n])\\
=&\Big[\cdots[[D_{t_2}D_{t_1}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big]\\
&+(-1)^{(|D_{t_1}|+|D_{t_2}|)|\mathsf{x}_1|}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), E_{t_2}E_{t_1}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big]\\
&+(-1)^{(|D_{t_1}|+|D_{t_2}|)(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], E_{t_2}E_{t_1}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big]\\
&+\cdots\\
&+(-1)^{(|D_{t_1}|+|D_{t_2}|)(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)]\cdots E_{t_2}E_{t_1}(\mathsf{x}_n)\Big].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&D_{t_1}D_{t_2}([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n])\\
=&\Big[\cdots[[[D_{t_1}, D_{t_2}](\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big]\\
&+(-1)^{(||D_{t_2}|+D_{t_1}|)|\mathsf{x}_1|}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), [E_{t_1}, E_{t_2}](\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big]\\
&+(-1)^{(|D_{t_1}|+|D_{t_2}|)(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], [E_{t_1}, E_{t_2}](\mathsf{x}_3)], \cdots, \tau^{s+k}(\mathsf{x}_n)\Big]\\
&+\cdots\\
&+(-1)^{(|D_{t_1}|+|D_{t_2}|)(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[\tau^{s+k}(\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2)], \tau^{s+k}(\mathsf{x}_3)]\cdots[E_{t_1}, E_{t_2}](\mathsf{x}_n)\Big].
\end{aligned}
$$

Hence, $[D_{t_1}, D_{t_2}] \in hGN\,\mathrm{Der}_{\tau^{k+s}}(\chi)$, i.e., $hGN\,\mathrm{Der}(\chi)$ is a Hom-Lie subalgebra of $hg(\chi)$. This establishes the result. □

Remark 2.1. *The sign computations in Propositions 2.1 and 2.3 follow from repeated applications of the graded Leibniz rule and the super-commutator identity*

$$[D_{t_1}, D_{t_2}] = D_{t_1}D_{t_2}(-1)^{|D_{t_1}||D_{t_2}|}D_{t_2}D_{t_1}.$$

Indeed, whenever a homogeneous operator of degree $|D|$ passes through a homogeneous element of degree $|x|$, a factor $(-1)^{|D||x|}$ is produced. Hence, all signs occurring in the expansions are determined inductively by the Koszul sign convention, and the closure properties of $hN\operatorname{Der}(\chi)$ and $hGN\operatorname{Der}(\chi)$ under the super Lie bracket are consequences of standard graded-commutator arguments.

Lemma 2.1. *For any $\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \cdots, \mathsf{x}_{n-1}, \mathsf{x}_n \in \chi, D_t \in hGT\operatorname{Der}_{\tau^k}(\chi)$, and E_t that is the corresponding n-derivation of D_t, it follows that*

$$\begin{aligned}
\textit{(i)}.\ & (D_t - E_t)([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]) \\
&= \Big[\cdots[[(D_t - E_t)(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
&= (-1)^{|D_t||\mathsf{x}_1|}\Big[\cdots[[\tau^k(\mathsf{x}_1), (D_t - E_t)(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
&= (-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], (D_t - E_t)(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
&= \cdots \\
&= (-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)]\cdots(D_t - E_t)(\mathsf{x}_n)\Big].
\end{aligned}$$

$$\begin{aligned}
\textit{(ii)}.\ D_t([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]) =& \Big[\cdots[[D_t(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_t||\mathsf{x}_1|}\Big[\cdots[[\tau^k(\mathsf{x}_1), D_t(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
&+ (-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], D_t(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
&+ \cdots \\
&+ (-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} \\
&\cdot \Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, D_t(\mathsf{x}_n)\Big].
\end{aligned}$$

$$\begin{aligned}
\textit{(iii)}.\ & (D_t - E_t)([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]) \\
&= \Big[\cdots[(D_t - E_t)(\mathsf{x}_1), [\tau^k(\mathsf{x}_2), \tau^k(\mathsf{x}_3)]], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
&= (-1)^{|D_t||\mathsf{x}_1|}\Big[\cdots[\tau^k(\mathsf{x}_1), [(D_t - E_t)(\mathsf{x}_2), \tau^k(\mathsf{x}_3)]], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
&= (-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[\tau^k(\mathsf{x}_1), [\tau^k(\mathsf{x}_2), (D_t - E_t)(\mathsf{x}_3)]], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
&= \cdots \\
&= (-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\Big[\cdots[\tau^k(\mathsf{x}_1), [\tau^k(\mathsf{x}_2), \tau^k(\mathsf{x}_3)]], \cdots, (D_t - E_t)(\mathsf{x}_n)\Big].
\end{aligned}$$

Proof. **(i)** Since $D_t \in hGDer^N_{\tau^k}(\chi)$, an associated τ^k-n-derivation E_t exists such that $|D_t| = |E_t|$ and

$$D_t\tau = \tau D_t, \qquad E_t\tau = \tau E_t.$$

By Definition 2.9, we have

$$\begin{aligned}
D_t([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]) =& [D_t(\mathsf{x}_1), \tau^k(\mathsf{x}_2), \ldots, \tau^k(\mathsf{x}_n)] \\
&+ \sum_{i=2}^{N}(-1)^{|D_t|(|\mathsf{x}_1|+\cdots+|\mathsf{x}_{i-1}|)}[\tau^k(\mathsf{x}_1), \ldots, \tau^k(\mathsf{x}_{i-1}), E_t(\mathsf{x}_i), \tau^k(\mathsf{x}_{i+1}), \ldots, \tau^k(\mathsf{x}_n)].
\end{aligned}$$

On the other hand, since E_t is a τ^k-n-derivation, we have

$$E_t([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]) = [E_t(\mathsf{x}_1), \tau^k(\mathsf{x}_2), \ldots, \tau^k(\mathsf{x}_n)] + \sum_{i=2}^{N} (-1)^{|E_t|(|\mathsf{x}_1|+\cdots+|\mathsf{x}_{i-1}|)} [\tau^k(\mathsf{x}_1), \ldots, \tau^k(\mathsf{x}_{i-1}), E_t(\mathsf{x}_i), \tau^k(\mathsf{x}_{i+1}), \ldots, \tau^k(\mathsf{x}_n)].$$

Because $|D_t| = |E_t|$, the signs in the two equations above are identical. Subtracting the two equations above, we obtain

$$(D_t - E_t)([\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n]) = [\cdots[[(D_t - E_t)(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \ldots, \tau^k(\mathsf{x}_n)]. \tag{4}$$

Now consider

$$\begin{aligned}
&(D_t - E_t)([\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n]) \\
&= -(-1)^{|\mathsf{x}_1||\mathsf{x}_2|}(D_t - E_t)([\cdots[[\tau^k(\mathsf{x}_2), \tau^k(\mathsf{x}_1)], \tau^k(\mathsf{x}_3)], \ldots, \tau^k(\mathsf{x}_n)]) \\
&= -(-1)^{|\mathsf{x}_1||\mathsf{x}_2|}[\cdots[[(D_t - E_t)(\mathsf{x}_2), \tau^k(\mathsf{x}_1)], \tau^k(\mathsf{x}_3)], \ldots, \tau^k(\mathsf{x}_n)] \\
&= -(-1)^{|\mathsf{x}_1||\mathsf{x}_2|+\left(|D_t|+|\mathsf{x}_2|\right)|\mathsf{x}_1|}[\cdots[[\tau^k(\mathsf{x}_1), (D_t - E_t)(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \ldots, \tau^k(\mathsf{x}_n)] \\
&= (-1)^{|D_t||\mathsf{x}_1|}[\cdots[[\tau^k(\mathsf{x}_1), (D_t - E_t)(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \ldots, \tau^k(\mathsf{x}_n)] \\
&= \cdots \\
&= (-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|+\cdots+|\mathsf{x}_{n-1}|)}[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \ldots, (D_t - E_t)(\mathsf{x}_n)]]
\end{aligned} \tag{5}$$

From Eq (4), together with the Hom-Jacobi identity, we deduced that:

$$\begin{aligned}
&(D_t - E_t)([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]) \\
&= -(-1)^{|\mathsf{x}_n|(|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} (D_t - E_t) [\cdots[\mathsf{x}_n, [\tau(\mathsf{x}_1), \mathsf{x}_2], \mathsf{x}_3], \cdots, \mathsf{x}_{n-1}] \\
&= -(-1)^{|\mathsf{x}_n|(|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} (D_t - E_t) ([\cdots[\mathsf{x}_n, [\mathsf{x}_1, \tau(\mathsf{x}_2)], \mathsf{x}_3], \cdots, \mathsf{x}_{n-1}] \\
&- \cdots - (-1)^{|\mathsf{x}_1||\mathsf{x}_2|} [\cdots[\mathsf{x}_n, [\tau(\mathsf{x}_1), \mathsf{x}_2], \mathsf{x}_3], \cdots, \mathsf{x}_{n-1}]) \\
&= -(-1)^{|\mathsf{x}_n|(|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} (D_t - E_t) \\
&\cdot \left(\left[\cdots[(D_t - E_t)(\mathsf{x}_n), [\tau^k(\mathsf{x}_1), \tau^{k+1}(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_{n-1}\right] \right. \\
&\left. - \cdots - (-1)^{|\mathsf{x}_1||\mathsf{x}_2|} \left[\cdots[(D_t - E_t)(\mathsf{x}_n), [\tau^{k+1}(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_{n-1}\right] \right) \\
&= -(-1)^{|\mathsf{x}_n|(|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} \left[\cdots[\tau(D_t - E_t)(\mathsf{x}_n), [\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_{n-1}\right] \\
&= (-1)^{|\mathsf{x}_n|(|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} \left[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau(D_t - E_t)(\mathsf{x}_{n-1})\right].
\end{aligned} \tag{6}$$

Eqs (4)–(6) give that

$$\begin{aligned}
&\textbf{(i)}\, (D_t - E_t)([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]) \\
&= \left[\cdots[[(D_t - E_t)(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\right] \\
&= (-1)^{|D_t||\mathsf{x}_1|} \left[\cdots[[\tau^k(\mathsf{x}_1), (D_t - E_t)(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\right] \\
&= (-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|)} \left[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], (D_t - E_t)(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\right] \\
&= (-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} \left[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots (D_t - E_t)(\mathsf{x}_n)\right].
\end{aligned} \tag{7}$$

(ii) Using the Hom-Jacobi identity:

$$[\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_i, \mathsf{x}_{i+1}, \cdots, \mathsf{x}_n] = -(-1)^{|\mathsf{x}_i||\mathsf{x}_{i+1}|} [\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_{i+1}, \mathsf{x}_i, \cdots, \mathsf{x}_n] = 0.$$

Now, we can write,

$$\begin{aligned} 0 \ = \ & (-1)^{|\mathsf{x}_1||\mathsf{x}_2|} D_t [\cdots [\tau(\mathsf{x}_1), [\mathsf{x}_2, \mathsf{x}_3]], \cdots, \mathsf{x}_n] \\ & +(-1)^{|\mathsf{x}_2||\mathsf{x}_3|} D_t [\cdots [\tau(\mathsf{x}_2), [\mathsf{x}_3, \mathsf{x}_4]] \cdots \mathsf{x}_1] \\ & +(-1)^{|\mathsf{x}_3||\mathsf{x}_4|} D_t [\cdots [\tau(\mathsf{x}_3), [\mathsf{x}_4, \mathsf{x}_5]] \cdots \mathsf{x}_2] \\ & \cdots \\ & +(-1)^{|\mathsf{x}_n||\mathsf{x}_{n+1}|} D_t [\cdots [\tau(\mathsf{x}_n), [\mathsf{x}_{n+1}, \mathsf{x}_{n+2}]] \cdots \mathsf{x}_{n-1}] . \end{aligned}$$

Next, applying $D_t \in hGT\, \mathrm{Der}(\chi)$ on identity above, we get

$$\begin{aligned} (-1)^{|\mathsf{x}_1||\mathsf{x}_2|} D_t [\tau(\mathsf{x}_1), \mathsf{x}_2, \mathsf{x}_3, \cdots, \mathsf{x}_n] \ = \ & -((-1)^{|\mathsf{x}_2||\mathsf{x}_3|} D_t [\cdots [\tau(\mathsf{x}_2), [\mathsf{x}_3, \mathsf{x}_4]] \cdots \mathsf{x}_1] \\ & +(-1)^{|\mathsf{x}_3||\mathsf{x}_4|} D_t [\cdots [\tau(\mathsf{x}_3), [\mathsf{x}_4, \mathsf{x}_5]] \cdots \mathsf{x}_2] \\ & \cdots \\ & +(-1)^{|\mathsf{x}_n||\mathsf{x}_{n+1}|} D_t [\cdots [\tau(\mathsf{x}_n), [\mathsf{x}_{n+1}, \mathsf{x}_{n+2}]] \cdots \mathsf{x}_{n-1}]). \end{aligned}$$

