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Research article

Symmetric nonnegative definite solutions of certain linear matrix equations on a subspace

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Abstract: For a given matrix $G \in \mathbb{R}^{l\times n}$, define a subspace Ω by $\Omega = \{z \in \mathbb{R}^n \mid Gz = 0\}$, and let $\mathcal{H} = \{K \in \mathbb{R}^{n\times n} \mid (x, Ky) = (Kx, y),\ x^\top Kx \geq 0,\ \forall x, y \in \Omega\}$. Thus, $\mathcal{H}$ consists of all symmetric nonnegative definite matrices on the subspace Ω. In this paper, we investigate the existence of $X \in \mathcal{H}$ satisfying the linear matrix equations $AX = C, XB = D$, and $AXB = D$. A necessary and sufficient condition for solvability and an explicit representation of the general solution are obtained. Finally, two numerical examples are provided to demonstrate the accuracy of the results.

Keywords: subspace; nonnegative matrix; generalized inverse; orthogonal projector
Mathematics Subject Classification: 15A24, 15A57

1. Introduction

In the paper, $\mathbb{R}^{m\times n}$ and $\mathbb{OR}^{m\times m}$ denote the sets of all real $m \times n$ matrices and all $m \times m$ orthogonal matrices, and $K^\top$, $K^\dagger$, $\mathcal{R}(K)$, $\|K\|_F$, and $\lambda(K)$ denote the transpose, the Moore-Penrose inverse, the range space, the Frobenius norm, and the spectrum of the matrix $K \in \mathbb{R}^{m\times n}$, respectively. K^- represents an arbitrary generalized inner inverse of K, that is, K^- satisfies $KK^-K = K$. I_m stands for the identity matrix of size m. $K \geq 0$ means that K is a symmetric nonnegative definite (SND) matrix. Furthermore, two orthogonal projectors (OPs) E_K and F_K are defined, respectively, by $E_K = I_m - KK^\dagger$ and $F_K = I_n - K^\dagger K$, where $K \in \mathbb{R}^{m\times n}$.

For a given matrix $G \in \mathbb{R}^{l\times n}$, $\Omega = \{z \in \mathbb{R}^n | Gz = 0\}$ then forms a subspace of $\mathbb{R}^n$. If we define

$$\mathcal{H} = \{K \in \mathbb{R}^{n\times n} | (x, Ky) = (Kx, y), x^\top Kx \geq 0, \forall x, y \in \Omega\},$$

then $\mathcal{H}$ represents the set of all SND matrices on the subspace Ω.

It is known that the linear matrix equations

$$AX = C,\ XB = D, \tag{1.1}$$

and the linear matrix equation

$$AXB = D \tag{1.2}$$

have been considered by many scholars. The various special solutions of Eqs (1.1) and (1.2) have been investigated. For example, Mitra [1] established the necessary and sufficient conditions for the solvability of Eq (1.1) and provided an explicit representation of its general solution using the generalized inverse. Wang [2] found the bisymmetric and centro-symmetric solutions to Eq (1.1) over the real quaternion algebra. Yuan [3] studied the least-squares solution and the least-squares symmetric solution with the minimum-norm of Eq (1.1) by applying the singular value decomposition. Dajić [4] studied the Hermitian positive definite solutions of Eq (1.1) for Hilbert space operators. For Eq (1.2), Penrose [5] derived both the solvability condition and an explicit form of the general solution of Eq (1.2). Dai [6], Khatri [7], and Peng [8] considered the symmetric solution of Eq (1.2). Khatri [7] obtained the solvability conditions for Eq (1.2) to have a Hermitian and Hermitian nonnegative definite solution. Cvetković [9] and Yuan and Zuo [10] discussed the Re-nonnegative definite and Re-positive definite solutions of Eq (1.2). In the context of solving constrained matrix equations, recent studies have also applied generalized inverses and matrix decomposition tools to equality-constrained least squares problems over commutative quaternion matrices, demonstrating further extensions of solution methods under constraints [11, 12].

We have noticed that the problem of solving SND solutions of Eqs (1.1) and (1.2) on a given subspace is rarely discussed. In 2005, Yuan and Dai [13] studied the antisymmetric solution of equation $AX = B$ on the subspace $\Omega = \left\{z \in \mathbb{R}^n | Gz = 0, G \in \mathbb{R}^{l\times n}\right\}$ by using the singular value decomposition. Recently, Hu and Yuan [14, 15] extended the results in [13] and put forward the solvability conditions for the existence of symmetric solutions of Eqs (1.1) and (1.2), as well as antisymmetric solutions of the equation $AXA^\top = B$ on the subspace Ω by virtue of the generalized inverse and matrix decompositions. Inspired by the works of papers [13–15], this paper will further investigate the solvability conditions for the matrix equations (1.1) and (1.2) to have SND solutions on a given subspace. Studying SND solutions is meaningful in practice. For example, the analysis mass matrix M_a, stiffness matrix K_a, and damping matrix G_a obtained by discretizing the damping vibration system using the finite element method usually exhibit symmetry and possess a banded structure. How to ensure that the coefficient matrices of the modified model preserve the same banded structure is an important issue [16]. All symmetric banded matrices form a subspace of $\mathbb{R}^{n\times n}$. Therefore, preserving the symmetric banded structure of the original model is equivalent to finding symmetric matrices $\hat{M}, \hat{K}$, and anti-symmetric matrices $\hat{G}$ in a specific subspace [15]. Ensuring the modified matrices are nonnegative definite guarantees that the kinetic energy of the system remains nonnegative, thereby enhancing stability. More broadly, subspace-constrained matrix formulations also underpin tasks such as community detection in networks [17], spectral clustering with graph structure learning [18], and nonnegative matrix factorization for dimensionality reduction [19]. We will consider the SND solutions of Eqs (1.1) and (1.2) on the subspace Ω, which can be described as follows:

Problem I. Given $A, C \in \mathbb{R}^{m\times n}$, $B, D \in \mathbb{R}^{n\times p}$, and $G \in \mathbb{R}^{l\times n}$, find $X \in \mathcal{H}$ such that Eq (1.1) is satisfied.

Problem II. Given $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{m\times p}$, and $G \in \mathbb{R}^{l\times n}$, find $X \in \mathcal{H}$ such that Eq (1.2) is satisfied.

The paper is outlined as follows: In Section 2, we present some preliminary lemmas. In Sections 3 and 4, by applying the generalized inverse of matrices and some OPs, we derive the solvability conditions for Problems I and II, and the explicit representations of their general solutions. Finally, two numerical examples are provided to validate the efficacy of our results.

2. Preliminary lemmas

Lemma 1. *[20] Let $A, C \in \mathbb{R}^{m\times n}$ and $B, D \in \mathbb{R}^{n\times p}$ be given. Then, Eq* (1.1) *has a solution $X \in \mathbb{R}^{n\times n}$ if and only if*

$$AA^{\dagger}C = C,\ DB^{\dagger}B = D,\ AD = CB. \tag{2.1}$$

In this case, the general solution is

$$X = A^{\dagger}C + F_A DB^{\dagger} + F_A W E_B, \tag{2.2}$$

where $W \in \mathbb{R}^{n\times n}$ is arbitrary.

