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Research article

A novel cubic circular picture fuzzy set framework with Bonferroni mean operators and an illustrative application to quantum computing technology selection

Haitham Qawaqneh[1], Abdallah Shihadeh[2], Wael Mahmoud Mohammad Salameh[3], Sultan Hussain[4], Muhammad Zeeshan[5], and Takaaki Fujita[6,*]

[1] Department of Basic Sciences, AI-Zaytoonah University of Jordan, Amman 11733, Jordan; h.alqawaqneh@zuj.edu.jo
[2] Department of Mathematics, Faculty of Science, The Hashemite University, Zarqa 13133, P.O. box 330127, Jordan; abdallaha_ka@hu.edu.jo
[3] Faculty of Information Technology, Abu Dhabi University, Abu Dhabi United Arab Emirates; Wael.salameh@adu.ac.ae
[4] Department of Mathematics and Statistics, College of Sciences, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11623, Saudi Arabia; SHDarwaish@imamu.edu.sa
[5] Departments of Mathematics, The University of Agriculture, Dera Ismail Khan, Pakistan; zeeshan.msc08@gmail.com
[6] Independent Researcher, (not affiliated with any university or research institute) Tokyo, Japan; Takaaki.fujita060@gmail.com

* **Correspondence:** Email: Takaaki.fujita060@gmail.com.

Abstract: Capturing hesitation and uncertainty in decision-making (DM) problems remains a critical challenge in many real-world applications. The picture fuzzy set (P-FS) model significantly represents positive, neutral, and negative information, and interval-valued PFSs (IVP-FSs) support additional flexibility by enabling interval-based evaluations. However, these approaches may still lack the ability to effectively model multilayered uncertainty and fluctuations of expert assessments. To handle such situations, the proposed study established an innovative fuzzy approach that combines P-FSs, IVP-FSs, and a radius parameter within a unified framework. The proposed model was termed cubic circular picture fuzzy sets (CuCP-FSs). In this proposed approach, the combination of P-FSs and IVP-FSs was crucial for capturing multilayered uncertainty, and the radius parameter represents the degree of reliability or fluctuations around the P-FS information. Several fundamental set-theoretic operations were demonstrated for the proposed CuCP-FSs framework. Moreover, Bonferroni operational laws were presented, and the corresponding aggregation operators (AOs) were designed to efficiently integrate assessment information. The essential characteristics of the proposed AOs are also investigated to ensure their validity. A multi-criteria decision-making (MCDM) technique was constructed based on these developments within the environment of CuCP-FSs. The proposed technique was employed for applications in the selection of quantum computing technology, presenting the effectiveness and practicality of the proposed MCDM approach. Furthermore, a comparative analysis with several existing fuzzy approaches was examined, which reflects the robustness and reliability of the newly defined CuCP-FS model in presenting complex uncertain information.

Keywords: innovative cubic picture fuzzy structure; score functions; Bonferroni operators; decision-making; quantum computing technology
Mathematics Subject Classification: 03E72, 28E10, 90B50, 94D05

1. Introduction

Uncertainty and vagueness are crucial characteristics of many real-world DM situations. In practical environments, experts frequently deal with problems where the assessments are imprecise, incomplete, or effected by subjective judgments. Several various factors including dynamic environmental conditions, conflicting expert judgments, and limited data make it difficult to represent a DM problem based on precise numerical values. Therefore, traditional fuzzy-based approaches based on exact data often fail in handling such complex DM situations. These challenges are addressed by various uncertainty modeling approaches, among which fuzzy sets (FSs) [34], proposed by Zadeh, offer a significant framework for analyzing and representing ambiguous or vague information. The FS model enables experts to describe their assessments based on MS values instead of strict binary classifications, and thus providing a more realistic and flexible modeling of real-world DM problems. However, when the FS approach has been employed in many DM problems, they only model the MS value and lack the ability to consider the non-membership (NMS) value or hesitation. Therefore, the FS model limits the ability to deal with more complex shapes of uncertainty. Later on, the intuitionistic fuzzy set (IFS) [4] approach was presented, which extends the notions of FSs by restricting the sum of the MS value and NMS value to be at most one. IFSs are very useful in many MCDM problems. Kumar et al. [21] presented the notions of Yager AOs within the IFS environments and demonstrated their practical utility in MCDM problems. Boczek et al. [7] proposed Choquet operators for IFSs and employed them for addressing real-world DM problems. Singh et al. [30] utilized IF information to propose weighted geometric AOs and applied the proposed approaches for solving MCDM problems. Garg and Kumar presented some modified approaches based on the IFS framework for solar energy selection [14]. Furthermore, Liu et al. [24] developed unified parametric divergence operators for Fermatean FSs and discussed its practical applications in machine learning and intelligent DM. Furthermore, Liu et al. [25] established a new approach of DM for addressing clustering problems within the complex Fermatean FS environments.

IFSs are still lack the ability to model neutral assessments, which are often faced in daily life evaluation processes. To further improve the modeling of uncertain information, P-FSs [11] were designed by integrating positive, neutral, and negative MS values within in a single framework such that their sum is at most 1. The P-FS approach facilitates experts to describe support, abstention, and opposition simultaneously and thus contributes a more reliable expression of expert opinions. The P-FS approach is commonly applied in many real-world applications. Chen and Hao developed some innovative AOs for P-FSs [10]. They discussed the practical applications of the proposed AOs in MCDM problems. Centillas et al. [9] employed P-FSs to propose Einstein AOs. They presented their effectiveness by solving real-life applications. Bhattacharya and Pal proposed some new programing approaches for P-FSs [6] and applied them for addressing real-world DM problems. Mishra et al. [28] designed an innovative DM approach for solving MCDM problems within the P-FS environments. Zhu [37] et al. presented innovative distance measures based on complex P-FSs and applied them for addressing real-world applications. Li et al. [23] demonstrated real-life practical applications using distance and similarity measures of P-FSs. However, in many DM problems, experts prefer to represent their judgments using interval-valued information to address uncertainty rather than precise numerical data. To overcome this limitation, the IVP-FS [16] was proposed, integrating interval-valued positive, neutral, and negative MS values. IVP-FSs are crucial to represent imprecise and incomplete information. They are applicable in many DM problems. Zhu and Liu designed several new distance measures for IVP-FSs and discussed their practical utility in daily life problems [35]. Ma et al. [26] constructed a new MCDM approach for solving DM problems within the framework of IVP-FSs. Gul and Sarfraz presented artificial intelligence approaches based on IVP-FSs and demonstrated their applications in real-world DM problems [16]. Zhu et al. [38] established the technique for order preference by similarity to ideal solution (TOPSIS) approach within the IVP-FSs and applied them for addressing risk assessment problems. Cao and Shen discussed pattern recognition problems based on similarity measures of IVP-FSs [8]. Ma et al. [27] proposed the grass rating points (GRP) method for evaluation processes within the environments of IVP-FSs. Khan et al. [19] presented IVP-FS hypergraphs for applications of MCDM problems. Kishorekumar et al. [20] demonstrated the notions of topological space within the IVP-FS framework and discussed their practical applications in MCDM problems. Zhu et al. [36] established new distance measures using P-FSs and IVP-FSs for solving diverse applications. Furthermore, advanced fuzzy frameworks are employed to propose DM approaches for addressing MCDM problems. Kamran et al. [17] applied Fermatean fuzzy rough sets to improve the confidence level of the DM method for solving real-life practical applications. Kamran et al. [18] designed several optimization approaches for addressing real-world applications using interval-valued Fermatean neutrosophic numbers.

Many uncertainty problems are multilayered in nature and therefore cannot be effectively addressed using single-layered models**.** To address such multilayered uncertainty problems, cubic P-FSs (CuP-FSs) [3] were proposed integrating the notions of P-FSs and IVP-FSs within a unified framework. CuP-FSs are crucial in representing precise information, interval-valued assessments, and multilayered uncertainty simultaneously. This approach is very useful for addressing higher levels of uncertain DM problems. Tanoli et al. [31] demonstrated Einstein AOs for CuP-FSs. They solved MCDM problems based on the proposed approaches. Al-Quran et al. [1] designed some new AOs based on CuP-FSs and employed them for handling real-world uncertainty problems. Ashraf et al. [2] solved DM problems based on innovative AOs of CuP-FSs. Tanoli et al. [32] proposed geometric AOs using CuP-FSs and highlighted their practical utility in real-life DM problems. Riaz et al. [29] applied CuP-FS information to address supply-chain management problems.

However, all these approaches lack the capability to model fluctuations or variability around the reference assessment. To handle this limitation, a circular PFS (CP-FS) [33] model was proposed by associating the radius parameter with the P-FS structure, allowing fluctuations or variability around the reference assessment. This approach is crucial for solving MCDM problems. Gao et al. [15] proposed a DM algorithm for real-life applications based on CP-FSs. Deng and Liu developed an evaluations mechanism for solutions of physical health within the CP-FS environments [12]. Li and Sun utilized CP-FSs to propose the MACROS method for solving MCDM problems [22].

This approach lacks the ability to model multilayered uncertainty. To overcome this limitation, the proposed study demonstrates a new model integrating P-FSs, IVP-FSs, and the radius parameter within a unified single framework, where the circular radius represents the fluctuations or tolerance present in expert judgments. The proposed approach models precise, information, interval-valued evaluations, multilayered uncertainty, and fluctuations around reference assessments simultaneously. Figure 1 represents the base models, such as FSs [34], IFSs [4], P-FSs [11], and IVP-FSs [16], of the proposed framework.

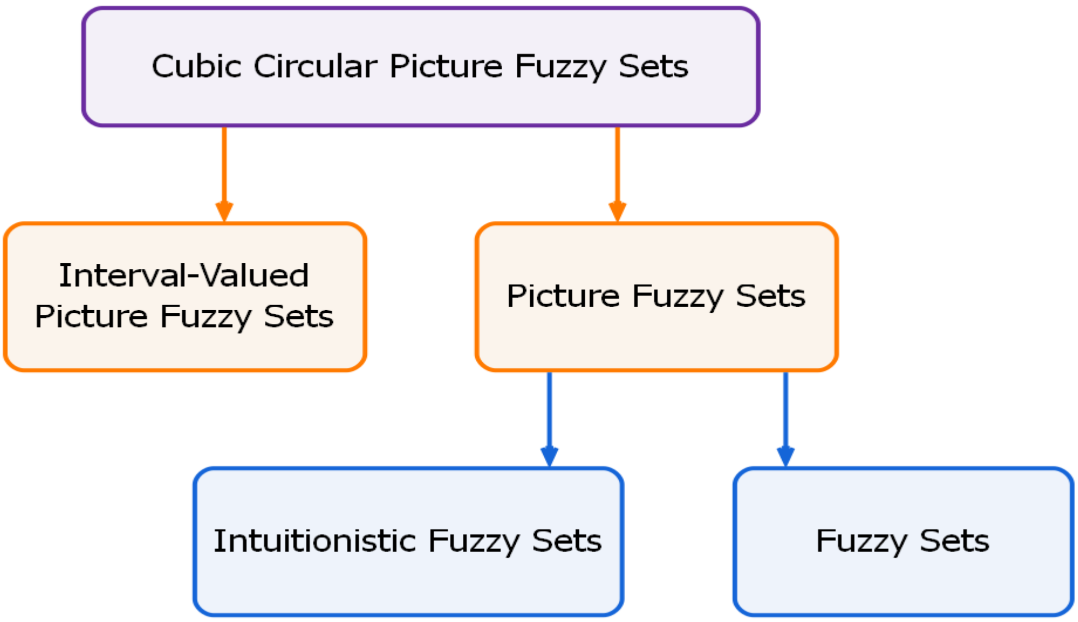

Figure 1. Base models of the proposed framework.

1.1. Cubic circular picture fuzzy sets

Here, an innovative CuCP-FS model is introduced integrating the frameworks of P-FSs and IVP-FSs together with the radius parameter in a unified single framework. The proposed model is crucial for allowing experts to represent precise and interval-valued positive, neutral, and negative MS degrees simultaneously, together with fluctuations or variability present in decision evaluations. The proposed model extends the existing picture fuzzy-based frameworks, thus enhancing the effectiveness and reliability of the DM process.

Let $\mathbb{V}$ be any non-empty universal of discourse. Then, a CuCP-FS $\mathscr{M}$ on $\mathbb{V}$ is modeled as

$$\mathcal{M}=\left\{\left(\mathbb{v},\left(\left(\mathfrak{u}_{\mathcal{M}}(\mathbb{v}),\mathrm{v}_{\mathcal{M}}(\mathbb{v}),\varepsilon_{\mathcal{M}}(\mathbb{v})\right),\begin{pmatrix}[\mathfrak{u}_{\mathcal{M}}{}^{lower}(\mathbb{v}),\mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v})],[\mathrm{v}_{\mathcal{M}}{}^{lower}(\mathbb{v}),\mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v})],\\ [\varepsilon_{\mathcal{M}}{}^{lower}(\mathbb{v}),\varepsilon_{\mathcal{M}}{}^{upper}(\mathbb{v})]\end{pmatrix}\right);\mathbb{r}_{\mathcal{M}}\right)|\mathbb{v}\in\mathbb{V}\right\}, \quad (1)$$

where for each $\mathbb{v}\in\mathbb{V}$:

i. The information $\left(\mathfrak{u}_{\mathcal{M}}(\mathbb{v}),\mathrm{v}_{\mathcal{M}}(\mathbb{v}),\varepsilon_{\mathcal{M}}(\mathbb{v})\right)$ indicates a P-FS structure where $\mathfrak{u}_{\mathcal{M}}(\mathbb{v}),\mathrm{v}_{\mathcal{M}}(\mathbb{v}),\varepsilon_{\mathcal{M}}(\mathbb{v})\in[0,1]$ represent crisp positive, neutral, and negative MS values such that $0\leq\mathfrak{u}_{\mathcal{M}}(\mathbb{v})+\mathrm{v}_{\mathcal{M}}(\mathbb{v})+\varepsilon_{\mathcal{M}}(\mathbb{v})\leq 1$;

ii. The information $\begin{pmatrix}[\mathfrak{u}_{\mathcal{M}}{}^{lower}(\mathbb{v}),\mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v})],[\mathrm{v}_{\mathcal{M}}{}^{lower}(\mathbb{v}),\mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v})],\\ [\varepsilon_{\mathcal{M}}{}^{low\varrho r}(\mathbb{v}),\varepsilon_{\mathcal{M}}{}^{\mathbb{upp}\varrho r}(\mathbb{v})]\end{pmatrix}$ defines an IVP-FS where $[\mathfrak{u}_{\mathcal{M}}{}^{lower}(\mathbb{v}),\mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v})],[\mathrm{v}_{\mathcal{M}}{}^{lower}(\mathbb{v}),\mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v})],[\varepsilon_{\mathcal{M}}{}^{lower}(\mathbb{v}),\varepsilon_{\mathcal{M}}{}^{upper}(\mathbb{v})]\subseteq[0,1]$ represent interval valued positive, neutral, and negative MS values such that $0\leq\mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v})+\mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v})+\varepsilon_{\mathcal{M}}{}^{upper}(\mathbb{v})\leq 1$;

iii. $\mathbb{r}_{\mathcal{M}}\in[0,1]$ indicates the circular radius reflecting fluctuations or variability around P-FS information.