We have

$$\begin{aligned} & D_t [\tau(\mathsf{x}_1), \mathsf{x}_2, \mathsf{x}_3, \cdots, \mathsf{x}_n] \\ =& D_t [\cdots [\mathsf{x}_1, [\mathsf{x}_2, \mathsf{x}_3]], \cdots, \tau(\mathsf{x}_n)] \\ & - (-1)^{|\mathsf{x}_1||\mathsf{x}_2|} D_t [[\cdots [\mathsf{x}_1, \mathsf{x}_3], \cdots, \mathsf{x}_n], \tau(\mathsf{x}_2)] \\ =& \left[\cdots [D_t(\mathsf{x}_1), [\tau^k(\mathsf{x}_2), \tau^k(\mathsf{x}_3)]], \cdots, \tau^{k+1}(\mathsf{x}_n)\right] \\ & + (-1)^{|\mathsf{x}_1||D_t|} \left[\cdots [\tau^k(\mathsf{x}_1), [E_t(\mathsf{x}_2), \tau^k(\mathsf{x}_3)]], \cdots, \tau^{k+1}(\mathsf{x}_n)\right] \\ & + (-1)^{(|\mathsf{x}_1|+|\mathsf{x}_2|)|D_t|} \left[\cdots [\tau^k(\mathsf{x}_1), [\tau^k(\mathsf{x}_2), E_t(\mathsf{x}_3)]], \cdots, \tau^{k+1}(\mathsf{x}_n)\right] \\ & + \cdots \\ & + (-1)^{(|\mathsf{x}_1|+|\mathsf{x}_2|+\cdots+|\mathsf{x}_{n-1}|)|D_t|} \\ & \cdot \left[\cdots [\tau^k(\mathsf{x}_1), [\tau^k(\mathsf{x}_2), \tau^k(\mathsf{x}_3)]], \cdots, \tau(E_t(\mathsf{x}_n))\right] \\ & - (-1)^{|\mathsf{x}_1||\mathsf{x}_2|} \Big(\left[[\cdots [D_t(\mathsf{x}_1), \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)], \tau^{k+1}(\mathsf{x}_2)\right] \\ & + (-1)^{|\mathsf{x}_1||D_t|} \left[[\cdots [\tau^k(\mathsf{x}_1), E_t(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)], \tau^{k+1}(\mathsf{x}_2)\right] \\ & + (-1)^{(|\mathsf{x}_1|+|\mathsf{x}_3|)|D_t|} \left[[\cdots [[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_3)], E_t(\mathsf{x}_4)], \cdots, \tau^k(\mathsf{x}_n)], \tau^{k+1}(\mathsf{x}_2)\right] \\ & + \cdots \\ & + (-1)^{(|\mathsf{x}_1|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)|D_t|} \left[[\cdots [\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)], \tau(E_t(\mathsf{x}_2))\right] \Big) \\ =& [\cdots [[\tau(D_t(\mathsf{x}_1)), \mathsf{x}_2], \mathsf{x}_3], \cdots, \mathsf{x}_n] \\ & + (-1)^{|D_t||\mathsf{x}_1|} [\cdots [[\tau(\mathsf{x}_1), E_t(\mathsf{x}_2)], \mathsf{x}_3], \cdots, \mathsf{x}_n] \\ & + (-1)^{|D_t||\mathsf{x}_1||\mathsf{x}_2|} [\cdots [\tau(\mathsf{x}_1), [\mathsf{x}_2, E_t(\mathsf{x}_3)]], \cdots, \mathsf{x}_n] \\ & + \cdots \\ & + (-1)^{|D_t||\mathsf{x}_1||\mathsf{x}_2||\mathsf{x}_3|\cdots|\mathsf{x}_{n-1}|} [\cdots [[\tau(\mathsf{x}_1), \mathsf{x}_2], \mathsf{x}_3], \cdots, [\mathsf{x}_{n-1}, E_t(\mathsf{x}_n)]] . \end{aligned}$$

Hence,

$$\begin{aligned}
D_t\left[\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n\right] = & \Big[\cdots[[D_t(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
& + (-1)^{|D_t||\mathsf{x}_1|}\Big[\cdots[[\tau^k(\mathsf{x}_1), E_t(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
& + (-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], E_t(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
& + \cdots \\
& + (-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)} \\
& \Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, E_t(\mathsf{x}_n)\Big].
\end{aligned}$$

(iii) This identity can be straightforwardly derived from Eq (7). □

Remark 2.2. *To avoid ambiguity, every exponent appearing in the proof is interpreted in the ring $\mathbb{Z}_2$. The sign changes arise from repeated applications of the graded skew-symmetry of the Hom-Lie superalgebra bracket and the homogeneity of the derivation $D_t - E_t$. In particular, the factor obtained when transferring $D_t - E_t$ across a homogeneous element x_i is*

$$(-1)^{|D_t||\mathsf{x}_i|},$$

and hence the cumulative sign after passing through $\mathsf{x}_1, \ldots, \mathsf{x}_{n-1}$ is

$$(-1)^{|D_t|(|\mathsf{x}_1|+\cdots+|\mathsf{x}_{n-1}|)}.$$

Definition 2.10. *[48] A linear transformation $D_t \in hg(\chi)$ is called a Hom-Lie τ^k-central n-derivation of χ if it fulfills the following condition:*

$$\begin{aligned}
D_t([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]) & = D_t([\mathsf{x}_1, \mathsf{x}_2, \cdots, \mathsf{x}_n]) \\
& = \Big[\cdots[[D_t(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
& = \Big[\cdots[[\tau^k(\mathsf{x}_1), D_t(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
& = \Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], D_t(\mathsf{x}_3)], \cdots, \tau^k(\mathsf{x}_n)\Big] \\
& = \cdots \\
& = \Big[\cdots[[\tau^k(\mathsf{x}_1), \tau^k(\mathsf{x}_2)], \tau^k(\mathsf{x}_3)], \cdots, D_t(\mathsf{x}_n)\Big] \\
& = 0.
\end{aligned}$$

for all $\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \cdots, \mathsf{x}_n \in \chi$. In this context, the collection of all τ^k-central n-derivations is represented by $hZT\,\mathrm{Der}(\chi)$.

Proposition 2.4. *Prove that $hZT\,\mathrm{Der}(\chi)$ is a Hom-Lie super ideal of $hN\,\mathrm{Der}(\chi)$ and $hGN\,\mathrm{Der}(\chi)$.*

Proof. It is already established that $hZN\,\mathrm{Der}(\chi)$ forms a subalgebra of both $hN\,\mathrm{Der}(\chi)$ and $hGN\,\mathrm{Der}(\chi)$. Therefore, to complete the argument, we need only to verify that $hZN\,\mathrm{Der}(\chi)$ constitutes an ideal. Let $D_{t1} \in hZN\,\mathrm{Der}_{\tau^k}(\chi)$ and $D_{t2} \in hN\,\mathrm{Der}_{\tau^s}(\chi)$. Then the commutator $[D_{t1}, D_{t2}] \in hZN\,\mathrm{Der}_{\tau^{s+k}}(\chi)$. The remaining part of the proposition can be proven analogously. For any $\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3, \ldots, \mathsf{x}_{n-1} \in hg(\chi)$ and $u_n \in \chi$, we have:

$$\Big[[D_{t_1} D_{t_2}](\mathsf{x}_1), \tau^{s+k}(\mathsf{x}_2), \tau^{s+k}(\mathsf{x}_3), \cdots, \tau^{s+k}(\mathsf{x}_n)\Big]$$

$$
\begin{aligned}
&=\Big[\cdots[[D_{t_1}D_{t_2}(\mathsf{x}_1),\tau^{s+k}(\mathsf{x}_2)],\tau^{s+k}(\mathsf{x}_3)],\cdots,\tau^{s+k}(\mathsf{x}_n)\Big]\\
&\quad-(-1)^{|D_{t_1}||D_{t_2}|}\Big[\cdots[[D_{t_1}D_{t_2}(\mathsf{x}_1),\tau^{s+k}(\mathsf{x}_2)],\tau^{s+k}(\mathsf{x}_3)],\cdots,\tau^{s+k}(\mathsf{x}_n)\Big]\\
&=0-(-1)^{|D_{t_1}||D_{t_2}|}D_{t_2}(\Big[\cdots[[D_{t_1}(\mathsf{x}_1),\tau^{k}(\mathsf{x}_2)],\tau^{k}(\mathsf{x}_3)],\cdots,\tau^{k}(\mathsf{x}_n)\Big])\\
&\quad-(-1)^{(|s|+|D_{t_1}|)|D_{t_2}|}\Big[\cdots[[\tau^{s}D_{t_1}(\mathsf{x}_1),\tau^{k}D_{t_2}(\mathsf{x}_2)],\tau^{s+k}(\mathsf{x}_3)],\cdots,\tau^{s+k}(\mathsf{x}_n)\Big]\\
&\quad-(-1)^{(|\mathsf{x}_1|+|D_{t_1}|+|\mathsf{x}_2|)|D_{t_2}|}\Big[\cdots[[\tau^{s}D_{t_1}(\mathsf{x}_1),\tau^{s+k}(\mathsf{x}_2)],\tau^{k}D_{t_2}(\mathsf{x}_3)],\cdots,\tau^{s+k}(\mathsf{x}_n)\Big]\\
&\quad-\cdots\\
&\quad-(-1)^{(|\mathsf{x}_1|+|D_{t_1}|+|\mathsf{x}_2|+\cdots+|\mathsf{x}_{n-1}|)|D_{t_2}|}\Big[\cdots[[\tau^{s}D_{t_1}(\mathsf{x}_1),\tau^{s+k}(\mathsf{x}_2)],\tau^{s+k}(\mathsf{x}_3)],\cdots,\tau^{k}D_{t_2}(\mathsf{x}_n)\Big]\\
&=0.
\end{aligned}
$$

Similarly, we can get

$$[D_{t_1},D_{t_2}]\,([\mathsf{x}_1,\mathsf{x}_2,\cdots,\mathsf{x}_n])=0.$$

Thus, we can conclude that $[D_{t_1},D_{t_2}]\in hZNDer_{\tau^{s+k}}(\chi)$. This guarantees the evidence. □

We can define, $hN\,\mathrm{Der}(\chi)=\oplus_{k\geq 0}hN\,\mathrm{Der}_{\tau^k}(\chi)$, $hGN\,\mathrm{Der}(\chi)=\oplus_{k\geq 0}hNG\,\mathrm{Der}_{\tau^k}(\chi)$, and $hZN\,\mathrm{Der}(\chi)=\oplus_{k\geq 0}hZN\,\mathrm{Der}_{\tau^k}(\chi)$.

3. Centroids and quasi centroids

In this part, we present the concepts of the n-centroid and the quasi n-centroid associated with the Hom-Lie algebra χ. We then explore how these structures relate to $h\,\mathrm{NDer}(\chi)$ and $hGN\,\mathrm{Der}(\chi)$. Additionally, we examine the connection between the center $hZ(\chi)$ and the centroids. To proceed toward our main results, we begin by presenting the necessary definitions.

Definition 3.1. *[48] A Hom linear map $D_t\in hg(\chi)$ is referred to as a Hom-Lie τ^k-n-centroid of χ, if it fulfills the subsequent axiom:*

$$
\begin{aligned}
D_t\tau&=\tau D_t,\\
D_t\left([\mathsf{x}_1,\mathsf{x}_2,\cdots,\mathsf{x}_n]\right)&=\left[D\left(\mathsf{x}_1\right),\tau^k\left(\mathsf{x}_2\right),\cdots,\tau^k\left(\mathsf{x}_n\right)\right]\\
&=(-1)^{D_t|\mathsf{x}_{i-1}|}\left[\tau^k\left(\mathsf{x}_1\right),\cdots,\tau^k\left(\mathsf{x}_{i-1}\right),D_t\left(\mathsf{x}_i\right),\tau^k\left(\mathsf{x}_{i+1}\right),\cdots,\tau^k\left(\mathsf{x}_n\right)\right]\\
&=0,
\end{aligned}
$$

for any $\mathsf{x}_i\in hg(\chi), i\in J$.

$$
\begin{aligned}
D_t\left[\mathsf{x}_1,\mathsf{x}_2,\cdots,\mathsf{x}_n\right]&=\Big[\cdots[[D_t(\mathsf{x}_1),\tau^k(\mathsf{x}_2)],\tau^k(\mathsf{x}_3)],\cdots,\tau^k(\mathsf{x}_n)\Big]\\
&=(-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|)}\Big[\cdots[[\tau^k(\mathsf{x}_1),D_t(\mathsf{x}_2)],\tau^k(\mathsf{x}_3)],\cdots,\tau^k(\mathsf{x}_n)\Big]\\
&=(-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|)}\Big[\cdots[[\tau^k(\mathsf{x}_1),\tau^k(\mathsf{x}_2)],D_t(\mathsf{x}_3)],\cdots,\tau^k(\mathsf{x}_n)\Big]\\
&=\cdots\\
&=(-1)^{|D_t|(|\mathsf{x}_1|+|\mathsf{x}_2|+|\mathsf{x}_3|+\cdots+|\mathsf{x}_{n-1}|)}\\
&\quad\cdot\Big[\cdots[[\tau^k(\mathsf{x}_1),\tau^k(\mathsf{x}_2)],\tau^k(\mathsf{x}_3)],\cdots,D_t(\mathsf{x}_n)\Big],
\end{aligned}
$$

for any $\mathsf{x}_1,\mathsf{x}_2,\mathsf{x}_3,\cdots,\mathsf{x}_{n-1}\in hg(\chi), u_n\in\chi$.