Based on Lemma 1, the set Ω can be expressed as $\Omega = \{F_G s \mid s \in \mathbb{R}^n\}$. Thus, for any $x, y \in \Omega$, there exist $s, t \in \mathbb{R}^n$ such that $x = F_G s$ and $y = F_G t$. Then it follows that

$$x^{\top}Kx \geq 0 \Leftrightarrow s^{\top}F_G K F_G s \geq 0 \Leftrightarrow F_G K F_G \geq 0,$$
$$(Kx, y) = (x, Ky) \Leftrightarrow t^{\top}F_G K F_G s = t^{\top}F_G K^{\top}F_G s \Leftrightarrow F_G K F_G = F_G K^{\top} F_G.$$

From the above results, $\mathcal{H}$ can be equivalently expressed as

$$\mathcal{H} = \{K \in \mathbb{R}^{n\times n} | F_G K F_G = F_G K^{\top} F_G,\ F_G K F_G \geq 0\}. \tag{2.3}$$

Lemma 2. *Let the spectral decomposition (SD) of F_G be $F_G = P\begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix}P^{\top}$, where $a = \operatorname{rank}(F_G)$, and $P = [\tilde{P}_1,\ \tilde{P}_2]$ is orthogonal. Then, $X \in \mathcal{H}$ if and only if X can be represented as*

$$X = P\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}P^{\top}, \tag{2.4}$$

where $X_1 \in \mathbb{R}^{a\times a}$ with $X_1 = X_1^{\top} \geq 0$ and $X_2 \in \mathbb{R}^{a\times(n-a)}, X_3 \in \mathbb{R}^{(n-a)\times a}$, and $X_4 \in \mathbb{R}^{(n-a)\times(n-a)}$ are arbitrary.

Proof. For $X \in \mathcal{H}$, it follows from (2.3) that $F_G X F_G = F_G X^{\top} F_G, F_G X F_G \geq 0$. Let

$$P^{\top}XP = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, X_1 \in \mathbb{R}^{a\times a}, X_4 \in \mathbb{R}^{(n-a)\times(n-a)}.$$

We have

$$F_G X F_G = P\begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix}P^{\top}XP\begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix}P^{\top} = P\begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}P^{\top} = \tilde{P}_1 X_1 \tilde{P}_1{}^{\top},$$
$$F_G X^{\top} F_G = P\begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix}P^{\top}X^{\top}P\begin{bmatrix} I_a & 0 \\ 0 & 0 \end{bmatrix}P^{\top} = P\begin{bmatrix} X_1^{\top} & 0 \\ 0 & 0 \end{bmatrix}P^{\top} = \tilde{P}_1 X_1^{\top} \tilde{P}_1{}^{\top}.$$

Then, we can obtain (2.4) from the above relationships, $X_1 = X_1^{\top} \geq 0$, and matrices $X_2,\ X_3,$ *and* X_4 are arbitrary. Conversely, for any X provided in (2.4), we can easily see that $X \in \mathcal{H}$ due to $X_1 = X_1^{\top} \geq 0$. □

Lemma 3. *[20] If $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, and $D \in \mathbb{R}^{m\times p}$, then Eq (1.2) is solvable if and only if*

$$AA^{\dagger}DB^{\dagger}B = D. \tag{2.5}$$

In this case, the solution X of Eq (1.2) is given by

$$X = A^{\dagger}DB^{\dagger} + F_A W + ZE_B, \tag{2.6}$$

where $W, Z \in \mathbb{R}^{n\times n}$ are arbitrary matrices.

Lemma 4. [21, 22] *Given $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times n}$. Then the equation $C\tilde{Y} = D$ has a solution $\tilde{Y} \geq 0$ if and only if*

$$DC^{\top} \geq 0,\ \mathcal{R}(D) = \mathcal{R}(DC^{\top}). \tag{2.7}$$

In this case, the general solution of the equation can be described as $\tilde{Y} = \tilde{Y}_0 + F_C S F_C$, where $\tilde{Y}_0 = C^{\dagger}D + F_C(C^{\dagger}D)^{\top} + F_C D^{\top}(DC^{\top})^{\dagger}DF_C$ and S is an arbitrary SND matrix.

Lemma 5. *[21] Given $N \in \mathbb{R}^{m\times n}$, $L \in \mathbb{R}^{m\times n}$. Then, the equation $\tilde{Y}N = L$ has a solution $\tilde{Y} \geq 0$ if and only if*

$$N^{\top}L = L^{\top}N \geq 0,\ \mathrm{rank}(N^{\top}L) = \mathrm{rank}(L). \tag{2.8}$$

In this case, the general solution of the equation can be described as $\tilde{Y} = \tilde{Y}_0 + E_N R E_N$, where $\tilde{Y}_0 = LN^{\dagger} + (LN^{\dagger})^{\top}E_N + E_N L(N^{\top}L)^{\dagger}L^{\top}E_N$ and R is an arbitrary SND matrix.

3. The solution of Problem I

According to Lemma 2, substituting (2.4) into Eq (1.1) gives

$$AP\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}P^{\top} = C,\ P\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}P^{\top}B = D,\ s.t.\ X_1 = X_1^{\top} \geq 0. \tag{3.1}$$

Define

$$A_1 = AP,\ C_1 = CP,\ B_1 = P^{\top}B,\ D_1 = P^{\top}D,\ \tilde{X} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \tag{3.2}$$

then Eq (3.1) can be rewritten as

$$A_1\tilde{X} = C_1,\ \tilde{X}B_1 = D_1. \tag{3.3}$$

By Lemma 1, there exists $\tilde{X} \in \mathbb{R}^{n\times n}$ such that Eq (3.3) is satisfied if and only if

$$A_1A_1^{\dagger}C_1 = C_1,\ D_1B_1^{\dagger}B_1 = D_1,\ A_1D_1 = C_1B_1. \tag{3.4}$$

In this case, the general solution admits the form

$$\tilde{X} = A_1^{\dagger}C_1 + F_{A_1}D_1B_1^{\dagger} + F_{A_1}WE_{B_1}, \tag{3.5}$$

where $W \in \mathbb{R}^{n\times n}$ is arbitrary. Let

$$U_1 = \begin{bmatrix} I_a & 0 \end{bmatrix},\ U_2 = \begin{bmatrix} 0 & I_{n-a} \end{bmatrix},\ V_1 = \begin{bmatrix} I_a \\ 0 \end{bmatrix},\ V_2 = \begin{bmatrix} 0 \\ I_{n-a} \end{bmatrix}; \tag{3.6}$$

$$A_2 = U_1 F_{A_1},\ A_3 = U_2 F_{A_1},\ B_2 = E_{B_1} V_1,\ B_3 = E_{B_1} V_2. \tag{3.7}$$

From (3.5)–(3.7), we can obtain

$$\begin{aligned} X_1 &= U_1 \tilde{X} V_1 = U_1 A_1^\dagger C_1 V_1 + A_2 D_1 B_1^\dagger V_1 + A_2 W B_2, \\ X_2 &= U_1 \tilde{X} V_2 = U_1 A_1^\dagger C_1 V_2 + A_2 D_1 B_1^\dagger V_2 + A_2 W B_3, \\ X_3 &= U_2 \tilde{X} V_1 = U_2 A_1^\dagger C_1 V_1 + A_3 D_1 B_1^\dagger V_1 + A_3 W B_2, \\ X_4 &= U_2 \tilde{X} V_2 = U_2 A_1^\dagger C_1 V_2 + A_3 D_1 B_1^\dagger V_2 + A_3 W B_3. \end{aligned}$$

It follows from Lemma 2 that X_1 must satisfy $X_1 = X_1^\top \geq 0$, that is,

$$U_1 A_1^\dagger C_1 V_1 + A_2 D_1 B_1^\dagger V_1 + A_2 W B_2 = (U_1 A_1^\dagger C_1 V_1 + A_2 D_1 B_1^\dagger V_1)^\top + B_2^\top W^\top A_2^\top \geq 0. \tag{3.8}$$