The newly defined fuzzy structure integrates the advantages of P-FSs, IVP-FSs, and a circular radius-based uncertainty representation into a single unified framework. In CuCP-FSs, the picture fuzzy information allows the simultaneous handling of positive, neutral, and negative MS values, thereby providing reliable human judgment evaluation. The interval-valued representation further improves the robustness of the proposed model by enabling these MS values to vary within bounded intervals, representing real-world imprecision and ambiguity in expert opinions. Moreover, the integration of a circular radius establishes an explicit way to deal with hesitation and local uncertainty around the reference assessment. This parameter facilitates a geometric description of fuzziness, where the radius parameter reflects the tolerance region of fluctuations around the assessed information. Therefore, the innovative CuCP-FS framework simultaneously handles multi-dimensional uncertainty arising from linguistic hesitation, interval variability, and geometric dispersion; making it a more realistic and comprehensive approach for complex DM problems. Table 1 reflects the distinguishing features and key advantages of the proposed model over existing approaches.

Table 1. Distinguishing features and key advantages of the proposed model over existing approaches.

Approaches/Key features	Positive MS value	Neutral MS value	Negative MS value	Interval-valued representation	Cubic nature	Circular radius
FSs	Yes	No	No	No	No	No
IFSs	Yes	No	Yes	No	No	No
P-FSs	Yes	Yes	Yes	No	No	No
IVP-FSs	Yes	Yes	Yes	Yes	No	No
CuP-FSs	Yes	Yes	Yes	Yes	Yes	No
Proposed Model	Yes	Yes	Yes	Yes	Yes	Yes

Note: Equation (1) can be simply written as:

$$\mathcal{M} = \left\{\left(\mathbb{v}, \left((\mathfrak{u}, \mathrm{v}, \varepsilon), ([\bar{\mathfrak{a}}, \mathfrak{b}], [\mathfrak{c}, \mathfrak{d}], [\varrho, f])\right); \mathbb{r}\right) | \mathbb{v} \in \mathbb{V}\right\}. \tag{2}$$

1.1.1. Interpretation of the CuCP-FS model

The proposed CuCP-FS model can be interpreted as follows:

i) The information $(\mathfrak{u}, \mathrm{v}, \varepsilon)$ in Eq (2) establishes reference evaluations.
ii) The interval-valued information $([\bar{\mathfrak{a}}, \mathfrak{b}], [\mathfrak{c}, \mathfrak{d}], [\varrho, f])$ demonstrates uncertain assessments.
iii) The radius $\mathbb{r}$ reflects fluctuations around the reference evaluations $(\mathfrak{u}, \mathrm{v}, \varepsilon)$.

1.1.2. Geometric interpretation of the proposed model

Geometrically, the proposed model (Figure 2) can be interpreted as follows:

i) The triplet $(\mathfrak{u}, \mathrm{v}, \varepsilon)$ serves as a center;
ii) The interval-valued information $([\bar{\mathfrak{a}}, \mathfrak{b}], [\mathfrak{c}, \mathfrak{d}], [\varrho, f])$ demonstrates a bounding box;
iii) The radius $\mathbb{r}$ facilitates a circular neighborhood around the reference evaluations $(\mathfrak{u}, \mathrm{v}, \varepsilon)$.

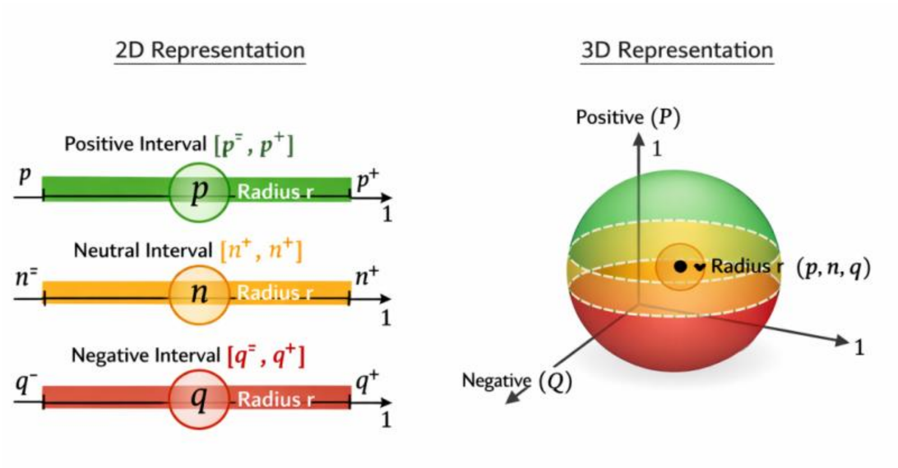

Figure 2. Geometrical shape of the proposed model.

1.1.3. Special cases of the CuCP-FS model

i) If interval-valued information collapse with crisp information and $\mathbb{r} = 0$, then the CuCP-FS framework reduces to P-FSs.
ii) If $\mathbb{r} = 0$, then the proposed model reduces to CuP-FSs.

1.2. Research gaps and motivations

Traditional FSs deal with uncertainty by associating each element a single MS value between 0 and 1. However, FSs play a significant role in addressing simple real-world uncertainty problems. They often fail to capture NMS values or neutral assessments, which are common in daily life DM problems. IFSs improve the FS model by involving NMS values, enabling a more flexible model for dealing with uncertain problems. This approach still lacks the ability to model neutral judgments, which are frequently crucial when expert opinion is indecisive. P-FSs further improve the IFS model by allowing a neutral MS value, contributing representation of positive, neutral, and negative evaluations in a unified framework. However, P-FSs are unable to model interval-valued or higher levels of uncertainty. Further, IVP-FSs were presented to improve the P-FS model by offering interval representations for all three dimensions. But, the IVP-FS framework fails to represent multilayered uncertainty, which often exists in real-world DM problems. To overcome this limitation, CuCP-FSs were demonstrated to solve multilayered uncertainty problems. This approach combines the frameworks of P-FSs and IVP-FSs within a unified single framework. However, all these picture fuzzy-based approaches ignore the radius parameter, which is critical for representing fluctuations or variability present in expert judgment. This gap can result to unreliable and unstable decision outcomes**.** Motivated by this research gap, the proposed study demonstrates a new model, combining the structures of P-FSs and IVP-FSs together with the radius parameter within a unified single framework. The proposed approach is crucial for handling multilayered uncertainty and fluctuations present in expert opinions simultaneously. Incorporating the radius parameter further improves the expression of expert reliability, bridging the gap between theoretical modeling and practical DM. By integrating P-FSs, IVP-FSs, and a radius parameter within in a single structure reduces the information loss that occurs in single picture fuzzy-based models. Thus, the proposed approach enhances the robustness, accuracy, and confidence of DM outcomes.

1.3. Major contributions

P-FSs and IVP-FSs are comprehensive fuzzy approaches that integrate positive, neutral, and negative assessments in a single framework to handle uncertain information. CuP-FSs combine these two approaches within a unified framework for representing multilayered uncertainty existing in real-world DM problems. The key contributions of this paper are highlighted as follows.

1) The proposed work demonstrates a new model, called CuCP-FSs, integrating the frameworks of P-FSs and IVP-FSs together radius parameter in a single framework.
2) Several set theoretic operations for the proposed model are designed.
3) A score function for ranking alternatives is introduced within the CuCP-FS environments.
4) Some basic AOs, such as weighted arithmetic and weighted geometric are developed based on CuCP-FSs.
5) Several key characteristics of the proposed approaches are investigated.
6) The proposed approaches such as score functions and AOs are employed to construct a DM method for solving real-world uncertainty problems.
7) A practical case study is solved based on the constructed DM algorithm.
8) A comparative study between the newly defined model and existing picture fuzzy-based models is conducted.

1.4. Summary of the manuscript

The proposed work is briefly illustrated as follows. In Section 2, we review the basic models that provide the foundation of the proposed work. Section 3 presents the new picture fuzzy-based model, together with basic operations. Section 4 designs weighted arithmetic and weighted geometric AOs for CuCP-FSs. In Section 5, we proposed an MCDM approach using the newly defined tools. Section 6 discusses their practical applications in real-world uncertainty problems. A comparative study between the newly defined model and existing picture fuzzy-based models is conducted in Section 7. Section 8 contributes the conclusion of the proposed work.

2. Preliminaries

This section demonstrates the basic notions and mathematical structures crucial for defining the proposed approaches. Here, we review the basic definitions including IFSs, P-FSs, IVP-FSs, CuP-FSs, and CPFSs, which provide the theoretical foundation of the proposed study.

Definition 1. [4] The IFS framework $\mathfrak{T}$ on a universe of discourse $\mathbb{V}$ is described as follows:

$$\mathfrak{T} = \left\{\mathbb{v}, \left(\mathrm{T}_{\mathfrak{T}}(\mathbb{v}), \mathrm{F}_{\mathfrak{T}}(\mathbb{v})\right) : \mathbb{v} \in \mathbb{V}\right\}, \tag{3}$$

where $\mathrm{T}_{\mathfrak{T}}(\mathbb{v}), \mathrm{F}_{\mathfrak{T}}(\mathbb{v}) : \mathbb{V} \to [0,1]$ reflects MS and NMS values of the element $\mathbb{v} \in \mathbb{V}$ such that $0 \leq \mathrm{T}_{\mathfrak{T}}(\mathbb{v}) + \mathrm{F}_{\mathfrak{T}}(\mathbb{v}) \leq 1$.

Definition 2. [11] A P-FS framework $\mathcal{M}$ on a universe of discourse $\mathbb{V}$ is described as follows:

$$\mathcal{M} = \{\langle \mathbb{v}, (\mathfrak{u}_{\mathcal{M}}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}(\mathbb{v}), \varepsilon_{\mathcal{M}}(\mathbb{v}))\rangle | \mathbb{v} \in \mathbb{V}\}, \tag{4}$$

where $\mathfrak{u}_{\mathcal{M}}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}(\mathbb{v}), \varepsilon_{\mathcal{M}}(\mathbb{v}) : \mathbb{V} \to [0,1]$ indicates positive, neutral, and negative MS values of the element $\mathbb{v} \in \mathbb{V}$, such that $0 \leq \mathfrak{u}_{\mathcal{M}}(\mathbb{v}) + \mathrm{v}_{\mathcal{M}}(\mathbb{v}) + \varepsilon_{\mathcal{M}}(\mathbb{v}) \leq 1$.

Definition 3. [33] A CP-FS framework $\mathcal{M}$ on a universe of discourse $\mathbb{V}$ is described as follows:

$$\mathcal{M} = \{\langle \mathbb{v}, (\mathfrak{u}_{\mathcal{M}}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}(\mathbb{v}), \varepsilon_{\mathcal{M}}(\mathbb{v})); \mathbb{r}_{\mathcal{M}}\rangle | \mathbb{v} \in \mathbb{V}\}, \tag{5}$$

where $\mathfrak{u}_{\mathcal{M}}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}(\mathbb{v}), \varepsilon_{\mathcal{M}}(\mathbb{v}) : \mathbb{V} \to [0,1]$ indicates positive, neutral, and negative MS values of the element $\mathbb{v} \in \mathbb{V}$ and $\mathbb{r}_{\mathcal{M}}$ reflects the circular radius around picture fuzzy information, such that $0 \leq \mathfrak{u}_{\mathcal{M}}(\mathbb{v}) + \mathrm{v}_{\mathcal{M}}(\mathbb{v}) + \varepsilon_{\mathcal{M}}(\mathbb{v}) \leq 1$.

Definition 4. [16] An IVP-FS model $\mathfrak{T}$ on a universe of discourse $\mathbb{V}$ is described as follows:

$$\mathfrak{T} = \left\{\left(\mathbb{v}, \left(\begin{matrix}[\mathfrak{u}_{\mathfrak{T}}{}^{lower}(\mathbb{v}), \mathfrak{u}_{\mathfrak{T}}{}^{upper}(\mathbb{v})], [\mathrm{v}_{\mathfrak{T}}{}^{lower}(\mathbb{v}), \mathrm{v}_{\mathfrak{T}}{}^{upper}(\mathbb{v})], \\ [\varepsilon_{\mathfrak{T}}{}^{lower}(\mathbb{v}), \varepsilon_{\mathfrak{T}}{}^{upper}(\mathbb{v})]\end{matrix}\right)\right) | \mathbb{v} \in \mathbb{V}\right\}, \tag{6}$$

where for each $\mathbb{v} \in \mathbb{V}$, $[\mathfrak{u}_{\mathfrak{T}}{}^{lower}(\mathbb{v}), \mathfrak{u}_{\mathfrak{T}}{}^{upper}(\mathbb{v})]$, $[\mathrm{v}_{\mathfrak{T}}{}^{lower}(\mathbb{v}), \mathrm{v}_{\mathfrak{T}}{}^{upper}(\mathbb{v})]$, $[\varepsilon_{\mathfrak{T}}{}^{lower}(\mathbb{v}), \varepsilon_{\mathfrak{T}}{}^{upper}(\mathbb{v})] \subseteq [0,1]$ reflects interval-valued positive, neutral, and negative MS values such that $0 \leq \mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v}) + \mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v}) + \varepsilon_{\mathcal{M}}{}^{upper}(\mathbb{v}) \leq 1$.

