Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 11(5): 14474–14486.
DOI: 10.3934/math.2026593
Received: 24 March 2026
Revised: 06 May 2026
Accepted: 11 May 2026
Published: 22 May 2026

Research article

The uniqueness of solutions for a singular Kirchhoff equation with the Kohn–Laplace operator

Yu-Cheng An*

School of Sciences, Guizhou University of Engineering Science, Bijie 551700, China

* **Correspondence:** Email: anyucheng@126.com.

Abstract: In this paper, we use the minimax method and certain analysis techniques to establish the uniqueness of solutions for a class of (strongly) singular Kirchhoff equations with the Kohn–Laplace operator in the n-Heisenberg group and further deduce that the solution is cylindrically symmetric under some necessary structural conditions for a Kohn–Laplace equation with singularity.

Keywords: Heisenberg group; Kohn–Laplacian; singular nonlinearity; uniqueness of solutions
Mathematics Subject Classification: 35B09, 35B32

1. Introduction

In this paper, we are interested in the uniqueness of solutions for the following Kirchhoff equation with singularity:

$$\begin{cases} -\left(a + b \int_\Omega |\nabla_H u|^2 d\xi\right) \Delta_H u = g(\xi)u^{-\gamma} - \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{H}^n (n \geq 1)$ is a smooth bounded domain; $\lambda \geq 0$; $\gamma > 0$; $a, b \geq 0$; $a + b > 0$; and

$$0 < p \leq 2_Q^* - 1 = 1 + 2/n;$$

g is a positive function in $L^1(\Omega)$, and Δ_H and ∇_H are the Kohn–Laplace operator and the horizontal gradient in the n-Heisenberg group (for whose details see Section 3), respectively. Problem (1.1) is a classic Kirchhoff equation with the Kohn–Laplace operator, and this type of equation has its own physical origin (see [1–3]). In addition, Kirchhoff equations are often referred to as being nonlocal because of the presence of the term $\left(\int_\Omega |\nabla_H u|^2 d\xi\right) \Delta_H u$, which means that problem (1.1) is no longer a pointwise identity. Moreover, it is obvious that if $a = 1$ and $b = 0$, problem (1.1) reduces to the following Kohn–Laplace equation with singularity:

$$\begin{cases} -\Delta_H u = g(\xi)u^{-\gamma} - \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

It is well-known that Fulks–Maybee [4] initiated the study of singular elliptic problems, which later attracted widespread interest following Crandall–Rabinowitz–Tartar's work [5]. For instance, in the Euclidean case with $\gamma \in (0,1)$, bifurcation-theoretic results for nonlinear singular equations were obtained in [6–8]; two positive solutions were established for singular elliptic problems in [9–11], and these results were later generalized to the Heisenberg (or Carnot) group [12, 13]. Furthermore, the authors of [14–16] investigated the existence and properties of solutions for nonlinear nonlocal elliptic problems. In addition, we would like to mention that Liao et al. in [17] studied the Kirchhoff Eq (1.1) in the Euclidean setting and proved the uniqueness of solutions for $\gamma \in (0,1)$. Moreover, using approximation methods and Schauder's fixed-point theorem, the uniqueness of solutions to problem (1.2) with $\lambda = 0$ was probed in [18], and it was further shown that the solution is cylindrically symmetric under some appropriate structural conditions on Ω and $g(\xi)$. Therefore, a natural question arising from [17, 18] is whether their results extend to the case $\gamma \in [1, +\infty)$, $\lambda \neq 0$ and the Heisenberg group. In this paper, we provide an affirmative answer to this question.

2. Main results

In this section, we will present our main results. First, we say that a function u is a solution of problem (1.1) if $u \in S_0^1(\Omega)$ such that $u > 0$ almost every $x \in \Omega$, and $g(\xi)u^{-\gamma}\varphi \in L^1(\Omega)$,

$$\left(a + b\int_\Omega |\nabla_H u|^2 d\xi\right)\int_\Omega \nabla_H u \nabla_H \varphi d\xi + \lambda \int_\Omega u^p \varphi d\xi = \int_\Omega g(\xi)u^{-\gamma}\varphi d\xi \tag{2.1}$$

for all $\varphi \in S_0^1(\Omega)$, where $S_0^1(\Omega)$ is the Folland–Stein Sobolev space (see Section 3). Now, let

$$D = \left\{u \in S_0^1(\Omega) : g(\xi)G(|u|) \in L^1(\Omega)\right\},$$

where the function G is defined as follows:

- If $0 < \gamma < 1$,

$$G(s) = \frac{|s|^{1-\gamma}}{1-\gamma}, \quad \forall s \in \mathbb{R}.$$

- If $\gamma = 1$,

$$G(s) = \begin{cases} \log s, & \text{if } s > 0, \\ +\infty, & \text{if } s = 0. \end{cases}$$

- If $\gamma > 1$,

$$G(s) = \begin{cases} \frac{s^{1-\gamma}}{1-\gamma}, & \text{if } s > 0, \\ +\infty, & \text{if } s = 0. \end{cases}$$

The main results of this paper are as follows:

Theorem 2.1. *Let $\Omega \subset \mathbb{H}^n (n \geq 1)$ be a smooth bounded domain; $a, b \geq 0$; $a + b > 0$; $\lambda \geq 0$; and $0 < p \leq 1 + 2/n$; $\gamma \geq 1$. Assume that $D \neq \emptyset$ and $g > 0$, with $g \in L^\infty(\Omega)$ if $\gamma = 1$ and $g \in L^1(\Omega)$ if $\gamma > 1$. Then, problem* (1.1) *possesses a unique solution. Moreover, the solution is a global minimizer solution.*