Definition 3.2. *[48] A Hom linear map $D_t \in hg(\chi)$ is termed as a Hom-Lie τ^k-quasi centroid of χ if it fulfills the following condition: $D_t\tau = \tau D_t$,*

$$\left[D_t(\mathrm{y}_1), \tau^k(\mathrm{y}_2), \cdots, \tau^k(\mathrm{y}_n)\right] = (-1)^{D_t|\mathrm{y}_{i-1}|}\left[\tau^k(\mathrm{y}_1), \cdots, \tau^k(\mathrm{y}_{i-1}), D_t(\mathrm{y}_i), \tau^k(\mathrm{y}_{i+1}), \cdots, \tau^k(\mathrm{y}_n)\right]$$

for any $\mathrm{y}_i \in hg(\chi), i \in J$.

$$[[[D_t(\mathrm{y}_1), \mathrm{y}_2], \mathrm{y}_3], \cdots, \mathrm{y}_n] = (-1)^{|D_t|(|\mathrm{y}_1|+|\mathrm{y}_2|+|\mathrm{y}_3|+\cdots+|\mathrm{y}_{n-1}|)}\left[\cdots[[\tau^k(\mathrm{y}_1), \tau^k(\mathrm{y}_2)], \tau^k(\mathrm{y}_3)], \cdots, D_t(\mathrm{y}_n)\right],$$

for any $\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \cdots, \mathrm{y}_{n-1} \in hg(\chi), u_n \in \chi$.

We can define, $hNC(\chi) = \oplus_{k\geq 0} hNC_{\tau^k}(\chi)$ and $hNQC(\chi) = \oplus_{k\geq 0} hNQC_{\tau^k}(\chi)$.

Proposition 3.1. *For the Hom-Lie superalgbera χ, prove that $hNC(\chi)$ is a Hom-Lie subalgebra of $hg(\chi)$.*

Proof. Consider $D_{t_1} \in hNC_{\tau^s}(\chi), D_{t_2} \in hNC_{\tau^k}(\chi)$, for $hnC(\chi)$ to be subalgebra of $hg(\chi)$, we must have $[D_{t_1}, D_{t_2}] \in hNC_{\tau^{s+k}}(\chi)$,

$$\begin{aligned}
&\left[[D_{t_1}D_{t_2}](\mathrm{y}_1), \tau^{s+k}(\mathrm{y}_2), \tau^{s+k}(\mathrm{y}_3), \cdots, \tau^{s+k}(\mathrm{y}_n)\right] \\
=&\left[\cdots[[D_{t_1}D_{t_2}(\mathrm{y}_1), \tau^{s+k}(\mathrm{y}_2)], \tau^{s+k}(\mathrm{y}_3)], \cdots, \tau^{s+k}(\mathrm{y}_n)\right] \\
&-(-1)^{|D_{t_1}||D_{t_2}|}\left[\cdots[[D_{t_1}D_{t_2}(\mathrm{y}_1), \tau^{s+k}(\mathrm{y}_2)], \tau^{s+k}(\mathrm{y}_3)], \cdots, \tau^{s+k}(\mathrm{y}_n)\right] \\
=&D_{t_1}\left[\cdots[[D_{t_2}(\mathrm{y}_1), \tau^k(\mathrm{y}_2)], \tau^k(\mathrm{y}_3)], \cdots, \tau^k(\mathrm{y}_n)\right] \\
&-(-1)^{|D_{t_1}||D_{t_2}|}D_{t_1}\left[\cdots[[D_{t_2}(\mathrm{y}_1), \tau^k(\mathrm{y}_2)], \tau^k(\mathrm{y}_3)], \cdots, \tau^k(\mathrm{y}_n)\right] \\
=&D_{t_1}D_{t_2}\left[\cdots[[\mathrm{y}_1, \mathrm{y}_2], \mathrm{y}_3], \cdots, \mathrm{y}_n\right] \\
&-(-1)^{|D_{t_1}||D_{t_2}|}D_{t_1}D_{t_2}\left[\cdots[[\mathrm{y}_1, \mathrm{y}_2], \mathrm{y}_3], \cdots, \mathrm{y}_n\right] \\
=&[D_{t_1}, D_{t_2}]([\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \cdots, \mathrm{y}_n]).
\end{aligned}$$

It follows that, $[D_{t_1}, D_{t_2}] \in hTC_{\tau^{s+k}}(\chi)$, thus the argument is established. □

Proposition 3.2. *Let χ be a Hom-Lie superalgebra. If $D_t \in hGNDer(\chi)$ represents the set of generalized n-derivations, and $E_t \in hN\,\mathrm{Der}(\chi)$ denotes its corresponding n-derivations set, then it follows that $(D_t - E_t) \in hNC(\chi)$.*

Proof. Consider $D_t \in hGNDer_{\tau^k}\chi$, for any $E_t \in hN\,\mathrm{Der}_{\tau^k}(\chi)$, and thus

$$\begin{aligned}
D_t([\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3, \cdots, \mathrm{y}_n]) =&\left[\cdots[[D_t(\mathrm{y}_1), \tau^k(\mathrm{y}_2)], \tau^k(\mathrm{y}_3)], \cdots, \tau^k(\mathrm{y}_n)\right] \\
&+(-1)^{|D_t||\mathrm{y}_1|}\left[\cdots[[\tau^k(\mathrm{y}_1), E_t(\mathrm{y}_2)], \tau^k(\mathrm{y}_3)], \cdots, \tau^k(\mathrm{y}_n)\right] \\
&+(-1)^{|D_t|(|\mathrm{y}_1|+|\mathrm{y}_2|)}\left[\cdots[[\tau^k(\mathrm{y}_1), \tau^k(\mathrm{y}_2)], E_t(\mathrm{y}_3)], \cdots, \tau^k(\mathrm{y}_n)\right] \\
&+\cdots \\
&+(-1)^{|D_t|(|\mathrm{y}_1|+|\mathrm{y}_2|+|\mathrm{y}_3|+\cdots|\mathrm{y}_{n-1}|)}\left[\cdots[[\tau^k(\mathrm{y}_1), \tau^k(\mathrm{y}_2)], \tau^k(\mathrm{y}_3)], \cdots, E_t(\mathrm{y}_n)\right] \\
=&\left[\cdots[[D_t(\mathrm{y}_1), \tau^k(\mathrm{y}_2)], \tau^k(\mathrm{y}_3)], \cdots, \tau^k(\mathrm{y}_n)\right]
\end{aligned}$$

$$+ E_t\left([\cdots[[y_1, y_2], y_3], \cdots, y_n]\right)$$
$$- \left[\cdots[[E_t(y_1), \tau^k(y_2)], \tau^k(y_3)], \cdots, \tau^k(y_n)\right].$$

This implies that

$$(D_t - E_t)\left([y_1, y_2, y_3, \cdots, y_n]\right) = \left[\cdots[(D_t - E_t)(y_1), [\tau^k(y_2), \tau^k(y_3)]], \cdots, (y_n)\right].$$

Therefore, by Definition 3.1 , we say $(D_t - E_t) \in hNC(\chi)$. This completes the proof. □

Proposition 3.3. *Let χ be a Hom-Lie superalgebra. Suppose $D_t \in hGNDer(\chi)$ represents the set of generalized n-derivations, and $E_t \in hN\,\mathrm{Der}(\chi)$ denotes the corresponding n-derivations set. It then follows that $(D_t - E_t) \in hNQC(\chi)$.*

Proof. Applying Lemma 2.1, it is evident that

$$(D_t - E_t)\left([y_1, y_2, y_3, \cdots, y_n]\right)$$
$$= (-1)^{|D_t|(|y_1|+|y_2|+|y_2|+\cdots|y_{n-1}|)}\left[\cdots[[\tau^k(y_1), \tau^k(y_2)], \tau^k(y_3)], \cdots, (D_t - E_t)(y_n)\right].$$

Consequently by Definition 3.2 , we obtain $(D_t - E_t) \in hNQC(\chi)$. This establishes the result. □

Proposition 3.4. *If $hZ(\chi) = 0$, then $hTC(\chi)$ and $hNQC(\chi)$ are commutative Hom-Lie superalgebras.*

Proof. Consider the initial scenario where, for every $s, t \in hg(\chi)$, $u \in \chi, D_{t_1} \in TC\tau^s(\chi)$, and $D_{t_2} \in TC_{\tau^k}(\chi)$, we deduce that

$$\left[D_{t_1}D_{t_2}(y_1), \tau^{s+k}(y_2), \tau^{s+k}(y_3), \cdots, \tau^{s+k}(y_n)\right]$$
$$= D_{t_1}(\left[\cdots[[D_{t_2}(y_1), \tau^k(y_2)], \tau^k(y_3)], \cdots, \tau^k(y_n)\right])$$
$$= (-1)^{|y_2|(|y_1|+|D_{t_2}|)}D_{t_1}(\left[\cdots[[\tau^k(y_2), D_{t_2}(y_1)], \tau^k(y_3)], \cdots, \tau^k(y_n)\right])$$
$$= (-1)^{|y_2|(|y_1|+|D_{t_2}|)}(\left[\cdots[[\tau^k D_{t_1}(y_2), \tau^s D_{t_2}(y_1)], \tau^{s+k}(y_3)], \cdots, \tau^{s+k}(y_n)\right])$$
$$= (-1)^{|y_2|(|y_1|+|D_{t_2}|)+(|y_1|+|D_{t_2}|)(|y_2|+|D_{t_1}|)}(\left[\cdots[[\tau^s D_{t_2}(y_1), \tau^k D_{t_1}(y_2)], \tau^{s+k}(y_3)], \cdots, \tau^{s+k}(y_n)\right])$$
$$= (-1)^{|D_{t_1}|(|y_1|+|D_{t_2}|)}D_{t_2}(\left[\cdots[[\tau^s(y_1), D_{t_1}(y_2)], \tau^{s+k}(y_3)], \cdots, \tau^{s+k}(y_n)\right])$$
$$= (-1)^{|y_1|(|y_2|+|D_{t_1}|)+|D_{t_1}|(|y_1|+|D_{t_2}|)}D_{t_2}([\cdots[[D_{t_1}(y_2), \tau^s(y_1)], \tau^s(y_3)], \cdots, \tau^s(y_n)])$$
$$= (-1)^{|y_1||y_2|+|D_{t_1}||D_{t_2}|}\left[D_{t_2}, D_{t_1}\right]([\cdots[[y_2, y_1], y_3], \cdots, y_n])$$
$$= (-1)^{|D_{t_1}||D_{t_2}|}\left[D_{t_2}, D_{t_1}\right]([y_1, y_2, y_3, \cdots, y_n])$$
$$= (-1)^{|D_{t_1}||D_{t_2}|}\left(\left[\cdots[[D_{t_2}D_{t_1}(y_1), \tau^{s+k}(y_2)], \tau^{s+k}(y_3)], \cdots, \tau^{s+k}(y_n)\right]\right).$$

Hence, we have $\left[\cdots[[[D_{t_2}, D_{t_1}](y_1), \tau^{s+k}(y_2)], \tau^{s+k}(y_3)], \cdots, \tau^{s+k}(y_n)\right] = 0$. Because of $hZ(\chi) = 0$, we obtain that $[D_{t_1}, D_{t_2}] = 0$. This concludes the proof. The remaining case can be handled in an analogous manner. □

Theorem 3.3. *Let $(\chi, [\cdot\cdot], \tau)$ be a multiplicative Hom-Lie superalgebra, let τ be a surjection and let $hZ(\chi)$ be the center of χ. Then, $[C(\chi), QC(\chi)] \subseteq \mathrm{End}(\chi, hZ(\chi))$. Moreover, if $hZ(\chi) = 0$, then $[hNC(\chi), hNQC(\chi)] = (0)$.*

Proof. Consider $D_{t_1} \in NC_{\tau^k}(\chi)$ and $D_{t_2} \in NQC_{\tau^s}(\chi)$, for $y_1, y_2, y_3, \cdots, y_{n-1} \in hg(\chi)$ and $y_n \in \chi$. Then, we have

$$
\begin{aligned}
&\Big[[D_{t_1}, D_{t_2}](y_1), \tau^{s+k}(y_2), \tau^{s+k}(y_3), \cdots, \tau^{s+k}(y_n)\Big] \\
=&\Big[\cdots[[D_{t_1}D_{t_2}(y_1), \tau^{s+k}(y_2)], \tau^{s+k}(y_3)], \cdots, \tau^{s+k}(y_n)\Big] \\
&-(-1)^{|D_{t_1}||D_{t_2}|}\Big[\cdots[[D_{t_1}D_{t_2}(y_1), \tau^{s+k}(y_2)], \tau^{s+k}(y_3)], \cdots, \tau^{s+k}(y_n)\Big] \\
=&D_{t_1}\left([\cdots[[D_{t_2}(y_1), \tau^s(y_2)], \tau^s(y_3)], \cdots, \tau^s(y_n)]\right) \\
&-(-1)^{|D_{t_1}||D_{t_2}|+(|D_{t_1}|+|y_1|+|y_2|+\cdots+|y_{n-1}|)|D_{t_2}|}\left([\cdots[[D_{t_1}(y_1), y_2], y_3], \cdots, D_{t_2}(y_n)]\right) \\
=&D_{t_1}\left([\cdots[[D_{t_2}(y_1), y_2], y_3], \cdots, y_n]\right) \\
&-(-1)^{(|y_1|+|y_2|+\cdots+|y_{n-1}|)|D_{t_2}|}D_{t_1}\left([\cdots[[y_1, y_2], y_3], \cdots, D_{t_2}(y_n)]\right) \\
=&D_{t_1}\Big([\cdots[[D_{t_2}(y_1), y_2], y_3], \cdots, y_n] - (-1)^{(|y_1|+|y_2|+\cdots+|y_{n-1}|)|D_{t_2}|}[\cdots[[y_1, y_2], y_3], \cdots, D_{t_2}(y_n)]\Big) \\
=&0.
\end{aligned}
$$

Hence, $[D_{t_1}, D_{t_2}] \in hZ_{\tau^{s+k}}(\chi)$ as desired. Furthermore, if $hZ(\chi) = 0$, then evidently $[hNC(\chi), hNQC(\chi)] = 0$. □

Theorem 3.4. *Let χ be a Hom-Lie superalgebra. Prove that*
(1) $[hN\,\mathrm{Der}(\chi), hNC(\chi)] \subseteq hNC(\chi)$.
(2) $[hGN\,\mathrm{Der}(\chi), hNC(\chi)] \subseteq hNC(\chi)$.

