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Research article

Inverse Laplace transform and spectral method based on Laguerre-Sobolev-type polynomials

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Abstract: In this contribution, given an analytic function defined in a suitable region of the complex plane, we established the existence of its inverse Laplace transform and showed that it admits a series representation of the form

$$e^{-bx}\sum_{n=0}^{\infty} c_n S_n(2bx), \quad x \geq 0, \quad b > 0,$$

where $\{S_n(2bx)\}_{n\geq 0}$ denotes the sequence of Laguerre-Sobolev-type polynomials orthogonal with respect to the inner product

$$\langle f, g\rangle_{s,b} = \int_0^\infty f(x)g(x)e^{-2bx}dx + \frac{\lambda}{2b} f(0)g(0), \quad \lambda \geq 0,$$

defined on a suitable weighted Sobolev space. Furthermore, we proved that the series converges absolutely and uniformly on compact subsets of $\mathbb{R}^+$. As a consequence, the inverse Laplace transform could be efficiently approximated by truncated series. We also proved that the corresponding approximation error exhibits exponential decay, and that parameter λ improves the approximation at $x = 0$ without increasing the degree of the truncated series. Finally, we proposed a spectral method based on these Laguerre-Sobolev-type polynomials for the numerical solution of differential equations whose Laplace transform can be determined explicitly.

Keywords: Laguerre-Sobolev-type polynomials; Laplace transform; spectral method
Mathematics Subject Classification: 33C45, 42C05, 44A10, 65M70

1. Introduction

The Laplace transform (LT) has historically been a fundamental linear operator in mathematics, with applications to differential equations and wide use in engineering [1]. However, as is often the case with inverse operators, the explicit computation of the inverse Laplace transform (ILT) is challenging due to its inherently ill-posed nature, making it a significant problem in applied mathematics and engineering. Consequently, numerous numerical techniques have been developed to address this problem; a comprehensive survey of these methods can be found in [2].

Among these numerical approaches, particular attention has been given to those that yield an analytic representation of the target function. In this context, methods based on expansions in orthogonal polynomials, denoted by $\{P_n\}_{n\geq 0}$, of the form $e^{-bx}\sum_{n=0}^{\infty} c_n P_n(2bx)$, with $b > 0$, have proven to be especially effective. Classical examples, in which classical Laguerre polynomials (CLPs) constitute the underlying orthogonal basis, include Tricomi's method and Weeks' method; see [3–5], among many others.

In contrast to these approaches based on classical orthogonal polynomials, we introduce a method based on Laguerre-Sobolev-type polynomials (LSTPs), denoted by S_n, which arise from a modified inner product incorporating boundary information. These polynomials are orthogonal with respect to the inner product [6]

$$\langle f, g\rangle_{\lambda,b} = \int_0^{\infty} f(x)g(x)e^{-2bx}dx + \frac{\lambda}{2b} f(0)g(0), \quad \lambda \geq 0, \tag{1.1}$$

where f and g belong to a suitable weighted Sobolev space. This extension introduces an additional degree of freedom that can enhance the approximation of the ILT, particularly in situations where the behavior of the function at the origin plays a significant role. More generally, the main motivation for using the Sobolev polynomials is their ability to mitigate convergence issues near the boundary of the support of the measure caused by the Gibbs phenomenon; see, for example, [7].

On the other hand, spectral methods for solving differential equations are high-accuracy numerical techniques in which the solution is approximated by series expansions in global basis functions, typically orthogonal polynomials or trigonometric functions. Historically, the most commonly used bases in these methods are the classical orthogonal polynomials (see [8]), such as Jacobi polynomials (including Legendre and Chebyshev polynomials as particular cases), as well as Laguerre and Hermite polynomials. Methods based on these classical families have been widely studied due to their high accuracy and their typical exponential convergence; see, for example, [9–12]. However, spectral methods based on non-classical orthogonal polynomials may, in some cases, yield more accurate approximations while preserving exponential convergence; see, for instance, [5].

A well-known class of non-classical orthogonal polynomials is given by Sobolev polynomials. A comprehensive overview, along with extensive references, can be found in [13]. These polynomials have been used in function approximation through Fourier-Sobolev series; see [7, 14–16]. They also play an important role in the construction of spectral methods for boundary value problems of differential equations [9, 17–19], as well as for problems associated with certain elliptic differential operators [12,20], where, as mentioned above, they may exhibit improved performance compared with classical orthogonal polynomials.

Moreover, spectral methods based on properties of the LT associated with its ordinary differential equation were considered in [21] using CLPs. Moreover, a spectral method based on Laguerre-Sobolev polynomials was proposed in [22]. These Laguerre-Sobolev polynomials are orthogonal with respect to the inner product

$$\langle f, g\rangle_s = \int_0^\infty f(x)g(x)e^{-x}dx + \lambda \int_0^\infty f'(x)g'(x)e^{-x}dx, \quad \lambda \geq 0,$$

where f and g belong to an appropriate weighted Sobolev space.

Within this framework, a further contribution of this work is the introduction of a spectral method for the numerical solution of differential equations with initial conditions at the origin, based on LSTPs, denoted by S_n and orthogonal with respect to (1.1), through approximations of the form

$$e^{-bx} \sum_{n=0}^{N} c_n S_n(2bx), \quad x \geq 0, \quad N \geq 0, \quad b > 0, \tag{1.2}$$

The structure of the manuscript is as follows: In the next section, we first establish several properties of the CLPs, including its LT. We then introduce the LSTPs and discuss their main properties, in particular their relationship with the CLPs. Moreover, the advantage of using LSTPs, together with the role of parameter λ in the approximation of functions by truncated series at $x = 0$, is described. A numerical example illustrating this advantage is also presented. Finally, we compute the LT of the LSTPs.

In Section 3, we introduce the Laguerre-Sobolev-type series (LSTS) and show that $F(s)$ is analytic in a suitable region of the complex plane if and only if its ILT exists and admits a representation of the form (1.2), which converges absolutely and uniformly on compact subsets of $\mathbb{R}^+$. Consequently, the ILT can be approximated by truncated LSTS of the form (1.2). Moreover, we prove that the corresponding approximation error exhibits exponential decay. The error analysis is carried out with respect to both the Laguerre-Sobolev-type norm associated with the inner product (1.1) and the classical L^2-norm. Furthermore, we show that parameter λ improves the approximation of the ILT near the boundary without increasing the degree of the truncated LSTS. Finally, we conclude the section with two numerical examples that illustrate the theoretical results.

In the final section, we use the results developed in the prior section to introduce a spectral method based on truncated LSTS of the form (1.2). The method is mainly based on approximating the ILT associated with the differential equation whenever it can be determined explicitly. Several plots and tables are included to illustrate the accuracy and convergence rate of the proposed spectral method. All plots, together with the numerical results reported in the tables, are generated using Matlab R2026a.

2. Laguerre-Sobolev-type polynomials and their properties

In this section, we introduce the CLPs and the LSTPs, along with several of their fundamental properties, including its LT.

2.1. Classical Laguerre polynomials

The CLPs, denoted by $L_n^{(\alpha)}$, arise as one of the solutions of the Sturm-Liouville equation (see [8, Equation (2.20)])

$$xy''(x) + (\alpha + 1 - x)y(x) + ny(x) = 0,$$

where n is a non-negative integer and $\alpha > -1$. Each polynomial admits the explicit representation (see [8, Equation (2.11)])

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \quad x \geq 0. \tag{2.1}$$

From this expression, it follows that the leading coefficient of $L_n^{(\alpha)}$ is

$$\kappa_n = \frac{(-1)^n}{n!}, \;\; n \geq 0. \tag{2.2}$$

Furthermore, the derivative of a CLP is again a CLP (see [8, Equation (2.22)]), namely, for $n \geq 1$,

$$\frac{dL_n^{(\alpha)}(x)}{dx} = -L_{n-1}^{(\alpha+1)}(x). \tag{2.3}$$

As a consequence, the following relation holds (see [23, Equation (2.2)])

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}(x) = \frac{dL_n^{(\alpha)}(x)}{dx} - \frac{dL_{n+1}^{(\alpha)}(x)}{dx}, \;\; n \geq 1. \tag{2.4}$$

In particular, when $\alpha = 0$, we write $L_n = L_n^{(0)}$. The sequence $\{L_n(2bx)\}_{n\geq 0}$ forms a system of orthogonal polynomials with respect to the positive weight function $\omega(x) = e^{-2bx}$ on the interval $(0, \infty)$, with $b > 0$ (see [8, 24]). More precisely, for all non-negative integers n and m, it holds that

$$\int_0^\infty L_n(2bx)L_m(2bx)e^{-2bx}dx = \frac{1}{2b}\delta_{n,m}, \tag{2.5}$$

where $\delta_{n,m}$ denotes the Kronecker delta.

Finally, the LT of a function $f : [0, \infty) \to \mathbb{C}$ is defined, for complex values s, by

$$\mathcal{L}\{f(x)\}(s) = \int_0^\infty f(x)e^{-sx}dx.$$

provided that the integral converges. In particular, the LT of the CLPs (see [25, Equation (7)]) is given by

$$\mathcal{L}\{L_n(2bx)\}(s) = \frac{1}{s}\left(1 - \frac{2b}{s}\right)^n = \frac{1}{s}\left(\frac{s-2b}{s}\right)^n, \; b > 0. \tag{2.6}$$

Moreover, by the first shifting property of the LT, we obtain

$$\mathcal{L}\{e^{-bx}L_n(2bx)\}(s) = \frac{1}{s+b}\left(1 - \frac{2b}{s+b}\right)^n = \frac{1}{s+b}\left(\frac{s-b}{s+b}\right)^n, \; b > 0. \tag{2.7}$$

2.2. Laguerre-Sobolev-type polynomials

The LSTP, denoted by S_n, were introduced in [6] and can be expressed in terms of the CLPs. An equivalent formulation is given in the following result.