Theorem 2.2. *Let $\Omega \subset \mathbb{H}^n (n \geq 1)$ be a smooth bounded domain; $a, b \geq 0$; $a + b > 0$; $\lambda \geq 0$; and $0 < p \leq 1 + \frac{2}{n}$; $0 < \gamma < 1$. Assume that g is a positive function in $L^\infty(\Omega)$. Then, problem* (1.1) *possesses a unique solution. Moreover, the solution is a global minimizer solution.*

By the uniqueness of solutions, we will directly deduce a symmetry result for problem (1.2). Next, we say that a function $u(z, t)$ is cylindrically symmetric if there exists $\xi_0 \in \mathbb{H}^n$ such that $v(\xi) = u(\xi_0 \circ \xi)$ is a two-variable function, that is, $v(z, t) = v(|z|, t)$. Without loss of generality, this paper always assumes that ξ_0 occurring in the definition is 0. In addition, we say that a domain Ω of $\mathbb{H}^n$ is a cylinder if there exists a cylindrical function Φ such that $\xi \in \Omega$ if and only if $\Phi(\xi) < 0$. As an example, the Heisenberg ball $B_r(\xi_0)$ is a cylinder (see [19]).

Corollary 2.3. *Let Ω be a bounded cylinder defined by a cylindrical function Φ, and let $0 < p \leq 1+2/n$. Assume that u_* is a solution of problem* (1.2). *Then, u_* is cylindrically symmetric, provided one of the following assumptions holds:*

(i) $\gamma \geq 1$, $D \neq \emptyset$, and $g > 0$, with $g \in L^\infty(\Omega)$ if $\gamma = 1$ and $g \in L^1(\Omega)$ if $\gamma > 1$;

(ii) $0 < \gamma < 1$, and g is a positive function in $L^\infty(\Omega)$.

Further, if

$$g(|z|, t) = g(|z|, -t) \quad \textit{and} \quad \Phi(|z|, t) = \Phi(|z|, -t),$$

then $u_*(|z|, t) = u_*(|z|, -t)$ for any $\xi = (z, t) \in \Omega$.

Remark 2.4. *It seems to be the same as in the Euclidean setting: When $\gamma \geq 1$ and $g(\xi) \in L^1(\Omega)$, it may happen that $D = \emptyset$ (see [20, Theorem 2]). Moreover, it is also easy to verify that $D = S^1_0(\Omega)$ when $\gamma \in (0, 1)$ and $g \in L^\infty(\Omega)$. Therefore, in Theorem 2.1, we assume that $D \neq \emptyset$ is meaningful, and in Theorem 2.2, this assumption is not necessary.*

Remark 2.5. *To the best of our knowledge, Theorems 2.1 and 2.2 represent a novel extension of singular elliptic problems to a non-Euclidean framework. As previously noted, Liao et al. [17] only established solution uniqueness in the Euclidean case for $\gamma \in (0, 1)$, whereas our work extends this result to cover the regimes where $\gamma \geq 1$.*

Remark 2.6. *It is a challenging problem to study the symmetry of solutions to Kohn–Laplace equations because the classical moving plane method does not apply directly to the Kohn–Laplace operator. In addition, Birindell and Prajapat in [19] established a partial symmetry result of solutions for the equation $-\Delta_H u = f(u)$ in a bounded cylindrical domain of $\mathbb{H}^n$, assuming f to be Lipschitz and u to be a cylindrical solution, respectively. However, we want to emphasize that Corollary 2.3 has not any condition on u_*, as we are able to obtain the uniqueness of solutions for problem* (1.2). *Further, Corollary 2.3 extends the result of [18] to the case $\lambda \neq 0$.*

3. Preliminaries

The nth Heisenberg group $\mathbb{H}^n$, whose points are denoted by $\xi = (z, t) = (x, y, t)$, is the Lie group $(\mathbb{R}^{2n+1}, \circ)$ with composition law defined by

$$\xi \circ \xi' = (x + x', y + y', t + t' + 2(x' \cdot y - y' \cdot x)).$$

Let $X_i = \partial_{x_i} + 2y_i\partial_t$ and $Y_i = \partial_{y_i} - 2x_i\partial_t$ $(i = 1, 2, \cdots, n)$. The Kohn–Laplace operator and the horizontal gradient are defined as

$$\Delta_H = \sum_{i=1}^{n}(X_i^2 + Y_i^2) \text{ and } \nabla_H = (X_1, \cdots, X_n, Y_1, \cdots, Y_n),$$

respectively. The Kohn–Laplace operator Δ_H is very degenerate, but it satisfies the Hörmander rank condition (see [2]), which means that it is hypoelliptic, and Bony's maximum principle is satisfied (see [21]). In addition, the Folland–Stein Sobolev space $S_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| = \sqrt{\int_\Omega |\nabla_H u|^2 d\xi}.$$