Definition 5. [3] The CuP-FS model $\mathcal{M}$ on a universe of discourse $\mathbb{V}$ is described as follows:

$$\mathcal{M} = \left\{\left(\mathbb{v}, \left(\left(\mathfrak{u}_{\mathcal{M}}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}(\mathbb{v}), \varepsilon_{\mathcal{M}}(\mathbb{v})\right), \left(\begin{matrix}[\mathfrak{u}_{\mathcal{M}}{}^{lower}(\mathbb{v}), \mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v})], [\mathrm{v}_{\mathcal{M}}{}^{lower}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v})], \\ [\varepsilon_{\mathcal{M}}{}^{lower}(\mathbb{v}), \varepsilon_{\mathcal{M}}{}^{upper}(\mathbb{v})]\end{matrix}\right)\right)\right) | \mathbb{v} \in \mathbb{V}\right\}, \tag{7}$$

where for each $\mathbb{v} \in \mathbb{V}$:

i) The information $\left(\mathfrak{u}_{\mathcal{M}}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}(\mathbb{v}), \varepsilon_{\mathcal{M}}(\mathbb{v})\right)$ indicates a P-FS structure, such that $0 \leq$

$\mathfrak{u}_{\mathcal{M}}(\mathbb{v}) + \mathrm{v}_{\mathcal{M}}(\mathbb{v}) + \varepsilon_{\mathcal{M}}(\mathbb{v}) \leq 1$,

ii) The information $\begin{pmatrix} [\mathfrak{u}_{\mathcal{M}}{}^{lower}(\mathbb{v}), \mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v})], [\mathrm{v}_{\mathcal{M}}{}^{lower}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v})] \\ [\varepsilon_{\mathcal{M}}{}^{lower}(\mathbb{v}), \varepsilon_{\mathcal{M}}{}^{upper}(\mathbb{v})] \end{pmatrix}$ defines an IVP-FS such that $0 \leq \mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v}) + \mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v}) + \varepsilon_{\mathcal{M}}{}^{upper}(\mathbb{v}) \leq 1$.

3. Cubic circular picture fuzzy sets

This section presents the innovative framework of CuCP-FSs. The CuCP-FS model extends the P-FS theory by integrating P-FSs, IVP-FSs, and a circular radius within a unified single framework. The proposed methods are crucial for handling precise information, interval-valued assessment, multilayered uncertainty, and fluctuations around the reference assessment. Moreover, several basic operations are demonstrated for the newly defined approach. The proposed approach can be structured as follows:

Definition 6. Let $\mathbb{V}$ be any non-empty universal of discourse. Then, a CuCP-FS $\mathscr{M}$ on $\mathbb{V}$ is modeled as

$$\mathcal{M} = \left\{ \left(\mathbb{v}, \left((\mathfrak{u}_{\mathcal{M}}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}(\mathbb{v}), \varepsilon_{\mathcal{M}}(\mathbb{v})), \begin{pmatrix} [\mathfrak{u}_{\mathcal{M}}{}^{lower}(\mathbb{v}), \mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v})], [\mathrm{v}_{\mathcal{M}}{}^{lower}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v})], \\ [\varepsilon_{\mathcal{M}}{}^{lower}(\mathbb{v}), \varepsilon_{\mathcal{M}}{}^{upper}(\mathbb{v})] \end{pmatrix} \right); \mathbb{r}_{\mathcal{M}} \right) | \mathbb{v} \in \mathbb{V} \right\}, \quad (8)$$

where for each $\mathbb{v} \in \mathbb{V}$:

i) The information $(\mathfrak{u}_{\mathcal{M}}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}(\mathbb{v}), \varepsilon_{\mathcal{M}}(\mathbb{v}))$ indicates a P-FS structure where $\mathfrak{u}_{\mathcal{M}}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}(\mathbb{v}), \varepsilon_{\mathcal{M}}(\mathbb{v}) \in [0,1]$ represent crisp positive, neutral, and negative MS values such that $0 \leq \mathfrak{u}_{\mathcal{M}}(\mathbb{v}) + \mathrm{v}_{\mathcal{M}}(\mathbb{v}) + \varepsilon_{\mathcal{M}}(\mathbb{v}) \leq 1$.

ii) The information $\begin{pmatrix} [\mathfrak{u}_{\mathcal{M}}{}^{lower}(\mathbb{v}), \mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v})], [\mathrm{v}_{\mathcal{M}}{}^{lower}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v})], \\ [\varepsilon_{\mathcal{M}}{}^{lower}(\mathbb{v}), \varepsilon_{\mathcal{M}}{}^{upper}(\mathbb{v})] \end{pmatrix}$ defines an IVP-FS where $[\mathfrak{u}_{\mathcal{M}}{}^{lower}(\mathbb{v}), \mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v})], [\mathrm{v}_{\mathcal{M}}{}^{lower}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v})], [\varepsilon_{\mathcal{M}}{}^{lower}(\mathbb{v}), \varepsilon_{\mathcal{M}}{}^{upper}(\mathbb{v})] \subseteq [0,1]$ represent interval valued positive, neutral, and negative MS values such that $0 \leq \mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v}) + \mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v}) + \varepsilon_{\mathcal{M}}{}^{upper}(\mathbb{v}) \leq 1$.

iii) $\mathbb{r}_{\mathcal{M}} \in [0,1]$ indicates the circular radius reflecting fluctuations or variability around P-FS information such that

$$max(|\mathfrak{u}_{\mathcal{M}}(\mathbb{v}) - \mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v})|, |\mathrm{v}_{\mathcal{M}}(\mathbb{v}) - \mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v})|, |\varepsilon_{\mathcal{M}}(\mathbb{v}) - \varepsilon_{\mathcal{M}}{}^{upper}(\mathbb{v})|) \leq \mathbb{r}_{\mathcal{M}}.$$

Note: Equation (1) can be simply represented as:

$$\mathcal{M} = \left\{ \left(\mathbb{v}, \begin{pmatrix} (\mathfrak{u}, \mathrm{v}, \varepsilon), \\ ([\bar{a}, \mathfrak{b}], [\varsigma, \mathfrak{d}], [\varrho, \mathfrak{f}]) \end{pmatrix}; \mathbb{r} \right) | \mathbb{v} \in \mathbb{V} \right\}. \quad (9)$$

Note: In Eq (8), the pair

$$\left(\mathbb{v}, \left((\mathfrak{u}_{\mathcal{M}}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}(\mathbb{v}), \varepsilon_{\mathcal{M}}(\mathbb{v})), \begin{pmatrix} [\mathfrak{u}_{\mathcal{M}}{}^{lower}(\mathbb{v}), \mathfrak{u}_{\mathcal{M}}{}^{upper}(\mathbb{v})], [\mathrm{v}_{\mathcal{M}}{}^{lower}(\mathbb{v}), \mathrm{v}_{\mathcal{M}}{}^{upper}(\mathbb{v})], \\ [\varepsilon_{\mathcal{M}}{}^{lower}(\mathbb{v}), \varepsilon_{\mathcal{M}}{}^{upper}(\mathbb{v})] \end{pmatrix} \right); \mathbb{r}_{\mathcal{M}} \right)$$

is termed as cubic circular picture fuzzy number. It can be simply written as

$$\mathcal{M} = \left((\mathfrak{u}, \mathrm{v}, \varepsilon), ([\bar{a}, b], [c, d], [\varrho, f]); \mathbb{r}\right). \tag{10}$$

Definition 7. Let $\mathcal{M}_1 = \left(\left(\begin{matrix}\langle \mathfrak{u}_1, \mathrm{v}_1, \varepsilon_1 \rangle, \\ \langle [\bar{a}_1, b_1], [c_1, d_1], [\varrho_1, f_1] \rangle\end{matrix}\right); \mathbb{r}_1\right)$ and $\mathcal{M}_2 = \left(\left(\begin{matrix}\langle \mathfrak{u}_2, \mathrm{v}_2, \varepsilon_2 \rangle, \\ \langle [\bar{a}_2, b_2], [c_2, d_2], [\varrho_2, f_2] \rangle\end{matrix}\right); \mathbb{r}_2\right)$ be two CuCP-FSs, and then

i) $\mathcal{M}_1 = \mathcal{M}_2$ iff $\mathfrak{u}_1 = \mathfrak{u}_2, \mathrm{v}_1 = \mathrm{v}_2$, $\varepsilon_1 = \varepsilon_2, \bar{a}_1 = \bar{a}_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$, $\varrho_1 = \varrho_2, f_1 = f_2$, and $\mathbb{r}_1 = \mathbb{r}_2$ for all $\mathbb{v} \in \mathbb{V}$.

ii) $\mathcal{M}_1 \subset_1 \mathcal{M}_2$ iff ($\mathbb{r}_1 = \mathbb{r}_2$ for all $\mathbb{v} \in \mathbb{V}$) and

$$(\mathfrak{u}_1 < \mathfrak{u}_2, \mathrm{v}_1 < \mathrm{v}_2, \varepsilon_1 \geq \varepsilon_2, \bar{a}_1 < \bar{a}_2, b_1 < b_2, c_1 < c_2, d_1 < d_2, \varrho_1 \geq \varrho_2, f_1 \geq f_2),$$

$$\vee\, (\mathfrak{u}_1 \leq \mathfrak{u}_2, \mathrm{v}_1 \leq \mathrm{v}_2, \varepsilon_1 > \varepsilon_2, \bar{a}_1 \leq \bar{a}_2, b_1 \leq b_2, c_1 \leq c_2, d_1 \leq d_2, \varrho_1 > \varrho_2, f_1 > f_2),$$

$$\vee\, (\mathfrak{u}_1 < \mathfrak{u}_2, \mathrm{v}_1 < \mathrm{v}_2, \varepsilon_1 > \varepsilon_2, \bar{a}_1 < \bar{a}_2, b_1 < b_2, c_1 < c_2, d_1 < d_2, \varrho_1 > \varrho_2, f_1 > f_2).$$

iii) $\mathcal{M}_1 \subset_2 \mathcal{M}_2$ iff ($\mathbb{r}_1 < \mathbb{r}_2$ for all $\mathbb{v} \in \mathbb{V}$) and

$$\mathfrak{u}_1 = \mathfrak{u}_2, \mathrm{v}_1 = \mathrm{v}_2, \varepsilon_1 = \varepsilon_2, \bar{a}_1 = \bar{a}_2, b_1 = b_2, c_1 = c_2, d_1 = d_2, \varrho_1 = \varrho_2, f_1 = f_2.$$

iv) $\mathcal{M}_1 \subset \mathcal{M}_2$ iff ($\mathbb{r}_1 < \mathbb{r}_2$ for all $\mathbb{v} \in \mathbb{V}$); and

$$(\mathfrak{u}_1 < \mathfrak{u}_2, \mathrm{v}_1 < \mathrm{v}_2, \varepsilon_1 \geq \varepsilon_2, \bar{a}_1 < \bar{a}_2, b_1 < b_2, c_1 < c_2, d_1 < d_2, \varrho_1 \geq \varrho_2, f_1 \geq f_2),$$

$$\vee\, (\mathfrak{u}_1 \leq \mathfrak{u}_2, \mathrm{v}_1 \leq \mathrm{v}_2, \varepsilon_1 > \varepsilon_2, \bar{a}_1 \leq \bar{a}_2, b_1 \leq b_2, c_1 \leq c_2, d_1 \leq d_2, \varrho_1 > \varrho_2, f_1 > f_2),$$

$$\vee\, (\mathfrak{u}_1 < \mathfrak{u}_2, \mathrm{v}_1 < \mathrm{v}_2, \varepsilon_1 > \varepsilon_2, \bar{a}_1 < \bar{a}_2, b_1 < b_2, c_1 < c_2, d_1 < d_2, \varrho_1 > \varrho_2, f_1 > f_2).$$

v) $\mathcal{M}_1 \subseteq_1 \mathcal{M}_2$ iff ($\mathbb{r}_1 = \mathbb{r}_2$ for all $\mathbb{v} \in \mathbb{V}$) and

$$(\mathfrak{u}_1 \leq \mathfrak{u}_2, \mathrm{v}_1 \leq \mathrm{v}_2, \varepsilon_1 \geq \varepsilon_2, \bar{a}_1 \leq \bar{a}_2, b_1 \leq b_2, c_1 \leq c_2, d_1 \leq d_2, \varrho_1 \geq \varrho_2, f_1 \geq f_2),$$

$$\vee\, (\mathfrak{u}_1 \leq \mathfrak{u}_2, \mathrm{v}_1 \leq \mathrm{v}_2, \varepsilon_1 > \varepsilon_2, \bar{a}_1 \leq \bar{a}_2, b_1 \leq b_2, c_1 \leq c_2, d_1 \leq d_2, \varrho_1 > \varrho_2, f_1 > f_2),$$

$$\vee\, (\mathfrak{u}_1 < \mathfrak{u}_2, \mathrm{v}_1 < \mathrm{v}_2, \varepsilon_1 > \varepsilon_2, \bar{a}_1 < \bar{a}_2, b_1 < b_2, c_1 < c_2, d_1 < d_2, \varrho_1 > \varrho_2, f_1 > f_2).$$

vi) $\mathcal{M}_1 \subseteq_2 \mathcal{M}_2$ iff ($\mathbb{r}_1 \leq \mathbb{r}_2$ for all $\mathbb{v} \in \mathbb{V}$) and

$$\mathfrak{u}_1 = \mathfrak{u}_2, \mathrm{v}_1 = \mathrm{v}_2, \varepsilon_1 = \varepsilon_2, \bar{a}_1 = \bar{a}_2, b_1 = b_2, c_1 = c_2, d_1 = d_2, \varrho_1 = \varrho_2, f_1 = f_2.$$

vii) $\mathcal{M}_1 \subseteq \mathcal{M}_2$ iff ($\mathbb{r}_1 \leq \mathbb{r}_2$ for all $\mathbb{v} \in \mathbb{V}$) and

$$(\mathfrak{u}_1 \leq \mathfrak{u}_2, \mathrm{v}_1 \leq \mathrm{v}_2, \varepsilon_1 \geq \varepsilon_2, \bar{a}_1 \leq \bar{a}_2, b_1 \leq b_2, c_1 \leq c_2, d_1 \leq d_2, \varrho_1 \geq \varrho_2, f_1 \geq f_2).$$

viii) $\neg\mathcal{M}_1 = \left(\left(\begin{matrix}\langle \varepsilon_1, \mathrm{v}_1, \mathfrak{u}_1 \rangle, \\ \langle [\varrho_1, f_1], [c_1, d_1], [\bar{a}_1, b_1] \rangle\end{matrix}\right); \mathbb{r}_1\right)$.