Proposition 2.1. *The LSTPs satisfy*

$$\begin{aligned} S_0(2bx) &= L_0(2bx) = 1, \\ S_n(2bx) &= L_n^{(1)}(2bx) - b_n L_{n-1}^{(1)}(2bx),\ n \geq 1, \end{aligned} \tag{2.8}$$

where $L_n^{(1)}$ *denotes the CLP of degree* n *with parameter* $\alpha = 1$*, as defined in* (2.1)*. The coefficients* $b_n \equiv b_n(\lambda)$ *are given by*

$$b_n = 1 + \frac{\lambda}{1 + \lambda n},\ \ n \geq 0. \tag{2.9}$$

Proof. It was shown in [6, Proposition 6] that the monic LSTPs satisfy $\tilde{S}_n(x) = \tilde{L}_n^{(1)}(x) + d_n \tilde{L}_{n-1}^{(1)}(x)$, where $d_n = n\left(1 + \frac{\lambda}{1+\lambda n}\right)$. Since the leading coefficient of $L_n^{(1)}$ is given by (2.2), it follows that the above relation can be rewritten in terms of the standard (non-monic) polynomials as

$$S_n(x) = L_n^{(1)}(x) - \frac{d_n}{n} L_{n-1}^{(1)}(x),\ \ n \geq 1.$$

Setting $b_n = \frac{d_n}{n}$ and evaluating the identity at $2bx$, we obtain (2.8).

Setting $\alpha = 0$ in (2.4), we obtain

$$L_n^{(1)}(x) = L_n(x) + L_{n-1}^{(1)}(x) = L_n(x) + L_{n-1}(x) + L_{n-2}^{(1)}(x) = \cdots = \sum_{k=0}^{n} L_k(x),\ \ n \geq 1. \tag{2.10}$$

Substituting this relation into (2.8), we obtain, for $n \geq 1$,

$$\begin{aligned} S_n(2bx) &= \sum_{k=0}^{n} L_k(2bx) - b_n \sum_{k=0}^{n-1} L_k(2bx) \\ &= L_n(2bx) + (1 - b_n) \sum_{k=0}^{n-1} L_k(2bx) = L_n(2bx) - \frac{\lambda}{1 + \lambda n} \sum_{k=0}^{n-1} L_k(2bx). \end{aligned}$$

Consequently, the LSTPs admit a representation in terms of classical Laguerre polynomials.

Corollary 2.1. *The LSTPs admit the representation*

$$S_n(2bx) = \sum_{k=0}^{n} \mathbf{A}(k,n) L_k(2bx),\ n \geq 0, \tag{2.11}$$

where the coefficients $\mathbf{A}(k,n)$ *are given by*

$$\mathbf{A}(k,n) = \begin{cases} 1, & \text{if } k = n, \\ 1 - b_n = -\frac{\lambda}{1+\lambda n}, & \text{if } k = 0, 1, \ldots, n-1. \end{cases} \tag{2.12}$$

Evaluating (2.11) at $x = 0$ and using the identity $L_k(0) = 1$, we obtain

$$S_n(0) = \sum_{k=0}^{n} \mathbf{A}(k,n) L_k(0) = L_n(0) - \frac{\lambda}{1 + \lambda n} \sum_{k=0}^{n-1} L_k(0) = \frac{1}{1 + \lambda n} = \frac{b_n - 1}{\lambda},\ \ n \geq 0. \tag{2.13}$$

In particular, it follows that $S_n(0) \to 0$ as $n \to \infty$. Moreover, from (2.11), the first LSTPs are given by

$$\begin{aligned}
S_0(x) &= 1,\\
S_1(x) &= -x + \frac{1}{1+\lambda},\\
S_2(x) &= \frac{x^2}{2} - \frac{2+3\lambda}{1+2\lambda}x + \frac{1}{1+2\lambda},\\
S_3(x) &= -\frac{x^3}{6} + \frac{3+8\lambda}{2(1+3\lambda)}x^2 - \frac{3(1+2\lambda)}{1+3\lambda}x + \frac{1}{1+3\lambda}.
\end{aligned} \tag{2.14}$$

The following result provides a representation of the CLPs in terms of the LSTPs.

Corollary 2.2. *The CLPs admit the representation*

$$L_k(2bx) = \sum_{n=0}^{k} \mathbf{C}(n,k) S_n(2bx), \quad k \geq 0, \tag{2.15}$$

where the coefficients $\mathbf{C}(n,k)$ *are given by*

$$\mathbf{C}(n,k) = \begin{cases} 1, & \text{if } n = k,\\ b_{n+1} - 1 = \frac{\lambda}{1+\lambda(n+1)}, & \text{if } n = 0,1,\ldots,k-1. \end{cases} \tag{2.16}$$

Proof. From (2.8), for $k \geq 1$, we recursively obtain

$$\begin{aligned}
L_k^{(1)}(2bx) &= S_k(2bx) + b_k L_{k-1}^{(1)}(2bx)\\
&= S_k(2bx) + b_k S_{k-1}(2bx) + b_k b_{k-1} L_{k-1}^{(1)}(2bx)\\
&\;\;\vdots\\
&= S_k(2bx) + b_k S_{k-1}(2bx) + b_k b_{k-1} S_{k-2}(2bx) + \cdots + b_k b_{k-1} \cdots b_1 S_0(2bx).
\end{aligned}$$

On the other hand, using (2.10), we have $L_k(2bx) = L_k^{(1)}(2bx) - L_{k-1}^{(1)}(2bx)$. Substituting the above expansions for $L_k^{(1)}(2bx)$ and $L_{k-1}^{(1)}(2bx)$, and simplifying term by term, yields

$$\begin{aligned}
L_k(2bx) &= S_k(2bx) + b_k S_{k-1}(2bx) + b_k b_{k-1} S_{k-2}(2bx) + \cdots + b_k b_{k-1} \cdots b_1 S_0(2bx)\\
&\quad - (S_{k-1}(2bx) + b_{k-1} S_{k-2}(2bx) + b_{k-1} b_{k-2} S_{k-3}(2bx) + \cdots + b_{k-1} b_{k-2} \cdots b_1 S_0(2bx))\\
&= S_k(2bx) + (b_k - 1) S_{k-1}(2bx) + b_{k-1}(b_k - 1) S_{k-2}(2bx) + \cdots + b_{k-1} \cdots b_1 (b_k - 1) S_0(2bx).
\end{aligned}$$

Using (2.9) and simplifying the coefficients, this expression becomes

$$L_k(2bx) = S_k(2bx) + \frac{\lambda}{1+\lambda k} S_{k-1}(2bx) + \frac{\lambda}{1+\lambda(k-1)} S_{k-2}(2bx) + \cdots + \frac{\lambda}{1+\lambda} S_0(2bx).$$

Finally, introducing the notation in (2.16), the desired representation follows.

It follows from (2.12) and (2.16) that

$$\begin{aligned}
-1 < \mathbf{A}(k,n) < 0, &\quad k = 0,\ldots,n-1, \quad n \geq 1,\\
0 < \mathbf{C}(n,k) < 1, &\quad n = 0,\ldots,k-1, \quad k \geq 1.
\end{aligned} \tag{2.17}$$

On the other hand, for $\lambda \geq 0$, we consider the Sobolev-type space

$$\mathbf{L}^2_\lambda([0,\infty), e^{-x}dx) = \left\{ f : [0,\infty) \to \mathbb{R} \,\middle|\, \int_0^\infty |f(x)|^2 e^{-x}dx + \lambda |f(0)|^2 < \infty \right\}.$$

This space is endowed with the Sobolev-type inner product

$$\langle f, g\rangle_\lambda = \int_0^\infty f(x)g(x)e^{-x}dx + \lambda f(0)g(0), \quad f, g \in \mathbf{L}^2_\lambda([0,\infty), e^{-x}dx), \tag{2.18}$$

with associated norm $\|f\|_\lambda = \sqrt{\langle f, f\rangle_\lambda}$.

Proposition 2.2. *The sequence of LSTPs is orthogonal with respect to the Sobolev-type inner product* (2.18)*. More precisely, for all non-negative integers n and m, it holds that*

$$\langle S_n(x), S_m(x)\rangle_\lambda = b_n \delta_{n,m},$$

where b_n is defined in (2.9)*.*

Proof. Let n and m be non-negative integers. Using (2.11) with $b = \frac{1}{2}$, together with (2.13), we obtain

$$\langle S_n, S_m\rangle_\lambda = \int_0^\infty \left(L_n(x) + (1-b_n)\sum_{k=0}^{n-1} L_k(x)\right)\left(L_m(x) + (1-b_m)\sum_{k=0}^{m-1} L_k(x)\right)e^{-x}dx \;\; + \frac{(b_n-1)(b_m-1)}{\lambda}.$$

We first consider case $m < n$. Using the orthogonality of the CLPs (2.5) with $b = \frac{1}{2}$, it follows that

$$\begin{aligned}
\langle S_n, S_m\rangle_\lambda &= (1-b_n)\int_0^\infty L_m^2(x)e^{-x}dx + (1-b_n)(1-b_m)\sum_{k=0}^{m-1}\int_0^\infty L_k^2(x)e^{-x}dx + \frac{(b_n-1)(b_m-1)}{\lambda} \\
&= (1-b_n)\left(1 + (1-b_m)m + \frac{1-b_m}{\lambda}\right) \\
&= (1-b_n)\left(1 + (1-b_m)\left(\frac{1+\lambda m}{\lambda}\right)\right).
\end{aligned}$$

Since $1 - b_m = -\frac{\lambda}{1+\lambda m}$, it follows that $\langle S_n, S_m\rangle_\lambda = 0$.