It follows from the celebrated work of Folland–Stein [22] that $(S_0^1(\Omega), \|\cdot\|)$ is a Hilbert space. Let $\Omega \subset \mathbb{H}^n (n \geq 1)$ be a smooth bounded domain,

$$Q = 2n + 2 \quad \text{and} \quad 2_Q^* = \frac{2Q}{Q-2} = 2 + \frac{2}{n}.$$

Then, the following holds:

$$\begin{cases} S_0^1(\Omega) \hookrightarrow L^s(\Omega) \text{ is compact when } 1 \leq s < 2_Q^*; \\ S_0^1(\Omega) \hookrightarrow L^s(\Omega) \text{ is only continuous when } s = 2_Q^*. \end{cases}$$

Hence, there exists a constant $S_s > 0$ such that

$$\|u\|_s \leq S_s \|u\|, \quad \forall u \in S_0^1(\Omega), \tag{3.1}$$

where $\|u\|_s$ denote the norm of $L^s(\Omega)$. In addition, let us define the energy functional J: $S_0^1(\Omega) \to \mathbb{R}$ corresponding to problem (1.1) by

$$J(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{p+1}\int_\Omega |u|^{p+1} d\xi - \int_\Omega g(\xi)G(|u|)d\xi. \tag{3.2}$$

Obviously, we have $J(u) = J(|u|)$. Let

$$D^+ = \{u \in D : u \geq 0 \text{ almost every } \xi \in \Omega\}.$$

Hence, if J is bounded from below on $S_0^1(\Omega)$, then

$$\inf_{v \in D} J(v) = \inf_{v \in D^+} J(v). \tag{3.3}$$

Beside, for every $\gamma > 0$ and $p > 0$, it is easy to obtain the following elementary inequalities:

$$(s^{-\gamma} - t^{-\gamma})(s - t) \leq 0, \quad (s^p - t^p)(s - t) \geq 0, \quad \forall s, t > 0. \tag{3.4}$$

4. Proof of main results

This section will be divided into three subsections. In Subsection 4.1, we prove the existence of a global minimum for the functional (3.2). In Subsections 4.2 and 4.3, we prove Theorems 2.1 and 2.2 and Corollary 2.3, respectively. In addition, we will use c_i and $C_i (i = 1, 2, \cdots)$ to denote some positive constants, which are possibly different from line to line.

4.1. Existence of a global minimum

In this subsection, we prove the existence of a global minimum of the functional (3.2). That is to say, we have the following lemma.

Lemma 4.1. *Under the assumption of Theorem 2.1 or 2.2, there exists $u_* \in D_+$ such that*

$$J(u_*) = m = \inf_{v\in D_+} J_\lambda(v) = \inf_{v\in D} J(v).$$

Proof. By the range of the parameter γ, we have three cases to consider.

Case 1. $0 < \gamma < 1$. In this case, it follows from (3.1) and (3.2) and the definition of G that

$$\begin{aligned} J(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}d\xi - \frac{1}{1-\gamma}\int_\Omega g(\xi)|u|^{1-\gamma}d\xi \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{1-\gamma}\|g\|_\infty \int_\Omega |u|^{1-\gamma}d\xi \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - C_1\|u\|^{1-\gamma}. \end{aligned} \tag{4.1}$$

By Remark 2.4 and Eq (4.1), we know that the functional J is coercive and bounded from below on $D = S^1_0(\Omega)$. Hence, by (3.3), we may choose a sequence $\{u_n\} \subset D^+$ such that

$$J(u_n) \to \inf_{v\in D^+} J(v) = m \tag{4.2}$$

as $n \to \infty$, which means that there exists $0 \leq u_* \in S^1_0(\Omega)$ such that

$$\begin{cases} u_n \rightharpoonup u_*, & \text{in } S^1_0(\Omega), \\ u_n \to u_*, & \text{in } L^s(\Omega), \quad s \in [1, 2^*_Q), \\ u_n(\xi) \to u_*(\xi), & \text{a.e. in } \Omega, \end{cases} \tag{4.3}$$

as $n \to \infty$. By the Vitali's convergence theorem (see [23]) and Eq (4.3), it is easy to get

$$\lim_{n\to\infty} \int_\Omega g(\xi)|u_n|^{1-\gamma}\mathrm{d}\xi = \int_\Omega g(\xi)|u_*|^{1-\gamma}\mathrm{d}\xi. \tag{4.4}$$

On the other hand, it follows from the Brézis–Lieb lemma (see [24]) that

$$\|u_n\|^2 = \|w_n\|^2 + \|u_*\|^2 + o(1) \geq \|u_*\|^2 + o(1), \tag{4.5}$$

$$\|u_n\|^4 = \|w_n\|^4 + \|u_*\|^4 + 2\|w_n\|^2\|u_*\|^2 + o(1) \geq \|u_*\|^4 + o(1), \tag{4.6}$$

$$\|u_n\|_{2_Q^*}^{2_Q^*} = \|w_n\|_{2_Q^*}^{2_Q^*} + \|u_*\|_{2_Q^*}^{2_Q^*} + o(1) \geq \|u_*\|_{2_Q^*}^{2_Q^*} + o(1), \tag{4.7}$$

where $w_n = u_n - u_*$ and $\lim_{n\to\infty} o(1) = 0$. Then, it follows from (4.3)–(4.6) that for $0 < p < 2_Q^*$,

$$\begin{aligned} m &= \lim_{n\to\infty} J(u_n) \\ &= \lim_{n\to\infty}\left(\frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 + \frac{\lambda}{p+1}\int_\Omega |u_n|^{p+1}d\xi - \frac{1}{1-\gamma}\int_\Omega g(\xi)|u_n|^{1-\gamma}d\xi\right) \\ &\geq \frac{a}{2}\|u_*\|^2 + \frac{b}{4}\|u_*\|^4 + \frac{\lambda}{p+1}\int_\Omega |u_*|^{p+1}d\xi - \frac{1}{1-\gamma}\int_\Omega g(\xi)|u_*|^{1-\gamma}d\xi \\ &= J(u_*) \geq m. \end{aligned} \tag{4.8}$$