Proof. Clearly, it suffices to establish the first case; the remaining cases can be shown in an analogous manner. Since $D_{t_1} \in hN\,\mathrm{Der}_{\tau^k}(\chi), D_{t_2} \in hNC_{\tau^s}(\chi)$, for any $y_1, y_2, y_3, \cdots, y_{n-1} \in hg(\chi), y_n \in \chi$, we can obtain that

$$
\begin{aligned}
&\Big[D_{t_1}D_{t_2}(y_1), \tau^{s+k}(y_2), \tau^{s+k}(y_3), \cdots, \tau^{s+k}(y_n)\Big] \\
=&D_{t_1}([\cdots[[D_{t_2}(y_1), \tau^s(y_2)], \tau^s(y_3)], \cdots, \tau^s(y_n)]) \\
&-(-1)^{|D_{t_1}|(|D_{t_2}|+|y_1|)}D_{t_1}(\Big[\cdots[[\tau^k D_{t_2}(y_1), \tau^s D_{t_1}(y_2)], \tau^{s+k}(y_3)], \cdots, \tau^{s+k}(y_n)\Big]) \\
&-(-1)^{|D_{t_1}|(|D_{t_2}|+|y_1|+|y_2|)}D_{t_1}(\Big[\cdots[[\tau^k D_{t_2}(y_1), \tau^{s+k}(y_2)], \tau^s D_{t_1}(y_3)], \cdots, \tau^{s+k}(y_n)\Big]) \\
&-\cdots \\
&-(-1)^{|D_{t_1}|(|D_{t_2}|+|y_1|+|y_2|+\cdots+|y_{n-1}|)}D_{t_1}(\Big[\cdots[[\tau^k D_{t_2}(y_1), \tau^{s+k}(y_2)], \tau^{s+k}(y_3)], \cdots, \tau^s D_{t_1}(y_n)\Big]) \\
=&[D_{t_2}, D_{t_1}]([y_1, y_2, y_3, \cdots, y_n]) \\
&-(-1)^{|D_{t_1}|(|D_{t_2}|+|y_1|)}D_{t_2}\left(\Big[\cdots[[\tau^k(y_1), D_{t_1}(y_2)], \tau^k(y_3)], \cdots, \tau^k(y_n)\Big]\right) \\
&-(-1)^{|D_{t_1}|(|D_{t_2}|+|y_1|+|y_2|)}D_{t_2}\left(\Big[\cdots[[\tau^k(y_1), \tau^k(y_2)], D_{t_1}(y_3)], \cdots, \tau^k y_n\Big]\right) \\
&-\cdots \\
&-(-1)^{|D_{t_1}|(|D_{t_2}|+|y_1|+|y_2|+|y_3|+\cdots+|y_{n-1}|)}D_{t_2}\left(\Big[\cdots[[\tau^k(y_1), \tau^k(y_2)], \tau^k(y_3)], \cdots, D_{t_1}(y_n)\Big]\right).
\end{aligned}
$$

Similarly, we have

$$
\Big[D_{t_2}D_{t_1}(y_1), \tau^{s+k}(y_2), \tau^{s+k}(y_3), \cdots, \tau^{s+k}(y_n)\Big]
$$

$$
\begin{aligned}
=&D_{t_2}(\Big[\cdots[[D_{t_1}(y_1),\tau^k(y_2)],\tau^k(y_3)],\cdots,\tau^k(y_n)\Big])\\
=&D_{t_2}D_{t_1}([\cdots[[y_1,y_2],y_3],\cdots,y_n])-(-1)^{|y_1||D_{t_1}|}D_{t_2}(\Big[\cdots[[\tau^k(y_1),D_{t_1}(y_2)],\tau^k(y_3)],\cdots,\tau^k(y_n)\Big])\\
&-(-1)^{(|y_1|+|y_2|)|D_{t_1}|}D_{t_2}(\Big[\cdots[[\tau^k(y_1),\tau^k(y_2)],D_{t_1}(y_3)],\cdots,\tau^k(y_n)\Big])\\
&-\cdots\\
&-(-1)^{(|y_1|+|y_2|+|y_3|+\cdots+|y_{n-1}|)|D_{t_1}|}D_{t_2}(\Big[\cdots[[\tau^k(y_1),\tau^k(y_2)],\tau^k(y_3)],\cdots,D_{t_1}(y_n)\Big]).
\end{aligned}
$$

Applying Definition 2.5, we deduce that

$$
\Big[\cdots[[[D_{t_2},D_{t_1}](y_1),\tau^{s+k}(y_2)],\tau^{s+k}(y_3)],\cdots,\tau^{s+k}(y_n)\Big]=[D_{t_2},D_{t_1}]\left([\cdots[[y_1,y_2],y_3],\cdots,y_n]\right).
$$

Hence, we obtain $[D_{t_1},D_{t_2}]\in hNC_{\tau^{s+k}}(\chi)$. The proof is now complete. □

Theorem 3.5. *Let χ be a Hom-Lie superalgebra. Prove that*
(1) $[hN\,\mathrm{Der}(\chi),hNQC(\chi)]\subseteq hNQC(\chi)$.
(2) $[hGN\,\mathrm{Der}(\chi),hNQC(\chi)]\subseteq hNQC(\chi)$.

Proof. As in the earlier results, we present only the proof of the second case; the first case can be established in an analogous way. Since $D_{t_1}\in hGN\,\mathrm{Der}_{\tau^k}(\chi),D_{t_2}\in hNQC_{\tau^k}(\chi)$, for any $y_1,y_2,y_3,\cdots,y_{n-1}\in hg(\chi),y_n\in\chi$, we can obtain that

$$
\begin{aligned}
&\Big[D_{t_1}D_{t_2}(y_1),\tau^{s+k}(y_2),\tau^{s+k}(y_3),\cdots,\tau^{s+k}(y_n)\Big]\\
=&D_{t_1}\left(\Big[\cdots[[D_{t_2}(y_1),\tau^k(y_2)],\tau^k(y_3)],\cdots,\tau^k(y_n)\Big]\right)\\
&-(-1)^{|D_{t_1}|(|D_{t_2}|+|y_1|)}\Big[\cdots[[\tau^s(D_{t_2}(y_1)),\tau^k(E_{t_1}(y_2))],\tau^{s+k}(y_3)],\cdots,\tau^{s+k}(y_n)\Big]\\
&-(-1)^{|D_{t_1}|(|D_{t_2}|+|y_1|+|y_2|)}\Big[\cdots[[\tau^s(D_{t_2}(y_1)),\tau^{s+k}y_2],\tau^k(E_{t_1}(y_3))],\cdots,\tau^{s+k}(y_n)\Big]\\
&-\cdots\\
&-(-1)^{|D_{t_1}|(|D_{t_2}|+|y_1|+|y_2|+|y_3|+\cdots+|y_{n-1}|)}\Big[\cdots[[\tau^s(D_{t_2}(y_1)),\tau^{s+k}y_2],\tau^{s+k}(y_3)],\cdots,\tau^k(E_{t_1}(y_n))\Big]\\
=&(-1)^{|D_{t_2}|(|y_1|+|y_2|+|y_3|+\cdots|y_{n-1}|)}D_{t_1}\left(\Big[\cdots[[\tau^k(y_1),\tau^k(y_2)],\tau^k(y_3)],\cdots,D_{t_2}(y_n)\Big]\right)\\
&+(-1)^{|D_{t_2}|(|y_1|+|y_2|+|y_3|+\cdots|y_{n-1}|)}(-1)^{|D_{t_1}||y_1|}\Big[\cdots[[\tau^{s+k}(y_1),\tau^kE_{t_1}(y_2)],\tau^k(y_3)],\cdots,\tau^sD_{t_2}(y_n)\Big]\\
&+(-1)^{|D_{t_2}|(|y_1|+|y_2|+|y_3|+\cdots|y_{n-1}|)}(-1)^{|D_{t_1}|(|y_1|+|y_2|)}\\
&\cdot\Big[\cdots[[\tau^{s+k}(y_1),\tau^{s+k}(y_2)],\tau^kE_{t_1}(y_3)],\cdots,\tau^sD_{t_2}(y_n)\Big]\\
&+\cdots\\
&+(-1)^{|D_{t_2}|(|y_1|+|y_2|+|y_3|+\cdots|y_{n-1}|)}(-1)^{|D_{t_1}|(|y_1|+|y_2|+|y_3|+\cdots|y_{n-1}|)}\\
&\cdot\Big[\cdots[[\tau^{s+k}(y_1),\tau^{s+k}(y_2)],\tau^{s+k}(y_3)],\cdots,D_{t_2}E_{t_1}(y_n)\Big].
\end{aligned}
$$

However, we obtain

$$
\begin{aligned}
&\Big[D_{t_2}D_{t_1}(y_1),\tau^{s+k}(y_2),\tau^{s+k}(y_3),\cdots,\tau^{s+k}(y_n)\Big]\\
=&(-1)^{|D_{t_2}|(|D_{t_1}|+|y_1|+|y_2|+|y_3|+\cdots+|y_{n-1}|)}\Big[\cdots[[\tau^kD_{t_2}(y_1),\tau^{s+k}y_2],\tau^{s+k}(y_3)],\cdots,\tau^sD_{t_1}(y_n)\Big]\\
=&(-1)^{|D_{t_2}|(|y_1|+|y_2|+|y_3|+\cdots|y_{n-1}|)}D_{t_1}\left(\Big[\cdots[[\tau^k(y_1),\tau^k(y_2)],\tau^k(y_3)],\cdots,D_{t_2}(y_n)\Big]\right)
\end{aligned}
$$

$$+(-1)^{|D_{t_2}|(|y_1|+|y_2|+|y_3|+\cdots|y_{n-1}|)}(-1)^{|D_{t_1}||y_1|}\Big[\cdots[[\tau^{s+k}(y_1),\tau^k E_{t_1}(y_2)],\tau^k(y_3)],\cdots,\tau^s D_{t_2}(y_n)\Big]$$
$$+(-1)^{|D_{t_2}|(|y_1|+|y_2|+|y_3|+\cdots|y_{n-1}|)}(-1)^{|D_{t_1}|(|y_1|+|y_2|)}$$
$$\cdot\Big[\cdots[[\tau^{s+k}(y_1),\tau^{s+k}(y_2)],\tau^k E_{t_1}(y_3)],\cdots,\tau^s D_{t_2}(y_n)\Big]$$
$$+\cdots$$
$$+(-1)^{|D_{t_2}|(|y_1|+|y_2|+|y_3|+\cdots|y_{n-1}|)}(-1)^{|D_{t_1}|(|y_1|+|y_2|+|y_3|+\cdots|y_{n-1}|)}$$
$$\cdot\Big[\cdots[[\tau^{s+k}(y_1),\tau^{s+k}(y_2)],\tau^{s+k}(y_3)],\cdots,D_{t_2}E_{t_1}(y_n)\Big].$$

By using Definition 3.2, we find that

$$\Big[[D_{t_1},D_{t_2}](y_1),\tau^{s+k}(y_2),\tau^{s+k}(y_3),\cdots,\tau^{s+k}(y_n)\Big]$$
$$=(-1)^{(D_{t_1}+D_{t_2})(y_1+y_2+y_3+\cdots+y_{n-1})}\Big[\cdots[[\tau^{s+k}(y_1),\tau^{s+k}(y_2)],\tau^{s+k}(y_3)],\cdots,[E_{t_1},E_{t_2}](y_n)\Big].$$

Thus, we have $[D_{t_1},D_{t_2}]\in hNQC_{\tau^{s+k}}(\chi)$. This establishes the result. □

4. Categorization of n-derivations on simple Hom-Lie superalgebras

This section investigates how to classify generalized Lie n-derivations for every simple Hom-Lie superalgebra. To this end, we first introduce a suitable definition.

Definition 4.1. *Let $p\in\chi$ such that $\tau(p)=p$. We define the adjoint derivation $\mathrm{ad}_{\tau^k}:\chi\to\chi$ by $\mathrm{ad}_{\tau^k}(p)(q)=-(-1)^{|p||q|}[p,\tau^k(q)]$, for all $q\in\chi$. The collection of these derivations is denoted by $h\,\mathrm{Inn}(\chi)$, and it holds that $h\,\mathrm{Inn}(\chi)=\bigoplus_{k\geq 0}h\,\mathrm{Inn}_{\tau^k}(\chi)$. Throughout, we assume that χ is a finite-dimensional simple Hom-Lie superalgebra. It can be readily verified that both $h\,\mathrm{Der}(\chi)$ and $h\,\mathrm{Inn}(\chi)$ form Hom-Lie subalgebras of $hN\,\mathrm{Der}(\chi)$. We now state and prove some useful lemmas.*

Lemma 4.1. *$h\,\mathrm{Inn}(\chi)$ is an ideal of the Hom-Lie superalgebra $hN\,\mathrm{Der}(\chi)$.*

Proof. Assume that $D_t\in hN\,\mathrm{Der}_{\tau^s}(\chi)$ and $ad(s)\in h\,\mathrm{Inn}_{\tau^k}(\chi)$ for $s\in\chi$. Since χ is a simple Hom-Lie superalgebra, then some finite index set I and $y_{i1},y_{i2}.\cdots,y_{i(n-1)}\in\chi$ exist such that $p=\sum_{i\in I}[\cdots[[y_{i1},y_{i2}],y_{i3}],\cdots,y_{i(n-1)}]$.