Next, consider case $n = m$. Proceeding similarly, we obtain

$$\langle S_n, S_n\rangle_\lambda = \int_0^\infty L_n^2(x)e^{-x}dx + (1-b_n)^2\sum_{k=0}^{n-1}\int_0^\infty L_k^2(x)e^{-x}dx + \frac{(b_n-1)^2}{\lambda} = 1 + (1-b_n)^2\left(\frac{1+\lambda n}{\lambda}\right).$$

Since $\frac{1+\lambda n}{\lambda} = \frac{1}{b_n - 1}$, we conclude that $\langle S_n, S_n\rangle_\lambda = 1 + b_n - 1 = b_n$.

Corollary 2.3. *The sequence of LSTPs, denoted by $\{S_n(2bx)\}_{n\geq 0}$, is orthogonal with respect to the inner product* (1.1)*. More precisely, for all non-negative integers n and m, it holds that*

$$\langle S_n(2bx), S_m(2bx)\rangle_{\lambda,b} = \frac{1}{2b}\langle S_n(x), S_m(x)\rangle_\lambda \,\delta_{n,m} = \frac{1}{2b} b_n \delta_{n,m}, \tag{2.19}$$

where b_n is defined in (2.9)*.*

The norm associated with the inner product (1.1) is defined by

$$\|f\|_{\lambda,b} = \sqrt{\langle f, f\rangle_{\lambda,b}}. \tag{2.20}$$

For $n \geq 1$, we have $b_n = 1 + \frac{\lambda}{1+\lambda n} < 2$. Therefore, it follows from (2.19) that

$$\|S_n(2bx)\|^2_{\lambda,b} = \frac{1}{2b} b_n < \frac{1}{b}. \tag{2.21}$$

Finally, any function $f \in L^2([0,\infty), dx)$ can be approximated by truncated Laguerre series and truncated LSTS, respectively, defined by

$$\Xi_{N,b} f(x) := e^{-bx} \sum_{n=0}^{N} a_n L_n(2bx), \quad \Pi^{(\lambda)}_{N,b} f(x) := e^{-bx} \sum_{n=0}^{N} c_n S_n(2bx), \quad N \geq 1, \ x \geq 0. \tag{2.22}$$

Notice that $\Pi^{(0)}_{N,b} f(x) = \Xi_{N,b} f(x)$. Using the orthogonality of the CLPs together with (2.5), the coefficients a_n are given by

$$a_n = \frac{\int_0^\infty e^{bx} f(x) L_n(2bx) e^{-2bx} dx}{\int_0^\infty L_n^2(2bx) e^{-2bx} dx} = 2b \int_0^\infty f(x) L_n(2bx) e^{-bx} dx, \quad n = 0, \ldots, N.$$

Similarly, using the orthogonality of the LSTPs together with (2.19) and (2.13), the coefficients c_n are given by

$$\begin{aligned} c_n &= \frac{\langle e^{bx} f(x), S_n(2bx)\rangle_{\lambda,b}}{\langle S_n(2bx), S_n(2bx)\rangle_{\lambda,b}} \\ &= \frac{1}{1+\lambda(n+1)} \left(2b(1+\lambda n) \int_0^\infty f(x) S_n(2bx) e^{-bx} dx + \lambda f(0) \right), \quad n = 0, \ldots, N. \end{aligned}$$

In particular, for $n = 0$,

$$c_0 = \frac{2b}{1+\lambda} \int_0^\infty f(x) e^{-bx} dx + \frac{\lambda}{1+\lambda} f(0). \tag{2.23}$$

Therefore, $c_0 \to f(0)$ as $\lambda \to \infty$. Furthermore, using (2.13), we obtain

$$\Pi^{(\lambda)}_{N,b} f(0) := \sum_{n=0}^{N} c_n S_n(0) = \sum_{n=0}^{N} \frac{c_n}{1+\lambda n} = c_0 + \frac{c_1}{1+\lambda} + \cdots + \frac{c_n}{1+\lambda N}. \tag{2.24}$$

Therefore, from (2.23) and (2.24), we obtain the following result.

Proposition 2.3. *Let $f \in L^2([0,\infty), dx)$. Then, for fixed b and $N \geq 0$,*

$$\lim_{\lambda\to\infty} \Pi^{(\lambda)}_{N,b} f(0) = f(0).$$

The above result guarantees that, when λ is sufficiently large, the truncated LSTS evaluated at $x = 0$ provides an accurate approximation of $f(0)$. In contrast, such a property cannot generally be guaranteed for the classical truncated Laguerre series. This behavior is illustrated in the following example, where the analysis is carried out using the maximum norm error (MNE) on the interval $[0, R]$, defined by

$$\|f - g\|_{[0,R]} = \max_{x\in[0,R]} |f(x) - g(x)|, \quad f \in C([0, R]).$$

Example 2.1. Consider the function $f(x) = \frac{\cos(3x)}{x+1} \in \mathbf{L}^2([0,\infty), dx)$. Figures 1 and 2 show the plots of $\Xi_{N,b} f(x)$ and $\Pi_{N,b}^{(\lambda)} f(x)$, respectively, with $N = 15$, $b = 2$, and $\lambda = \frac{1}{10}$, 10.

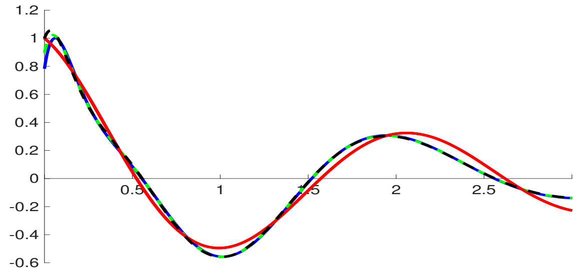

Figure 1. Plots of f (—), together with its approximations $\Xi_{N,b} f$ (—), and $\Pi_{N,b}^{(\lambda)} f$ for $\lambda = \frac{1}{10}$ (—) and $\lambda = 10$ (—), on the interval $[0, 3]$, with $N = 10$ and $b = 2$.

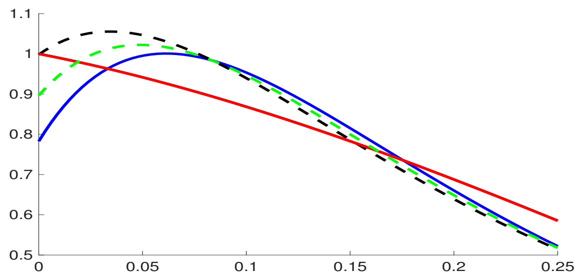

Figure 2. Plots of f (—), together with its approximations $\Xi_{N,b} f$ (—), and $\Pi_{N,b}^{(\lambda)} f$ for $\lambda = \frac{1}{10}$ (—) and $\lambda = 10$ (—), on the interval $[0, 0.25]$, with $N = 10$ and $b = 2$.

In general, the approximations are accurate on the interval $[0, 3]$. However, near the origin, the truncated Laguerre series $\Xi_{N,b} f(x)$ exhibits a poor approximation. In contrast, as λ increases, the truncated LSTS provides a significantly better approximation at this point.

Moreover, Table 1 shows that, for $\lambda = 0$, the MNE occurs near the origin. As λ increases, this error decreases, illustrating the role of parameter λ in improving the approximation near $x = 0$.

Table 1. MNE between f and $\Pi_{N,b}^{(\lambda)} f$ for $N = 10$, $b = 2$, and $\lambda = 0, \frac{1}{100}, 1, 10, 100$.

λ	$\|f - \Pi_{N,b}^{(\lambda)} f\|_{[0,0.5]}$	$\|f - \Pi_{N,b}^{(\lambda)} f\|_{[0,1]}$	$\|f - \Pi_{N,b}^{(\lambda)} f\|_{[0,3]}$	$\|f - \Pi_{N,b}^{(\lambda)} f\|_{[0,10]}$
0	2.1739×10^{-1}	2.1739×10^{-1}	2.1739×10^{-1}	2.1739×10^{-1}
$\frac{1}{100}$	1.9585×10^{-1}	1.9585×10^{-1}	1.9585×10^{-1}	1.9585×10^{-1}
1	1.0244×10^{-1}	1.0244×10^{-1}	1.0243×10^{-1}	1.6857×10^{-1}
10	1.0535×10^{-1}	1.0535×10^{-1}	1.0533×10^{-1}	1.6867×10^{-1}
100	1.0569×10^{-1}	1.0569×10^{-1}	1.0568×10^{-1}	1.6868×10^{-1}

2.3. Laplace transform of Laguerre-Sobolev-type polynomials

Applying the LT to (2.11), and using (2.6) together with (2.7), we obtain the following expression for the LT of the LSTPs.