When $p = 2_Q^*$, it follows from (4.4)–(4.7) that

$$\begin{aligned} m &= \lim_{n\to\infty} J(u_n) \\ &\geq \frac{a}{2}\|u_*\|^2 + \frac{b}{4}\|u_*\|^4 + \frac{\lambda}{2_Q^*}\int_\Omega |u_*|^{2_Q^*}d\xi - \frac{1}{1-\gamma}\int_\Omega g(\xi)|u_*|^{1-\gamma}d\xi \\ &= J(u_*) \geq m. \end{aligned} \tag{4.9}$$

It follows from (4.8) and (4.9) that the statement of Lemma 4.1 is true if $\gamma \in (0, 1)$.

Case 2. $\gamma = 1$. In this case, because $a(x) \in L^\infty(\Omega)$ and $\log s \leq s$ for all $s > 0$, it follows from Eqs (3.1) and (3.2) and the definition of G that

$$\begin{aligned} J(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}d\xi - \int_\Omega g(\xi)\log|u|d\xi \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \|g\|_\infty \int_\Omega |u|d\xi \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - C_2\|u\| \end{aligned} \tag{4.10}$$

for every $u \in D^+$, which implies that the functional J is coercive in D^+. Therefore, we can choose a sequence $\{u_n\} \subset D^+$ such that (4.2) holds. Meanwhile, as in the proof of Case 1, we may assume that there exists $0 \leq u_* \in S_0^1(\Omega)$ such that (4.3) holds. It follows from $\inf_{v\in D^+} J(v) = m$ that the sequence

$$\left\{\int_\Omega g(\xi)G(|u_n|)d\xi\right\} = \left\{\int_\Omega g(\xi)\log|u_n|d\xi\right\}$$

is bounded, which implies that

$$\limsup_{n\to\infty}\int_\Omega g(\xi)\log|u_n|d\xi < \infty.$$

It follows from the Fatou's lemma (see [23]) and $\log|u_n| \leq |u_n|$ for all n that

$$-\infty < \limsup_{n\to\infty}\int_\Omega g(\xi)\log|u_n|d\xi \leq \int_\Omega \limsup_{n\to\infty} g(\xi)\log|u_n|d\xi = \int_\Omega g(\xi)\log|u_*|d\xi < +\infty,$$

which proves that $g(\xi)\log|u_*| \in L^1(\Omega)$. That is, $u_* \in D^+$. Moreover, when $0 < p \le 2^*_Q$, it follows from (4.3)–(4.7) that

$$\begin{aligned} m &\ge \liminf_{n\to\infty} J(u_n) \\ &\ge \frac{a}{2}\|u_*\|^2 + \frac{b}{4}\|u_*\|^4 + \frac{\lambda}{p+1}\int_\Omega |u_*|^{p+1}d\xi - \limsup_{n\to\infty}\int_\Omega g(\xi)\log|u_n|d\xi \\ &\ge \frac{a}{2}\|u_*\|^2 + \frac{b}{4}\|u_*\|^4 + \frac{\lambda}{p+1}\int_\Omega |u_*|^{p+1}d\xi - \int_\Omega g(\xi)\log|u_*|d\xi \\ &\ge \inf_{v\in D^+} J(v). \end{aligned} \tag{4.11}$$

Hence, it follows from (4.11) that the statement of Lemma 4.1 is true if $\gamma = 1$.

Case 3. $\gamma > 1$. In this case, it follows from (3.2) and the definition of G that

$$\begin{aligned} J(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{\lambda}{p+1}\int_\Omega |u|^{p+1}d\xi + \frac{1}{\gamma-1}\int_\Omega g(\xi)|u|^{1-\gamma}d\xi \\ &\ge \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4, \end{aligned} \tag{4.12}$$

for all $u \in D^+$, which implies that J is coercive in D^+. Therefore, we can also choose a sequence $\{u_n\} \subset D^+$ such that (4.2) holds. Meanwhile, as in the proof of Case 1, we may assume that there is $0 \le u_* \in S^1_0(\Omega)$ such that (4.3) holds. Similarly, it follows from (4.3)–(4.7) and Fatou's lemma (see [23]) that

$$\begin{aligned} m &\ge \liminf_{n\to\infty} J(u_n) \\ &\ge \frac{a}{2}\|u_*\|^2 + \frac{b}{4}\|u_*\|^4 + \frac{\lambda}{p+1}\int_\Omega |u_*|^{p+1}d\xi + \frac{1}{\gamma-1}\int_\Omega g(\xi)|u_*|^{1-\gamma}d\xi \\ &\ge \inf_{v\in D^+} J(v), \end{aligned} \tag{4.13}$$

which implies that $u_* \in D^+$ and

$$J(u_*) = \inf_{v\in D^+} J(v)$$

in the case $\gamma > 1$.

All in all, for any $\gamma > 0$, we may conclude that the statement of Lemma 4.1 is true. This completes the proof of Lemma 4.1. □

4.2. Proof of Theorems 2.1 and 2.2

In this subsection, we shall give the proof of Theorems 2.1 and 2.2. First, let u_* be obtained in Lemma 4.1. In the following, we prove that u_* is the unique solution of problem (1.1) under the conditions of Theorems 2.1 and 2.2, respectively.