For any $q\in\chi$, we have

$$[D_t,\mathrm{ad}\,p](q)$$
$$=D_t\,\mathrm{ad}\,p(q)-(-1)^{|D_t||p|}\,\mathrm{ad}\,q\,(D_t(q))$$
$$=-(-1)^{|p||q|}\Big(D_t\Big(\Big[p,\tau^k(q)\Big]\Big)-(-1)^{|D_t||p|}\Big[\tau^s(p),D_t\Big(\tau^k(q)\Big)\Big]\Big)$$
$$=-(-1)^{|p||q|}D_t\left(\left[\sum_{i\in I}[\cdots[[y_{i1},y_{i2}],y_{i3}],\cdots,y_{i(n-1)}],\tau^k(q)\right]\right)$$
$$-(-1)^{|D_t||p|}\left[\sum_{i\in I}\tau^s[\cdots[[y_{i1},y_{i2}],y_{i3}],\cdots,y_{i(n-1)}],D_t\Big(\tau^k(q)\Big)\right]$$
$$=-(-1)^{|p||q|}\sum_{i\in I}([[\cdots[[D_t(y_{i1}),\tau^s(y_{i2})],\tau^s(y_{i3})],\cdots,\tau^s(y_{i(n-1)})],\tau^{s+k}(q)]$$
$$+(-1)^{|D_t||p|}[[\cdots[[\tau^s(y_{i1}),D_t(y_{i2})],\tau^s(y_{i3})],\cdots,\tau^s(y_{i(n-1)})],\tau^{s+k}(q)]$$

$$
\begin{aligned}
&+ (-1)^{|D_t|(|\mathsf{y}_{i1}|+|\mathsf{y}_{i2}|)}[[\cdots[[\tau^s(\mathsf{y}_{i1}),\tau^s(\mathsf{y}_{i2})],D_t(\mathsf{y}_{i3})],\cdots,\tau^s(\mathsf{y}_{i(n-1)})],\tau^{s+k}(q)]\\
&+\cdots\\
&+ (-1)^{|D_t|(|\mathsf{y}_{i1}|+|\mathsf{y}_{i2}|+\cdots+|\mathsf{y}_{i(n-1)}|)}\\
&\cdot[[\cdots[[\tau^s(\mathsf{y}_{i1}),\tau^s(\mathsf{y}_{i2})],\tau^s(\mathsf{y}_{i3})],\cdots,\tau^s(\mathsf{y}_{i(n-1)})],D_t(\tau^k(q))])\\
&- (-1)^{|D_t||p|}\left[\sum_{i\in I}\tau^s\left[\cdots[[\mathsf{y}_{i1},\mathsf{y}_{i2}],\mathsf{y}_{i3}],\cdots,\mathsf{y}_{i(n-1)}\right],D_t\left(\tau^k(q)\right)\right]\\
=&-(-1)^{|p||q|}(\sum_{i\in I}[[\cdots[[D_t(\mathsf{y}_{i1}),\tau^s(\mathsf{y}_{i2})],\tau^s(\mathsf{y}_{i3})],\cdots,\tau^s(\mathsf{y}_{i(n-1)})],\tau^{s+k}(q)]\\
&+ (-1)^{|D_t||\mathsf{y}_{i1}|}[[\cdots[[\tau^s(\mathsf{y}_{i1}),D_t(\mathsf{y}_{i2})],\tau^s(\mathsf{y}_{i3})],\cdots,\tau^s(\mathsf{y}_{i(n-1)})],\tau^{s+k}(q)]\\
&+ (-1)^{|D_t|(|\mathsf{y}_{i1}|+|\mathsf{y}_{i2}|)}[[\cdots[[\tau^s(\mathsf{y}_{i1}),\tau^s(\mathsf{y}_{i2})],D_t(\mathsf{y}_{i3})],\cdots,\tau^s(\mathsf{y}_{i(n-1)})],\tau^{s+k}(q)]\\
&+\cdots\\
&+ (-1)^{|D_t|(|\mathsf{y}_{i1}|+|\mathsf{y}_{i2}|+|\mathsf{y}_{i3}|+\cdots+|\mathsf{y}_{i(n-1)}|)}[[\cdots[[\tau^s(\mathsf{y}_{i1}),\tau^s(\mathsf{y}_{i2})],\tau^s(\mathsf{y}_{i3})],\cdots,D_t(\mathsf{y}_{i(n-1)})],\tau^{s+k}(q)])\\
=&\,\mathrm{ad}_{\tau^{s+k}}(\sum_{i\in I}[[\cdots[[D_t(\mathsf{y}_{i1}),\tau^s(\mathsf{y}_{i2})],\tau^s(\mathsf{y}_{i3})],\cdots,\tau^s(\mathsf{y}_{i(n-1)})],\tau^{s+k}(q)]\\
&+ (-1)^{|D_t||\mathsf{y}_{i1}|}[[\cdots[[\tau^s(\mathsf{y}_{i1}),D_t(\mathsf{y}_{i2})],\tau^s(\mathsf{y}_{i3})],\cdots,\tau^s(\mathsf{y}_{i(n-1)})],\tau^{s+k}(q)]\\
&+ (-1)^{|D_t|(|\mathsf{y}_{i1}|+|\mathsf{y}_{i2}|)}[[\cdots[[\tau^s(\mathsf{y}_{i1}),\tau^s(\mathsf{y}_{i2})],D_t(\mathsf{y}_{i3})],\cdots,\tau^s(\mathsf{y}_{i(n-1)})],\tau^{s+k}(q)]\\
&+\cdots\\
&+ (-1)^{|D_t|(|\mathsf{y}_{i1}|+|\mathsf{y}_{i2}|+|\mathsf{y}_{i3}|+\cdots+|\mathsf{y}_{i(n-1)}|)}[[\cdots[[\tau^s(\mathsf{y}_{i1}),\tau^s(\mathsf{y}_{i2})],\tau^s(\mathsf{y}_{i3})],\cdots,D_t(\mathsf{y}_{i(n-1)})],\tau^{s+k}(q)]).
\end{aligned}
$$

Since q was chosen arbitrarily, it follows that $h\,\mathrm{Inn}(\chi)$ is indeed an ideal of the Hom-Lie superalgebra $hN\,\mathrm{Der}(\chi)$, which concludes the proof. □

Moreover, there exists a deeper relationship between $h\,\mathrm{Der}(\chi)$ and $h\,\mathrm{TDer}(\chi)$, beyond what is established. Lemma 4.1 leads us to the following conclusion.

Lemma 4.2. *For al given linear map $d : hN\,\mathrm{Der}(\chi) \to \mathrm{Der}(\chi), D_t \mapsto \mathrm{d}_{D_t}$ where, for each $p \in \chi, D_t \in hN\,\mathrm{Der}(\chi)$, one has $[D_t, \mathrm{ad}\, p] = \mathrm{ad}\,(\mathrm{d}_{D_t})\,(p)$.*

Proof. In view of Lemma 4.1, we are able to introduce a linear endomorphism d_{D_t} on χ, such that for any $p = \sum_{i\in I}[\cdots[[\mathsf{y}_{i1},\mathsf{y}_{i2}],\mathsf{y}_{i3}],\cdots,\mathsf{y}_{i(n-1)}] \in \chi$, we have

$$
\begin{aligned}
dD_t(p) = \sum_{i\in I}\Big(&[\cdots[[D_t(\mathsf{y}_{i1}),\mathsf{y}_{i2}],\mathsf{y}_{i3}],\cdots,\mathsf{y}_{i(n-1)}]\\
&+ (-1)^{|D_t||\mathsf{y}_{i1}|}[\cdots[[\tau^s(\mathsf{y}_{i1}),D_t(\mathsf{y}_{i2})],\tau^s(\mathsf{y}_{i3})],\cdots,\tau^s(\mathsf{y}_{i(n-1)})]\\
&+ (-1)^{|D_t|(|\mathsf{y}_{i1}|+|\mathsf{y}_{i2}|)}[\cdots[[\tau^s(\mathsf{y}_{i1}),\tau^s(\mathsf{y}_{i2})],D_t(\mathsf{y}_{i3})],\cdots,\tau^s(\mathsf{y}_{i(n-1)})]\\
&+\cdots\\
&+ (-1)^{|D_t|(|\mathsf{y}_{i1}|+|\mathsf{y}_{i2}|+|\mathsf{y}_{i3}|+\cdots+|\mathsf{y}_{i(n-2)}|)}[\cdots[[\tau^s(\mathsf{y}_{i1}),\tau^s(\mathsf{y}_{i2})],\tau^s(\mathsf{y}_{i3})],\cdots,D_t(\mathsf{y}_{i(n-1)})]\Big).
\end{aligned}
$$

Initially, $d_{D_t}(p)$ is independent of the selected representation of p. To demonstrate it, consider

$$
\mathbb{A} = \sum_{i\in I}\Big([\cdots[[D_t(\mathsf{y}_{i1}),\mathsf{y}_{i2}],\mathsf{y}_{i3}],\cdots,\mathsf{y}_{i(n-1)}]
$$

$$+ (-1)^{|D_t||y_{i1}|}[\cdots[[\tau^s(y_{i1}), D_t(y_{i2})], \tau^s(y_{i3})], \cdots, \tau^s(y_{i(n-1)})]$$
$$+ (-1)^{|D_t|(|y_{i1}|+|y_{i2}|)}[\cdots[[\tau^s(y_{i1}), \tau^s(y_{i2})], D_t(y_{i3})], \cdots, \tau^s(y_{i(n-1)})]$$
$$+\cdots$$
$$+ (-1)^{|D_t|(|y_{i1}|+|y_{i2}|+|y_{i3}|+\cdots+|y_{i(n-2)}|)}[\cdots[[\tau^s(y_{i1}), \tau^s(y_{i2})], \tau^s(y_{i3})], \cdots, D_t(y_{i(n-1)})]\Big).$$

$$\begin{aligned}\mathbb{B} = \sum_{i\in I}\Big(&[\cdots[[D_t(z_{i1}), z_{i2}], z_{i3}]\cdots z_{i(n-1)}]\\ &+ (-1)^{|D_t||z_{i1}|}[\cdots[[\tau^s(z_{1i}), D_t(z_{2i})], \tau^s(z_{3i})], \cdots, \tau^s(z_{i(n-1)})]\\ &+ (-1)^{|D_t|(|z_{i1}|+|z_{2i}|)}[\cdots[[\tau^s(z_{1i}), \tau^s(z_{2i})], D_t(z_{3i})], \cdots, \tau^s(z_{i(n-1)})]\\ &+\cdots\\ &+ (-1)^{|D_t|(|z_{i1}|+|z_{i2}|+|z_{i3}|+\cdots+|z_{i(n-2)}|)}[\cdots[[\tau^s(z_{1i}), \tau^s(z_{2i})], \tau^s(z_{3i})], \cdots, D_t(z_{i(n-1)})]\Big),\end{aligned}$$

where p can also be expressed in the form $p = \sum_{i\in I}[\cdots[[z_{i1}, z_{i2}], z_{i3}], \cdots, z_{i(n-1)}] \in \chi$. As $D_t \in hN\,\mathrm{Der}(\chi)$, for any $q \in \chi$, we conclude that

$$[\mathbb{A}, \tau^s q] = D_t([p, q]) - [\tau^s(p), D_t(q)] = [\mathbb{B}, \tau^s(q)].$$

Thus, we obtain that $[(\mathbb{A} - \mathbb{B}), \tau^s(p)] = 0$ for all $p \in \chi$, which implies $\mathbb{A} - \mathbb{B} \in Z(\chi)$. Since the center of a simple Hom-Lie superalgebra is trivial, we conclude that $\mathbb{A} = \mathbb{B}$. Therefore, d_{D_t} is well-defined. Additionally, as shown in the proof of Lemma 4.1, it holds that

$$[D, \mathrm{ad}_p] = \mathrm{ad}(\mathrm{d}_{D_t}(p)).$$

To finalize the proof, we need only to verify that d_{D_t} satisfies the derivation property. From the discussion above, for $\tau^s D_t = D_t \in hN\,\mathrm{Der}(\chi)$ and for all $p, l \in \chi$, we have

$$[D, \mathrm{ad}_{[p,l]}] = \mathrm{ad}(\mathrm{d}_{D_t}([p, l])).$$

Meanwhile, applying the Hom-Lie superalgebra version of the Jacobi identity gives us the following:

If we suppose $D \in \mathrm{NDer}(L)$, from Lemma 4.1, we have $p = \sum_{i\in I}[\cdots[[y_{i1}, y_{i2}], y_{i3}], \cdots, y_{i(n-1)}] \in hg(L)$, $w \in L$. We then have

$$\begin{aligned}&\Big[D_t,\ \mathrm{ad}\Big(\Big[\sum_{i\in I}[\cdots[[y_{i1}, y_{i2}], y_{i3}], \ldots, y_{i(n-1)}],\ w\Big]\Big)\Big]\\ &= \mathrm{ad}\,\delta_{D_t}\Big(\Big[\sum_{i\in I}[\cdots[[y_{i1}, y_{i2}], y_{i3}], \ldots, y_{i(n-1)}],\ w\Big]\Big).\end{aligned}$$