Let

$$K = A_4 + A_2 W B_2, \tag{3.9}$$

where $A_4 = U_1 A_1^\dagger C_1 V_1 + A_2 D_1 B_1^\dagger V_1$. Then, solving Eq (3.8) is equivalent to finding a matrix W such that $K = K^\top \geq 0$. By Lemma 3, Eq (3.9) is solvable with respect to W if and only if

$$A_2 A_2^\dagger Q B_2^\dagger B_2 = Q. \tag{3.10}$$

In this case, the general solution is $W = A_2^\dagger Q B_2^\dagger + F_{A_2} Z_1 + Z_2 E_{B_2}$, where

$$Q = K - A_4, \tag{3.11}$$

and $Z_1, Z_2 \in \mathbb{R}^{n \times n}$ are arbitrary. Substituting (3.11) into Eq (3.10), we can get that

$$A_5 K B_5 - K = C_5, \tag{3.12}$$

where $A_5 = A_2 A_2^\dagger,\ B_5 = B_2^\dagger B_2, and\ C_5 = A_2 A_2^\dagger A_4 B_2^\dagger B_2 - A_4$. Obviously, A_5 and B_5 are two OPs, then there exist orthogonal matrices $P_1 = [P_{11}, P_{12}] \in \mathbb{OR}^{a \times a}$ and $Q_1 = [Q_{11}, Q_{12}] \in \mathbb{OR}^{a \times a}$ such that

$$A_5 = P_1 \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} P_1^\top = P_{11} P_{11}^\top,\ \ B_5 = Q_1 \begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix} Q_1^\top = Q_{11} Q_{11}^\top,$$

where $r = \operatorname{rank}(A_5),\ t = \operatorname{rank}(B_5)$, and $P_{11} \in \mathbb{R}^{a \times r}$, $Q_{11} \in \mathbb{R}^{a \times t}$ are column orthogonal.

Notice that Eq (3.12), when multiplied on the left by matrix $P_{12}^\top$ on the left and on the right by matrix Q_{12}, can be transformed into $P_{12}^\top K = -P_{12}^\top C_5$ and $K Q_{12} = -C_5 Q_{12}$, respectively. If Eq (3.12) is simultaneously left-multiplied by $P_{11}^\top$ and right-multiplied by Q_{11}, it can be obtained that $P_{11}^\top C_5 Q_{11} = 0$. On the contrary, we found that

$$\begin{aligned} & A_5 K B_5 \\ =& (I - P_{12} P_{12}^\top) K (I - Q_{12} Q_{12}^\top) \\ =& K - K Q_{12} Q_{12}^\top - P_{12} P_{12}^\top K + P_{12} P_{12}^\top K Q_{12} Q_{12}^\top \\ =& K + C_5 Q_{12} Q_{12}^\top + P_{12} P_{12}^\top C_5 - P_{12} P_{12}^\top C_5 Q_{12} Q_{12}^\top \\ =& K + C_5 Q_{12} Q_{12}^\top + P_{12} P_{12}^\top C_5 Q_{11} Q_{11}^\top \end{aligned}$$

$$
\begin{aligned}
&= K + C_5 - C_5 Q_{11} Q_{11}^\top + P_{12} P_{12}^\top C_5 Q_{11} Q_{11}^\top \\
&= K + C_5.
\end{aligned}
$$

Then, it is easily evident that Eq (3.12) is equivalent to

$$P_{12}^\top K = J_1, \tag{3.13}$$
$$K Q_{12} = J_2, \tag{3.14}$$
$$P_{11}^\top C_5 Q_{11} = 0, \tag{3.15}$$

where $J_1 = -P_{12}^\top C_5$, $J_2 = -C_5 Q_{12}$. By Lemma 4, Eq (3.13) has a solution $K \geq 0$ if and only if

$$\Gamma \geq 0,\ \mathcal{R}(J_1) = \mathcal{R}(\Gamma), \tag{3.16}$$

where $\Gamma = J_1 P_{12}$; the solution of Eq (3.13) can be formulated as

$$K = K_0 + P_{11} S_1 P_{11}^\top, \tag{3.17}$$

where $K_0 = P_{12} J_1 + P_{11} P_{11}^\top (P_{12} J_1)^\top + P_{11} P_{11}^\top J_1^\top \Gamma^\dagger J_1 P_{11} P_{11}^\top$ and the SND matrix S_1 is arbitrary. After substituting (3.17) into Eq (3.14), the following equation is derived:

$$P_{11} S_1 P_{11}^\top Q_{12} = J_2 - K_0 Q_{12}. \tag{3.18}$$

Let

$$N_1 = P_{11}^\top Q_{12},\ J_3 = J_2 - K_0 Q_{12},\ L_1 = P_{11}^\top J_3, \tag{3.19}$$

then Eq (3.18) can be equivalently rewritten as

$$S_1 N_1 = L_1, \tag{3.20}$$
$$P_{12}^\top J_3 = 0. \tag{3.21}$$

From Lemma 5, Eq (3.20) has a solution $H_1 \geq 0$ if and only if

$$\Upsilon = \Upsilon^\top \geq 0,\ \operatorname{rank}(\Upsilon) = \operatorname{rank}(L_1), \tag{3.22}$$

where $\Upsilon = N_1^\top L_1$; the solution can be formulated as

$$S_1 = S_0 + E_{N_1} R_1 E_{N_1}, \tag{3.23}$$

where $S_0 = L_1 N_1^\dagger + (L_1 N_1^\dagger)^\top E_{N_1} + E_{N_1} L_1 \Upsilon^\dagger L_1^\top E_{N_1}$ and R_1 is an arbitrary SND matrix.

To summarize the above discussion, we have arrived at the following conclusion. For convenience, Table 1 lists the key orthogonal matrices from Theorem 1 and their origins.

Table 1. Table of notations.

Symbol	Size	Definition / Role
P	$n \times n$	Orthogonal matrix from spectral decomposition of F_G: $F_G = P\,\mathrm{diag}(I_a, 0)\,P^\top$
P_1	$a \times a$	Orthogonal matrix from spectral decomposition of A_5: $A_5 = P_1\,\mathrm{diag}(I_r, 0)\,P_1^\top$
Q_1	$a \times a$	Orthogonal matrix from spectral decomposition of B_5: $B_5 = Q_1\,\mathrm{diag}(I_t, 0)\,Q_1^\top$

Theorem 1. *For given $A, C \in \mathbb{R}^{m\times n}$, $B, D \in \mathbb{R}^{n\times p}$ and $G \in \mathbb{R}^{l\times n}$. Problem I is solvable if and only if the conditions* (3.4)*,* (3.15)*,* (3.16)*,* (3.21)*, and* (3.22) *are satisfied. In this case, the SND solution of Eq (1.1) on the subspace Ω can be described as*

$$X = P\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} P^\top,$$

where

$$X_1 = U_1A_1^\dagger C_1V_1 + A_2D_1B_1^\dagger V_1 + A_2A_2^\dagger QB_2^\dagger B_2, \tag{3.24}$$

$$X_2 = U_1A_1^\dagger C_1V_2 + A_2D_1B_1^\dagger V_2 + A_2A_2^\dagger QB_2^\dagger B_3 + A_2Z_2E_{B_2}B_3, \tag{3.25}$$

$$X_3 = U_2A_1^\dagger C_1V_1 + A_3D_1B_1^\dagger V_1 + A_3A_2^\dagger QB_2^\dagger B_2 + A_3F_{A_2}Z_1B_2, \tag{3.26}$$

$$X_4 = U_2A_1^\dagger C_1V_2 + A_3D_1B_1^\dagger V_2 + A_3A_2^\dagger QB_2^\dagger B_3 + A_3F_{A_2}Z_1B_3 + A_3Z_2E_{B_2}B_3, \tag{3.27}$$

and A_1, C_1, B_1, D_1, U_i, V_i, A_j, B_j, $i = 1,2$; $j = 2,3$ and Q are given by Eqs (3.2)*,* (3.6)*,* (3.7)*, and* (3.11)*, respectively, and $Z_1, Z_2 \in \mathbb{R}^{n\times n}$ are arbitrary.*

According to Theorem 1, we propose Algorithm 1 to solve Problem I.