Proposition 2.4. *The LT of the LSTPs is given by*

$$\begin{aligned}\mathcal{L}\{S_n(2bx)\}(s) &= \sum_{k=0}^{n} \mathbf{A}(k,n)\frac{1}{s}\left(\frac{s-2b}{s}\right)^k, \quad n \geq 0,\\ \mathcal{L}\{e^{-bx}S_n(2bx)\}(s) &= \sum_{k=0}^{n} \mathbf{A}(k,n)\frac{1}{s+b}\left(\frac{s-b}{s+b}\right)^k, \quad n \geq 0.\end{aligned} \tag{2.25}$$

An alternative representation can be derived from (2.8) and the differentiation formula (2.3). Indeed, for $n \geq 1$, $S_n(2bx) = -\frac{d}{dx}L_n(2bx) + b_n\frac{d}{dx}L_{n-1}(2bx)$. Applying the LT and using the differentiation property, together with the identity $L_n(0) = 1$, we obtain

$$\begin{aligned}\mathcal{L}\{S_n(2bx)\} &= -s\mathcal{L}\{L_n(2bx)\} + L_n(0) + b_n\left(s\mathcal{L}\{L_{n-1}(2bx)\} - L_{n-1}(0)\right)\\ &= s\left(b_n\mathcal{L}\{L_{n-1}(2bx)\} - \mathcal{L}\{L_n(2bx)\}\right) + (1-b_n).\end{aligned}$$

Using (2.6), this expression simplifies to

$$\mathcal{L}\{S_n(2bx)\} = \frac{1}{s}\left(\frac{s-2b}{s}\right)^{n-1}(s(b_n-1)-2b) + (1-b_n), \quad n \geq 1.$$

Finally, by the first shifting property of the LT, $\mathcal{L}\{e^{-bx}S_n(2bx)\} = \mathcal{L}\{S_n(2bx)\}_{s\to s+b}$, and therefore

$$\mathcal{L}\{e^{-bx}S_n(2bx)\} = \frac{1}{s+b}\left(\frac{s-b}{s+b}\right)^{n-1}((s+b)(b_n-1)-2b) + (1-b_n), \quad n \geq 1.$$

This yields an alternative explicit representation of the Laplace transform of the LSTPs.

3. Laguerre-Sobolev-type series and their properties

In this section, we introduce the LSTS and study its relationship with the ILT of functions $F(s)$ that are analytic in a suitable region of the complex plane. In addition, we analyze the approximation error with respect to two norms and show that parameter λ helps improve the approximation of the ILT at $x = 0$. Finally, the theoretical results are illustrated with two numerical examples.

For the remainder of this work, we consider the auxiliary function Φ, which plays a central role, and is defined by

$$\Phi(z) = (s+b)F(s) \;\Leftrightarrow\; \Phi(z) = \frac{2b}{1-z}F\left(\frac{1+z}{1-z}b\right), \tag{3.1}$$

where $F(s)$ denotes the LT of a function $f : [0,\infty) \to \mathbb{R}$. The complex variables z and s are related through the conformal mapping

$$z = \frac{s-b}{s+b} \Leftrightarrow s = \frac{1+z}{1-z}b. \tag{3.2}$$

3.1. Convergence of the Laguerre-Sobolev-type series

A preliminary result establishing the connection between the classical Laguerre series and the LSTS is presented below.

Proposition 3.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a real-valued function. Then f admits a LSTS representation*

$$f(x) = e^{-bx} \sum_{n=0}^{\infty} c_n S_n(2bx),\ b > 0,\ x \geq 0, \tag{3.3}$$

if and only if f admits a Laguerre series representation

$$f(x) = e^{-bx} \sum_{k=0}^{\infty} a_k L_k(2bx),\ b > 0,\ x \geq 0. \tag{3.4}$$

Moreover, the corresponding coefficients are related by

$$a_k = \sum_{n=k}^{\infty} c_n \mathbf{A}(k,n) = c_k - \sum_{n=k+1}^{\infty} c_n \frac{\lambda}{1+\lambda n} = c_k - \sum_{n=k+1}^{\infty} c_n (b_n - 1) < \infty,\ \ k \geq 0, \tag{3.5}$$

and, conversely,

$$c_n = \sum_{k=n}^{\infty} a_k \mathbf{C}(n,k) = a_n + \frac{\lambda}{1+\lambda(n+1)} \sum_{k=n+1}^{\infty} a_k = a_n + (b_{n+1} - 1) \sum_{k=n+1}^{\infty} a_k < \infty,\ \ n \geq 0. \tag{3.6}$$

Proof. Assume that f admits the LSTS representation (3.3). Substituting the expansion (2.11) into (3.3) and rearranging the order of the sums, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n S_n(2bx) &= \sum_{n=0}^{\infty} c_n \left(\sum_{k=0}^{n} \mathbf{A}(k,n) L_k(2bx) \right) \\
&= \left(\sum_{n=0}^{\infty} c_n \mathbf{A}(0,n) \right) L_0(2bx) + \left(\sum_{n=1}^{\infty} c_n \mathbf{A}(1,n) \right) L_1(2bx) + \cdots \\
&= \sum_{k=0}^{\infty} \left(\sum_{n=k}^{\infty} c_n \mathbf{A}(k,n) \right) L_k(2bx) = \sum_{k=0}^{\infty} a_k L_k(2bx),
\end{aligned}$$

where the coefficients a_k are given by (3.5). Moreover, evaluating (3.3) at $x = 0$ and using (2.13), we obtain

$$f(0) = \sum_{n=0}^{\infty} c_n S_n(0) = \sum_{n=0}^{\infty} \frac{c_n}{1+\lambda n} < \infty.$$

This ensures that the coefficients a_k defined in (3.5) are finite.

Conversely, suppose that f admits the Laguerre representation (3.4). Substituting (2.15) into (3.4), we obtain

$$\begin{aligned}
\sum_{k=0}^{\infty} a_k L_k(2bx) &= \sum_{k=0}^{\infty} a_k \left(\sum_{n=0}^{k} \mathbf{C}(n,k) S_n(2bx) \right) \\
&= \left(\sum_{k=0}^{\infty} a_k \mathbf{C}(0,k) \right) S_0(2bx) + \left(\sum_{k=1}^{\infty} a_k \mathbf{C}(1,k) \right) S_1(2bx) + \cdots \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=n}^{\infty} a_k \mathbf{C}(n,k) \right) S_n(2bx) = \sum_{n=0}^{\infty} c_n S_n(2bx),
\end{aligned}$$

where the coefficients c_n are given by (3.6). Furthermore, since $f(0) = \sum_{k=0}^{\infty} a_k$ is finite, it follows that each c_n is also finite.

Suppose that function f admits a representation of the form (3.3). Applying the LT to the LSTSs (3.3) and using (2.25), we obtain $F(s) = \frac{1}{s+b}\sum_{k=0}^{\infty} a_k \left(\frac{s-b}{s+b}\right)^k$, where the coefficients a_k are defined in (3.5). The converse follows by the same arguments. Hence, we obtain the following result.

Proposition 3.2. *Let $F(s)$ be the Laplace transform of $f(x)$. Then f admits the LSTS representation* (3.3) *if and only if its LT F admits the representation*

$$F(s) = \frac{1}{s+b}\sum_{k=0}^{\infty} a_k \left(\frac{s-b}{s+b}\right)^k, \quad b > 0. \tag{3.7}$$

The coefficients a_k are defined in (3.5).

In [21, Property 2.3], it was shown that the series (3.7) converges uniformly to $F(s)$ on every compact subset of the half-plane $\Re(s) > 0$.

The following theorem constitutes one of the main results of this work. It establishes that if $F(s)$ is analytic in a suitable region of the complex plane, then its ILT exists and admits a representation in terms of the LSTPs of form (3.3).

Theorem 3.1. *Let $F(s)$ be the LT of $f(x)$. If $F(s)$ is analytic in the half-plane $\Re(s) \geq 0$ and $(s+b)F(s)$ is analytic at infinity, then the LSTS*

$$e^{-bx}\sum_{n=0}^{\infty} c_n S_n(2bx), \quad b > 0, \quad x \geq 0, \tag{3.8}$$

converges absolutely for every $x \in \mathbb{R}^+$ and uniformly to $f(x)$ on every compact subset of $\mathbb{R}^+$.

Proof. Assume that $F(s)$ is analytic in the half-plane $\Re(s) \geq 0$. If $F(s)$ has singularities, then they must lie in the half-plane $\Re(s) < 0$. Consequently, the corresponding singularities of $\Phi(z)$, defined in (3.1), lie outside the unit disk, since the variables s and z are related by the conformal mapping (3.2). Moreover, because $(s + b)F(s)$ is regular at infinity, which is mapped to $z = 1$, it follows that $z = 1$ is not a singular point of $\Phi(z)$. Therefore, $\Phi(z)$ admits a Taylor expansion

$$\Phi(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{3.9}$$

with radius of convergence $R > 1$. Furthermore, from (3.1), it follows that

$$F(s) = \frac{1}{s+b}\sum_{k=0}^{\infty} a_k z^k = \frac{1}{s+b}\sum_{k=0}^{\infty} a_k \left(\frac{s-b}{s+b}\right)^k, \tag{3.10}$$

which is precisely the representation given in (3.7). Therefore, by Proposition 3.2, this implies that the ILT $f(x)$ admits the LSTS representation (3.3).

On the other hand, let ρ be any positive constant such that $1 < \rho < R$. By [26, Theorem 4.1], there exists a constant $\mathbf{K}$, depending on ρ, such that

$$|a_k| \leq \frac{\mathbf{K}}{\rho^k}, \quad k \geq 0. \tag{3.11}$$

Under these assumptions, it was shown in [21, Property 2.5] that the Laguerre series satisfies

$$\left| e^{-bx} \sum_{k=0}^{\infty} a_k L_k(2bx) \right| \leq \mathbf{K} \frac{\rho}{\rho - 1} \exp\left(-bx + \frac{2bx}{\rho - 1} \right).$$

Therefore, using Proposition 3.1, we obtain the same bound for the LSTS, that is

$$\left| e^{-bx} \sum_{n=0}^{\infty} c_n S_n(2bx) \right| = \left| e^{-bx} \sum_{k=0}^{\infty} a_k L_k(2bx) \right| \leq \mathbf{K} \frac{\rho}{\rho - 1} \exp\left(-bx + \frac{2bx}{\rho - 1} \right).$$

This estimate guarantees that the series (3.8) converges absolutely for any $x \in \mathbb{R}^+$ and uniformly on every compact subset of $\mathbb{R}^+$. Finally, combining Proposition 3.2 with representation (3.10), we conclude that expansion (3.8) coincides with $f(x)$.