Alternatively,

$$\begin{aligned}&\Big[D_t, \mathrm{ad}\Big(\Big[\sum_{i\in I}[\cdots[[y_{i1}, y_{i2}], y_{i3}], \ldots, y_{i(n-1)}],\ w\Big]\Big)\Big]\\ =&\Big[D_t, \Big[\sum_{i\in I}[\cdots[[y_{i1}, y_{i2}], y_{i3}], \ldots, y_{i(n-1)}],\ w\Big]\Big].\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i\in I} [[\cdots[[[\tau^s(D_t), \mathrm{ad}(\mathsf{y}_{i1})], \tau^s(\mathrm{ad}(\mathsf{y}_{i2}))], \tau^s(\mathrm{ad}(\mathsf{y}_{i3}))], \cdots, \tau^s(\mathrm{ad}(\mathsf{y}_{i(n-1)}))], \tau^s(\mathrm{ad}(w))] \\
&\quad + \sum_{i\in I}(-1)^{|D_t||\mathsf{y}_{i1}|} [[\cdots[\tau^s(\mathrm{ad}(\mathsf{y}_{i1})), [D_t, \mathrm{ad}(\mathsf{y}_{i2})], \tau^s(\mathrm{ad}(\mathsf{y}_{i3}))], \cdots, \tau^s(\mathrm{ad}(\mathsf{y}_{i(n-1)}))], \tau^s(\mathrm{ad}(w))] \\
&\quad + \sum_{i\in I}(-1)^{|D_t|(|\mathsf{y}_{i1}|+|\mathsf{y}_{i2}|)} [[\cdots[\tau^s(\mathrm{ad}(\mathsf{y}_{i1})), [\mathrm{ad}(\mathsf{y}_{i2}), [D_t, \mathrm{ad}(\mathsf{y}_{i3})]]], \cdots, \tau^s(\mathrm{ad}(\mathsf{y}_{i(n-1)}))], \tau^s(\mathrm{ad}(w))] \\
&\quad + \cdots \\
&\quad + \sum_{i\in I}(-1)^{|D_t||p|} [[\cdots[[\tau^s(\mathrm{ad}(\mathsf{y}_{i1})), \tau^s(\mathrm{ad}(\mathsf{y}_{i2}))], \tau^s(\mathrm{ad}(\mathsf{y}_{i3}))], \cdots, \tau^s(\mathsf{y}_{i(n-1)})], [D_t, \mathrm{ad}(w)]] \\
&= \sum_{i\in I} [[\cdots[[\mathrm{ad}(D_t)(\mathsf{y}_{i1}), \tau^s(\mathrm{ad}(\mathsf{y}_{i2}))], \tau^s(\mathrm{ad}(\mathsf{y}_{i3}))], \cdots, \tau^s(\mathrm{ad}(\mathsf{y}_{i(n-1)}))], \tau^s(\mathrm{ad}(w))] \\
&\quad + \sum_{i\in I}(-1)^{|D_t||\mathsf{y}_{i1}|} [[\cdots[[\tau^s(\mathrm{ad}(\mathsf{y}_{i1})), \mathrm{ad}(D_t)(\mathsf{y}_{i2})], \tau^s(\mathrm{ad}(\mathsf{y}_{i3}))], \cdots, (\mathrm{ad}(\mathsf{y}_{i(n-1)}))], \tau^s(\mathrm{ad}(w))] \\
&\quad + \sum_{i\in I}(-1)^{|D_t|(|\mathsf{y}_{i1}|+|\mathsf{y}_{i2}|)} \\
&\quad \cdot [[\cdots[\tau^s(\mathrm{ad}(\mathsf{y}_{i1})), [\tau^s(\mathrm{ad}(\mathsf{y}_{i2})), \mathrm{ad}(D_t)(\mathsf{y}_{i3})]], \cdots, \tau^s(\mathrm{ad}(\mathsf{y}_{i(n-1)}))], \tau^s(\mathrm{ad}(w))] \\
&\quad + \cdots \\
&\quad + \sum_{i\in I}(-1)^{|D_t||p|} [\cdots[[\tau^s(\mathrm{ad}(\mathsf{y}_{i1})), \tau^s(\mathrm{ad}(\mathsf{y}_{i2}))], \tau^s(\mathrm{ad}(\mathsf{y}_{i3}))], \cdots, [\tau^s\, \mathrm{ad}(\mathsf{y}_{i(n-1)}), \mathrm{ad}(\mathrm{d}_{D_t})(w)]].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&= \mathrm{ad}\Bigg(\sum_{i\in I} [[\cdots[[(\mathrm{d}_{D_t})(\mathsf{y}_{i1}), \tau^s(\mathsf{y}_{i2})], \tau^s(\mathsf{y}_{i3})], \cdots, \tau^s(\mathsf{y}_{i(n-1)})], \tau^s(w)] \\
&\quad + \sum_{i\in I}(-1)^{|D_t||\mathsf{y}_{i1}|} [[\cdots[[\tau^s(\mathsf{y}_{i1}), (\mathrm{d}_{D_t})(\mathsf{y}_{i2})], \tau^s(\mathsf{y}_{i3})], \cdots, \tau^s(\mathsf{y}_{i(n-1)})], \tau^s(w)] \\
&\quad + \sum_{i\in I}(-1)^{|D_t|(|\mathsf{y}_{i1}|+|\mathsf{y}_{i2}|)} [[\cdots[\tau^s(\mathsf{y}_{i1}), [\tau^s(\mathsf{y}_{i2}), (\mathrm{d}_{D_t})(\mathsf{y}_{i3})]], \cdots, \tau^s(\mathsf{y}_{i(n-1)})], \tau^s(w)] \\
&\quad + \cdots \\
&\quad + \sum_{i\in I}(-1)^{|D_t||p|} [\cdots[[\tau^s(\mathsf{y}_{i1}), \tau^s(\mathsf{y}_{i2})], \tau^s(\mathsf{y}_{i3})], \cdots, [\tau^s(\mathsf{y}_{i(n-1)}), (\mathrm{d}_{D_t})(w)]] \Bigg).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathrm{ad}(\delta_{D_t})\Bigg(\Bigg[\sum_{i\in I} [\cdots[[\tau^s(\mathsf{y}_{i1}), \tau^s(\mathsf{y}_{i2})], \tau^s(\mathsf{y}_{i3})], \cdots, \tau^s(\mathsf{y}_{i(n-1)})], \tau^s(w) \Bigg] \Bigg) \\
&= \mathrm{ad}\Bigg(\sum_{i\in I} [[\cdots[[(\delta_{D_t})(\mathsf{y}_{i1}), \tau^s(\mathsf{y}_{i2})], \tau^s(\mathsf{y}_{i3})], \cdots, \tau^s(\mathsf{y}_{i(n-1)})], \tau^s(w)] \\
&\quad + \sum_{i\in I}(-1)^{|D_t||\mathsf{y}_{i1}|} [[\cdots[[\tau^s(\mathsf{y}_{i1}), (\delta_{D_t})(\mathsf{y}_{i2})], \tau^s(\mathsf{y}_{i3})], \cdots, \tau^s(\mathsf{y}_{i(n-1)})], \tau^s(w)] \\
&\quad + \sum_{i\in I}(-1)^{|D_t|(|\mathsf{y}_{i1}|+|\mathsf{y}_{i2}|)} [[\cdots[\tau^s(\mathsf{y}_{i1}), [\tau^s(\mathsf{y}_{i2}), (\delta_{D_t})(\mathsf{y}_{i3})]], \cdots, \tau^s(\mathsf{y}_{i(n-1)})], \tau^s(w)] \\
&\quad + \cdots
\end{aligned}
$$

$$+\sum_{i\in I}(-1)^{|D_t||p|}\left[\cdots[[\tau^s(\mathsf{y}_{i1}),\tau^s(\mathsf{y}_{i2})],\tau^s(\mathsf{y}_{i3})],\cdots,[\tau^s(\mathsf{y}_{i(n-1)}),(\delta_{D_t})(w)]\right]\Big).$$

We therefore have

$$\begin{aligned}
&\delta_{D_t}\Big(\Big[\sum_{i\in I}[\cdots[[\tau^s(\mathsf{y}_{i1}),\tau^s(\mathsf{y}_{i2})],\tau^s(\mathsf{y}_{i3})],\cdots,\tau^s(\mathsf{y}_{i(n-1)})],\tau^s(w)\Big]\Big)\\
=&\sum_{i\in I}\left[[\cdots[[(\delta_{D_t})(\mathsf{y}_{i1}),\mathsf{y}_{i2}],\tau^s(\mathsf{y}_{i3})],\cdots,\tau^s(\mathsf{y}_{i(n-1)})\right],\tau^s(w)]\\
&+\sum_{i\in I}(-1)^{|D_t||\mathsf{y}_{i1}|}\left[[\cdots[[\tau^s(\mathsf{y}_{i1}),(\delta_{D_t})(\mathsf{y}_{i2})],\mathsf{y}_{i3}],\cdots,\tau^s(\mathsf{y}_{i(n-1)})\right],\tau^s(w)]\\
&+\sum_{i\in I}(-1)^{|D_t|(|\mathsf{y}_{i1}|+|\mathsf{y}_{i2}|)}\left[[\cdots[\tau^s(\mathsf{y}_{i1}),[\mathsf{y}_{i2},(\delta_{D_t})(\mathsf{y}_{i3})]],\cdots,\tau^s(\mathsf{y}_{i(n-1)})\right],\tau^s(w)]\\
&+\cdots\\
&+\sum_{i\in I}(-1)^{|D_t||p|}\left[\cdots[[\tau^s(\mathsf{y}_{i1}),\tau^s(\mathsf{y}_{i2})],\tau^s(\mathsf{y}_{i3})],\cdots,[\mathsf{y}_{i(n-1)},(\delta_{D_t})(w)]\right].
\end{aligned}$$

Since $\mathsf{y}_{i1},\mathsf{y}_{i2}.\cdots,\mathsf{y}_{i(n-1)}$, we can deduce that $\mathrm{d}_{D_t}\in hDer_{\tau^s}(\chi)$. Thus, the proof is complete. □

Lemma 4.3. *Given the trivial centralizer of $h\,\mathrm{Inn}(\chi)$ in $hN\,\mathrm{Der}(\chi)$, i.e., $C_{hN\,\mathrm{Der}(\chi)}(h\,\mathrm{Inn}(\chi))=0$. Prove that the center of $hN\,\mathrm{Der}(\chi)$ is 0.*

Proof. Assume that $D_t\in hC_{NDer_{\tau^s}(\chi)}(\mathrm{Inn}_{\tau^k}(\chi))$. Then, for any $P\in\chi$, it holds that $[D_t,\mathrm{ad}_P]=0$. Consequently, for any $P\in\chi$ with $\tau^k(P)=P$, we deduce that

$$D_t([\mathsf{z}_1,\mathsf{z}_2])=(-1)^{|D_t||\mathsf{z}_1|}[\tau^s(\mathsf{z}_1),D_t(\mathsf{z}_2)]=[D_t(\mathsf{z}_1),\tau^s(\mathsf{z}_2)].$$

On one side, we have the following for any $\mathsf{z}_1,\mathsf{z}_2,\mathsf{z}_3\in\chi$:

$$\begin{aligned}
D_t([[\mathsf{z}_1,\mathsf{z}_2],\mathsf{z}_3])&=(-1)^{|D_t||\mathsf{z}_1|}[[\tau^s(\mathsf{z}_1),D_t(\mathsf{z}_2)],\tau^s(\mathsf{z}_3)]\\
&=(-1)^{|D_t|(|\mathsf{z}_1|+|\mathsf{z}_2|)}[[\tau^s(\mathsf{z}_1),\tau^s(\mathsf{z}_2)],D_t(\mathsf{z}_3)]\\
&=[[D_t(\mathsf{z}_1),\tau^s(\mathsf{z}_2)],\tau^s(\mathsf{z}_3)].
\end{aligned}$$

Moreover, on the flip side, for $\mathsf{z}_1,\mathsf{z}_2,\mathsf{z}_2\cdots\mathsf{z}_{n-1},\mathsf{z}_n\in\chi$, we establish that, $D_t\in hN\,\mathrm{Der}(\chi)$

$$\begin{aligned}
D_t([\mathsf{z}_1,\mathsf{z}_2,\mathsf{z}_3,\cdots,\mathsf{z}_n])&=(-1)^{|D_t|(|\mathsf{z}_1|+|\mathsf{z}_2|+|\mathsf{z}_2|+\cdots|\mathsf{z}_{n-1}|)}\left[\cdots[[\tau^k(\mathsf{z}_1),\tau^k(\mathsf{z}_2)],\tau^k(\mathsf{z}_3)],\cdots,D_t(\mathsf{z}_n)\right]\\
&=\cdots\\
&=(-1)^{|D_t|(|\mathsf{z}_1|+|\mathsf{z}_2|)}\left[\cdots[[\tau^k(\mathsf{z}_1),\tau^k(\mathsf{z}_2)],D_t(\mathsf{z}_3)],\cdots,\tau^k(\mathsf{z}_n)\right]\\
&=(-1)^{|D_t||\mathsf{z}_1|}\left[\cdots[[\tau^k(\mathsf{z}_1),D_t(\mathsf{z}_2)],\tau^k(\mathsf{z}_3)],\cdots,\tau^k(\mathsf{z}_n)\right]\\
&=\left[\cdots[[D_t(\mathsf{z}_1),\tau^k(\mathsf{z}_2)],\tau^k(\mathsf{z}_3)],\cdots,\tau^k(\mathsf{z}_n)\right].
\end{aligned}$$

Therefore,

$$D_t([\mathsf{z}_1,\mathsf{z}_2,\mathsf{z}_3,\cdots,\mathsf{z}_n])=\left[\cdots[[D_t(\mathsf{z}_1),\tau^k(\mathsf{z}_2)],\tau^k(\mathsf{z}_3)],\cdots,\tau^k(\mathsf{z}_n)\right]$$

$$
\begin{aligned}
&= (-1)^{|D_t||z_1|}\Big[\cdots[[\tau^k(z_1), D_t(z_2)], \tau^k(z_3)], \cdots, \tau^k(z_n)\Big]\\
&= (-1)^{|D_t|(|z_1|+|z_2|)}\Big[\cdots[[\tau^k(z_1), \tau^k(z_2)], D_t(z_3)], \cdots, \tau^k(z_n)\Big]\\
&= \cdots\\
&= (-1)^{|D_t|(|z_1|+|z_2|+|z_2|+\cdots|z_{n-1}|)}\Big[\cdots[[\tau^k(z_1), \tau^k(z_2)], \tau^k(z_3)], \cdots, D_t(z_n)\Big]
\end{aligned}
$$

$$
nD_t(\,[\cdots[[z_1, z_2], z_3], \cdots, z_n]\,) = 0,
$$

which indicates that, as χ is a simple Hom-Lie superalgebra. Thus, its center is trivial, that is,

$$
Z(\chi) = 0.
$$

Therefore,

$$
D_t(\chi) \subseteq Z(\chi) = 0,
$$

and, consequently, $D_t = 0$. This completes the proof. □

We are now prepared to establish the principal result of this section. The assumption $\tau = \mathrm{id}_\chi$ is essential in the following result. Indeed, if $\tau \neq \mathrm{id}_\chi$, then a τ^s-derivation does not automatically reduce to an ordinary derivation. Hence, the equality with $\mathrm{Der}(\chi)$ holds only in the untwisted case.