Algorithm 1

1: Input matrices A, B, C, D, and G.
2: Compute the SD of matrix F_G.
3: Compute the matrices A_1, C_1, B_1, and D_1 by (3.2).
4: If the condition (3.4) holds, go to 5; or else, Eq (1.1) has no solutions, and stop.
5: Compute the matrices U_i, V_i, A_j, B_j, $i = 1,2$; $j = 2,3$ by (3.6), (3.7).
6: Compute the matrices $A_5 = A_2A_2^\dagger$, $B_5 = B_2^\dagger B_2$, $C_5 = A_2A_2^\dagger A_4B_2^\dagger B_2 - A_4$.
7: Compute the matrices $J_1 = -P_{12}^\top C_5$, $J_2 = -C_5Q_{12}$.
8: If the conditions (3.15), (3.16) hold, go to 9; or else, Eq (1.1) has no solutions, and stop.
9: Compute matrices $P_{11}^\top Q_{12} = N_1, J_2 - K_0Q_{12} = J_3, P_{11}^\top J_3 = L_1$.
10: If the conditions (3.21), (3.22) hold, go to 11; or else, Eq (1.1) has no solutions, and stop.
11: Choose a matrix $R_1 \geq 0$, and compute S_1, K and by (3.23), (3.17), then calculate $Q = K - A_4$, respectively.
12: Choose matrices Z_1, Z_2, and compute X_i, $i = 1,2,3,4$ by (3.24)–(3.27), respectively.

Example 1. Let $m = 10$, $n = 8$, $p = 10$, and $l = 9$. The matrices A, B, C, D, and G are given by

$$A = \begin{bmatrix} 0.8887 & 0.9018 & 0.2692 & 0.7512 & 0.8238 & 0.4619 & 0.8008 & 0.4892 \\ 0.7368 & 0.6472 & 0.2229 & 0.6226 & 0.7741 & 0.2759 & 0.5844 & 0.4008 \\ 0.5894 & 0.3159 & 0.1156 & 0.2394 & 0.5833 & 0.2581 & 0.4213 & 0.1684 \\ 0.3852 & 0.3994 & 0.1016 & 0.3577 & 0.3233 & 0.1442 & 0.2706 & 0.2432 \\ 0.7655 & 0.6554 & 0.2214 & 0.4872 & 0.7502 & 0.4663 & 0.7494 & 0.3114 \\ 0.9760 & 0.9563 & 0.2829 & 0.7696 & 0.8903 & 0.5260 & 0.8788 & 0.5047 \\ 0.6902 & 0.5170 & 0.2042 & 0.4921 & 0.7699 & 0.2780 & 0.5698 & 0.3109 \\ 0.5347 & 0.6518 & 0.2046 & 0.5892 & 0.5347 & 0.2741 & 0.5363 & 0.3698 \\ 0.4742 & 0.4558 & 0.1391 & 0.4148 & 0.4563 & 0.1941 & 0.3774 & 0.2720 \\ 0.3363 & 0.5790 & 0.2530 & 0.5929 & 0.5356 & 0.2394 & 0.5867 & 0.3230 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.6191 & 0.8411 & 0.8031 & 0.7975 & 0.6619 & 0.8356 & 0.5900 & 0.8081 & 0.8760 & 0.7275 \\ 0.3206 & 0.4279 & 0.5285 & 0.4941 & 0.3165 & 0.4071 & 0.1110 & 0.4832 & 0.5831 & 0.3930 \\ 0.3055 & 0.4160 & 0.3566 & 0.3113 & 0.2446 & 0.3571 & 0.3545 & 0.4199 & 0.3368 & 0.2625 \\ 0.5049 & 0.7046 & 0.7032 & 0.5006 & 0.5814 & 0.6816 & 0.6205 & 0.5380 & 0.6814 & 0.4434 \\ 0.3295 & 0.4519 & 0.4907 & 0.3528 & 0.2993 & 0.3920 & 0.2992 & 0.4307 & 0.4665 & 0.2662 \\ 0.4200 & 0.5505 & 0.3340 & 0.6015 & 0.3156 & 0.5251 & 0.4284 & 0.6977 & 0.4261 & 0.5921 \\ 0.4094 & 0.5466 & 0.5626 & 0.6209 & 0.4227 & 0.5492 & 0.2586 & 0.5992 & 0.6523 & 0.5512 \\ 0.3409 & 0.4487 & 0.3132 & 0.5818 & 0.3890 & 0.5145 & 0.3041 & 0.5101 & 0.4610 & 0.6085 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.7040 & -0.1955 & 0.8425 & 0.4497 & 1.3549 & 0.3537 & 0.3136 & -1.0692 \\ 0.1720 & -0.0873 & 0.8642 & 0.4022 & 1.7545 & -0.2054 & 0.1042 & -0.9254 \\ 0.6327 & -0.2878 & 0.4629 & -0.0026 & 0.8555 & 0.1233 & 0.3027 & -0.6743 \\ 0.1877 & -0.0348 & 0.3592 & 0.2514 & 0.6210 & 0.0605 & 0.0512 & -0.4210 \\ 0.9115 & -0.3186 & 0.6551 & 0.1719 & 0.9781 & 0.4682 & 0.4503 & -0.9493 \\ 0.9165 & -0.2685 & 0.8630 & 0.4214 & 1.3169 & 0.4949 & 0.4066 & -1.1581 \\ 0.2514 & -0.1530 & 0.8065 & 0.2591 & 1.6715 & -0.2010 & 0.1710 & -0.8945 \\ 0.1644 & -0.0078 & 0.6500 & 0.4374 & 1.1497 & 0.0601 & 0.0886 & -0.7009 \\ 0.2036 & -0.0617 & 0.5009 & 0.2721 & 0.9341 & 0.0010 & 0.0881 & -0.5687 \\ -0.3775 & 0.1533 & 0.8026 & 0.5202 & 1.6490 & -0.3090 & -0.0417 & -0.6752 \end{bmatrix},$$