3.2. Numerical approximation of the inverse Laplace transform

Let $F(s)$ be analytic in the half-plane $\Re(s) \geq 0$ and let $(s + b)F(s)$ be analytic at infinity. Then, by Theorem 3.1, $F(s)$ admits an ILT $f(x)$, which can be represented by the LSTS (3.3). Consequently, an approximation of f is given by

$$\Pi_{N,b}^{(\lambda)} f(x) = e^{-bx} \sum_{n=0}^{N} c_n S_n(2bx), \ N \geq 0, \ x \geq 0, \tag{3.12}$$

where the coefficients c_n satisfy (3.6), and the coefficients $a_k = \frac{\Phi^{(k)}(0)}{k!}$ are explicitly known.

In particular, evaluating (3.12) at $x = 0$, it follows from (2.24) and (3.6) with $n = 0$ that

$$\lim_{\lambda \to \infty} \Pi_{N,b}^{(\lambda)} f(0) = c_0, \ \text{ and } \ c_0 = a_0 + \lim_{\lambda \to \infty} \frac{\lambda}{1 + \lambda} \sum_{k=1}^{\infty} a_k = \sum_{k=0}^{\infty} a_k.$$

Moreover, since $L_k(0) = 1$, it follows from (3.4) that $f(0) = \sum_{k=0}^{\infty} a_k$. Consequently, we obtain the following result.

Proposition 3.3. *Let $F(s)$ be the LT of $f(x)$. If $F(s)$ is analytic in the half-plane $\Re(s) \geq 0$ and $(s + b)F(s)$ is analytic at infinity, then for fixed b and $N \geq 0$,*

$$\lim_{\lambda \to \infty} \Pi_{N,b}^{(\lambda)} f(0) = f(0).$$

Therefore, Proposition 3.3 shows that parameter λ improves the approximation of the ILT at $x = 0$ without increasing the degree N of the approximation (3.12).

On the one hand, we provide an error analysis for the approximation of f by the truncated LSTS (3.12). We consider the error with respect to the Laguerre-Sobolev-type norm (2.20) and the error measured in the L^2-norm, defined by $\|f\|_{\mathbf{L}^2} = \left(\int_0^{\infty} |f(x)|^2 \, dx \right)^{\frac{1}{2}}$. This norm quantifies the average discrepancy between the function and its approximation in the classical L^2 sense and is commonly referred to as the mean square error (MSE).

Proposition 3.4. *Let $F(s)$ denote the LT of $f(x)$. Assume that $F(s)$ is analytic in the half-plane $\Re(s) \geq 0$ and that $(s+b)F(s)$ is analytic at infinity. Then, for any $N \geq 1$, there exist constants $\rho > 1$ and $\mathbf{K} > 0$, depending on ρ, such that*

$$\left\| \frac{1}{e^{-bx}} \left(f(x) - \Pi_{N,b}^{(\lambda)} f(x) \right) \right\|_{s,b} \leq \frac{\mathbf{L}}{\rho^N}, \tag{3.13}$$

where $\mathbf{L} = \mathbf{K} \frac{1}{\sqrt{b}} \frac{\rho}{\rho-1} \frac{1}{\sqrt{\rho^2-1}}$, and $\Pi_{N,b}^{(\lambda)} f(x)$ is the truncated LSTS (3.12).

Proof. From the orthogonality property of the LSTPs (2.19), we obtain

$$\left\| \frac{1}{e^{-bx}} \left(f(x) - \Pi_{N,b}^{(\lambda)} f(x) \right) \right\|_{s,b}^2 = \left\| \sum_{n=N+1}^{\infty} c_n S_n(2bx) \right\|_{s,b}^2 = \sum_{n=N+1}^{\infty} |c_n|^2 \|S_n(2bx)\|_{s,b}^2. \tag{3.14}$$

Using (3.11) and taking into account that the coefficients $\mathbf{C}(n,k)$ satisfy (2.17), we obtain

$$|c_n| = \left| \sum_{k=n}^{\infty} a_k \mathbf{C}(n,k) \right| \leq \mathbf{K} \sum_{k=n}^{\infty} \frac{1}{\rho^k} = \frac{\mathbf{K}}{\rho^n} \sum_{k=0}^{\infty} \frac{1}{\rho^k} = \frac{\mathbf{K}}{\rho^n} \frac{\rho}{(\rho-1)}. \tag{3.15}$$

Consequently, $|c_n|^2 \leq \frac{\mathbf{K}^2}{\rho^{2n}} \frac{\rho^2}{(\rho-1)^2}$. Therefore,

$$\begin{aligned} \sum_{n=N+1}^{\infty} |c_n|^2 &\leq \mathbf{K}^2 \frac{\rho^2}{(\rho-1)^2} \sum_{n=N+1}^{\infty} \frac{1}{\rho^{2n}} = \mathbf{K}^2 \frac{\rho^2}{(\rho-1)^2} \frac{1}{\rho^{2(N+1)}} \sum_{n=0}^{\infty} \frac{1}{\rho^{2n}} \\ &= \mathbf{K}^2 \frac{\rho^2}{(\rho-1)^2} \frac{1}{\rho^{2(N+1)}} \frac{\rho^2}{\rho^2-1} = \mathbf{K}^2 \frac{\rho^2}{(\rho-1)^2} \frac{1}{\rho^2-1} \frac{1}{\rho^{2N}}. \end{aligned}$$

Substituting this bound, together with (2.21), into (3.14), we obtain

$$\left\| \frac{1}{e^{-bx}} \left(f(x) - \Pi_{N,b}^{(\lambda)} f(x) \right) \right\|_{s,b}^2 \leq \frac{1}{b} \mathbf{K}^2 \frac{\rho^2}{(\rho-1)^2} \frac{1}{\rho^2-1} \frac{1}{\rho^{2N}},$$

which is equivalent to (3.13).

Proposition 3.5. *Let $F(s)$ denote the LT of $f(x)$. Assume that $F(s)$ is analytic in the half-plane $\Re(s) \geq 0$ and that $(s+b)F(s)$ is analytic at infinity. Then, for any $N \geq 1$, there exist constants $\rho > 1$ and $\mathbf{K} > 0$, depending on ρ, such that*

$$\left\| f(x) - \Pi_{N,b}^{(\lambda)} f(x) \right\|_{\mathbf{L}^2} \leq \frac{\mathbf{M}\sqrt{N+2}}{\rho^N}, \tag{3.16}$$

where $\mathbf{M} = \mathbf{K} \sqrt{\frac{1}{2b} \left(\frac{\rho^2}{(\rho-1)^4} + \frac{1}{\rho^2(\rho^2-1)} \right)}$, and $\Pi_{N,b}^{(\lambda)} f(x)$ is the truncated LSTS (3.12).

Proof. Using (2.11) and rearranging the terms, we obtain

$$\left\| f(x) - \Pi_{N,b}^{(\lambda)} f(x) \right\|_{\mathbf{L}^2}^2 = \left\| e^{-bx} \sum_{n=N+1}^{\infty} c_n S_n(2bx) \right\|_{\mathbf{L}^2}^2 = \int_0^{\infty} \left| \sum_{n=N+1}^{\infty} c_n \left(\sum_{k=0}^{n} \mathbf{A}(k,n) L_k(2bx) \right) \right|^2 e^{-2bx} dx.$$

Since

$$\sum_{n=N+1}^{\infty} c_n \left(\sum_{k=0}^{n} \mathbf{A}(k,n) L_k(2bx) \right) = \sum_{k=0}^{N+1} \left(\sum_{n=N+1}^{\infty} c_n \mathbf{A}(k,n) \right) L_k(2bx) + \sum_{k=N+2}^{\infty} \left(\sum_{n=k}^{\infty} c_n \mathbf{A}(k,n) \right) L_k(2bx)$$
$$= \sum_{k=0}^{N+1} e_k L_k(2bx) + \sum_{k=N+2}^{\infty} a_k L_k(2bx)$$

where a_k is defined in (3.5) and $e_k = \sum_{n=N+1}^{\infty} c_n \mathbf{A}(k,n)$. By the orthogonality property of the CLP (2.5), it follows that

$$\left\| f(x) - \Pi_{N,b}^{(\lambda)} f(x) \right\|_{\mathbf{L}^2}^2 = \frac{1}{2b} \left(\sum_{k=0}^{N+1} |e_k|^2 + \sum_{k=N+2}^{\infty} |a_k|^2 \right). \tag{3.17}$$

Using (3.11), we obtain

$$\sum_{k=N+2}^{\infty} |a_k|^2 \leq \mathbf{K}^2 \sum_{k=N+2}^{\infty} \frac{1}{(\rho^2)^k} = \frac{\mathbf{K}^2}{\rho^{2(N+2)}} \sum_{k=0}^{\infty} \frac{1}{\rho^{2k}} = \frac{\mathbf{K}^2}{\rho^{2(N+2)}} \frac{\rho^2}{\rho^2 - 1} = \frac{\mathbf{K}^2}{\rho^{2(N+1)}(\rho^2 - 1)}.$$

Moreover, by (3.15) and the fact that the coefficients $\mathbf{A}(k,n)$ satisfy (2.17), we obtain

$$|e_k| \leq \sum_{n=N+1}^{\infty} |c_n \mathbf{A}(k,n)| \leq \frac{\mathbf{K}\rho}{\rho - 1} \sum_{n=N+1}^{\infty} \frac{1}{\rho^n} \leq \frac{\mathbf{K}\rho}{\rho - 1} \frac{1}{\rho^{N+1}} \sum_{n=0}^{\infty} \frac{1}{\rho^n} = \frac{\mathbf{K}\rho}{(\rho - 1)^2} \frac{1}{\rho^N}.$$