ix) $\mathcal{M}_1 \sqcap_{min} \mathcal{M}_2 = \left(\begin{pmatrix} \langle min(u_1, u_2), min(v_1, v_2), max(\varepsilon_1, \varepsilon_2) \rangle, \\ [min(\bar{\dot{a}}_1, \bar{\dot{a}}_2), min(ɓ_1, ɓ_2)], \\ \langle [min(ꞓ_1, ꞓ_1), min(ɗ_1, ɗ_2)], \rangle \\ [max(\varrho_1, \varrho_1), max(ʄ_1, ʄ_2)] \end{pmatrix}; min(𝕣_1, 𝕣_2) \right).$

x) $\mathcal{M}_1 \sqcap_{max} \mathcal{M}_2 = \left(\begin{pmatrix} \langle min(u_1, u_2), min(v_1, v_2), max(\varepsilon_1, \varepsilon_2) \rangle, \\ [min(\bar{\dot{a}}_1, \bar{\dot{a}}_2), min(ɓ_1, ɓ_2)], \\ \langle [min(ꞓ_1, ꞓ_1), min(ɗ_1, ɗ_2)], \rangle \\ [max(\varrho_1, \varrho_1), max(ʄ_1, ʄ_2)] \end{pmatrix}; max(𝕣_1, 𝕣_2) \right).$

xi) $\mathcal{M}_1 \sqcup_{min} \mathcal{M}_2 = \left(\begin{pmatrix} \langle max(u_1, u_2), max(v_1, v_2), min(\varepsilon_1, \varepsilon_2) \rangle, \\ [max(\bar{\dot{a}}_1, \bar{\dot{a}}_2), max(ɓ_1, ɓ_2)], \\ \langle [max(ꞓ_1, ꞓ_1), max(ɗ_1, ɗ_2)], \rangle \\ [min(\varrho_1, \varrho_1), min(ʄ_1, ʄ_2)] \end{pmatrix}; min(𝕣_1, 𝕣_2) \right).$

xii) $\mathcal{M}_1 \sqcup_{max} \mathcal{M}_2 = \left(\begin{pmatrix} \langle max(u_1, u_2), max(v_1, v_2), min(\varepsilon_1, \varepsilon_2) \rangle, \\ [max(\bar{\dot{a}}_1, \bar{\dot{a}}_2), max(ɓ_1, ɓ_2)], \\ \langle [max(ꞓ_1, ꞓ_1), max(ɗ_1, ɗ_2)], \rangle \\ [min(\varrho_1, \varrho_1), min(ʄ_1, ʄ_2)] \end{pmatrix}; max(𝕣_1, 𝕣_2) \right).$

Theorem 1. Let $\mathcal{M}_1 = \left(\begin{pmatrix} \langle u_1, v_1, \varepsilon_1 \rangle, \\ \langle [\bar{\dot{a}}_1, ɓ_1], [ꞓ_1, ɗ_1], [\varrho_1, ʄ_1] \rangle \end{pmatrix}; 𝕣_1 \right)$ and $\mathcal{M}_2 = \left(\begin{pmatrix} \langle u_2, v_2, \varepsilon_2 \rangle, \\ \langle [\bar{\dot{a}}_2, ɓ_2], [ꞓ_2, ɗ_2], [\varrho_2, ʄ_2] \rangle \end{pmatrix}; 𝕣_2 \right)$ be two CuCP-FSs, and then

i) $\neg(\neg \mathcal{M}_1) = \mathcal{M}_1$;
ii) $\mathcal{M}_1 \sqcap_{min} \mathcal{M}_2 = \mathcal{M}_2 \sqcap_{min} \mathcal{M}_1$;
iii) $\mathcal{M}_1 \sqcap_{max} \mathcal{M}_2 = \mathcal{M}_2 \sqcap_{max} \mathcal{M}_1$;
iv) $\mathcal{M}_1 \sqcup_{min} \mathcal{M}_2 = \mathcal{M}_2 \sqcup_{min} \mathcal{M}_1$;
v) $\mathcal{M}_1 \sqcup_{max} \mathcal{M}_2 = \mathcal{M}_2 \sqcup_{max} \mathcal{M}_1$.

Proof. i) We have $\neg \mathcal{M}_1 = \left(\begin{pmatrix} \langle \varepsilon_1, v_1, u_1 \rangle, \\ \langle [\varrho_1, ʄ_1], [ꞓ_1, ɗ_1], [\bar{\dot{a}}_1, ɓ_1] \rangle \end{pmatrix}; 𝕣_1 \right)$ this implies that $\neg(\neg \mathcal{M}_1) = \left(\begin{pmatrix} \langle u_1, v_1, \varepsilon_1 \rangle, \\ \langle [\bar{\dot{a}}_1, ɓ_1], [ꞓ_1, ɗ_1], [\varrho_1, ʄ_1] \rangle \end{pmatrix}; 𝕣_1 \right)$. Thus, $\neg(\neg \mathcal{M}_1) = \mathcal{M}_1$.

ii) We have

$$\mathcal{M}_1 \sqcap_{min} \mathcal{M}_2 = \left(\begin{pmatrix} \langle min(\mathrm{u}_1, \mathrm{u}_2), min(\mathrm{v}_1, \mathrm{v}_2), max(\varepsilon_1, \varepsilon_2)\rangle, \\ [min(\dot{\bar{\mathrm{a}}}_1, \dot{\bar{\mathrm{a}}}_2), min(\text{ɓ}_1, \text{ɓ}_2)], \\ \langle [min(\text{ɕ}_1, \text{ɕ}_2), min(\text{ɗ}_1, \text{ɗ}_2)], \rangle \\ [max(\varrho_1, \varrho_2), max(\text{ʄ}_1, \text{ʄ}_2)] \end{pmatrix}; min(\text{𝕣}_1, \text{𝕣}_2)\right)$$

$$= \left(\begin{pmatrix} \langle min(\mathrm{u}_2, \mathrm{u}_1), min(\mathrm{v}_2, \mathrm{v}_1), max(\varepsilon_2, \varepsilon_1)\rangle, \\ [min(\dot{\bar{\mathrm{a}}}_2, \dot{\bar{\mathrm{a}}}_1), min(\text{ɓ}_2, \text{ɓ}_1)], \\ \langle [min(\text{ɕ}_2, \text{ɕ}_1), min(\text{ɗ}_2, \text{ɗ}_1)], \rangle \\ [max(\varrho_2, \varrho_1), max(\text{ʄ}_2, \text{ʄ}_1)] \end{pmatrix}; min(\text{𝕣}_2, \text{𝕣}_1)\right) = \mathcal{M}_2 \sqcap_{min} \mathcal{M}_1.$$

iii) We have

$$\mathcal{M}_1 \sqcap_{max} \mathcal{M}_2 = \left(\begin{pmatrix} \langle min(\mathrm{u}_1, \mathrm{u}_2), min(\mathrm{v}_1, \mathrm{v}_2), max(\varepsilon_1, \varepsilon_2)\rangle, \\ [min(\dot{\bar{\mathrm{a}}}_1, \dot{\bar{\mathrm{a}}}_2), min(\text{ɓ}_1, \text{ɓ}_2)], \\ \langle [min(\text{ɕ}_1, \text{ɕ}_2), min(\text{ɗ}_1, \text{ɗ}_2)], \rangle \\ [max(\varrho_1, \varrho_2), max(\text{ʄ}_1, \text{ʄ}_2)] \end{pmatrix}; max(\text{𝕣}_1, \text{𝕣}_2)\right)$$

$$= \left(\begin{pmatrix} \langle min(\mathrm{u}_2, \mathrm{u}_1), min(\mathrm{v}_2, \mathrm{v}_1), max(\varepsilon_2, \varepsilon_1)\rangle, \\ [min(\dot{\bar{\mathrm{a}}}_2, \dot{\bar{\mathrm{a}}}_1), min(\text{ɓ}_2, \text{ɓ}_1)], \\ \langle [min(\text{ɕ}_2, \text{ɕ}_1), min(\text{ɗ}_2, \text{ɗ}_1)], \rangle \\ [max(\varrho_2, \varrho_1), max(\text{ʄ}_2, \text{ʄ}_1)] \end{pmatrix}; max(\text{𝕣}_2, \text{𝕣}_1)\right) = \mathcal{M}_2 \sqcap_{max} \mathcal{M}_1.$$

iv) We have

$$\mathcal{M}_1 \sqcup_{min} \mathcal{M}_2 = \left(\begin{pmatrix} \langle max(\mathrm{u}_1, \mathrm{u}_2), max(\mathrm{v}_1, \mathrm{v}_2), min(\varepsilon_1, \varepsilon_2)\rangle, \\ [max(\dot{\bar{\mathrm{a}}}_1, \dot{\bar{\mathrm{a}}}_2), max(\text{ɓ}_1, \text{ɓ}_2)], \\ \langle [max(\text{ɕ}_1, \text{ɕ}_2), max(\text{ɗ}_1, \text{ɗ}_2)], \rangle \\ [min(\varrho_1, \varrho_2), min(\text{ʄ}_1, \text{ʄ}_2)] \end{pmatrix}; min(\text{𝕣}_1, \text{𝕣}_2)\right)$$

$$= \left(\begin{pmatrix} \langle max(\mathrm{u}_2, \mathrm{u}_1), max(\mathrm{v}_2, \mathrm{v}_1), min(\varepsilon_2, \varepsilon_1)\rangle, \\ [max(\dot{\bar{\mathrm{a}}}_2, \dot{\bar{\mathrm{a}}}_1), max(\text{ɓ}_2, \text{ɓ}_1)], \\ \langle [max(\text{ɕ}_2, \text{ɕ}_1), max(\text{ɗ}_2, \text{ɗ}_1)], \rangle \\ [min(\varrho_2, \varrho_1), min(\text{ʄ}_2, \text{ʄ}_1)] \end{pmatrix}; min(\text{𝕣}_2, \text{𝕣}_1)\right) = \mathcal{M}_2 \sqcup_{min} \mathcal{M}_1.$$

v) We have

$$\mathcal{M}_1 \sqcup_{max} \mathcal{M}_2 = \left(\begin{pmatrix} \langle max(\mathrm{u}_1, \mathrm{u}_2), max(\mathrm{v}_1, \mathrm{v}_2), min(\varepsilon_1, \varepsilon_2)\rangle, \\ [max(\dot{\bar{\mathrm{a}}}_1, \dot{\bar{\mathrm{a}}}_2), max(\text{ɓ}_1, \text{ɓ}_2)], \\ \langle [max(\text{ɕ}_1, \text{ɕ}_2), max(\text{ɗ}_1, \text{ɗ}_2)], \rangle \\ [min(\varrho_1, \varrho_2), min(\text{ʄ}_1, \text{ʄ}_2)] \end{pmatrix}; max(\text{𝕣}_1, \text{𝕣}_2)\right)$$

$$= \left(\begin{pmatrix} \langle max(\mathrm{u}_2, \mathrm{u}_1), max(\mathrm{v}_2, \mathrm{v}_1), min(\varepsilon_2, \varepsilon_1)\rangle, \\ [max(\dot{\bar{\mathrm{a}}}_2, \dot{\bar{\mathrm{a}}}_1), max(\text{ɓ}_2, \text{ɓ}_1)], \\ \langle [max(\text{ɕ}_2, \text{ɕ}_1), max(\text{ɗ}_2, \text{ɗ}_1)], \rangle \\ [min(\varrho_2, \varrho_1), min(\text{ʄ}_2, \text{ʄ}_1)] \end{pmatrix}; max(\text{𝕣}_2, \text{𝕣}_1)\right) = \mathcal{M}_2 \sqcup_{max} \mathcal{M}_1.$$

Theorem 2. Let $\mathcal{M}_1 = \left(\left(\begin{matrix}\langle \mathrm{u}_1, \mathrm{v}_1, \varepsilon_1\rangle,\\ \langle[\dot{\bar{\mathrm{a}}}_1, \text{ɓ}_1], [\text{ɕ}_1, \text{ɗ}_1], [\varrho_1, \text{ʄ}_1]\rangle\end{matrix}\right); \mathbb{r}_1\right)$, $\mathcal{M}_2 = \left(\left(\begin{matrix}\langle \mathrm{u}_2, \mathrm{v}_2, \varepsilon_2\rangle,\\ \langle[\dot{\bar{\mathrm{a}}}_2, \text{ɓ}_2], [\text{ɕ}_2, \text{ɗ}_2], [\varrho_2, \text{ʄ}_2]\rangle\end{matrix}\right); \mathbb{r}_2\right)$, and $\mathcal{M}_3 = \left(\left(\begin{matrix}\langle \mathrm{u}_3, \mathrm{v}_3, \varepsilon_3\rangle,\\ \langle[\dot{\bar{\mathrm{a}}}_3, \text{ɓ}_3], [\text{ɕ}_3, \text{ɗ}_3], [\varrho_3, \text{ʄ}_3]\rangle\end{matrix}\right); \mathbb{r}_3\right)$ be three CuCP-FSs. Then

i) $(\mathcal{M}_1 \sqcap_{min} \mathcal{M}_2) \sqcap_{min} \mathcal{M}_3 = \mathcal{M}_1 \sqcap_{min} (\mathcal{M}_2 \sqcap_{min} \mathcal{M}_3)$;
ii) $(\mathcal{M}_1 \sqcap_{max} \mathcal{M}_2) \sqcap_{max} \mathcal{M}_3 = \mathcal{M}_1 \sqcap_{max} (\mathcal{M}_2 \sqcap_{max} \mathcal{M}_3)$;
iii) $(\mathcal{M}_1 \sqcup_{min} \mathcal{M}_2) \sqcup_{min} \mathcal{M}_3 = \mathcal{M}_1 \sqcup_{min} (\mathcal{M}_2 \sqcup_{min} \mathcal{M}_3)$;
iv) $(\mathcal{M}_1 \sqcup_{max} \mathcal{M}_2) \sqcup_{max} \mathcal{M}_3 = \mathcal{M}_1 \sqcup_{max} (\mathcal{M}_2 \sqcup_{max} \mathcal{M}_3)$.