$$D = \begin{bmatrix} 1.1669 & 1.5309 & 0.9724 & 1.8841 & 1.1581 & 1.6549 & 1.1200 & 1.8293 & 1.3995 & 1.9508 \\ 0.0544 & 0.0976 & 0.3299 & -0.1426 & 0.0302 & -0.0133 & 0.0324 & -0.0350 & 0.1650 & -0.2731 \\ -0.1902 & -0.2414 & -0.0244 & -0.4470 & -0.2963 & -0.3771 & -0.2230 & -0.2939 & -0.2200 & -0.5607 \\ 0.2048 & 0.3029 & 0.4245 & 0.0617 & 0.2459 & 0.2424 & 0.2851 & 0.1251 & 0.3146 & -0.0169 \\ -0.8870 & -1.1100 & -0.2142 & -1.9234 & -0.9360 & -1.4404 & -0.8284 & -1.6408 & -0.9184 & -2.2236 \\ 1.0444 & 1.3605 & 0.7136 & 1.7518 & 1.0330 & 1.5131 & 1.0651 & 1.6820 & 1.1604 & 1.8798 \\ 0.6045 & 0.7922 & 0.5340 & 0.9563 & 0.5540 & 0.8208 & 0.5449 & 0.9731 & 0.7284 & 0.9581 \\ -0.4248 & -0.5821 & -0.7181 & -0.5034 & -0.4200 & -0.5186 & -0.2868 & -0.5527 & -0.7049 & -0.3658 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.4750 & 0.3485 & 0.1999 & 0.5090 & 0.1386 & 0.2727 & 0.3755 & 0.4122 \\ 0.2292 & 0.1584 & 0.0733 & 0.2554 & 0.0704 & 0.1399 & 0.1647 & 0.1899 \\ 0.3158 & 0.2082 & 0.0771 & 0.3619 & 0.1006 & 0.2013 & 0.2099 & 0.2524 \\ 0.5373 & 0.3828 & 0.1990 & 0.5872 & 0.1609 & 0.3182 & 0.4054 & 0.4557 \\ 0.4508 & 0.3885 & 0.3263 & 0.4257 & 0.1109 & 0.2102 & 0.4536 & 0.4442 \\ 0.1951 & 0.1497 & 0.0976 & 0.2026 & 0.0546 & 0.1065 & 0.1652 & 0.1753 \\ 0.6167 & 0.4876 & 0.3426 & 0.6260 & 0.1674 & 0.3245 & 0.5466 & 0.5674 \\ 0.6197 & 0.4721 & 0.3018 & 0.6469 & 0.1746 & 0.3412 & 0.5191 & 0.5537 \\ 0.3473 & 0.2210 & 0.0660 & 0.4059 & 0.1135 & 0.2281 & 0.2174 & 0.2703 \end{bmatrix}.$$

A direct computation shows that the conditions (3.4), (3.15), (3.16), (3.21), and (3.22) are satisfied:

$$\|A_1A_1^{\dagger}C_1 - C_1\|_F = 2.4299\times 10^{-15},\ \|D_1B_1^{\dagger}B_1 - D_1\|_F = 4.1004\times 10^{-15},$$
$$\|A_1D_1 - C_1B_1\|_F = 6.1317\times 10^{-15},\ \|P_{11}^{\top}C_5Q_{11}\|_F = 1.3204\times 10^{-15},$$
$$\Gamma = \begin{bmatrix} 1.8212 & -1.1041 \\ -1.1041 & 5.5377 \end{bmatrix} \geq 0,\ \lambda(\Gamma) = \{\ 1.5179,\ 5.8409\},$$
$$\mathrm{rank}(\Gamma) = \mathrm{rank}(J_1) = 2,\ \|P_{12}^{\top}J_3\|_F = 7.6580\times 10^{-16},$$

$$\Upsilon = \begin{bmatrix} 0.4612 & -0.1315 \\ -0.1315 & 0.4703 \end{bmatrix} \geq 0, \lambda(\Upsilon) = \{\ 0.3342,\ 0.5974\},$$
$$\operatorname{rank}(\Upsilon) = \operatorname{rank}(L_1) = 2.$$

By Algorithm 1, and choosing matrices Z_1, Z_2, we can obtain

$$X = \begin{bmatrix} 2.9013 & -1.0311 & -0.0211 & -0.9190 & -2.3708 & 1.3845 & 1.0766 & -0.2032 \\ -0.4977 & 1.0964 & -0.2143 & 0.5607 & 1.1685 & -0.0988 & -1.0870 & -0.3085 \\ -0.0929 & 0.4254 & 1.5671 & -0.7094 & 0.9744 & -0.9358 & -0.5812 & -0.2336 \\ -1.1124 & -0.0012 & 0.5551 & 1.0457 & 0.4320 & -0.4403 & 0.5408 & 0.0510 \\ -2.4846 & 0.6894 & 0.7078 & 0.6589 & 4.6735 & -2.3413 & -1.0179 & -0.7589 \\ 1.4049 & -0.7238 & -0.9019 & 0.1517 & -2.3530 & 2.2202 & 0.9859 & 0.1317 \\ 1.0764 & -0.5411 & 0.2195 & -0.5228 & -0.8696 & 0.7158 & 0.8690 & -0.1799 \\ -0.0589 & -0.3721 & -0.2590 & -0.0572 & -0.5025 & 0.2560 & -0.4604 & 0.2508 \end{bmatrix}.$$

The absolute errors are estimated by

$$||AX - C||_F = 6.3907 \times 10^{-15},\ ||XB - D||_F = 8.1400 \times 10^{-15},$$
$$||F_G X F_G - F_G X^\top FG||_F = 4.2407 \times 10^{-15},$$

and $\lambda(F_G X F_G) = \{8.3282,\ 1.8359,\ 1.1792,\ 0.8066,\ 0.3872,\ 0.5375,\ 0.0000,\ 0.0000\}$, which implies that X is an SND solution of Eq (1.1) on the subspace Ω.

4. The solution of Problem II

To solve Problem II, we substitute (2.4) into Eq (1.2) to get

$$AP \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} P^\top B = D,\ s.t.\ X_1 = X_1^\top \geq 0. \tag{4.1}$$

Define

$$A_1 = AP,\ B_1 = P^\top B,\ \tilde{X} = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \tag{4.2}$$

then, Eq (4.1) can be rewritten as

$$A_1 \tilde{X} B_1 = D. \tag{4.3}$$

By Lemma 3, Eq (4.3) has a solution $\tilde{X} \in \mathbb{R}^{n\times n}$ if and only if

$$A_1 A_1^\dagger D B_1^\dagger B_1 = D. \tag{4.4}$$

In this case, the general solution is

$$\tilde{X} = A_1^\dagger D B_1^\dagger + F_{A_1} W + Z E_{B_1}, \tag{4.5}$$

where $W, Z \in \mathbb{R}^{n\times n}$ are arbitrary. Partition the matrices W, Z as $W = [W_1, W_2]$, $Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ with $W_1 \in \mathbb{R}^{n\times a}$, $Z_1 \in \mathbb{R}^{a\times n}$, and let

$$U_1 = \begin{bmatrix} I_a & 0 \end{bmatrix},\ U_2 = \begin{bmatrix} 0 & I_{n-a} \end{bmatrix},\ V_1 = \begin{bmatrix} I_a \\ 0 \end{bmatrix},\ V_2 = \begin{bmatrix} 0 \\ I_{n-a} \end{bmatrix}, \tag{4.6}$$

$$A_2 = U_1 F_{A_1},\ A_3 = U_2 F_{A_1},\ B_2 = E_{B_1} V_1,\ B_3 = E_{B_1} V_2. \tag{4.7}$$

From (4.5)–(4.7), we can obtain

$$\begin{aligned} X_1 &= U_1 \tilde{X} V_1 = U_1 A_1^{\dagger} D B_1^{\dagger} V_1 + A_2 W_1 + Z_1 B_2, \\ X_2 &= U_1 \tilde{X} V_2 = U_1 A_1^{\dagger} D B_1^{\dagger} V_2 + A_2 W_2 + Z_1 B_3, \\ X_3 &= U_2 \tilde{X} V_1 = U_2 A_1^{\dagger} D B_1^{\dagger} V_1 + A_3 W_1 + Z_2 B_2, \\ X_4 &= U_2 \tilde{X} V_2 = U_2 A_1^{\dagger} D B_1^{\dagger} V_2 + A_3 W_2 + Z_2 B_3. \end{aligned}$$