Hence, $\sum_{k=0}^{N+1} |e_k|^2 \leq (N+2) \frac{\mathbf{K}^2 \rho^2}{(\rho-1)^4} \frac{1}{\rho^{2N}}$. Substituting these bounds into (3.17), we obtain

$$\left\| f(x) - \Pi_{N,b}^{(\lambda)} f(x) \right\|_{\mathbf{L}^2}^2 \leq \frac{1}{2b} \left((N+2) \frac{\mathbf{K}^2 \rho^2}{(\rho - 1)^4} \frac{1}{\rho^{2N}} + \frac{\mathbf{K}^2}{\rho^{2(N+1)}(\rho^2 - 1)} \right)$$
$$= \frac{\mathbf{K}^2}{2b} \left(\frac{(N+2)\rho^2}{(\rho - 1)^4} + \frac{1}{\rho^2(\rho^2 - 1)} \right) \frac{1}{\rho^{2N}}$$
$$\leq \frac{\mathbf{K}^2}{2b} \left(\frac{\rho^2}{(\rho - 1)^4} + \frac{1}{\rho^2(\rho^2 - 1)} \right) \frac{N+2}{\rho^{2N}},$$

which is equivalent to (3.16).

Notice that the convergence of $\Pi_{N,b}^{(\lambda)} f$ is exponential in both norms. Furthermore, the error estimates (3.13) and (3.16) are independent of parameter λ, although they depend on parameter b.

On the other hand, the problem of determining an optimal value of parameter b for the CLPs was investigated in [21, Section 3]. Since parameter λ affects convergence near the origin, the same optimal parameter b remains valid for the CLPs and the LSTPs.

In particular, when $F(s)$ has a unique singularity μ such that $-\mu$ lies in the right half-plane, it was established in [21, Property 3.2.] that:

- If $sF(s)$ has no singularity at infinity, then the optimal choice is $b = -\Re(\mu)$.
- If $sF(s)$ has a singularity at infinity , then the optimal choice is $b = -\Re(\mu)$.

Moreover, if $sF(s)$ has two singularities, μ_1 and μ_2, such that μ_1 and μ_2 belong to the right half-plane, then the optimal choice is $b^2 = \mu_1\mu_1$. This is possible if and only if μ_1 and μ_2 are real, or if they are complex conjugates with opposite arguments.

Finally, since the coefficients c_n, defined in (3.6), involve infinite sums, we introduce the truncated coefficients

$$c_n^m = a_n + \frac{\lambda}{1+\lambda(n+1)} \sum_{k=n+1}^{m} a_k, \quad m \geq N. \tag{3.18}$$

Consequently, the approximation (3.12) takes the form

$$\Pi_{N,b}^{(m,\lambda)} f(x) = e^{-bx} \sum_{n=0}^{N} c_n^m S_n(2bx), \; m \geq N \geq 0, \; x \geq 0. \tag{3.19}$$

In particular, when $m = N$, we obtain

$$\Pi_{N,b}^{(N,\lambda)} f(x) = \Xi_{N,b} f(x), \tag{3.20}$$

where $\Xi_{N,b} f(x)$ is defined in (2.22). Thus, using (2.11) together with (2.16), we obtain

$$\begin{aligned} \sum_{n=0}^{N} a_n L_n(2bx) &= \sum_{n=0}^{N} \left(a_n + \sum_{k=n+1}^{N} a_k \mathbf{C}(n,k) \right) S_n(2bx) \\ &= \sum_{n=0}^{N} \left(a_n + \frac{\lambda}{1+\lambda(n+1)} \sum_{k=n+1}^{N} a_k \right) S_n(2bx) = \sum_{n=0}^{N} c_n^m S_n(2bx), \end{aligned}$$

which is equivalent to (3.20). Therefore, when $m = N$, parameter λ does not affect the approximation, and hence one must take $m \geq N+1$. Indeed, by taking $m = N+1$ and λ sufficiently large, the approximation $\Pi_{N,b}^{(m,\lambda)} f(x)$ provides high accuracy near at $x = 0$. The following examples illustrate the above theoretical results.

Example 3.1. Consider the function $F(s) = \frac{4}{s+2}$, which satisfies the hypothesis of Theorem 3.1. In this case, the function $\Phi(z)$, defined in (3.1), is given by

$$\Phi(z) = \frac{2b}{1-z} F\left(\frac{1+z}{1-z} b\right) = \frac{8b}{(b-2)z + b + 2}, \quad b > 0.$$

Consequently, the coefficients a_k are explicitly given by

$$a_k = \frac{\Phi^{(k)}(0)}{k!} = (-1)^k \frac{8b(b-2)^k}{(b+2)^{k+1}}, \quad k \geq 0.$$

Moreover, since $F(s)$ has a unique singularity at $\mu = -2$, the optimal choice is $b = 2$. Therefore, $c_0^m = 4$ and $c_n^m = 0$ for $n \geq 1$. Consequently, the approximation (3.19) coincides exactly with the ILT of $F(s)$. Furthermore, (3.19) is independent of λ and m, and coincides with the truncated classical Laguerre series. More precisely,

$$f(x) = \mathcal{L}^{-1}\{F(s)\}(x) = \Xi_{0,2} f(x) = \Pi_{0,2}^{(m,\lambda)} f(x) = 4e^{-2x}.$$

On the other hand, if we consider different values of b, Tables 2 and 3 show that $\Xi_{N,b} f(x)$ and $\Pi_{N,b}^{(m,\lambda)} f(x)$ provide good average approximations and yields very similar results. However, Tables 4 and 5 indicate that near the boundary, the approximation obtained using the LSTPs is better that obtained using the CLPs.

Table 2. MSE between f and $\Xi_{N,b} f$, and between f and $\Pi_{N,b}^{(N+1,\lambda)} f$, for $b = \frac{1}{2}$, and $\lambda = 10$.

N	$\left\Vert f - \Xi_{N,\frac{1}{2}} f \right\Vert_{\mathbf{L}^2}$	$\left\Vert f - \Pi_{N,\frac{1}{2}}^{(N+1,10)} f \right\Vert_{\mathbf{L}^2}$
0	1.2000×10^{0}	1.4837×10^{0}
1	7.1795×10^{-1}	7.1472×10^{-1}
2	4.1862×10^{-1}	4.1469×10^{-1}
3	2.3903×10^{-1}	2.4138×10^{-1}
4	1.4341×10^{-1}	1.4948×10^{-1}
5	8.3304×10^{-2}	8.2743×10^{-2}

Table 3. MSE between f and $\Xi_{N,b} f$, and between f and $\Pi_{N,b}^{(N+1,\lambda)} f$, for $b = 4$, and $\lambda = 10$.

N	$\left\Vert f - \Xi_{N,4} f \right\Vert_{[0,0.01]]}$	$\left\Vert f - \Pi_{N,4}^{(N+1,10)} f \right\Vert_{[0,0.01]]}$
0	6.6667×10^{-1}	6.6782×10^{-1}
1	2.2223×10^{-1}	2.2131×10^{-1}
2	7.4075×10^{-2}	7.4509×10^{-2}
3	2.4691×10^{-2}	2.4826×10^{-2}
4	8.2307×10^{-3}	8.2373×10^{-3}
5	2.7436×10^{-3}	2.7296×10^{-3}

Table 4. MNE between f and $\Xi_{N,b} f$, and between f and $\Pi_{N,b}^{(N+1,\lambda)} f$, for $b = \frac{1}{2}$, and $\lambda = 10$ on interval $[0, 0.2]$.

N	$\left\Vert f - \Xi_{N,\frac{1}{2}} f \right\Vert_{[0,0.2]}$	$\left\Vert f - \Pi_{N,\frac{1}{2}}^{(N+1,10)} f \right\Vert_{[0,0.2]}$
0	2.4000×10^{0}	1.5273×10^{0}
1	1.4400×10^{0}	8.9143×10^{-1}
2	8.6400×10^{-1}	5.2955×10^{-1}
3	5.1840×10^{-1}	3.1607×10^{-1}
4	3.1104×10^{-1}	1.8906×10^{-1}
5	1.8662×10^{-1}	1.1319×10^{-1}

Table 5. MNE between f and $\Xi_{N,b}f$, and between f and $\Pi_{N,b}^{(N+1,\lambda)}f$, for $b = 4$ and $\lambda = 10$ on interval $[0, 0.2]$.

N	$\left\|f - \Xi_{N,4}f\right\|_{[0,0.2]}$	$\left\|f - \Pi_{N,4}^{(N+1,10)}f\right\|_{[0,0.2]}$
0	1.3333×10^{0}	1.0110×10^{0}
1	4.4444×10^{-1}	2.9135×10^{-1}
2	1.4815×10^{-1}	9.2532×10^{-2}
3	4.9383×10^{-2}	3.0193×10^{-2}
4	1.6461×10^{-2}	9.9449×10^{-3}
5	5.4869×10^{-3}	3.2899×10^{-3}

Example 3.2. Consider the function $F(s) = \frac{10(s+1)}{(s+1)^2+4}$, which satisfies the hypothesis of Theorem 3.1. In this case, the function $\Phi(z)$, defined in (3.1), is given by

$$\Phi(z) = \frac{2b}{1-z}F\left(\frac{1+z}{1-z}b\right) = \frac{20b(b+1+(b-1)z)}{(b+1+(b-1)z)^2+4(1-z)^2}, \quad b > 0,$$

and the ILT is $f(x) = 10e^{-x}\cos(2x)$. Since $F(s)$ has two singularities $\mu_1 = -1+2i$ and $\mu_1 = -1-2i$, the optimal value of parameter b is $b = \sqrt{\mu_1\mu_2} = \sqrt{5}$.