Proof of Theorem 2.1. First, we prove that u_* is a solution of problem (1.1). Now, let $\phi \in S^1_0(\Omega)$ with $\phi \ge 0$ in Ω. Then, for any $\varepsilon > 0$, we claim that $u_* + \varepsilon\phi \in D^+$.

In fact, when $\gamma = 1$, we have

$$G(s) = \log s \quad \text{and} \quad \log s < s$$

for all $s > 0$. Hence,

$$\begin{aligned} -\infty &< \int_\Omega g(\xi)\log u_* d\xi \le \int_\Omega g(\xi)\log(u_* + \varepsilon\phi)d\xi \\ &\le \int_\Omega g(\xi)(u_* + \varepsilon\phi)d\xi = \int_\Omega g(\xi)u_* d\xi + \varepsilon\int_\Omega g(\xi)\phi d\xi \\ &\le \int_\Omega g(\xi)u_* d\xi + \varepsilon\|g\|_\infty \int_\Omega \phi d\xi < +\infty, \end{aligned}$$

which means that $u_* + \varepsilon\phi \in D^+$.

In addition, when $\gamma > 1$, we have $u_* + \varepsilon\phi \ge u_* \ge 0$ and $(u_* + \varepsilon\phi)^{1-\gamma} \le u_*^{1-\gamma}$. Hence,

$$\int_\Omega g(\xi)|u_* + \varepsilon\phi|^{1-\gamma} d\xi \le \int_\Omega g(\xi)|u_*|^{1-\gamma} d\xi < +\infty,$$

that is to say, $u_* + \varepsilon\phi \in D^+$. Therefore, for any $\gamma \ge 1$, we have

$$\begin{aligned} \int_\Omega g(\xi)\left(G(u_* + \varepsilon\phi) - G(u_*)\right)d\xi \le{}& \frac{a}{2}\|u_* + \varepsilon\phi\|^2 - \frac{a}{2}\|u_*\|^2 + \frac{b}{4}\|u_* + \varepsilon\phi\|^4 - \frac{b}{4}\|u_*\|^4 \\ &+ \frac{\lambda}{p+1}\int_\Omega |u_* + \varepsilon\phi|^{p+1} d\xi - \frac{\lambda}{p+1}\int_\Omega |u_*|^{p+1} d\xi. \end{aligned} \tag{4.14}$$

Letting $\varepsilon \to 0$, it follows from Fatou's Lemma (see [23]) and (4.14) that

$$\int_\Omega g(\xi)u_*^{-\gamma}\phi d\xi \le \left(a + \|u_*\|^2\right)\int_\Omega \nabla_H u_* \nabla_H \phi d\xi + \lambda\int_\Omega u_*^p \phi d\xi \tag{4.15}$$

for every $\phi \in H_0^1(\Omega)$, with $\phi \ge 0$.

Next, let $\psi(t) = J(tu_*)$. Then, the function $\psi \in C^1((0,\infty),\mathbb{R})$ holds, noting that $tu_* \in D^+$ for every $t > 0$. Therefore, it follows from Lemma 4.1 that $\psi(t)$ has a global minimum at $t = 1$, which means that

$$0 = \psi'(1) = a\|u_*\|^2 + b\|u_*\|^4 + \lambda\int_\Omega u_*^{p+1} d\xi - \int_\Omega g(\xi)|u_*|^{1-\gamma} d\xi. \tag{4.16}$$

Now, for all $\phi \in S_0^1(\Omega)$, let

$$\Psi(x) = (u_*(x) + \varepsilon\phi(x))^+, \quad \Omega_1 = \{x \in \Omega : u_* + \varepsilon\phi \le 0\}.$$

It follows from (4.15) and (4.16) that

$$\begin{aligned} 0 &\le \left(a + b\|u_*\|^2\right)\int_\Omega \nabla_H u_* \nabla_H \Psi d\xi + \lambda\int_\Omega u_*^p \Psi d\xi - \int_\Omega g(\xi)u_*^{-\gamma}\Psi d\xi \\ &= \varepsilon\left\{\left(a + b\|u_*\|^2\right)\int_\Omega \nabla_H u_* \nabla_H \phi d\xi + \lambda\int_\Omega u_*^p \phi d\xi - \int_\Omega g(\xi)u_*^{-\gamma}\phi d\xi\right\} \\ &- \left(a + b\|u_*\|^2\right)\int_{\Omega_1} \nabla_H u_* \nabla_H (u_* + \varepsilon\phi) d\xi - \lambda\int_{\Omega_1} u_*^p (u_* + \varepsilon\phi) d\xi + \int_{\Omega_1} g(\xi)u_*^{-\gamma}(u_* + \varepsilon\phi) d\xi \\ &\le \varepsilon\left\{\left(a + b\|u_*\|^2\right)\int_\Omega \nabla_H u_* \nabla_H \phi d\xi + \lambda\int_\Omega u_*^p \phi d\xi - \int_\Omega g(\xi)u_*^{-\gamma}\phi d\xi\right\} \end{aligned}$$

$$- \varepsilon \left\{ \left(a + b\|u_*\|^2\right) \int_{\Omega_1} \nabla_H u_* \nabla_H \phi d\xi + \lambda \int_{\Omega_1} u_*^p \phi d\xi \right\}. \tag{4.17}$$