Proof. i) We have

$$
\begin{aligned}
&(\mathcal{M}_1 \sqcap_{min} \mathcal{M}_2) \sqcap_{min} \mathcal{M}_3\\
&= \left(\left(\begin{matrix}\langle min(\mathrm{u}_1, \mathrm{u}_2), min(\mathrm{v}_1, \mathrm{v}_2), max(\varepsilon_1, \varepsilon_2)\rangle,\\ [min(\dot{\bar{\mathrm{a}}}_1, \dot{\bar{\mathrm{a}}}_2), min(\text{ɓ}_1, \text{ɓ}_2)],\\ \langle[min(\text{ɕ}_1, \text{ɕ}_2), min(\text{ɗ}_1, \text{ɗ}_2)],\rangle\\ [max(\varrho_1, \varrho_2), max(\text{ʄ}_1, \text{ʄ}_2)]\end{matrix}\right); min(\mathbb{r}_1, \mathbb{r}_2)\right) \sqcap_{min} \left(\left(\begin{matrix}\langle \mathrm{u}_3, \mathrm{v}_3, \varepsilon_3\rangle,\\ \langle[\dot{\bar{\mathrm{a}}}_3, \text{ɓ}_3], [\text{ɕ}_3, \text{ɗ}_3], [\varrho_3, \text{ʄ}_3]\rangle\end{matrix}\right); \mathbb{r}_3\right)\\
&= \left(\left(\begin{matrix}\langle min(min(\mathrm{u}_1, \mathrm{u}_2), \mathrm{u}_3), min(min(\mathrm{v}_1, \mathrm{v}_2), \mathrm{v}_3), max(max(\varepsilon_1, \varepsilon_2), \varepsilon_3)\rangle,\\ [min(min(\dot{\bar{\mathrm{a}}}_1, \dot{\bar{\mathrm{a}}}_2), \dot{\bar{\mathrm{a}}}_3), min(min(\text{ɓ}_1, \text{ɓ}_2), \text{ɓ}_3)],\\ \langle[min(min(\text{ɕ}_1, \text{ɕ}_2), \text{ɕ}_3), min(min(\text{ɗ}_1, \text{ɗ}_2), \text{ɗ}_3)],\rangle\\ [max(max(\varrho_1, \varrho_2), \varrho_3), max(max(\text{ʄ}_1, \text{ʄ}_2), \text{ʄ}_3)]\end{matrix}\right); min(min(\mathbb{r}_1, \mathbb{r}_2), \mathbb{r}_3)\right)\\
&= \left(\left(\begin{matrix}\langle min\big(\mathrm{u}_1, min(\mathrm{u}_2, \mathrm{u}_3)\big), min\big(\mathrm{v}_1, min(\mathrm{v}_2, \mathrm{v}_3)\big), max\big(\varepsilon_1, max(\varepsilon_2, \varepsilon_3)\big)\rangle,\\ \big[min\big(\dot{\bar{\mathrm{a}}}_1, min(\dot{\bar{\mathrm{a}}}_2, \dot{\bar{\mathrm{a}}}_3)\big), min\big(\text{ɓ}_1, min(\text{ɓ}_2, \text{ɓ}_3)\big)\big],\\ \langle\big[min\big(\text{ɕ}_1, min(\text{ɕ}_2, \text{ɕ}_3)\big), min\big(\text{ɗ}_1, min(\text{ɗ}_2, \text{ɗ}_3)\big)\big],\rangle\\ \big[max\big(\varrho_1, max(\varrho_2, \varrho_3)\big), max\big(\text{ʄ}_1, max(\text{ʄ}_2, \text{ʄ}_3)\big)\big]\end{matrix}\right); min\big(\mathbb{r}_1, min(\mathbb{r}_2, \mathbb{r}_3)\big)\right)\\
&= \left(\left(\begin{matrix}\langle \mathrm{u}_1, \mathrm{v}_1, \varepsilon_1\rangle,\\ \langle[\dot{\bar{\mathrm{a}}}_1, \text{ɓ}_1], [\text{ɕ}_1, \text{ɗ}_1], [\varrho_1, \text{ʄ}_1]\rangle\end{matrix}\right); \mathbb{r}_1\right) \sqcap_{min} \left(\left(\begin{matrix}\langle min(\mathrm{u}_2, \mathrm{u}_3), min(\mathrm{v}_2, \mathrm{v}_3), max(\varepsilon_2, \varepsilon_3)\rangle,\\ [min(\dot{\bar{\mathrm{a}}}_2, \dot{\bar{\mathrm{a}}}_3), min(\text{ɓ}_2, \text{ɓ}_3)],\\ \langle[min(\text{ɕ}_2, \text{ɕ}_3), min(\text{ɗ}_2, \text{ɗ}_3)],\rangle\\ [max(\varrho_2, \varrho_3), max(\text{ʄ}_2, \text{ʄ}_3)]\end{matrix}\right); min(\mathbb{r}_2, \mathbb{r}_3)\right)\\
&= \mathcal{M}_1 \sqcap_{min} (\mathcal{M}_2 \sqcap_{min} \mathcal{M}_3).
\end{aligned}
$$

ii) We have

$$
\begin{aligned}
&(\mathcal{M}_1 \sqcap_{max} \mathcal{M}_2) \sqcap_{max} \mathcal{M}_3\\
&= \left(\left(\begin{matrix}\langle min(\mathrm{u}_1, \mathrm{u}_2), min(\mathrm{v}_1, \mathrm{v}_2), max(\varepsilon_1, \varepsilon_2)\rangle,\\ [min(\dot{\bar{\mathrm{a}}}_1, \dot{\bar{\mathrm{a}}}_2), min(\text{ɓ}_1, \text{ɓ}_2)],\\ \langle[min(\text{ɕ}_1, \text{ɕ}_2), min(\text{ɗ}_1, \text{ɗ}_2)],\rangle\\ [max(\varrho_1, \varrho_2), max(\text{ʄ}_1, \text{ʄ}_2)]\end{matrix}\right); max(\mathbb{r}_1, \mathbb{r}_2)\right) \sqcap_{max} \left(\left(\begin{matrix}\langle \mathrm{u}_3, \mathrm{v}_3, \varepsilon_3\rangle,\\ \langle[\dot{\bar{\mathrm{a}}}_3, \text{ɓ}_3], [\text{ɕ}_3, \text{ɗ}_3], [\varrho_3, \text{ʄ}_3]\rangle\end{matrix}\right); \mathbb{r}_3\right)\\
&= \left(\left(\begin{matrix}\langle min(min(\mathrm{u}_1, \mathrm{u}_2), \mathrm{u}_3), min(min(\mathrm{v}_1, \mathrm{v}_2), \mathrm{v}_3), max(max(\varepsilon_1, \varepsilon_2), \varepsilon_3)\rangle,\\ [min(min(\dot{\bar{\mathrm{a}}}_1, \dot{\bar{\mathrm{a}}}_2), \dot{\bar{\mathrm{a}}}_3), min(min(\text{ɓ}_1, \text{ɓ}_2), \text{ɓ}_3)],\\ \langle[min(min(\text{ɕ}_1, \text{ɕ}_2), \text{ɕ}_3), min(min(\text{ɗ}_1, \text{ɗ}_2), \text{ɗ}_3)],\rangle\\ [max(max(\varrho_1, \varrho_2), \varrho_3), max(max(\text{ʄ}_1, \text{ʄ}_2), \text{ʄ}_3)]\end{matrix}\right); max(max(\mathbb{r}_1, \mathbb{r}_2), \mathbb{r}_3)\right)
\end{aligned}
$$

$$= \left(\left(\begin{array}{c} \langle min(\mathrm{u}_1, min(\mathrm{u}_2, \mathrm{u}_3)), min(\mathrm{v}_1, min(\mathrm{v}_2, \mathrm{v}_3)), max(\varepsilon_1, max(\varepsilon_2, \varepsilon_3))\rangle, \\ [min(\text{ȁ}_1, min(\text{ȁ}_2, \text{ȁ}_3)), min(\text{ɓ}_1, min(\text{ɓ}_2, \text{ɓ}_3))], \\ \langle [min(\text{ɕ}_1, min(\text{ɕ}_2, \text{ɕ}_3)), min(\text{ɗ}_1, min(\text{ɗ}_2, \text{ɗ}_3))], \rangle \\ [max(\varrho_1, max(\varrho_2, \varrho_3)), max(\text{ʄ}_1, max(\text{ʄ}_2, \text{ʄ}_3))] \end{array}\right); max(\text{𝕣}_1, max(\text{𝕣}_2, \text{𝕣}_3))\right)$$

$$= \left(\left(\begin{array}{c} \langle \mathrm{u}_1, \mathrm{v}_1, \varepsilon_1\rangle, \\ \langle [\text{ȁ}_1, \text{ɓ}_1], [\text{ɕ}_1, \text{ɗ}_1], [\varrho_1, \text{ʄ}_1]\rangle \end{array}\right); \text{𝕣}_1\right) \sqcap_{max} \left(\left(\begin{array}{c} \langle min(\mathrm{u}_2, \mathrm{u}_3), min(\mathrm{v}_2, \mathrm{v}_3), max(\varepsilon_2, \varepsilon_3)\rangle, \\ [min(\text{ȁ}_2, \text{ȁ}_3), min(\text{ɓ}_2, \text{ɓ}_3)], \\ \langle [min(\text{ɕ}_2, \text{ɕ}_3), min(\text{ɗ}_2, \text{ɗ}_3)], \rangle \\ [max(\varrho_2, \varrho_3), max(\text{ʄ}_2, \text{ʄ}_3)] \end{array}\right); max(\text{𝕣}_2, \text{𝕣}_3)\right)$$

$$= \mathcal{M}_1 \sqcap_{max} (\mathcal{M}_2 \sqcap_{max} \mathcal{M}_3).$$

iii) We have

$$(\mathcal{M}_1 \sqcup_{min} \mathcal{M}_2) \sqcup_{min} \mathcal{M}_3$$

$$= \left(\left(\begin{array}{c} \langle max(\mathrm{u}_1, \mathrm{u}_2), max(\mathrm{v}_1, \mathrm{v}_2), min(\varepsilon_1, \varepsilon_2)\rangle, \\ [max(\text{ȁ}_1, \text{ȁ}_2), max(\text{ɓ}_1, \text{ɓ}_2)], \\ \langle [max(\text{ɕ}_1, \text{ɕ}_2), max(\text{ɗ}_1, \text{ɗ}_2)], \rangle \\ [min(\varrho_1, \varrho_2), min(\text{ʄ}_1, \text{ʄ}_2)] \end{array}\right); min(\text{𝕣}_1, \text{𝕣}_2)\right) \sqcup_{min} \left(\left(\begin{array}{c} \langle \mathrm{u}_3, \mathrm{v}_3, \varepsilon_3\rangle, \\ \langle [\text{ȁ}_3, \text{ɓ}_3], [\text{ɕ}_3, \text{ɗ}_3], [\varrho_3, \text{ʄ}_3]\rangle \end{array}\right); \text{𝕣}_3\right)$$

$$= \left(\left(\begin{array}{c} \langle max(max(\mathrm{u}_1, \mathrm{u}_2), \mathrm{u}_3), max(max(\mathrm{v}_1, \mathrm{v}_2), \mathrm{v}_3), min(min(\varepsilon_1, \varepsilon_2), \varepsilon_3)\rangle, \\ [max(max(\text{ȁ}_1, \text{ȁ}_2), \text{ȁ}_3), max(max(\text{ɓ}_1, \text{ɓ}_2), \text{ɓ}_3)], \\ \langle [max(max(\text{ɕ}_1, \text{ɕ}_2), \text{ɕ}_3), max(max(\text{ɗ}_1, \text{ɗ}_2), \text{ɗ}_3)], \rangle \\ [min(min(\varrho_1, \varrho_2), \varrho_3), min(min(\text{ʄ}_1, \text{ʄ}_2), \text{ʄ}_3)] \end{array}\right); min(min(\text{𝕣}_1, \text{𝕣}_2), \text{𝕣}_3)\right),$$

$$(\mathcal{M}_1 \sqcup_{min} \mathcal{M}_2) \sqcup_{min} \mathcal{M}_3$$

$$= \left(\left(\begin{array}{c} \langle max(\mathrm{u}_1, max(\mathrm{u}_2, \mathrm{u}_3)), max(\mathrm{v}_1, max(\mathrm{v}_2, \mathrm{v}_3)), min(\varepsilon_1, min(\varepsilon_2, \varepsilon_3))\rangle, \\ [max(\text{ȁ}_1, max(\text{ȁ}_2, \text{ȁ}_3)), max(\text{ɓ}_1, max(\text{ɓ}_2, \text{ɓ}_3))], \\ \langle [max(\text{ɕ}_1, max(\text{ɕ}_2, \text{ɕ}_3)), max(\text{ɗ}_1, max(\text{ɗ}_2, \text{ɗ}_3))], \rangle \\ [min(\varrho_1, min(\varrho_2, \varrho_3)), min(\text{ʄ}_1, min(\text{ʄ}_2, \text{ʄ}_3))] \end{array}\right); min(\text{𝕣}_1, min(\text{𝕣}_2, \text{𝕣}_3))\right)$$