Figures 3 and 4 show that the approximations are accurate on the interval $[0, 3]$. However, at $f(0)$, the truncated Laguerre series $\Xi_{N,b}f(x)$ exhibits poor accuracy. In contrast, as λ increases, the truncated LSTS provides a significantly better approximation near this point.

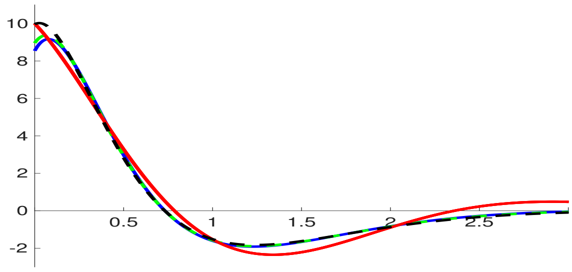

Figure 3. Plots of $f(x)$ (__), together with its approximations $\Xi_{N,b}f$ (__), and $\Pi_{N,b}^{(m,\lambda)}f$ for $\lambda = \frac{1}{10}$ (__) and $\lambda = 10$ (__), on the interval $[0, 3]$, with $N = 3$, $m = N+1$ and $b = \sqrt{5}$.

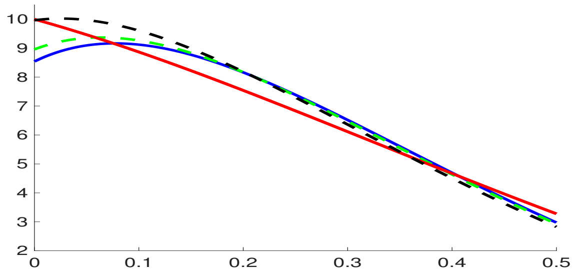

Figure 4. Plots of $f(x)$ (__), together with its approximations $\Xi_{N,b}f$ (__), and $\Pi_{N,b}^{(m,\lambda)}f$ for $\lambda = \frac{1}{10}$ (__) and $\lambda = 10$ (__), on the interval $[0, 0.5]$, with $N = 3$, $m = N+1$ and $b = \sqrt{5}$

Moreover, Figures 5 and 6 show that taking $m = N+1$ or $m = N+10$ yields practically the same results. In addition, although the classical Laguerre series improves as N increases, the value of

$\Xi_{N,b} f(0)$ remains far from $f(0)$. In contrast, $\Pi^{(m,\lambda)}_{N,b} f(0)$ yields values much closer to $f(0) = 10$. Indeed,

$$\left|10 - \Xi_{3,\sqrt{5}} f(0)\right| = 1.4589, \quad \left|10 - \Pi^{(4,1/10)}_{3,\sqrt{5}} f(0)\right| = 1.0422, \quad \left|10 - \Pi^{(13,10)}_{3,\sqrt{5}} f(0)\right| = 0.0241,$$

and

$$\left|10 - \Xi_{5,\sqrt{5}} f(0)\right| = 0.5572, \quad \left|10 - \Pi^{(6,1/10)}_{5,\sqrt{5}} f(0)\right| = 0.3483, \quad \left|10 - \Pi^{(15,10)}_{5,\sqrt{5}} f(0)\right| = 0.0047.$$

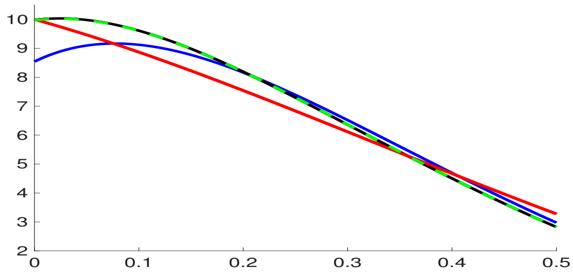

Figure 5. Plots of f (___), together with its approximations $\Xi_{N,b} f$ (___), and $\Pi^{(m,\lambda)}_{N,b} f$ for $m = 4$ (___) and $m = 13$ (___), on the interval $[0, 0.5]$, with $N = 3$, $\lambda = 10$, and $b = \sqrt{5}$.

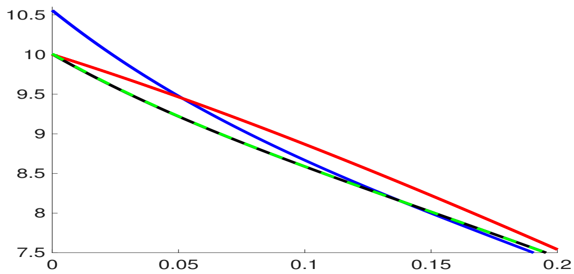

Figure 6. Plots of f (___), together with its approximations $\Xi_{N,b} f$ (___), and $\Pi^{(m,\lambda)}_{N,b} f$ for $m = 6$ (___) and $m = 15$ (___), on the interval $[0, 0.2]$, with $N = 5$, $\lambda = 10$, and $b = \sqrt{5}$.

The above examples show that parameter λ improves the approximation of $f(0)$. In particular, for sufficiently large values of λ, it is enough to take $m = N + 1$, and parameter b has only a minor influence on the approximation of $f(0)$. Moreover, the truncated LSTS provides a much better approximation of $f(0)$ than the truncated classical Laguerre series.

4. Application to differential equations: a spectral method

In this section, we illustrate how the results of the previous section can be applied to the numerical solution of differential equations with initial conditions at $x = 0$, using a spectral method based on the LSTPs.

Consider a differential equation whose solution is defined on $[0, \infty)$ with initial conditions at $x = 0$, and whose LT can be explicitly obtained in the form $F(s) = \mathcal{L}\{f(x)\}(s)$. If $F(s)$ satisfies the conditions of Theorem 3.1, then an approximate solution can be expressed in terms of the LSTPs, as given in (3.19), where the coefficients c^m_n are determined by (3.18). This approach can be regarded as a spectral method, since the sequence $\{S_n(2bx)\}_{n\geq 0}$ forms a system of orthogonal polynomials, as shown in Corollary 2.3.

Example 4.1. Consider the first-order linear differential equation with a piecewise forcing term:

$$f'(x) + f(x) = g(x), \quad f(0) = 5, \quad \text{where } \; g(x) = \begin{cases} 0, & \text{if } \; 0 \leq x < \pi, \\ 3\cos(x), & \text{if } \; \pi \leq x. \end{cases} \tag{4.1}$$

This equation admits several physical interpretations. For instance, it models a first-order electrical circuit, where $f(x)$ represents the capacitor charge and $g(x)$ is an external voltage applied in two stages: zero before $x = \pi$ and oscillatory afterward. Alternatively, it can describe a damped mechanical system subject to a piecewise external force, with $f(x)$ representing velocity, $f'(x)$ acceleration, and $g(x)$ the applied forcing.

Applying properties of the LT to (4.1), we obtain that the LT and associated auxiliary function are given by

$$F(s) = \frac{5(s^2+1) - 3se^{-\pi s}}{s^3 + s^2 + s + 1}, \quad \Phi(z) = \frac{2b}{1-z} F\left(\frac{1+z}{1-z} b\right).$$

Moreover, the ILT, which coincides with the exact solution of (4.1), is

$$f(x) = \begin{cases} 5e^{-x}, & \text{if } \; 0 \leq x < \pi, \\ 5e^{-x} + \frac{3}{2}\left(e^{\pi - x} + \sin(x) + \cos(x)\right), & \text{if } \; \pi \leq x. \end{cases}$$

The numerical solutions of the differential equation (4.1) through $\Xi_{N,b} f$ and $\Pi_{N,b}^{(m,\lambda)} f$, with $\lambda = 20$, $b = 2$, $N = 12$, and $m = N + 3$ are presented in the following figures.

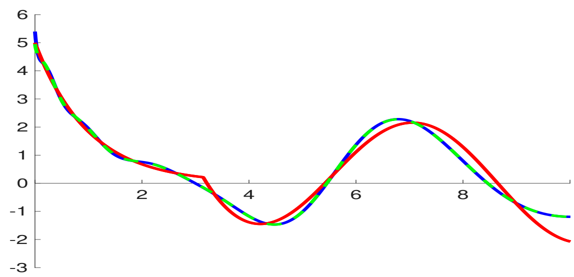

Figure 7. Plots of f (red), together with its approximations $\Xi_{N,b} f$ (blue) and $\Pi_{N,b}^{(m,\lambda)} f$ (green) on the interval $[0, 10]$, with $N = 12$, $m = N + 1$, $\lambda = 20$, and $b = 2$.

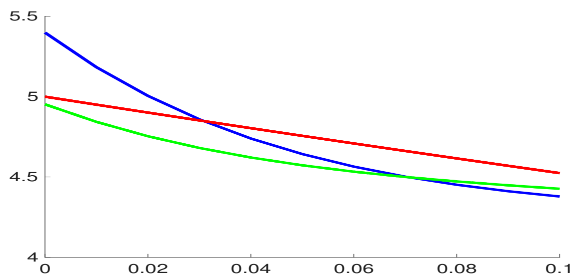

Figure 8. Plots of f (red), together with its approximations $\Xi_{N,b} f$ (blue) and $\Pi_{N,b}^{(m,\lambda)} f$ (green) on the interval $[0, 0.1]$, with $N = 12$, $m = N + 1$, $\lambda = 20$, and $b = 2$.