Similarly, X_1 must satisfy $X_1 = X_1^{\top} \geq 0$; we have

$$U_1 A_1^{\dagger} D B_1^{\dagger} V_1 + A_2 W_1 + Z_1 B_2 = (U_1 A_1^{\dagger} D B_1^{\dagger} V_1)^{\top} + W_1^{\top} A_2^{\top} + B_2^{\top} Z_1^{\top} \geq 0. \tag{4.8}$$

Let

$$K = A_4 + A_2 W_1 + Z_1 B_2, \tag{4.9}$$

where $A_4 = U_1 A_1^{\dagger} D B_1^{\dagger} V_1$. Then to solve Eq (4.8) is equivalent to finding matrices W_1, Z_1 such that $K = K^{\top} \geq 0$. By Lemma 1, Eq (4.9) is solvable with respect to W_1 if and only if

$$E_{A_2} Z_1 B_2 = E_{A_2} Q. \tag{4.10}$$

In this case, the solution can be expressed as $W_1 = A_2^{\dagger} Q - A_2^{\dagger} Z_1 B_2 + F_{A_2} M_1$, where

$$Q = K - A_4, \tag{4.11}$$

and $M_1 \in \mathbb{R}^{n \times a}$ is arbitrary. Once again, by using Lemma 3, there exists $Z_1 \in \mathbb{R}^{a \times n}$ such that Eq (4.10) is satisfied if and only if

$$E_{A_2} Q F_{B_2} = 0. \tag{4.12}$$

In this case, the general solution is

$$Z_1 = E_{A_2} Q B_2^{\dagger} + A_2 A_2^{\dagger} M_2 + M_3 E_{B_2}, \tag{4.13}$$

where $M_2, M_3 \in \mathbb{R}^{a \times n}$ are arbitrary. By substituting (4.11) into Eq (4.12), we have

$$E_{A_2} K F_{B_2} = E_{A_2} A_4 F_{B_2}. \tag{4.14}$$

E_{A_2} and F_{B_2} are two OPs; then, there exist $P_2 = [P_{21}, P_{22}] \in \mathbb{OR}^{a \times a}$ and $Q_2 = [Q_{21}, Q_{22}] \in \mathbb{OR}^{a \times a}$ such that

$$E_{A_2} = P_2 \begin{bmatrix} I_b & 0 \\ 0 & 0 \end{bmatrix} P_2^{\top} = P_{21} P_{21}^{\top},\ \ F_{B_2} = Q_2 \begin{bmatrix} I_c & 0 \\ 0 & 0 \end{bmatrix} Q_2^{\top} = Q_{21} Q_{21}^{\top},$$

where $b = \operatorname{rank}(E_{A_2})$, $c = \operatorname{rank}(F_{B_2})$ and $P_{21} \in \mathbb{R}^{a \times b}$, $Q_{21} \in \mathbb{R}^{a \times c}$ are column orthogonal. Then, Eq (4.14) is equivalent to

$$P_{21}^{\top} K Q_{21} = J_1, \tag{4.15}$$

where $J_1 = P_{21}^\top A_4 Q_{21}$. Equation (4.15) is a standard problem of finding a nonnegative definite solution of a matrix equation. Applying the results of Khatri and Mitra [7] as well as Young et al. [23], there exists a solution $K \geq 0$ to Eq (4.15) if and only if

$$T \geq 0,\ \operatorname{rank}(T) = \operatorname{rank}(P_{21}P_{21}^\top O_1^- Q_{21}J_1^\top P_{21}^\top) = \operatorname{rank}(Q_{21}Q_{21}^\top O_1^- P_{21}J_1Q_{21}^\top), \tag{4.16}$$

where $T = Q_{21}Q_{21}^\top O_1^- P_{21}J_1Q_{21}^\top O_1^- P_{21}P_{21}^\top$. The general solution is

$$K = O_1^=(P_{21}J_1Q_{21}^\top + Q_{21}J_1^\top P_{21}^\top + Y + Z)(O_1^=)^\top + (I - O_1^- O_1)R_1(I - O_1^- O_1)^\top, \tag{4.17}$$

where $O_1 = P_{21}P_{21}^\top + Q_{21}Q_{21}^\top$, and $O_1^=$ represents the class of g-inverse of O_1 given by

$$O_1^= = O_1^- + (I - O_1^- O_1)S_1[(P_{21}J_1Q_{21}^\top + Q_{21}J_1^\top P_{21}^\top + Y + Z)^{\frac{1}{2}}]^-. \tag{4.18}$$

Y, Z are the SND solutions of the following matrix equations:

$$YO_1^- Q_{21}Q_{21}^\top = P_{21}J_1Q_{21}^\top O_1^- P_{21}P_{21}^\top, \tag{4.19}$$
$$P_{21}P_{21}^\top O_1^- Z = Q_{21}Q_{21}^\top O_1^- P_{21}J_1Q_{21}^\top. \tag{4.20}$$

The general solutions are

$$\begin{aligned} Y =& P_{21}J_1Q_{21}^\top O_1^- P_{21}P_{21}^\top T^-(P_{21}J_1Q_{21}^\top O_1^- P_{21}P_{21}^\top)^\top \\ &+ (I - O_1^- Q_{21}Q_{21}^\top(O_1^- Q_{21}Q_{21}^\top)^-)^\top R_2(I - O_1^- Q_{21}Q_{21}^\top(O_1^- Q_{21}Q_{21}^\top)^-), \\ Z =& (Q_{21}Q_{21}^\top O_1^- P_{21}J_1Q_{21}^\top)^\top T^- Q_{21}Q_{21}^\top O_1^- P_{21}J_1Q_{21}^\top \\ &+ (I - (P_{21}P_{21}^\top O_1^-)^- P_{21}P_{21}^\top O_1^-)R_3(I - (P_{21}P_{21}^\top O_1^-)^- P_{21}P_{21}^\top O_1^-)^\top, \end{aligned}$$

which satisfies $(P_{21}J_1Q_{21}^\top + Q_{21}J_1^\top P_{21}^\top + Y + Z) \geq 0$, S_1 is arbitrary, and R_1, R_2, R_3 are arbitrary SND matrices.

To summarize the above discussion, we have arrived at the following conclusion. For convenience, Table 2 lists the key orthogonal matrices from Theorem 2 and their origins.

Table 2. Table of notations.