$$= \left(\left(\begin{array}{c} \langle \mathrm{u}_1, \mathrm{v}_1, \varepsilon_1\rangle, \\ \langle [\text{ȁ}_1, \text{ɓ}_1], [\text{ɕ}_1, \text{ɗ}_1], [\varrho_1, \text{ʄ}_1]\rangle \end{array}\right); \text{𝕣}_1\right) \sqcup_{min} \left(\left(\begin{array}{c} \langle max(\mathrm{u}_2, \mathrm{u}_3), max(\mathrm{v}_2, \mathrm{v}_3), min(\varepsilon_2, \varepsilon_3)\rangle, \\ [max(\text{ȁ}_2, \text{ȁ}_3), max(\text{ɓ}_2, \text{ɓ}_3)], \\ \langle [max(\text{ɕ}_2, \text{ɕ}_3), max(\text{ɗ}_2, \text{ɗ}_3)], \rangle \\ [min(\varrho_2, \varrho_3), min(\text{ʄ}_2, \text{ʄ}_3)] \end{array}\right); min(\text{𝕣}_2, \text{𝕣}_3)\right)$$

$$= \mathcal{M}_1 \sqcup_{min} (\mathcal{M}_2 \sqcup_{min} \mathcal{M}_3).$$

iv) We have

$$(\mathcal{M}_1 \sqcup_{max} \mathcal{M}_2) \sqcup_{max} \mathcal{M}_3$$

$$= \left(\left(\begin{array}{c} \langle max(\mathrm{u}_1, \mathrm{u}_2), max(\mathrm{v}_1, \mathrm{v}_2), min(\varepsilon_1, \varepsilon_2)\rangle, \\ [max(\text{ȁ}_1, \text{ȁ}_2), max(\text{ɓ}_1, \text{ɓ}_2)], \\ \langle [max(\text{ɕ}_1, \text{ɕ}_2), max(\text{ɗ}_1, \text{ɗ}_2)], \rangle \\ [min(\varrho_1, \varrho_2), min(\text{ʄ}_1, \text{ʄ}_2)] \end{array}\right); max(\text{𝕣}_1, \text{𝕣}_2)\right) \sqcup_{max} \left(\left(\begin{array}{c} \langle \mathrm{u}_3, \mathrm{v}_3, \varepsilon_3\rangle, \\ \langle [\text{ȁ}_3, \text{ɓ}_3], [\text{ɕ}_3, \text{ɗ}_3], [\varrho_3, \text{ʄ}_3]\rangle \end{array}\right); \text{𝕣}_3\right)$$

$$= \left(\begin{pmatrix} \langle max(max(\mathrm{u}_1, \mathrm{u}_2), \mathrm{u}_3), max(max(\mathrm{v}_1, \mathrm{v}_2), \mathrm{v}_3), min(min(\varepsilon_1, \varepsilon_2), \varepsilon_3) \rangle, \\ [max(max(\bar{\mathring{a}}_1, \bar{\mathring{a}}_2), \bar{\mathring{a}}_3), max(max(ɓ_1, ɓ_2), ɓ_3)], \\ \langle [max(max(ɕ_1, ɕ_2), ɕ_3), max(max(ɗ_1, ɗ_2), ɗ_3)], \rangle \\ [min(min(\varrho_1, \varrho_2), \varrho_3), min(min(ƒ_1, ƒ_2), ƒ_3)] \end{pmatrix}; max(max(𝕣_1, 𝕣_2), 𝕣_3) \right)$$

$$= \left(\begin{pmatrix} \langle max\big(\mathrm{u}_1, max(\mathrm{u}_2, \mathrm{u}_3)\big), max\big(\mathrm{v}_1, max(\mathrm{v}_2, \mathrm{v}_3)\big), min\big(\varepsilon_1, (\varepsilon_2, \varepsilon_3)\big) \rangle, \\ \big[max\big(\bar{\mathring{a}}_1, max(\bar{\mathring{a}}_2, \bar{\mathring{a}}_3)\big), max\big(ɓ_1, max(ɓ_2, ɓ_3)\big)\big], \\ \langle \big[max\big(ɕ_1, max(ɕ_2, ɕ_3)\big), max\big(ɗ_1, max(ɗ_2, ɗ_3)\big)\big], \rangle \\ \big[min\big(\varrho_1, min(\varrho_2, \varrho_3)\big), min\big(ƒ_1, min(ƒ_2, ƒ_3)\big)\big] \end{pmatrix}; max\big(𝕣_1, (𝕣_2, 𝕣_3)\big) \right)$$

$$= \left(\begin{pmatrix} \langle \mathrm{u}_1, \mathrm{v}_1, \varepsilon_1 \rangle, \\ \langle [\bar{\mathring{a}}_1, ɓ_1], [ɕ_1, ɗ_1], [\varrho_1, ƒ_1] \rangle \end{pmatrix}; 𝕣_1 \right) \sqcup_{max} \left(\begin{pmatrix} \langle max(\mathrm{u}_2, \mathrm{u}_3), max(\mathrm{v}_2, \mathrm{v}_3), min(\varepsilon_2, \varepsilon_3) \rangle, \\ [max(\bar{\mathring{a}}_2, \bar{\mathring{a}}_3), max(ɓ_2, ɓ_3)], \\ \langle [max(ɕ_2, ɕ_3), max(ɗ_2, ɗ_3)], \rangle \\ [min(\varrho_2, \varrho_3), min(ƒ_2, ƒ_3)] \end{pmatrix}; max(𝕣_2, 𝕣_3) \right)$$

$$= \mathcal{M}_1 \sqcup_{max} (\mathcal{M}_2 \sqcup_{max} \mathcal{M}_3).$$

Theorem 3. Let $\mathcal{M}_1 = \left(\begin{pmatrix} \langle \mathrm{u}_1, \mathrm{v}_1, \varepsilon_1 \rangle, \\ \langle [\bar{\mathring{a}}_1, ɓ_1], [ɕ_1, ɗ_1], [\varrho_1, ƒ_1] \rangle \end{pmatrix}; 𝕣_1 \right)$, $\mathcal{M}_2 = \left(\begin{pmatrix} \langle \mathrm{u}_2, \mathrm{v}_2, \varepsilon_2 \rangle, \\ \langle [\bar{\mathring{a}}_2, ɓ_2], [ɕ_2, ɗ_2], [\varrho_2, ƒ_2] \rangle \end{pmatrix}; 𝕣_2 \right)$, and $\mathcal{M}_3 = \left(\begin{pmatrix} \langle \mathrm{u}_3, \mathrm{v}_3, \varepsilon_3 \rangle, \\ \langle [\bar{\mathring{a}}_3, ɓ_3], [ɕ_3, ɗ_3], [\varrho_3, ƒ_3] \rangle \end{pmatrix}; 𝕣_3 \right)$ be three CuCP-FSs. Then

i) $(\mathcal{M}_1 \sqcap_{min} \mathcal{M}_2) \sqcup_{min} \mathcal{M}_3 = (\mathcal{M}_1 \sqcup_{min} \mathcal{M}_3) \sqcap_{min} (\mathcal{M}_2 \sqcup_{min} \mathcal{M}_3)$;
ii) $(\mathcal{M}_1 \sqcap_{min} \mathcal{M}_2) \sqcup_{max} \mathcal{M}_3 = (\mathcal{M}_1 \sqcup_{max} \mathcal{M}_3) \sqcap_{min} (\mathcal{M}_2 \sqcup_{max} \mathcal{M}_3)$;
iii) $(\mathcal{M}_1 \sqcap_{max} \mathcal{M}_2) \sqcup_{min} \mathcal{M}_3 = (\mathcal{M}_1 \sqcup_{min} \mathcal{M}_3) \sqcap_{max} (\mathcal{M}_2 \sqcup_{min} \mathcal{M}_3)$;
iv) $(\mathcal{M}_1 \sqcap_{max} \mathcal{M}_2) \sqcup_{max} \mathcal{M}_3 = (\mathcal{M}_1 \sqcup_{max} \mathcal{M}_3) \sqcap_{max} (\mathcal{M}_2 \sqcup_{max} \mathcal{M}_3)$;
v) $(\mathcal{M}_1 \sqcup_{min} \mathcal{M}_2) \sqcap_{min} \mathcal{M}_3 = (\mathcal{M}_1 \sqcap_{min} \mathcal{M}_3) \sqcup_{min} (\mathcal{M}_2 \sqcap_{min} \mathcal{M}_3)$;
vi) $(\mathcal{M}_1 \sqcup_{min} \mathcal{M}_2) \sqcap_{max} \mathcal{M}_3 = (\mathcal{M}_1 \sqcap_{max} \mathcal{M}_3) \sqcup_{min} (\mathcal{M}_2 \sqcap_{max} \mathcal{M}_3)$;
vii) $(\mathcal{M}_1 \sqcup_{max} \mathcal{M}_2) \sqcap_{min} \mathcal{M}_3 = (\mathcal{M}_1 \sqcap_{min} \mathcal{M}_3) \sqcup_{max} (\mathcal{M}_2 \sqcap_{min} \mathcal{M}_3)$;
viii) $(\mathcal{M}_1 \sqcup_{max} \mathcal{M}_2) \sqcap_{max} \mathcal{M}_3 = (\mathcal{M}_1 \sqcap_{max} \mathcal{M}_3) \sqcup_{max} (\mathcal{M}_2 \sqcap_{max} \mathcal{M}_3)$.

Proof. The proofs of (i)−(viii) are straightforward, and hence are omitted.

Here, we present a result involving the aggregation mechanism that is crucial for converting *n* CuCP-FNs into a single CuCP-FN. Specifically, this theorem is significant for the transformation of multiple expert evaluations into a single assessment.

Theorem 4. Let

$$\left\{ \left(\begin{pmatrix} \langle \mathrm{u}_1, \mathrm{v}_1, \varepsilon_1 \rangle, \\ \langle [\bar{\mathring{a}}_1, ɓ_1], [ɕ_1, ɗ_1], [\varrho_1, ƒ_1] \rangle \end{pmatrix}; \atop 𝕣_1 \right), \left(\begin{pmatrix} \langle \mathrm{u}_2, \mathrm{v}_2, \varepsilon_2 \rangle, \\ \langle [\bar{\mathring{a}}_2, ɓ_2], [ɕ_2, ɗ_2], [\varrho_2, ƒ_2] \rangle \end{pmatrix}; \atop 𝕣_2 \right), \dots, \left(\begin{pmatrix} \langle \mathrm{u}_n, \mathrm{v}_n, \varepsilon_n \rangle, \\ \langle [\bar{\mathring{a}}_n, ɓ_n], [ɕ_n, ɗ_n], [\varrho_n, ƒ_n] \rangle \end{pmatrix}; \atop 𝕣_n \right) \right\}$$

be a collection of *n* CuCP-FNs and then $\left(\begin{pmatrix} \langle \mathrm{u}, \mathrm{v}, \varepsilon \rangle, \\ \langle [\bar{\mathring{a}}, ɓ], [ɕ, ɗ], [\varrho, ƒ] \rangle \end{pmatrix}; \atop 𝕣 \right)$ is a CuCP-FN where $\mathrm{u} = \frac{\sum_{i=1}^{n} \mathrm{u}_i}{n}, \mathrm{v} = \frac{\sum_{i=1}^{n} \mathrm{v}_i}{n}, \varepsilon = \frac{\sum_{i=1}^{n} \varepsilon_i}{n}, \bar{\mathring{a}} = \frac{\sum_{i=1}^{n} \bar{\mathring{a}}_i}{n}, ɓ = \frac{\sum_{i=1}^{n} ɓ_i}{n}, ɕ = \frac{\sum_{i=1}^{n} ɕ_i}{n}, ɗ = \frac{\sum_{i=1}^{n} ɗ_i}{n}, \varrho = \frac{\sum_{i=1}^{n} \varrho_i}{n}, ƒ = \frac{\sum_{i=1}^{n} ƒ_i}{n}$,

and

$$\mathbb{r} = min\left\{max_{1\le i\le n}\sqrt{(ɥ-ɥ_i)^2+(v-v_i)^2+(ε-ε_i)^2+(\bar{ȧ}-\bar{ȧ}_i)^2+(ɓ-ɓ_i)^2+(ɕ-ɕ_i)^2+(ɗ-ɗ_i)^2+(ϱ-ϱ_i)^2+(ʄ-ʄ_i)^2},1\right\}.$$

Proof. We have

$$ɥ+v+ε=\frac{\sum_{i=1}^{n}ɥ_i}{n}+\frac{\sum_{i=1}^{n}v_i}{n}+\frac{\sum_{i=1}^{n}ε_i}{n},$$

$$ɥ+v+ε=\frac{\sum_{i=1}^{n}(ɥ_i+v_i+ε_i)}{n},$$

$$ɥ+v+ε\le\frac{\sum_{i=1}^{n}(1)}{n}=\frac{n}{n}=1.$$

Thus, $ɥ+v+ε\le 1$.

Similarly, $ɓ+ɗ+ʄ\le 1$.

Moreover, by the defined framework of $\mathbb{r}$, it is clear that $\mathbb{r}\in[0,1]$. Therefore, $\left(\left(\begin{matrix}\langle ɥ,v,v\rangle,\\ \langle[\bar{ȧ},ɓ],[ɕ,ɗ],[ϱ,ʄ]\rangle\end{matrix}\right);\atop\mathbb{r}\right)$ is a CuCP-FN.