From Figures 7 and 8, it follows that both approximations are accurate on the interval $[0, 10]$. However, as expected, $\Pi_{N,b}^{(m,\lambda)} f$ provides a better approximation of the initial condition $f(0) = 5$. Furthermore, as N increases, the approximation on the interval $[0, 0.1]$ becomes more uniform, as illustrated in Table 6. Moreover, The results presented in Table 7 indicate that $b = 2$ is an optimal choice of parameter b. Nevertheless, other values of b also yield good convergence and accurate approximations, though at a slightly slower rate.

Table 6. MNE between f and $\Pi_{N,2}^{(N+1,20)} f$, for $b = 2$, and $\lambda = 20$ on interval $[0, 0.1]$.

N	$\left\Vert f - \Pi_{N,2}^{(N+1,20)} f \right\Vert_{[0,0.1]}$
10	1.8502×10^{0}
12	1.8432×10^{-1}
13	1.4366×10^{-1}
14	1.0102×10^{-1}
15	4.9387×10^{-2}

Table 7. MSE between f and $\Pi_{N,b}^{(N+1,1)} f$ for $b = \frac{1}{2}, 1, \frac{3}{2}, 2$, and $\frac{5}{2}$ with $\lambda = 1$.

N	$\left\Vert f - \Pi_{N,\frac{1}{2}}^{(N+1,1)} f \right\Vert_{\mathbf{L}^2}$	$\left\Vert f - \Pi_{N,1}^{(N+1,1)} f \right\Vert_{\mathbf{L}^2}$	$\left\Vert f - \Pi_{N,\frac{3}{2}}^{(N+1,1)} f \right\Vert_{\mathbf{L}^2}$	$\left\Vert f - \Pi_{N,2}^{(N+1,1)} f \right\Vert_{\mathbf{L}^2}$	$\left\Vert f - \Pi_{N,\frac{5}{2}}^{(N+1,1)} f \right\Vert_{\mathbf{L}^2}$
1	3.6195	3.5556	3.5494	3.5515	3.5831
3	3.4141	3.5089	3.4791	3.5304	3.5393
6	2.9271	2.3884	2.5559	3.3377	3.3498
9	2.4142	1.5598	1.1097	2.1912	2.2939
12	1.9540	1.0586	7.9163×10^{-1}	8.7695×10^{-1}	1.8486
15	1.4169	1.0068	6.4233×10^{-1}	4.2159×10^{-1}	6.7935×10^{-1}

Finally, we present an example illustrating how the proposed method can be applied to the numerical solution of partial differential equations with initial conditions.

Example 4.2. We consider the transport equation with decay in the domain $x \geq 0$ and $y \geq 0$,

$$f_x(x, y) + f_y(x, y) + f(x, y) = 0, \quad f(x, 0) = \sin(x), \quad f(0, y) = 0. \tag{4.2}$$

This equation models the propagation of a quantity along diagonal characteristic curves in the plane, with a decay term proportional to the solution. The exact solution is given by

$$f(x, y) = \begin{cases} 0, & \text{if} \quad x < y, \\ \sin(x - y)e^{-y}, & \text{if} \quad y \leq x. \end{cases}$$

Applying the LT to (4.2) with respect to variable x, and denoting $F(s, y) = \mathcal{L}\{f(x, y)\}$, we obtain

$$sF(s, y) - f(0, y) + F_y(s, y) + F_y(s, y) = 0.$$

Since $f(0, y) = 0$, it follows that $F(s, y)$ satisfies the differential equation $F_y(s, y) + (s + 1)F(s, y) = 0$, with initial condition $F(s, 0) = \mathcal{L}\{f(x, 0)\} = \frac{1}{s^2+1}$. Hence, the solution of this equation and the associated auxiliary function are given by

$$F(s,y)=\frac{e^{-sy}}{s^2+1}e^{-y},\quad \Phi(z,y)=\frac{2b}{1-z}F\left(\frac{1+z}{1-z}b,y\right)=\frac{2b(1-z)e^{-\left(\frac{1+z}{1-z}\right)by}}{(1+z)^2b^2+(1-z)^2}\,e^{-y}.$$

The approximate solution is given by

$$\Pi_{N,b}^{(m,\lambda)}f(x,y)=e^{-bx}\sum_{n=0}^{N}c_n^m(y)S_n(2bx),\ m\geq N\geq 0,\ x,y\geq 0, \tag{4.3}$$

where the coefficients $c_n^m(y)$ are given by

$$c_n^m(y)=\frac{1}{n!}\frac{\partial^n\Phi(0,y)}{\partial z^n}+\frac{\lambda}{1+\lambda(n+1)}\sum_{k=n+1}^{m}\frac{1}{k!}\frac{\partial^k\Phi(0,y)}{\partial z^k},\quad n=0,1,\ldots,N.$$

For instance, taking $N=2$, $m=3$, $\lambda=1$, and $b=2$, we obtain

$$\begin{aligned}
c_0^3&=\frac{2}{1875}\left(-4000y^3+2400y^2-360y+627\right)e^{-3y},\\
c_1^3&=-\frac{4}{5625}\left(4000y^3-2400y^2+3360y+1773\right)e^{-3y},\\
c_2^3&=\frac{1}{1875}\left(-4000y^3+11400y^2+6540y+2547\right)e^{-3y}.
\end{aligned}$$

Consequently, the corresponding approximation is given by

$$\begin{aligned}
\Pi_{2,2}^{(3,1)}f(x,y)=\frac{1}{3750}\Big(&-4000x^2y^3+11400x^2y^2+6540x^2y+2547x^2+24000xy^3-44400xy^2\\
&-12840xy-3762x-24000y^3+20400y^2-1560y+1842\Big)e^{-(2x+3y)}.
\end{aligned}$$

On the other hand, Figures 9 and 10 display the approximations $\Xi_{N,b}f(x,\pi)$ and $\Pi_{N,b}^{(N+1,\lambda)}f(x,\pi)$ for $y=\pi$ with $N=15$. From these plots, it follows that both approximations are accurate and show good agreement with the exact solution. However, as expected, the approximation associated with the LSTPs is more accurate than $\Xi_{N,b}f(x,\pi)$ near the origin.

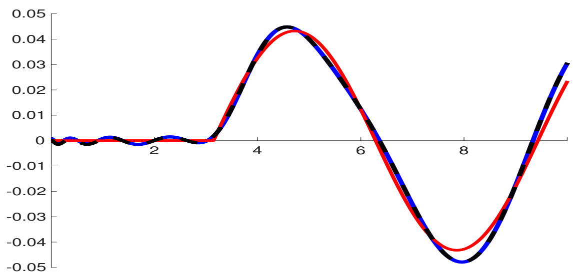

Figure 9. Plots of $f(x,\pi)$ (__), together with its approximations $\Xi_{N,b}f(x,\pi)$ (__) and $\Pi_{N,b}^{(m,\lambda)}f(x,\pi)$ (__) on the interval $[0,10]$, with $N=15$, $m=N+1$, $\lambda=1$, and $b=\frac{3}{2}$.

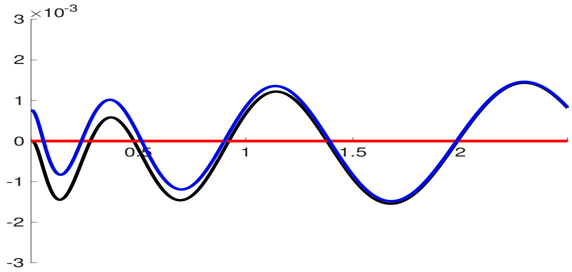

Figure 10. Plots of $f(x,\pi)$ (___), together with its approximations $\Xi_{N,b}f(x,\pi)$ (___) and $\Pi_{N,b}^{(m,\lambda)}f(x,\pi)$ (___) on the interval $[0, 2.5]$, with $N = 15$, $m = N + 1$, $\lambda = 1$, and $b = \frac{3}{2}$.

5. Conclusions

We derived the LT of the LSTPs and established their connection with the CLPs, together with several structural relations. In addition, Theorem 3.1 ensures that the ILT of an analytic function $F(s)$, defined in a suitable region of the complex plane, admits an approximation by LSTS that converges absolutely and uniformly on compact subsets of $\mathbb{R}^+$. Moreover, Proposition 3.3 guarantees that the truncated LSTS converges to $f(0)$ when λ is sufficiently large, which ensures an accurate approximation of the initial conditions associated with differential equations.

The numerical results presented in Examples 3.1, 3.2, and 4.1 indicate that the approximations using truncated Laguerre series and truncated LSTS are essentially the same, except near the boundary, where truncated LSTS provides a significantly better approximation. As a consequence, the parameter λ improves the approximation of the ILT at $f(0)$ without increasing the degree N of the approximation (3.12). In contrast, the MSE and the convergence rate are strongly influenced by parameter b, agreeing with the theoretical error estimates established in Propositions 3.4 and 3.5. A brief discussion on the optimal choice of b can be found in [21, Section 3].

Furthermore, a spectral method based on LSTPs was developed and successfully applied to the numerical solution of an ordinary and a partial differential equation. The results obtained in Examples 4.1 and 4.2 exhibit rapid convergence and high accuracy, demonstrating the robustness and effectiveness of the proposed approach. These findings highlight the potential of LSTPs as an efficient tool for approximating the ILT, and consequently for approximating solutions of differential equations with initial conditions.

Author contributions

Edinson Fuentes developed the theoretical results, wrote the Introduction and the Conclusions, and prepared the manuscript. Lida E. Riscanevo supervised the project, verified the analytical proofs, and reviewed and edited the manuscript for English language and style. Manuel A. Silva developed the numerical simulations, supervised the project, and verified the analytical proofs. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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