Symbol	Size	Definition / Role
P	$n \times n$	Orthogonal matrix from spectral decomposition of F_G: $F_G = P\,\mathrm{diag}(I_a, 0)\,P^\top$
P_2	$a \times a$	Orthogonal matrix from spectral decomposition of E_{A_2}: $E_{A_2} = P_2\,\mathrm{diag}(I_b, 0)\,P_2^\top$
Q_2	$a \times a$	Orthogonal matrix from spectral decomposition of F_{B_2}: $F_{B_2} = Q_2\,\mathrm{diag}(I_c, 0)\,Q_2^\top$

Theorem 2. *Given $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{m\times p}$, and $G \in \mathbb{R}^{l\times n}$. Problem II is solvable if and only if the conditions* (4.4) *and* (4.16) *are satisfied. In this case, the SND solution of Eq (1.2) on the subspace Ω can be described as*

$$X = P\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} P^\top,$$

where

$$X_1 = U_1A_1^\dagger DB_1^\dagger V_1 + A_2A_2^\dagger Q - A_2A_2^\dagger Z_1B_2 + E_{A_2}QB_2^\dagger B_2 + A_2A_2^\dagger M_2B_2, \tag{4.21}$$

$$X_2 = U_1 A_1^{\dagger} D B_1^{\dagger} V_2 + A_2 W_2 + E_{A_2} Q B_2^{\dagger} B_3 + A_2 A_2^{\dagger} M_2 B_3 + M_3 E_{B_2} B_3, \tag{4.22}$$
$$X_3 = U_2 A_1^{\dagger} D B_1^{\dagger} V_1 + A_3 A_2^{\dagger} Q - A_3 A_2^{\dagger} Z_1 B_2 + A_3 F_{A_2} M_1 + Z_2 B_2, \tag{4.23}$$
$$X_4 = U_2 A_1^{\dagger} D B_1^{\dagger} V_2 + A_3 W_2 + Z_2 B_3, \tag{4.24}$$

and A_1, B_1, U_i, V_i, A_j, B_j, $i = 1, 2$; $j = 2, 3$ *and* Q *are given by Eqs* (4.2), (4.6), (4.7), *and* (4.11), *respectively, and* $W_2 \in \mathbb{R}^{n\times(n-a)}$, $Z_2 \in \mathbb{R}^{(n-a)\times n}$, $M_i (i = 1, 2, 3)$ *are arbitrary.*

According to Theorem 2, we propose Algorithm 2 to solve Problem II.

Algorithm 2

1: Input matrices A, B, D, and G.
2: Compute the SD of matrix F_G.
3: Compute the matrices A_1, B_1 by (4.2).
4: If the condition (4.4) holds, go to 5; or else, Eq (1.2) has no solutions, and stop.
5: Compute the matrices U_i, V_i, A_j, B_j, $i = 1, 2$; $j = 2, 3$ by (4.6), (4.7).
6: Compute the SDs of the matrices E_{A_2} and F_{B_2}.
7: Compute the matrices $J_1 = P_{21}^{\top} A_4 Q_{21}$ and $O_1 = P_{21} P_{21}^{\top} + Q_{21} Q_{21}^{\top}$.
8: If the condition (4.16) holds, go to 9; or else, Eq (1.2) has no solutions, and stop.
9: Choose matrices $S_1, R_1 \geq 0$, and compute $O_1^{=}$, K and Q by (4.18), (4.17) and (4.11), respectively.
10: Choose matrices W_2, Z_2 and M_k, $k = 1, 2, 3$, and compute X_i, $i = 1, 2, 3, 4$ by (4.21)–(4.24), respectively.

Remark 1. *Both algorithms require* $O(n^2 + mn + np)$ *memory, dominated by a few dense* $n \times n$ *matrices* (P, X, *projectors*) *and the transformed coefficient matrices. The additional* $a \times a$ *blocks in Algorithm 2 do not change the asymptotic bound since* $a \leq n$. *In addition, the data in this matrix are all rounded to four decimal places.*

Example 2. Let $m = 9$, $n = 8$, $p = 9$, and $l = 9$. The matrices A, B, D, and G are given by

$$A = \begin{bmatrix} 0.7852 & 0.8507 & 0.9645 & 0.9137 & 0.8298 & 0.5004 & 0.4746 & 0.9070 \\ 1.0348 & 0.7336 & 1.1469 & 0.8269 & 0.8655 & 0.7034 & 0.7041 & 1.0026 \\ 0.9346 & 0.9852 & 0.9346 & 1.1498 & 0.6040 & 0.5534 & 0.6945 & 1.0910 \\ 0.8419 & 0.6236 & 0.8929 & 0.6576 & 0.7229 & 0.3655 & 0.3663 & 0.5892 \\ 0.8540 & 0.7374 & 1.0249 & 0.8259 & 0.7964 & 0.5159 & 0.6269 & 0.8879 \\ 1.0070 & 1.0052 & 1.1661 & 1.1538 & 0.9602 & 0.5062 & 0.6169 & 0.8428 \\ 0.8961 & 0.9931 & 1.1218 & 1.1176 & 0.9020 & 0.5849 & 0.6972 & 1.1009 \\ 0.7972 & 0.6278 & 0.7759 & 0.7050 & 0.6351 & 0.4309 & 0.3188 & 0.5210 \\ 0.9531 & 0.7130 & 1.0902 & 0.7614 & 0.8995 & 0.6830 & 0.5572 & 0.9478 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.1174 & 1.5307 & 0.9998 & 1.5048 & 1.2836 & 1.4061 & 1.2411 & 0.9652 & 1.3505 \\ 1.0494 & 1.4589 & 0.9034 & 1.5577 & 1.3724 & 1.5105 & 0.9999 & 1.0377 & 1.2570 \\ 0.7769 & 1.1538 & 0.8479 & 1.2679 & 1.1632 & 1.2573 & 0.8505 & 0.8255 & 1.0073 \\ 0.7111 & 0.8314 & 0.5809 & 0.7580 & 0.5892 & 0.6583 & 0.6404 & 0.4906 & 0.6920 \\ 0.9842 & 1.3181 & 0.7438 & 1.3312 & 1.1556 & 1.2763 & 0.8235 & 0.9140 & 1.1465 \\ 0.8871 & 1.2371 & 0.6687 & 1.0818 & 0.8913 & 0.9681 & 0.9759 & 0.7157 & 1.0923 \\ 0.8826 & 1.2543 & 0.6536 & 1.1291 & 0.8583 & 0.9287 & 1.0316 & 0.7338 & 0.9180 \\ 0.8997 & 1.2947 & 0.8205 & 1.3734 & 1.1753 & 1.2898 & 1.0612 & 0.8740 & 1.0791 \end{bmatrix},$$

$$D = \begin{bmatrix} 0.9527 & 1.1879 & 1.5355 & 2.2819 & 2.7098 & 3.0311 & -0.0565 & 1.6117 & 1.7913 \\ 0.1828 & 0.1565 & 0.8067 & 1.1366 & 1.6363 & 1.8449 & -0.7116 & 0.8438 & 0.7859 \\ 0.0933 & -0.0040 & 0.6760 & 0.8461 & 1.2907 & 1.4617 & -0.7186 & 0.6349 & 0.5522 \\ 1.2518 & 1.6318 & 1.7608 & 2.6835 & 3.0355 & 3.3830 & 0.3123 & 1.8730 & 2.1580 \\ 0.4393 & 0.5064 & 0.9852 & 1.4103 & 1.8221 & 2.0448 & -0.3894 & 1.0160 & 1.0429 \\ 1.4843 & 1.9005 & 2.1339 & 3.1700 & 3.5966 & 4.0097 & 0.3303 & 2.2108 & 2.5294 \\ 0.5974 & 0.6855 & 1.2347 & 1.7517 & 2.2242 & 2.4962 & -0.3977 & 1.2560 & 1.3115 \\ 1.1668 & 1.5135 & 1.6465 & 2.4970 & 2.8281 & 3.1529 & 0.2830 & 1.7427 & 2.0107 \\ 0.5499 & 0.6581 & 1.1761 & 1.7505 & 2.2490 & 2.5242 & -0.4438 & 1.2625 & 1.3196 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.6411 & 0.1087 & 0.2174 & 0.4024 & 0.0549 & 0.4999 & 0.1874 & 0.2093 \\ 0.3708 & 0.0579 & 0.0893 & 0.1675 & 0.0393 & 0.2799 & 0.1283 & 0.1411 \\ 0.1295 & 0.0250 & 0.0664 & 0.1215 & 0.0064 & 0.1067 & 0.0256 & 0.0299 \\ 0.6378 & 0.1068 & 0.2066 & 0.3830 & 0.0566 & 0.4949 & 0.1917 & 0.2136 \\ 0.5768 & 0.0933 & 0.1626 & 0.3030 & 0.0562 & 0.4414 & 0.1866 & 0.2065 \\ 0.5871 & 0.0984 & 0.1909 & 0.3539 & 0.0520 & 0.4557 & 0.1761 & 0.1962 \\ 0.1268 & 0.0238 & 0.0600 & 0.1100 & 0.0073 & 0.1032 & 0.0278 & 0.0321 \\ 0.2278 & 0.0365 & 0.0621 & 0.1159 & 0.0226 & 0.1738 & 0.0748 & 0.0827 \\ 0.6577 & 0.1136 & 0.2381 & 0.4397 & 0.0532 & 0.5167 & 0.1841 & 0.2065 \end{bmatrix}.$$