Definition 8. For any two CuCP-FNs $\mathcal{M}_1=\left(\left(\begin{matrix}\langle ɥ_1,v_1,ε_1\rangle,\\ \langle[\bar{ȧ}_1,ɓ_1],[ɕ_1,ɗ_1],[ϱ_1,ʄ_1]\rangle\end{matrix}\right);\mathbb{r}_1\right)$ and $\mathcal{M}_2=\left(\left(\begin{matrix}\langle ɥ_2,v_2,ε_2\rangle,\\ \langle[\bar{ȧ}_2,ɓ_2],[ɕ_2,ɗ_2],[ϱ_2,ʄ_2]\rangle\end{matrix}\right);\mathbb{r}_2\right)$, the admissible order "$\preccurlyeq$", represented by "$\mathcal{M}_1\preccurlyeq\mathcal{M}_2$", is defined by

$$(\mathbb{r}_1\le\mathbb{r}_2 \text{ for all } \mathbb{v}\in\mathbb{V}) \text{ and}$$

$$(ɥ_1\le ɥ_2, v_1\le v_2, ε_1\ge ε_2, \bar{ȧ}_1\le\bar{ȧ}_2, ɓ_1\le ɓ_2, ɕ_1\le ɕ_2, ɗ_1\le ɗ_2, ϱ_1\ge ϱ_2, ʄ_1\ge ʄ_2).$$

However, this approach does not hold for all CuCP-FNs. For example, let $\mathcal{M}_1=\left(\left(\begin{matrix}\langle 0.4,0.2,0.3\rangle,\\ \langle[0.1,0.4],[0.2,0.3],[0.1,0.2]\rangle\end{matrix}\right);0.6\right)$ and $\mathcal{M}_2=\left(\left(\begin{matrix}\langle 0.5,0.1,0.4\rangle,\\ \langle[0.3,0.5],[0.1,0.3],[0.1,0.2]\rangle\end{matrix}\right);0.8\right)$ indicate two CuCP-FNs, but here, the admissible order does not hold for $\mathcal{M}_2$ and $\mathcal{M}_1$. Thus, these two CuCP-FNs are not comparable based on the proposed approach of admissible order. To compare the CuCP-FNs, we convert the CuCP-FN into a single fuzzy value based on some other well-defined structure.

Let $\mathcal{M}_i=\left(\left(\begin{matrix}\langle ɥ_i,v_i,ε_i\rangle,\\ \langle[\bar{ȧ}_i,ɓ_i],[ɕ_i,ɗ_i],[ϱ_i,ʄ_i]\rangle\end{matrix}\right);\mathbb{r}_i\right)$ represents any CuCP-FN on a universe of discourse $\mathbb{V}$ and then

$$\mathcal{S}\mathrm{F}(\mathcal{M}_i)=\frac{3+\mathbb{r}_i+ɥ_i+v_i+\bar{ȧ}_i+ɓ_i+ɕ_i+ɗ_i-ε_i-ϱ_i-ʄ_i}{10}. \tag{11}$$

The proposed structure is said to be a score function of $\mathcal{M}_i$. It is constructed to convert the CuCP-FN into a scalar value for ranking purposes. In the proposed development, the terms

$u_i, v_i, \bar{\bar{a}}_i, b_i, c_i,$ and d_i reflect positive and neutral MS values, representing positively to overall evaluations, whereas $\varepsilon_i, \varrho_i,$ and f_i correspond to negative MS values, and are therefore subtracted. The normalization factor that is divided by 10 is guaranteed to confine the proposed score function within an interval [0,1], supporting significant ranking and comparison of alternatives. The proposed framework fulfills key features such as monotonicity and boundedness, thereby contributing a mathematically and rationally consistent basis for DM. Moreover, $\mathcal{M}_i \succ \mathcal{M}_j$ if $S\mathrm{F}(\mathcal{M}_i) > S\mathrm{F}(\mathcal{M}_j)$ for all j.

Definition 9. Let $\mathcal{M}_1 = \left(\left(\begin{matrix}\langle u_1, v_1, \varepsilon_1\rangle, \\ \langle[\bar{\bar{a}}_1, b_1], [c_1, d_1], [\varrho_1, f_1]\rangle\end{matrix}\right); r_1\right)$ and $\mathcal{M}_2 = \left(\left(\begin{matrix}\langle u_2, v_2, \varepsilon_2\rangle, \\ \langle[\bar{\bar{a}}_2, b_2], [c_2, d_2], [\varrho_2, f_2]\rangle\end{matrix}\right); r_2\right)$ be two CuCP-FNs and $\mathbb{w} > 0$ reflects any real number. Then several fundamental Bonferroni operational laws for CuCP-FNs are presented as follows:

i) $\mathcal{M}_1 \oplus \mathcal{M}_2$

$$= \left(\left(\begin{matrix}\langle (u_1{}^{\mathbb{p}}.\,u_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (v_1{}^{\mathbb{p}}.\,v_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (\varepsilon_1{}^{\mathbb{p}}.\,\varepsilon_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}\rangle, \\ \langle\left[(\bar{\bar{a}}_1{}^{\mathbb{p}}.\,\bar{\bar{a}}_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (b_1{}^{\mathbb{p}}.\,b_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right], \left[(c_1{}^{\mathbb{p}}.\,c_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (d_1{}^{\mathbb{p}}.\,d_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right], \left[(\varrho_1{}^{\mathbb{p}}.\,\varrho_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (f_1{}^{\mathbb{p}}.\,f_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}_1}}\right]\rangle\end{matrix}\right); \\ (r_1{}^{\mathbb{p}}.\,r_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right);$$

ii) $\mathbb{w}\mathcal{M}_1 \oplus \mathbb{w}\mathcal{M}_2$

$$= \left(\left(\begin{matrix}\langle (\mathbb{w}u_1{}^{\mathbb{p}}.\,\mathbb{w}u_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (\mathbb{w}v_1{}^{\mathbb{p}}.\,\mathbb{w}v_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (\mathbb{w}\varepsilon_1{}^{\mathbb{p}}.\,\mathbb{w}\varepsilon_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}\rangle, \\ \langle\begin{matrix}\left[(\mathbb{w}\bar{\bar{a}}_1{}^{\mathbb{p}}.\,\mathbb{w}\bar{\bar{a}}_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (\mathbb{w}b_1{}^{\mathbb{p}}.\,\mathbb{w}b_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right], \left[(\mathbb{w}c_1{}^{\mathbb{p}}.\,\mathbb{w}c_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (\mathbb{w}d_1{}^{\mathbb{p}}.\,\mathbb{w}d_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right], \\ \left[(\mathbb{w}\varrho_1{}^{\mathbb{p}}.\,\mathbb{w}\varrho_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (\mathbb{w}f_1{}^{\mathbb{p}}.\,\mathbb{w}f_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}_1}}\right]\end{matrix}\rangle\end{matrix}\right); \\ (\mathbb{w}r_1{}^{\mathbb{p}}.\,\mathbb{w}r_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right).$$

Theorem 5. Suppose $\mathcal{M}_1 = \left(\left(\begin{matrix}\langle u_1, v_1, \varepsilon_1\rangle, \\ \langle[\bar{\bar{a}}_1, b_1], [c_1, d_1], [\varrho_1, f_1]\rangle\end{matrix}\right); r_1\right)$ and $\mathcal{M}_2 = \left(\left(\begin{matrix}\langle u_2, v_2, \varepsilon_2\rangle, \\ \langle[\bar{\bar{a}}_2, b_2], [c_2, d_2], [\varrho_2, f_2]\rangle\end{matrix}\right); r_2\right)$ are two CuCP-FNs and $\mathbb{w} > 0$. Then

i) $\mathcal{M}_1 \oplus \mathcal{M}_2$ is a CuCP-FN;
ii) $\mathbb{w}\mathcal{M}_1 \oplus \mathbb{w}\mathcal{M}_2$ is a CuCP-FN.

Proof. i) Let $\mathcal{M}_1 = \left(\left(\begin{matrix}\langle u_1, v_1, \varepsilon_1\rangle, \\ \langle[\bar{\bar{a}}_1, b_1], [c_1, d_1], [\varrho_1, f_1]\rangle\end{matrix}\right); r_1\right)$ and $\mathcal{M}_2 = \left(\left(\begin{matrix}\langle u_2, v_2, \varepsilon_2\rangle, \\ \langle[\bar{\bar{a}}_2, b_2], [c_2, d_2], [\varrho_2, f_2]\rangle\end{matrix}\right); r_2\right)$ be two CuCP-FNs and $\mathbb{w} > 0$ reflects any real number. Then

$$
\begin{aligned}
&\mathcal{M}_1 \oplus \mathcal{M}_2 \\
&= \left(\left(\begin{array}{c} \langle (u_1{}^{\mathbb{p}}.u_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (v_1{}^{\mathbb{p}}.v_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (\varepsilon_1{}^{\mathbb{p}}.\varepsilon_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \rangle, \\ \langle \left[\left(\bar{a}_1{}^{\mathbb{p}}.\bar{a}_2{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (ɓ_1{}^{\mathbb{p}}.ɓ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \left[(ɕ_1{}^{\mathbb{p}}.ɕ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (ɗ_1{}^{\mathbb{p}}.ɗ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \left[(\varrho_1{}^{\mathbb{p}}.\varrho_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (ʄ_1{}^{\mathbb{p}}.ʄ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}{}_1 \right] \rangle \end{array} \right); \\ (\mathbb{r}_1{}^{\mathbb{p}}.\mathbb{r}_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right).
\end{aligned}
$$

For the CuCP-FN, we need to prove that the sum of positive, neutral, and negative MS values in picture fuzzy information is at most 1 and the sum of the supremum of positive, neutral, and negative MS values in IVP-F information is at most 1. Further, the radius part could be in [0,1]. Since, $0 \leq u_1 + v_1 + \varepsilon_1 \leq 1$ and $0 \leq u_2 + v_2 + \varepsilon_2 \leq 1$, then $u_1{}^{\mathbb{p}} \leq u_1, u_2{}^{\mathbb{q}} \leq u_2, v_1{}^{\mathbb{p}} \leq v_1, v_2{}^{\mathbb{q}} \leq v_2, \varepsilon_1{}^{\mathbb{p}} \leq \varepsilon_1, \varepsilon_2{}^{\mathbb{q}} \leq \varepsilon_2$. This implies that $0 \leq u_1{}^{\mathbb{p}} + v_1{}^{\mathbb{p}} + \varepsilon_1{}^{\mathbb{p}} \leq 1$ and $0 \leq u_2{}^{\mathbb{q}} + v_2{}^{\mathbb{q}} + \varepsilon_2{}^{\mathbb{q}} \leq 1$. Thus, $0 \leq u_1{}^{\mathbb{p}}.u_2{}^{\mathbb{q}} + v_1{}^{\mathbb{p}}.v_2{}^{\mathbb{q}} + \varepsilon_1{}^{\mathbb{p}}.\varepsilon_2{}^{\mathbb{q}} \leq 1$ and hence $0 \leq (u_1{}^{\mathbb{p}}.u_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} + (v_1{}^{\mathbb{p}}.v_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} + (\varepsilon_1{}^{\mathbb{p}}.\varepsilon_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \leq 1$.

Similarly, we can prove that $0 \leq (ɓ_1{}^{\mathbb{p}}.ɓ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} + (ɗ_1{}^{\mathbb{p}}.ɗ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} + (ʄ_1{}^{\mathbb{p}}.ʄ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \leq 1$.

Also since $\mathbb{r}_1, \mathbb{r}_2 \in [0,1]$, then $\mathbb{r}_1.\mathbb{r}_2 \in [0,1]$. But $\mathbb{r}_1{}^{\mathbb{p}} \leq \mathbb{r}_1$ and $\mathbb{r}_2{}^{\mathbb{q}} \leq \mathbb{r}_2$, therefore $\mathbb{r}_1{}^{\mathbb{p}}.\mathbb{r}_2{}^{\mathbb{q}} \in [0,1]$, and hence $(\mathbb{r}_1{}^{\mathbb{p}}.\mathbb{r}_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \in [0,1]$. Thus, the Bonferroni operation $\mathcal{M}_1 \oplus \mathcal{M}_2$ is a CuCP-FN.

ii) The Proof of (ii) is same and hence is omitted.

Theorem 6. Suppose $\mathcal{M}_1 = \left(\left(\begin{array}{c} \langle u_1, v_1, \varepsilon_1 \rangle, \\ \langle [\bar{a}_1, ɓ_1], [ɕ_1, ɗ_1], [\varrho_1, ʄ_1] \rangle \end{array} \right); \mathbb{r}_1 \right)$ and $\mathcal{M}_2 = \left(\left(\begin{array}{c} \langle u_2, v_2, \varepsilon_2 \rangle, \\ \langle [\bar{a}_2, ɓ_2], [ɕ_2, ɗ_2], [\varrho_2, ʄ_2] \rangle \end{array} \right); \mathbb{r}_2 \right)$ are two CuCP-FNs and $\mathbb{w} > 0$. Then

i) $\mathcal{M}_1 \oplus \mathcal{M}_2 = \mathcal{M}_2 \oplus \mathcal{M}_1$; ii) $\mathbb{w}\mathcal{M}_1 \oplus \mathbb{w}\mathcal{M}_2 = \mathbb{w}\mathcal{M}_2 \oplus \mathbb{w}\mathcal{M}_1$.

Proof. i) We have

$$
\begin{aligned}
&\mathcal{M}_1 \oplus \mathcal{M}_2 \\
&= \left(\left(\begin{array}{c} \langle (u_1{}^{\mathbb{p}}.u_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (v_1{}^{\mathbb{p}}.v_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (\varepsilon_1{}^{\mathbb{p}}.\varepsilon_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \rangle, \\ \langle \left[\left(\bar{a}_1{}^{\mathbb{p}}.\bar{a}_2{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (ɓ_1{}^{\mathbb{p}}.ɓ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \left[(ɕ_1{}^{\mathbb{p}}.ɕ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (ɗ_1{}^{\mathbb{p}}.ɗ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \left[(\varrho_1{}^{\mathbb{p}}.\varrho_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (ʄ_1{}^{\mathbb{p}}.ʄ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}{}_1 \right] \rangle \end{array} \right); \\ (\mathbb{r}_1{}^{\mathbb{p}}.\mathbb{r}_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right) \\
&= \left(\left(\begin{array}{c} \langle (u_2{}^{\mathbb{q}}.u_1{}^{\mathbb{p}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (v_2{}^{\mathbb{q}}.v_1{}^{\mathbb{p}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (\varepsilon_2{}^{\mathbb{q}}.\varepsilon_1{}^{\mathbb{p}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \rangle, \\ \langle \left[\left(\bar{a}_2{}^{\mathbb{q}}.\bar{a}_1{}^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(ɓ_2{}^{\mathbb{q}}.ɓ_1{}^{\mathbb{p}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \left[(ɕ_2{}^{\mathbb{q}}.ɕ_1{}^{\mathbb{p}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (ɗ_2{}^{\mathbb{q}}.ɗ_1{}^{\mathbb{p}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \left[(\varrho_2{}^{\mathbb{q}}.\varrho_1{}^{\mathbb{p}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (ʄ_2{}^{\mathbb{q}}.ʄ_1{}^{\mathbb{p}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}{}_1 \right] \rangle \end{array} \right); \\ (\mathbb{r}_2{}^{\mathbb{q}}.\mathbb{r}_1{}^{\mathbb{p}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right) \\
&= \mathcal{M}_2 \oplus \mathcal{M}_1.
\end{aligned}
$$

Thus,

$$\mathcal{M}_1 \oplus \mathcal{M}_2 = \mathcal{M}_2 \oplus \mathcal{M}_1.$$

ii) The proof of (ii) is the same and hence is omitted.