Noting that meas$(\Omega_1) \to 0$ as $\varepsilon \to 0^+$, dividing by ε and letting $\varepsilon \to 0^+$ in (4.17), we get

$$0 \leq \left(a + b\|u_*\|^2\right) \int_\Omega \nabla_H u_* \nabla_H \phi d\xi + \lambda \int_\Omega u_*^p \phi d\xi - \int_\Omega g(\xi) u_*^{-\gamma} \phi d\xi$$

for every $\phi \in S_0^1(\Omega)$. Hence, this inequality also holds equally well for $-\phi$. Thus, we have

$$\left(a + b\|u_*\|^2\right) \int_\Omega \nabla_H u_* \nabla_H \phi d\xi + \lambda \int_\Omega u_*^p \phi d\xi - \int_\Omega g(\xi) u_*^{-\gamma} \phi d\xi = 0 \tag{4.18}$$

for every $\phi \in S_0^1(\Omega)$. In addition, let $\phi_1 \in S_0^1(\Omega)$ be the first eigenfunction of the Kohn–Laplace operator $-\Delta_H$ with $\phi_1 > 0$ and $\|\phi_1\| = 1$. Then, taking $\phi = \phi_1$ in (4.2), one has

$$\int_\Omega g(\xi) u_*^{-\gamma} \phi_1 d\xi \leq \left(a + \|u_*\|^2\right) \int_\Omega \nabla_H u_* \nabla_H \phi_1 d\xi + \lambda \int_\Omega u_*^p \phi_1 d\xi < \infty, \tag{4.19}$$

which implies that $u_* > 0$ almost everywhere in Ω. If not, we have

$$\int_\Omega g(\xi) u_*^{-\gamma} \phi_1 d\xi = \infty,$$

which is in contradiction with (4.19). Hence, by (4.18), we know that u_* is a solution of (1.1).

Next, we prove that the solution of problem (1.1) is unique. In fact, assume that u_* and v_* are solutions of problem (1.1). Then, we have

$$\int_\Omega \left\{ \left(a + b\|u_*\|^2\right) \nabla_H u_* \nabla_H (u_* - v_*) + \lambda u_*^p (u_* - v_*) - g(\xi) u_*^{-\gamma} (u_* - v_*) \right\} d\xi = 0, \tag{4.20}$$

$$\int_\Omega \left\{ \left(a + b\|v_*\|^2\right) \nabla_H v_* \nabla_H (u_* - v_*) + \lambda v_*^p (u_* - v_*) - g(\xi) v_*^{-\gamma} (u_* - v_*) \right\} d\xi = 0. \tag{4.21}$$

Meanwhile, let

$$\Phi(u_*, v_*) = \|u_*\|^4 + \|v_*\|^4 - \|u_*\|^2 \int_\Omega \nabla_H u_* \nabla_H v_* d\xi - \|v_*\|^2 \int_\Omega \nabla_H u_* \nabla_H v_* d\xi.$$

Then, it follows from (4.20) and (4.21) that

$$a\|u_* - v_*\|^2 + b\Phi(u_*, v_*) + \lambda \int_\Omega (u_*^p - v_*^p)(u_* - v_*) d\xi - \int_\Omega g(\xi)(u_*^{-\gamma} - v_*^{-\gamma})(u_* - v_*) d\xi = 0. \tag{4.22}$$

By Hölder's inequality, we have

$$\begin{aligned} \Phi(u_*, v_*) &\geq \|u_*\|^4 - \|u_*\|^3 \|v_*\| - \|v_*\| \|u_*\| + \|v_*\|^4 \\ &= (\|u_*\| - \|v_*\|)^2 (\|u_*\|^2 + \|u_*\| \|v_*\| + \|v_*\|^2) \\ &\geq 0. \end{aligned} \tag{4.23}$$

It follows from (3.4), (4.8), and (4.9) that $u_* = v_*$. In fact, by (3.4), one has

$$\int_\Omega (u_*^p - v_*^p)(u_* - v_*)d\xi \geq 0 \text{ and } \int_\Omega g(\xi)(u_*^{-\gamma} - v_*^{-\gamma})(u_* - v_*)d\xi \leq 0. \tag{4.24}$$

Hence, if $a > 0$, from (4.22)–(4.24), we have

$$\|u_* - v_*\|^2 = 0. \tag{4.25}$$

If $a = 0$, from (4.22)–(4.24), we have $\Phi(u_*, v_*) = 0$. Therefore, it follows from (4.23) that $\|u_*\| = \|v_*\|$, which implies that

$$\begin{aligned}\Phi(u_*, v_*) &= 2\|u_*\|^4 - 2\|u_*\|^2 \int_\Omega \nabla_H u_* \nabla_H v_* d\xi \\ &= \|u_*\|^2 \left(2\|u_*\|^2 - 2\int_\Omega \nabla_H u_* \nabla_H v_* d\xi\right) \\ &= \|u_*\|^2 \left(\|u_*\|^2 - 2\int_\Omega \nabla_H u_* \nabla_H v_* d\xi + \|v_*\|^2\right) \\ &= \|u_*\|^2 \|u_* - v_*\|^2. \end{aligned} \tag{4.26}$$