Theorem 4.2. *Let $(\chi, [\cdot, \cdots, \cdot], \tau)$ be a simple Hom-Lie superalgebra over a field of characteristic zero. Assume that $\tau = \mathrm{id}_\chi$. We then prove that $h\,\mathrm{NDer}(\chi) = \mathrm{Der}(\chi)$.*

Proof. Since both $\mathrm{Der}(\chi)$ and $h\,\mathrm{NDer}(\chi)$ are graded subspaces, it is enough to prove the two inclusions for homogeneous linear maps. First, we show that

$$
\mathrm{Der}(\chi) \subseteq h\,\mathrm{NDer}(\chi).
$$

Let $D \in \mathrm{Der}(\chi)$ be homogeneous. Thus, for every homogeneous $x, y \in \chi$, we have

$$
D([x, y]) = [D(x), y] + (-1)^{|D||x|}[x, D(y)].
$$

For the homogeneous elements $x_1, x_2, \ldots, x_n \in \chi$, define the Lie bracket by

$$
[x_1, x_2, \ldots, x_n] = [\cdots[[x_1, x_2], x_3], \ldots, x_n].
$$

We claim that

$$
D([x_1, x_2, \ldots, x_n]) = \sum_{i=1}^{n} (-1)^{|D|(|x_1|+\cdots+|x_{i-1}|)}[x_1, \ldots, D(x_i), \ldots, x_n].
$$

This is precisely the n-derivation identity in the untwisted case.

We prove the claim by induction on n. For $n = 2$, the formula is exactly the usual derivation identity. Assume that the formula holds for $n - 1$. Put

$$
u = [x_1, x_2, \ldots, x_{n-1}].
$$

We then have

$$
[x_1, x_2, \ldots, x_n] = [u, x_n].
$$

Since, D is a derivation, we have

$$D([u, \mathsf{x}_n]) = [D(u), \mathsf{x}_n] + (-1)^{|D||u|}[u, D(\mathsf{x}_n)].$$

Because

$$|u| = |x_1| + \cdots + |x_{n-1}|,$$

By the induction hypothesis, we have

$$D(u) = \sum_{i=1}^{n-1} (-1)^{|D|(|\mathsf{x}_1|+\cdots+|\mathsf{x}_{i-1}|)}[\mathsf{x}_1, \ldots, D(\mathsf{x}_i), \ldots, \mathsf{x}_{n-1}].$$

Therefore, we have

$$\begin{aligned} D([\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n]) = &\sum_{i=1}^{n-1} (-1)^{|D|(|\mathsf{x}_1|+\cdots+|\mathsf{x}_{i-1}|)}[\mathsf{x}_1, \ldots, D(\mathsf{x}_i), \ldots, \mathsf{x}_{n-1}, \mathsf{x}_n] \\ &+ (-1)^{|D|(|\mathsf{x}_1|+\cdots+|\mathsf{x}_{n-1}|)}[\mathsf{x}_1, \ldots, \mathsf{x}_{n-1}, D(\mathsf{x}_n)]. \end{aligned}$$

Hence,

$$D([\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n]) = \sum_{i=1}^{n} (-1)^{|D|(|\mathsf{x}_1|+\cdots+|\mathsf{x}_{i-1}|)}[\mathsf{x}_1, \ldots, D(\mathsf{x}_i), \ldots, \mathsf{x}_n].$$

Since $\tau = \mathrm{id}_\chi$, this is precisely the Hom-n-derivation identity in the untwisted case. Thus, $D \in h\,\mathrm{NDer}(\chi)$. Consequently, we have

$$\mathrm{Der}(\chi) \subseteq h\,\mathrm{NDer}(\chi).$$

Next, we prove the reverse inclusion. Let

$$D_t \in h\,\mathrm{NDer}(\chi)$$

be homogeneous. Since $\tau = \mathrm{id}_\chi$, every homogeneous Hom-n-derivation is an ordinary n-derivation. By Lemma 4.2, a linear map $\mathrm{d}_{D_t} : \chi \longrightarrow \chi$ exists such that, for every homogeneous $p \in \chi$, we have

$$[D_t, \mathrm{ad}_p] = \mathrm{ad}_{\mathrm{d}_{D_t}(p)}\,.$$

Moreover, Lemma 4.2 shows that d_{D_t} is a Hom-derivation of degree τ^s. Since $\tau = \mathrm{id}_\chi$, this Hom-derivation identity reduces to the ordinary derivation identity. Explicitly, for every homogeneous $p, q \in \chi$, we have

$$\mathrm{d}_{D_t}([p, q]) = [\mathrm{d}_{D_t}(p), q] + (-1)^{|D_t||p|}[p, \mathrm{d}_{D_t}(q)].$$

Therefore,

$$\mathrm{d}_{D_t} \in \mathrm{Der}(\chi).$$

Since, d_{D_t} is an ordinary derivation, it satisfies

$$[\mathrm{d}_{D_t}, \mathrm{ad}_p] = \mathrm{ad}_{\mathrm{d}_{D_t}(p)}$$

for every homogeneous $p \in \chi$. Combining this identity with

$$[D_t, \mathrm{ad}_p] = \mathrm{ad}_{\mathrm{d}_{D_t}(p)},$$

we obtain

$$[D_t - \mathrm{d}_{D_t}, \mathrm{ad}_p] = 0$$

for every homogeneous $p \in \chi$. Hence, we obtain

$$D_t - \mathrm{d}_{D_t} \in C_{h\,\mathrm{NDer}(\chi)}(h\,\mathrm{Inn}(\chi)).$$

By Lemma 4.3, the centralizer of $h\,\mathrm{Inn}(\chi)$ in $h\,\mathrm{NDer}(\chi)$ is trivial. Therefore,

$$D_t - \mathrm{d}_{D_t} = 0.$$

Thus,

$$D_t = \mathrm{d}_{D_t}.$$

Since $\mathrm{d}_{D_t} \in \mathrm{Der}(\chi)$, it follows that

$$D_t \in \mathrm{Der}(\chi).$$

Therefore,

$$h\,\mathrm{NDer}(\chi) \subseteq \mathrm{Der}(\chi).$$

Combining the two inclusions, we obtain

$$h\,\mathrm{NDer}(\chi) = \mathrm{Der}(\chi).$$

This completes the proof. □

Connection with Hom-associative superalgebras

We now explain how the notion of Hom-n-derivation is compatible with Hom-associative superalgebras. This gives a direct link between the present results and the Hom-associative setting.

Let $(\mathcal{A}, \mu, \tau)$ be a multiplicative Hom-associative superalgebra. For homogeneous $x, y \in \mathcal{A}$, write $xy = \mu(x, y)$. The associated Hom-Lie superalgebra is denoted by $\mathcal{A}^- = (\mathcal{A}, [\cdot,\cdot], \tau)$, where the bracket is the graded commutator $[x, y] = xy - (-1)^{|x||y|}yx$. Since $(\mathcal{A}, \mu, \tau)$ is multiplicative, we have $\tau(xy) = \tau(x)\tau(y)$. Therefore, $\tau([x, y]) = [\tau(x), \tau(y)]$ for all homogeneous $x, y \in \mathcal{A}$.

Proposition 4.1. *Let $(\mathcal{A}, \mu, \tau)$ be a multiplicative Hom-associative superalgebra, and let $\mathcal{A}^- = (\mathcal{A}, [\cdot,\cdot], \tau)$ be its associated Hom-Lie superalgebra. Let $D : \mathcal{A} \to \mathcal{A}$ be a homogeneous linear map such that $D\tau = \tau D$. Assume that D is a τ^k-derivation of the Hom-associative product, that is,*

$$D(xy) = D(x)\tau^k(y) + (-1)^{|D||x|}\tau^k(x)D(y),$$

for all homogeneous $x, y \in \mathcal{A}$. Then, D is a τ^k-derivation of the associated Hom-Lie superalgebra $\mathcal{A}^-$. Consequently, for every $n \geq 2$, D is a τ^k-n-derivation of $\mathcal{A}^-$ with respect to the Lie bracket.

Proof. Let $x, y \in \mathcal{A}$ be homogeneous. By definition of the graded commutator, we have

$$[x, y] = xy - (-1)^{|x||y|}yx.$$

Applying D, we obtain

$$D([x, y]) = D(xy) - (-1)^{|x||y|}D(yx).$$

Since D is a τ^k-derivation of the Hom-associative product, we have

$$D(xy) = D(x)\tau^k(y) + (-1)^{|D||x|}\tau^k(x)D(y),$$

and

$$D(yx) = D(y)\tau^k(x) + (-1)^{|D||y|}\tau^k(y)D(x).$$

Therefore,

$$D([x, y]) = D(x)\tau^k(y) + (-1)^{|D||x|}\tau^k(x)D(y) - (-1)^{|x||y|}D(y)\tau^k(x) - (-1)^{|x||y|+|D||y|}\tau^k(y)D(x).$$

On the other hand, we have

$$[D(x), \tau^k(y)] = D(x)\tau^k(y) - (-1)^{(|D|+|x|)|y|}\tau^k(y)D(x),$$

and

$$[\tau^k(x), D(y)] = \tau^k(x)D(y) - (-1)^{|x|(|D|+|y|)}D(y)\tau^k(x).$$

Hence,

$$\begin{aligned}[D(x), \tau^k(y)] + (-1)^{|D||x|}[\tau^k(x), D(y)] &= D(x)\tau^k(y) - (-1)^{(|D|+|x|)|y|}\tau^k(y)D(x)\\ &\quad + (-1)^{|D||x|}\tau^k(x)D(y) - (-1)^{|D||x|+|x|(|D|+|y|)}D(y)\tau^k(x).\end{aligned}$$

Since

$$(|D| + |x|)|y| = |D||y| + |x||y|$$

and

$$|D||x| + |x|(|D| + |y|) = 2|D||x| + |x||y| \equiv |x||y| \pmod 2,$$

we get

$$\begin{aligned}[D(x), \tau^k(y)] + (-1)^{|D||x|}[\tau^k(x), D(y)] &= D(x)\tau^k(y) + (-1)^{|D||x|}\tau^k(x)D(y) - (-1)^{|x||y|}D(y)\tau^k(x)\\ &\quad - (-1)^{|x||y|+|D||y|}\tau^k(y)D(x).\end{aligned}$$

Therefore,

$$D([x, y]) = [D(x), \tau^k(y)] + (-1)^{|D||x|}[\tau^k(x), D(y)].$$

Thus, D is a τ^k-derivation of the associated Hom-Lie superalgebra $\mathcal{A}^-$.

It remains to show that D is a τ^k-N-derivation for every $N \geq 2$. For the homogeneous elements $x_1, \ldots, x_n \in \mathcal{A}$, we define the Lie bracket by

$$[x_1, \ldots, x_n] = [\cdots[[x_1, x_2], x_3], \ldots, x_n].$$

We prove by induction on n that

$$D([x_1,\dots,x_n]) = \sum_{i=1}^{n}(-1)^{|D|(|x_1|+\cdots+|x_{i-1}|)}[\tau^k(x_1),\dots,D(x_i),\dots,\tau^k(x_n)].$$

For $n = 2$, this is precisely the τ^k-derivation identity for the associated Hom-Lie bracket, which has just been proved.