A direct computation shows that the conditions (4.4), (4.16), and (4.17) are satisfied:

$$\|A_1A_1^{\dagger}DB_1^{\dagger}B_1 - D\|_F = 7.2890 \times 10^{-13},$$

$$T = \begin{bmatrix} 0.1869 & 0.2698 & -0.0844 & 0.2144 & 0.3570 & 0.0639 \\ 0.2698 & 0.3898 & -0.1213 & 0.3112 & 0.5200 & 0.0949 \\ -0.0844 & -0.1213 & 0.0388 & -0.0941 & -0.1541 & -0.0249 \\ 0.2144 & 0.3112 & -0.0941 & 0.2564 & 0.4373 & 0.0890 \\ 0.3570 & 0.5200 & -0.1541 & 0.4373 & 0.7552 & 0.1635 \\ 0.0639 & 0.0949 & -0.0249 & 0.0890 & 0.1635 & 0.0453 \end{bmatrix} \geq 0,$$

$$\lambda(T) = \{1.6246,\ 0.0478,\ 0.0000,\ 0.0000,\ 0.0000,\ 0.0000\},$$

$$\mathrm{rank}(T) = \mathrm{rank}(P_{21}P_{21}^{\top}O_1^{-}Q_{21}J_1^{\top}P_{21}^{\top}) = \mathrm{rank}(Q_{21}Q_{21}^{\top}O_1^{-}P_{21}J_1Q_{21}^{\top}) = 2.$$

Note that

$$O = \begin{bmatrix} 1.5593 & 0.4822 & 0.3495 & 0.0910 & -0.1088 & 0.0172 \\ 0.4822 & 1.0998 & -0.4217 & 0.1668 & 0.3173 & -0.4989 \\ 0.3495 & -0.4217 & 1.2725 & -0.3682 & 0.1610 & 0.1334 \\ 0.0910 & 0.1668 & -0.3682 & 1.2035 & 0.1672 & 0.2814 \\ -0.1088 & 0.3173 & 0.1610 & 0.1672 & 1.6872 & 0.3777 \\ 0.0172 & -0.4989 & 0.1334 & 0.2814 & 0.3777 & 1.1778 \end{bmatrix},$$

where $rank(O) = 6$, $I - O_1^- O_1 = 0$. Next, we substitute O into Eqs (4.20) and (4.21) to obtain

$$Y = \begin{bmatrix} 3.2624 & 1.2036 & 0.7637 & 1.9083 & 2.6156 & 0.5641 \\ 1.2036 & 0.9288 & -0.0350 & 0.9185 & 1.4102 & 0.2469 \\ 0.7637 & -0.0350 & 0.3955 & 0.2507 & 0.3722 & 0.1177 \\ 1.9083 & 0.9185 & 0.2507 & 2.2607 & 1.5791 & 0.5438 \\ 2.6156 & 1.4102 & 0.3722 & 1.5791 & 2.7983 & 0.5958 \\ 0.5641 & 0.2469 & 0.1177 & 0.5438 & 0.5958 & 0.2052 \end{bmatrix},$$

$$Z = \begin{bmatrix} 2.3917 & 2.1491 & 0.9386 & 0.9726 & 2.3623 & 0.9084 \\ 2.1491 & 2.0640 & 0.8085 & 0.9552 & 2.3021 & 0.7192 \\ 0.9386 & 0.8085 & 0.4440 & 0.3779 & 0.9932 & 0.4408 \\ 0.9726 & 0.9552 & 0.3779 & 0.4518 & 1.1048 & 0.3289 \\ 2.3623 & 2.3021 & 0.9932 & 1.1048 & 2.7789 & 0.8740 \\ 0.9084 & 0.7192 & 0.4408 & 0.3289 & 0.8740 & 0.4734 \end{bmatrix}.$$

Then, by Algorithm 2, and choosing matrices W_2, Z_2, M_k, $k = 1, 2, 3$, we can obtain

$$X = \begin{bmatrix} 0.7744 & 1.3133 & 0.4707 & -0.5324 & 0.8611 & -0.4092 & -2.0561 & -0.3293 \\ 0.9085 & 1.7919 & 1.0445 & 0.1845 & 1.2927 & -0.3446 & -3.0173 & -0.5020 \\ 0.4164 & 0.5937 & 1.2040 & 0.0597 & 0.5449 & -0.2746 & -1.3373 & -0.3242 \\ -0.4629 & -0.5579 & 0.3186 & 0.6899 & -0.3685 & 0.1762 & 0.5333 & 0.2706 \\ 0.9622 & 1.4405 & 0.4085 & -0.5829 & 1.3213 & -0.3025 & -2.2616 & -0.1767 \\ -0.3086 & -0.1796 & -0.3515 & -0.1403 & -0.2965 & -0.0627 & 0.1867 & 0.0404 \\ -1.6561 & -2.5262 & -1.2625 & 0.7363 & -2.3365 & 0.3662 & 4.8728 & 0.2127 \\ -0.5173 & -0.1102 & -0.5650 & -0.1079 & -0.2818 & -0.2727 & 0.0910 & 0.1234 \end{bmatrix}.$$

The absolute errors are estimated by

$$\|AXB - D\|_F = 7.7691 \times 10^{-13}, \ \|F_G X F_G - F_G X^\top FG\|_F = 3.2055 \times 10^{-13},$$

and $\lambda(F_G X F_G) = \{8.6028,\ 2.4713,\ 0.8839,\ 0.2028,\ 0.0000,\ 0.0000,\ 0.0000,\ 0.0000\}$, which implies that X is an SND solution of Eq (1.2) on the subspace Ω.

5. Conclusions

In this paper, we have obtained the solvability conditions of $AX = C, XB = D$ (Problem I) and $AXB = D$ (Problem II) by applying the generalized inverse of matrices, matrix decomposition and some OPs (see Theorems 1 and 2). Furthermore, two given numerical examples have verified the correctness of our results. In the future, we will further consider the symmetric nonnegative definite solutions of other matrix equations on the subspace Ω, as well as the corresponding optimal approximation solutions.

Author contributions

Yinlan Chen: Conceptualization, Methodology; Qingyi Yang: Investigation, Software, Validation, Writing-original draft; Honglin Zou: Supervision, review, editing. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in this article.

Conflict of interest

The authors declare that there are no conflicts of interest associated with this work.

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