Recalling $\Phi(u_*, v_*) = 0$, it follows from (4.26) that (4.25) holds for $a = 0$. Therefore, in both cases, it is easy to get $u_* = v_*$. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. As in the proof of Theorem 2.1, let u_* be as in the Lemma 4.1, and prove that u_* is a solution of problem (1.1) under the conditions of Theorem 2.2. To this aim, we first consider $\phi \in S_0^1(\Omega)$ such that $\phi \geq 0$ in Ω and $\varepsilon > 0$. Recalling $D = S_0^1(\Omega)$ for $0 < \gamma < 1$ by Remark 2.4, we then have

$$D^+ = \left\{u \in S_0^1(\Omega) : u \geq 0 \text{ a.e. in } \Omega\right\},$$

which implies that $u_* + \varepsilon\phi \in D^+$ and $J(u_* + \varepsilon\phi) \geq J(u_*)$. Therefore, for $0 < \gamma < 1$, we have

$$\begin{aligned}&\frac{a}{2}\left(\|u_* + \varepsilon\phi\|^2 - \|u_*\|^2\right) + \frac{b}{4}\left(\|u_* + \varepsilon\phi\|^4 - \|u_*\|^4\right) + \lambda \int_\Omega \left((u_* + \varepsilon\phi)^p - u_*^p\right) d\xi \\ &\geq \frac{1}{1-\gamma} \int_\Omega g(\xi) \left((u_* + \varepsilon\phi)^{1-\gamma} - u_*^{1-\gamma}\right) d\xi. \end{aligned} \tag{4.27}$$

Letting $\varepsilon \to 0$ in (4.27), it follows from Fatou's Lemma (see [23]) that

$$\left(a + \|u_*\|^2\right) \int_\Omega \nabla_H u_* \nabla_H \phi d\xi + \lambda \int_\Omega u_*^p \phi d\xi \geq \int_\Omega g(\xi) u_*^{-\gamma} \phi d\xi \tag{4.28}$$

for every $\phi \in S_0^1(\Omega)$, with $\phi \geq 0$.

Let $\psi(t) = J(tu_*)$. Noting that $tu_* \in D^+$ for all $t > 0$, $\psi \in C^1((0, \infty), \mathbb{R})$, and it has a global minimum at $t = 1$; therefore,

$$0 = \psi'(1) = a\|u_*\|^2 + b\|u_*\|^4 + \lambda \int_\Omega u_*^{p+1} d\xi - \int_\Omega g(\xi)|u_*|^{1-\gamma} d\xi.$$

In the following, as in the proof of Theorem 2.1, we can conclude that u_* is the unique solution of problem (1.1). □

4.3. Proof of Corollary 2.3

In this subsection, we prove Corollary 2.3. In fact, the proof is as in the proof of [18, Theorem 1.3]. Here, we give the details.

First, by Theorems 2.1 and 2.2, it is easy to get the uniqueness of solutions for problem (1.2) under the conditions of Corollary 2.3. Second, let u be a solution of problem (1.2) and $\mathcal{S}$ denote a unitary rotation in $\mathbb{C}^n$. Set

$$u_{\mathcal{S}}(z,t) = u(\mathcal{S}z,t), \ \forall (z,t) \in \mathbb{C}^n \times \mathbb{R} = \mathbb{H}^n.$$

Because Δ_H is invariant with respect to $\mathcal{S}$ and $g(\xi)$ is cylindrically symmetric, we have

$$\begin{aligned} -\Delta_H u_{\mathcal{S}}(z,t) = -\Delta_H u(\mathcal{S}z,t) &= \frac{g(\mathcal{S}z,t)}{u(\mathcal{S}z,t)^\gamma} - \lambda u(\mathcal{S}z,t)^p \\ &= \frac{g(z,t)}{u_{\mathcal{S}}(z,t)^\gamma} - \lambda u_{\mathcal{S}}(z,t)^p. \end{aligned}$$

This means that $u_{\mathcal{S}}$ is also a solution of (1.2). From this and the uniqueness of solutions, we conclude that $u = u_{\mathcal{S}}$ for any unitary rotation $\mathcal{S}$ in $\mathbb{C}^n$. This is to say that the solution u is a function of $(|z_1|, |z_2|, \cdots, |z_n|, t)$. Meanwhile, problem (1.2) becomes

$$\begin{cases} -(\Delta_z u + 4|z|^2 \partial_{tt} u) = g(\xi)u^{-\gamma} - \lambda u^p, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.29}$$

Now, let $\mathcal{R}$ denote a real rotation around t-axis in $\mathbb{R}^{2n}$ and $u_{\mathcal{R}}(z,t) = u(\mathcal{R}z,t)$. Recalling that the operator $-(\Delta_z + 4|z|^2\partial_{tt})$ is also invariant with respect to $\mathcal{R}$, once again, by the uniqueness of solutions, we may obtain that $u(z,t) = u(\mathcal{R}z,t)$ for any rotation $\mathcal{R}$ in $\mathbb{R}^{2n}$. This means that u is cylindrically symmetric. Furthermore, if

$$g(|z|,t) = g(|z|,-t) \quad \text{and} \quad \Phi(|z|,t) = \Phi(|z|,-t),$$

it then follows from the same reason that $u(|z|,-t) = u(|z|,t)$. This completes the proof of Corollary 2.3.

5. Conclusions

In this study, we employ the minimax method and certain analysis techniques to establish the uniqueness of solutions to problem (1.1). Furthermore, under some necessary structural conditions, we deduce that the solution to problem (1.2) is cylindrically symmetric. The obtained conclusion has extended and improved the relevant results of the existing literature.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Guizhou Provincial Basic Research Program (No. MS[2026]230) and Bijie Science and Technology Bureau Joint Fund project of China (Nos. [2025]138, [2025]141).