Assume that the formula holds for $n-1$. Put

$$u = [x_1,\dots,x_{n-1}].$$

We then have

$$[x_1,\dots,x_n] = [u, x_n].$$

Since D is a τ^k-derivation of the associated Hom-Lie bracket, we have

$$D([u,x_n]) = [D(u),\tau^k(x_n)] + (-1)^{|D||u|}[\tau^k(u), D(x_n)].$$

Because

$$|u| = |x_1| + \cdots + |x_{n-1}|,$$

and, by the induction hypothesis,

$$D(u) = \sum_{i=1}^{n-1}(-1)^{|D|(|x_1|+\cdots+|x_{i-1}|)}[\tau^k(x_1),\dots,D(x_i),\dots,\tau^k(x_{n-1})],$$

we get

$$\begin{aligned}[D(u),\tau^k(x_n)] = &\sum_{i=1}^{n-1}(-1)^{|D|(|x_1|+\cdots+|x_{i-1}|)}\\ &\times[\tau^k(x_1),\dots,D(x_i),\dots,\tau^k(x_{N-1}),\tau^k(x_n)].\end{aligned}$$

Moreover, since τ is multiplicative for the associated Hom-Lie bracket, we have

$$\tau^k(u) = [\tau^k(x_1),\dots,\tau^k(x_{n-1})].$$

Hence,

$$[\tau^k(u), D(x_n)] = [\tau^k(x_1),\dots,\tau^k(x_{n-1}),D(x_n)].$$

Therefore,

$$\begin{aligned}D([x_1,\dots,x_n]) = &\sum_{i=1}^{n-1}(-1)^{|D|(|x_1|+\cdots+|x_{i-1}|)}[\tau^k(x_1),\dots,D(x_i),\dots,\tau^k(x_n)]\\ &+(-1)^{|D|(|x_1|+\cdots+|x_{n-1}|)}[\tau^k(x_1),\dots,\tau^k(x_{n-1}),D(x_n)].\end{aligned}$$

This is exactly

$$D([x_1,\dots,x_n]) = \sum_{i=1}^{n}(-1)^{|D|(|x_1|+\cdots+|x_{i-1}|)}[\tau^k(x_1),\dots,D(x_i),\dots,\tau^k(x_n)].$$

Thus, D is a τ^k-n-derivation of $\mathcal{A}^-$ for every $n \geq 2$. □

Example 4.1. *Let $\mathcal{A} = M_{1|1}(\mathbb{K})$ be the associative superalgebra of 2×2 supermatrices, where*

$$\mathcal{A}_{\bar{0}} = \mathrm{span}\{E_{11}, E_{22}\}, \qquad \mathcal{A}_{\bar{1}} = \mathrm{span}\{E_{12}, E_{21}\}.$$

Take the usual matrix multiplication and let $\tau = \mathrm{id}_{\mathcal{A}}$. Then, $(\mathcal{A}, \mu, \tau)$ is a multiplicative Hom-associative superalgebra. Define an even linear map $D : \mathcal{A} \to \mathcal{A}$ as

$$D(X) = HX - XH, \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus,

$$D(E_{11}) = 0, \qquad D(E_{22}) = 0, \qquad D(E_{12}) = E_{12}, \qquad D(E_{21}) = -E_{21}.$$

Since $\tau = \mathrm{id}_{\mathcal{A}}$, we have $D\tau = \tau D$ and $\tau^k = \mathrm{id}_{\mathcal{A}}$ for every $k \geq 0$. Moreover, for a homogeneous $X, Y \in \mathcal{A}$, we have

$$D(XY) = D(X)Y + XD(Y),$$

so D is a τ^k-derivation of the Hom-associative product.

The associated Hom-Lie superalgebra $\mathcal{A}^-$ is given by the supercommutator

$$[X, Y] = XY - (-1)^{|X||Y|}YX.$$

For example, we have

$$[E_{12}, E_{21}] = E_{12}E_{21} + E_{21}E_{12} = E_{11} + E_{22}.$$

Furthermore,

$$D([E_{12}, E_{21}]) = D(E_{11} + E_{22}) = 0,$$

while

$$[D(E_{12}), E_{21}] + [E_{12}, D(E_{21})] = [E_{12}, E_{21}] - [E_{12}, E_{21}] = 0.$$

Hence, D is a τ^k-derivation of the associated Hom-Lie superalgebra $\mathcal{A}^-$. Consequently, for every $n \geq 2$, D is also a τ^k-n-derivation of $\mathcal{A}^-$.

5. Extension to non-simple Hom-Lie superalgebras

In this section, we develop one of the directions mentioned previously as a future problem. More precisely, we show that the equality between Hom-N-derivations and ordinary derivations is not restricted only to simple Hom-Lie superalgebras. The result remains true for finite direct sums of simple Hom-Lie superalgebras. We also provide an example showing that the equality does not hold for arbitrary non-simple Hom-Lie superalgebras.

Throughout this section, we assume that $\tau = \mathrm{id}_{\chi}$. Therefore, every Hom-$n$-derivation is an ordinary n-derivation.

Theorem 5.1. *Let $\chi = \chi_1 \oplus \chi_2 \oplus \cdots \oplus \chi_m$ be a finite direct sum of simple Hom-Lie superalgebras over a field of characteristic zero. Assume that $\tau = \mathrm{id}_{\chi}$. Then, $h\,\mathrm{NDer}(\chi) = \mathrm{Der}(\chi)$.*

Proof. We prove the equality by showing two inclusions.

First, let $D \in \mathrm{Der}(\chi)$ be homogeneous. Since $\tau = \mathrm{id}_\chi$, the Hom-N-derivation identity reduces to the ordinary N-derivation identity. As in the proof of Theorem 4.2, an induction on the length of the left-normed bracket gives

$$D([x_1, \dots, x_n]) = \sum_{i=1}^{n} (-1)^{|D|(|x_1|+\cdots+|x_{i-1}|)} [x_1, \dots, D(x_i), \dots, x_n]$$

for all homogeneous $x_1, \dots, x_n \in \chi$. Hence, we have

$$D \in h\,\mathrm{NDer}(\chi).$$

Therefore, we have

$$\mathrm{Der}(\chi) \subseteq h\,\mathrm{NDer}(\chi).$$

Conversely, let $D \in h\,\mathrm{NDer}(\chi)$ be homogeneous. We prove that D is an ordinary derivation. Since each χ_j is a non-abelian simple Hom-Lie superalgebra and $\tau = \mathrm{id}_\chi$, each χ_j is perfect, that is,

$$[\chi_j, \chi_j] = \chi_j.$$

Consequently, for every $n \geq 2$, we have

$$[\chi_j, \dots, \chi_j] = \chi_j,$$

where the bracket is left-normed.

First, we show that D preserves each simple summand χ_j. Let $x_1, \dots, x_n \in \chi_j$ be homogeneous and D is an n-derivation. We then have

$$D([x_1, \dots, x_n]) = \sum_{i=1}^{n} (-1)^{|D|(|x_1|+\cdots+|x_{i-1}|)} [x_1, \dots, D(x_i), \dots, x_n].$$

We can write

$$D(x_i) = \sum_{r=1}^{m} y_{i,r}, \qquad y_{i,r} \in \chi_r.$$

Because $\chi = \chi_1 \oplus \cdots \oplus \chi_m$ is a direct sum of ideals, we have

$$[\chi_j, \chi_r] = 0 \text{ whenever } j \neq r.$$

Therefore,

$$[x_1, \dots, D(x_i), \dots, x_n],$$

and only the component $y_{i,j} \in \chi_j$ can contribute. All components $y_{i,r}$ with $r \neq j$ vanish in the bracket. Hence,

$$D([x_1, \dots, x_N]) \in \chi_j.$$

Since

$$[\chi_j, \dots, \chi_j] = \chi_j,$$

it follows that $D(\chi_j) \subseteq \chi_j$. Thus, D preserves every summand χ_j. For each j, let

$$D_j = D|_{\chi_j}.$$

Since $D(\chi_j) \subseteq \chi_j$, the restriction D_j is well-defined as a map from χ_j to itself. Moreover, the N-derivation identity for a D being restricted to homogeneous elements of χ_j shows that

$$D_j \in h\,\mathrm{NDer}(\chi_j).$$

By Theorem 4.2, applied to the simple Hom-Lie superalgebra χ_j, we have

$$D_j \in \mathrm{Der}(\chi_j)$$

for every $j = 1, \ldots, m$.

Now, let $x, y \in \chi$ be homogeneous. we can write as

$$x = x_1 + \cdots + x_m, \qquad y = y_1 + \cdots + y_m,$$

where $x_j, y_j \in \chi_j$. We then have

$$[x, y] = \sum_{j=1}^{m} [x_j, y_j].$$

Using the fact that each D_j is a derivation of χ_j, we obtain

$$\begin{aligned} D([x, y]) &= \sum_{j=1}^{m} D_j([x_j, y_j]) \\ &= \sum_{j=1}^{m} \Big([D_j(x_j), y_j] + (-1)^{|D||x|} [x_j, D_j(y_j)] \Big) \\ &= [D(x), y] + (-1)^{|D||x|} [x, D(y)]. \end{aligned}$$

Therefore, D is an ordinary derivation of χ. Hence, we have $h\,\mathrm{NDer}(\chi) \subseteq \mathrm{Der}(\chi)$. Combining both inclusions, we conclude that $h\,\mathrm{NDer}(\chi) = \mathrm{Der}(\chi)$. □

The preceding theorem shows that the simplicity assumption in Theorem 4.2 can be weakened to a finite direct sum of simple ideals. However, the equality does not hold for arbitrary non-simple Hom-Lie superalgebras. The following example illustrates this point.

Example 5.1. *Let χ be the three-dimensional Heisenberg Lie algebra with the basis $\{e_1, e_2, e_3\}$ and the bracket $[e_1, e_2] = e_3$, with all other brackets determined by skew-symmetry and bilinearity. Regard χ as a purely even Hom-Lie superalgebra and take $\tau = \mathrm{id}_\chi$. In this cae, χ is two-step nilpotent, because $[\chi, [\chi, \chi]] = 0$. Therefore, every Lie bracket of length $n \geq 3$ is zero. Hence, for $n \geq 3$, every linear map $T : \chi \to \chi$ satisfies the N-derivation identity trivially. Indeed, for all $x_1, \ldots, x_n \in \chi$, we have*

$$[x_1, \ldots, x_n] = 0$$

and also

$$[x_1, \ldots, T(x_i), \ldots, x_n] = 0$$

for every $i = 1, \ldots, n$. Thus, we have

$$T([x_1, \ldots, x_n]) = \sum_{i=1}^{n} [x_1, \ldots, T(x_i), \ldots, x_n].$$

Therefore, $T \in h\,\mathrm{NDer}(\chi)$ for every linear map $T : \chi \to \chi$.

Now, we define $T : \chi \to \chi$ as

$$T(e_1) = 0, \quad T(e_2) = 0, \quad T(e_3) = e_1.$$

In this case, T is an n-derivation for every $N \geq 3$. However, T is not an ordinary derivation. Indeed,

$$T([e_1, e_2]) = T(e_3) = e_1,$$

whereas

$$[T(e_1), e_2] + [e_1, T(e_2)] = [0, e_2] + [e_1, 0] = 0.$$

Thus, we have

$$T([e_1, e_2]) \neq [T(e_1), e_2] + [e_1, T(e_2)].$$

Hence, $T \notin \mathrm{Der}(\chi)$. Consequently, $h\,\mathrm{NDer}(\chi) \neq \mathrm{Der}(\chi)$ for this non-simple Hom-Lie superalgebra when $n \geq 3$.

Example 5.1 shows that some structural assumption is necessary. Thus, Theorem 5.1 gives a proper non-simple extension of Theorem 4.2, while the nilpotent example demonstrates that the equality cannot be extended to all non-simple Hom-Lie superalgebras.

6. Conclusions

In the present paper, we investigated Lie n-derivations and generalized Lie n-derivations in the framework of Hom-Lie superalgebras. By introducing and studying the structures $hN\,\mathrm{Der}(\chi)$ and $hGN\,\mathrm{Der}(\chi)$, we have identified significant algebraic properties and analyzed how they relate to fundamental objects such as the n-centroids, the n-quasi centroids, and central n-derivations.

Through this analysis, we have identified foundational connections among these operators, offering a clearer view of the function and significance of generalized derivations in the Hom-Lie setting. In particular, we have proven that when the underlying Hom-Lie superalgebra is simple, the set of all Lie n-derivations coincides with that of the standard derivations as follows:

$$hN\,\mathrm{Der}(\chi) = h\,\mathrm{Der}(\chi).$$

This result highlights that, for simple structures, n-derivations do not extend beyond the ordinary derivation framework. Overall, these results deepen the understanding of derivation-related operators in Hom-Lie superalgebras and create a foundation for further study, including potential applications in cohomology and deformation theory.

While this paper offers a thorough classification of n-derivations for Hom-Lie superalgebras and establishes several fundamental structural results, a number of problems remain open. One natural direction is to broaden the classification to more general families of Hom-Lie superalgebras, such as non-simple or infinite-dimensional ones, where the relationship between $hN\,\mathrm{Der}(\chi)$ and $h\,\mathrm{Der}(\chi)$ may exhibit different behaviors. Another promising line of work is to study the cohomological aspects of n-derivations and their role in deformation theory, which could shed further light on the algebraic structure of these superalgebras. In addition, it would be worthwhile to analyze how n-derivations interact with other algebraic frameworks, including Hom-associative algebras and Hopf algebras.

Author contributions

Abu Zaid Ansari and Mukhtar Ahmed: Conceptualization, methodology, formal analysis; Shakir Ali and Intan Muchtadi-Alamsyah: Investigation, validation, supervision; Faiza Shujat: Writing–original draft and writing–review and editing. All authors jointly contributed to all aspects of the work and have read and approved the submitted version of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare have no conflict of interest.

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