Conflict of interest

The author declares no conflict of interest.

References

1. G. Kirchhoff, *Mechanik*, Teubner, 1883.
2. A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer, 2007. https://doi.org/10.1007/978-3-540-71897-0
3. G. M. Figueiredo, E. D. Lima, E. D. Silva, J. C. O. Junior, Positive solutions for a Kirchhoff type problem with critical growth via nonlinear Rayleigh quotient, *Calc. Var. Partial Differ. Equations*, **64** (2025), 16. https://doi.org/10.1007/s00526-024-02861-y
4. W. Fulks, J. S. Maybee, A singular non-linear equation, *Osaka Math. J.*, **12** (1960), 1–19.
5. M. G. Crandall, P. H. Rabinowitz, L. Tatar, On a Dirichlet problem with a singular nonlinearity, *Commun. Partial Differ. Equations*, **2** (1977), 193–222. https://doi.org/10.1080/03605307708820029
6. N. S. Papageorgius, P. Winkert, Singular p-Laplacian equations with superlinear perturbation, *J. Differ. Equations*, **266** (2019), 1462–1487. https://doi.org/10.1016/j.jde.2018.08.002
7. G. Anello, F. Faraci, On a singular semilinear elliptic problem with an asymptotically linear nonlinearity, *Proc. R. Soc. Edinburgh*, **146** (2016), 59–77. https://doi.org/10.1017/S0308210515000414
8. R. L. Alves, Existence of positive solution for a singular elliptic problem with an asymptotically linear nonlinearity, *Mediterr. J. Math.*, **18** (2021), 4. https://doi.org/10.1007/s00009-020-01646-9
9. N. Hirano, C. Saccon, N. Shioji, Existence of multiple positive solutions for singular elliptic problems with a concave and convex nonlinearities, *Adv. Differ. Equations*, **9** (2004), 197–220. https://doi.org/10.57262/ade/1355867973
10. N. S. Papageorgiou, V. D. Rădulescu, D. D. Repovš, Pairs of positive solutions for resonant singular equations with the p-Laplacian, *ArXiv*, 2017. https://doi.org/10.48550/arXiv.1710.02643
11. R. L. Alves, C. A. Santos, K. Silva, Multiplicity of negative-energy solutions for singular-superlinear Schrödinger equations with indefinite-sign potential, *Commun. Contemp. Math.*, **24** (2022), 2150042. https://doi.org/10.1142/S0219199721500425
12. S. Biagi, M. Galeotti, E. Vecchi, Critical singular problems in Carnot groups, *ArXiv*, 2025. https://doi.org/10.48550/arXiv.2506.07521
13. D. Kumar, Semilinear elliptic problems with singular terms on the Heisenberg group, *Complex Var. Elliptic Equ.*, **64** (2019), 1844–1853. https://doi.org/10.1080/17476933.2018.1557157
14. A. Benhana, J. Zhang, On a class of Choquard-Schrödinger-Poisson systems involving p-Laplacian, *Math. Methods Appl. Sci.*, **49** (2025), 2626–2638. https://doi.org/10.1002/mma.70274
15. N. S. Papageorgiou, V. D. Radulescu, W. Zhang, Nonlinear double phase eigenvalue problems, *Math. Mech. Solids*, **31** (2026), 580–595. https://doi.org/10.1177/10812865251397537

16. J. Zhang, W. Zhang, V. D. Radulescu, Localized concentration of semiclassical solutions for double phase problems with nonlocal reaction, *Anal. Math. Phys.*, **16** (2026), 35. https://doi.org/10.1007/s13324-026-01182-x

17. J. Liao, X. F. Ke, C. Y. Lei, C. L. Tang, A uniqueness result for Kirchhoff type problems with singularity, *Appl. Math. Lett.*, **59** (2016), 24–30. https://doi.org/10.1016/j.aml.2016.03.001

18. Y. C. An, H. Liu, L. Tian, The Dirichlet problem for a sub-elliptic equation with singular nonlinearity on the Heisenberg group, *J. Math. Inequal.*, **14** (2020), 67–82. https://doi.org/10.7153/jmi-2020-14-06

19. I. Birindelli, J. Prajapat, Monotonicity and symmetry results for degenerate elliptic equations on nilpotent Lie groups, *Pacific J. Math.*, **204** (2002), 1–17. https://doi.org/10.2140/pjm.2002.204.1

20. Y. Sun, D. Zhang, The role of the power 3 for elliptic equations with negative exponents, *Calc. Var. Partial Differ. Equations*, **49** (2014), 909–922. https://doi.org/10.1007/s00526-013-0724-5

21. J. M. Bony, Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les operateurs elliptiques dég´énérés, *Ann. Inst. Fourier Grenobles*, **19** (1969), 277–304. https://doi.org/10.5802/aif.319

22. G. B. Folland, E. M. Stein, Estimates for the $\overline{\partial}_b$ complex and analysis on the Heisenberg group, *Commun. Pure Appl. Math.*, **27** (1974), 429–522. https://doi.org/10.1002/cpa.3160270403

23. W. Rudin, *Real and complex analysis*, 3 Eds., McGraw-Hill, 1987.

24. H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, *Proc. Amer. Math. Soc.*, **88** (1983), 486–490. https://doi.org/10.1090/S0002-9939-1983-0699419-3

AIMS Press