Theorem 7. Let $\mathcal{M}_1 = \left(\left(\begin{matrix}\langle \mathfrak{u}_1, \mathfrak{v}_1, \varepsilon_1 \rangle, \\ \langle [\bar{\mathfrak{a}}_1, \mathfrak{b}_1], [\mathfrak{c}_1, \mathfrak{d}_1], [\varrho_1, \mathfrak{f}_1] \rangle\end{matrix}\right); \mathbb{r}_1\right)$, $\mathcal{M}_2 = \left(\left(\begin{matrix}\langle \mathfrak{u}_2, \mathfrak{v}_2, \varepsilon_2 \rangle, \\ \langle [\bar{\mathfrak{a}}_2, \mathfrak{b}_2], [\mathfrak{c}_2, \mathfrak{d}_2], [\varrho_2, \mathfrak{f}_2] \rangle\end{matrix}\right); \mathbb{r}_2\right)$, and $\mathcal{M}_3 = \left(\left(\begin{matrix}\langle \mathfrak{u}_3, \mathfrak{v}_3, \varepsilon_3 \rangle, \\ \langle [\bar{\mathfrak{a}}_3, \mathfrak{b}_3], [\mathfrak{c}_3, \mathfrak{d}_3], [\varrho_3, \mathfrak{f}_3] \rangle\end{matrix}\right); \mathbb{r}_3\right)$ be three CuCP-FNs and $\mathbb{w} > 0$. Then

i) $(\mathcal{M}_1 \oplus \mathcal{M}_2) \oplus \mathcal{M}_3 = \mathcal{M}_1 \oplus (\mathcal{M}_2 \oplus \mathcal{M}_3)$;

ii) $(\mathbb{w}\mathcal{M}_1 \oplus \mathbb{w}\mathcal{M}_2) \oplus \mathbb{w}\mathcal{M}_3 = \mathbb{w}\mathcal{M}_1 \oplus (\mathbb{w}\mathcal{M}_2 \oplus \mathbb{w}\mathcal{M}_3)$.

Proof. The proof is straightforward and hence is omitted.

4. Cubic circular picture fuzzy aggregation operators

This section demonstrates Bonferroni mean operators (BMOs) within the CuCP-FS environments. The BMOs are a class of AOs that are crucial for capturing interactions and interrelationships among multiple decision attributes. Unlike simple arithmetic or geometric AOs, which deal with attributes independently, BMOs integrate pairwise combinations of attributes. The parameters $\mathbb{p}$ and $\mathbb{q}$ offer flexibility in controlling the impact of joint and individual contributions, enabling adaptation to various DM situations. By incorporating interaction terms, the newly defined operators improve discrimination among decision alternatives that show similar individual criterion values but are distinct in joint behavior. Within the framework of CuCP-FSs, the proposed operators describe each alternative by picture fuzzy information including a reference assessment, IVP-F information, and a radius parameter that denotes the fluctuations around a reference assessment. The proposed BMOs contribute many advantages by capturing modeling interdependence and flexibility through parameters, preserving picture fuzzy characteristics, facilitating nonlinear impacts, addressing fluctuations around reference evaluations, and supporting multilayered uncertainty.

These BMOs for CuCP-FNs are presented as follows:

Definition 10. Let $\mathcal{M}_i = \left(\left(\begin{matrix}\langle \mathfrak{u}_i, \mathfrak{v}_i, \varepsilon_i \rangle, \\ \langle [\bar{\mathfrak{a}}_i, \mathfrak{b}_i], [\mathfrak{c}_i, \mathfrak{d}_i], [\varrho_i, \mathfrak{f}_i] \rangle\end{matrix}\right); \mathbb{r}_i\right); i = 1,2,3, \ldots, n,$ reflect a collection of n CuCP-FNs. Then, the cubic circular picture fuzzy Bonferroni mean operator CuCPFBMO is a map $\mathrm{CuCPFBMO}: {\mathcal{M}_i}^n \to \mathcal{M}_i$, demonstrated as

$$\mathrm{CuCPFBMO}(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \ldots, \mathcal{M}_n) = \oplus_{i=1}^{n} \mathcal{M}_i,$$

$$
\begin{aligned}
&\mathrm{CuCPFBMO}(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\ldots,\mathcal{M}_n)\\
&=\left(\left(\begin{array}{c}
\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{u}_i{}^{\mathbb{p}}.\,\text{u}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{v}_i{}^{\mathbb{p}}.\,\text{v}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon_i{}^{\mathbb{p}}.\,\varepsilon_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right),\\
\left(\begin{array}{c}
\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\bar{\text{a}}_i{}^{\mathbb{p}}.\,\bar{\text{a}}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{ɓ}_i{}^{\mathbb{p}}.\,\text{ɓ}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\\
\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{ɕ}_i{}^{\mathbb{p}}.\,\text{ɕ}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{ɗ}_i{}^{\mathbb{p}}.\,\text{ɗ}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\\
\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varrho_i{}^{\mathbb{p}}.\,\varrho_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{ʄ}_i{}^{\mathbb{p}}.\,\text{ʄ}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right]
\end{array}\right)
\end{array}\right);\\
&\qquad\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{ɼ}_i{}^{\mathbb{p}}.\,\text{ɼ}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right),
\end{aligned}
$$

where for each $\mathbb{v}\in\mathbb{V}$:

i) $\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{u}_i{}^{\mathbb{p}}.\,\text{u}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}$ and $\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\bar{\text{a}}_i{}^{\mathbb{p}}.\,\bar{\text{a}}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{ɓ}_i{}^{\mathbb{p}}.\,\text{ɓ}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right]$ reflect the positive MS part of picture fuzzy information and interval-valued picture fuzzy information, respectively;

ii) $\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{v}_i{}^{\mathbb{p}}.\,\text{v}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}$ and $\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{ɕ}_i{}^{\mathbb{p}}.\,\text{ɕ}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{ɗ}_i{}^{\mathbb{p}}.\,\text{ɗ}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right]$ reflect the neutral MS part of picture fuzzy information and interval-valued picture fuzzy information, respectively;

iii) $\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon_i{}^{\mathbb{p}}.\,\varepsilon_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}$ and $\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varrho_i{}^{\mathbb{p}}.\,\varrho_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{ʄ}_i{}^{\mathbb{p}}.\,\text{ʄ}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right]$ reflect the negative MS part of picture fuzzy information and interval-valued picture fuzzy information, respectively;

iv) $\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\text{ɼ}_i{}^{\mathbb{p}}.\,\text{ɼ}_j{}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}$ represents radius uncertainty; and

v) the parameters $\mathbb{p}$ and $\mathbb{q}$ indicate control parameters.

Example 1. Let $\mathcal{M}_1=\left(\left(\begin{array}{c}\langle 0.4,0.2,0.1\rangle,\\ \langle[0.2,0.4],[0.1,0.3],[0.1,0.2]\rangle\end{array}\right);0.2\right)$, $\mathcal{M}_2=\left(\left(\begin{array}{c}\langle 0.2,0.5,0.1\rangle,\\ \langle[0.1,0.3],[0.2,0.5],[0.1,0.2]\rangle\end{array}\right);0.3\right)$, and $\mathcal{M}_1=\left(\left(\begin{array}{c}\langle 0.3,0.1,0.3\rangle,\\ \langle[0.2,0.5],[0.1,0.2],[0.2,0.3]\rangle\end{array}\right);0.5\right)$ be three CuCP-FNs. Then, for $\mathbb{p},\mathbb{q}=1$, we have

$$
\begin{aligned}
&\mathrm{CuCPFBMO}(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3)\\
&=\left(\left(\begin{array}{c}
\left(\begin{array}{c}
\left(\frac{1}{3(3-1)}\left((\mathfrak{u}_1.\mathfrak{u}_2)+(\mathfrak{u}_1.\mathfrak{u}_3)+(\mathfrak{u}_2.\mathfrak{u}_1)+(\mathfrak{u}_2.\mathfrak{u}_3)+\ (\mathfrak{u}_3.\mathfrak{u}_1)+(\mathfrak{u}_3.\mathfrak{u}_2)\right)\right)^{\frac{1}{1+1}},\\
\left(\frac{1}{3(3-1)}\left((\mathrm{v}_1.\mathrm{v}_2)+(\mathrm{v}_1.\mathrm{v}_3)+(\mathrm{v}_2.\mathrm{v}_1)+(\mathrm{v}_2.\mathrm{v}_3)+\ (\mathrm{v}_3.\mathrm{v}_1)+(\mathrm{v}_3.\mathrm{v}_2)\right)\right)^{\frac{1}{1+1}},\\
\left(\frac{1}{3(3-1)}\left((\varepsilon_1.\varepsilon_2)+(\varepsilon_1.\varepsilon_3)+(\varepsilon_2.\varepsilon_1)+(\varepsilon_2.\varepsilon_3)+\ (\varepsilon_3.\varepsilon_1)+(\varepsilon_3.\varepsilon_2)\right)\right)^{\frac{1}{1+1}}
\end{array}\right),\\
\left(\begin{array}{c}
\left[\begin{array}{c}
\left(\frac{1}{3(3-1)}\left((\bar{\mathrm{a}}_1.\bar{\mathrm{a}}_2)+(\bar{\mathrm{a}}_1.\bar{\mathrm{a}}_3)+(\bar{\mathrm{a}}_2.\bar{\mathrm{a}}_1)+(\bar{\mathrm{a}}_2.\bar{\mathrm{a}}_3)+\ (\bar{\mathrm{a}}_3.\bar{\mathrm{a}}_1)+(\bar{\mathrm{a}}_3.\bar{\mathrm{a}}_2)\right)\right)^{\frac{1}{1+1}},\\
\left(\frac{1}{3(3-1)}\left((\mathrm{b}_1.\mathrm{b}_2)+(\mathrm{b}_1.\mathrm{b}_3)+(\mathrm{b}_2.\mathrm{b}_1)+(\mathrm{b}_2.\mathrm{b}_3)+\ (\mathrm{b}_3.\mathrm{b}_1)+(\mathrm{b}_3.\mathrm{b}_2)\right)\right)^{\frac{1}{1+1}}
\end{array}\right],\\
\left[\begin{array}{c}
\left(\frac{1}{3(3-1)}\left((\mathrm{c}_1.\mathrm{c}_2)+(\mathrm{c}_1.\mathrm{c}_3)+(\mathrm{c}_2.\mathrm{c}_1)+(\mathrm{c}_2.\mathrm{c}_3)+\ (\mathrm{c}_3.\mathrm{c}_1)+(\mathrm{c}_3.\mathrm{c}_2)\right)\right)^{\frac{1}{1+1}},\\
\left(\frac{1}{3(3-1)}\left((\mathrm{d}_1.\mathrm{d}_2)+(\mathrm{d}_1.\mathrm{d}_3)+(\mathrm{d}_2.\mathrm{d}_1)+(\mathrm{d}_2.\mathrm{d}_3)+\ (\mathrm{d}_3.\mathrm{d}_1)+(\mathrm{d}_3.\mathrm{d}_2)\right)\right)^{\frac{1}{1+1}}
\end{array}\right],\\
\left[\begin{array}{c}
\left(\frac{1}{3(3-1)}\left((\varrho_1.\varrho_2)+(\varrho_1.\varrho_3)+(\varrho_2.\varrho_1)+(\varrho_2.\varrho_3)+\ (\varrho_3.\varrho_1)+(\varrho_3.\varrho_2)\right)\right)^{\frac{1}{1+1}},\\
\left(\frac{1}{3(3-1)}\left((\mathrm{f}_1.\mathrm{f}_2)+(\mathrm{f}_1.\mathrm{f}_3)+(\mathrm{f}_2.\mathrm{f}_1)+(\mathrm{f}_2.\mathrm{f}_3)+\ (\mathrm{f}_3.\mathrm{f}_1)+(\mathrm{f}_3.\mathrm{f}_2)\right)\right)^{\frac{1}{1+1}}
\end{array}\right]
\end{array}\right)
\end{array}\right);\right.\\
&\qquad\left.\left(\frac{1}{3(3-1)}\left((\mathbb{r}_1.\mathbb{r}_2)+(\mathbb{r}_1.\mathbb{r}_3)+(\mathbb{r}_2.\mathbb{r}_1)+(\mathbb{r}_2.\mathbb{r}_3)+\ (\mathbb{r}_3.\mathbb{r}_1)+(\mathbb{r}_3.\mathbb{r}_2)\right)\right)^{\frac{1}{1+1}}\right).
\end{aligned}
$$

After aggregating these three CuCP-FNs, we get

$$\mathrm{CuCPFBMO}(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3)=\left(\left(\begin{array}{c}\langle 0.2943,0.238,0.1528\rangle,\\ \langle[0.1633,0.3958],[0.1291,0.3215],[0.1291,0.231]\rangle\end{array}\right);0.3215\right).$$

Definition 11. Let $\mathcal{M}_i=\left(\left(\begin{array}{c}\langle\mathfrak{u}_i,\mathrm{v}_i,\varepsilon_i\rangle,\\ \langle[\bar{\mathrm{a}}_i,\mathrm{b}_i],[\mathrm{c}_i,\mathrm{d}_i],[\varrho_i,\mathrm{f}_i]\rangle\end{array}\right);\mathbb{r}_i\right);i=1,2,3,\ldots,n,$ reflect a collection of n CuCP-FNs and $\mathbb{w}_i>0$ shows any real number such that $\sum_{i=1}^{n}\mathbb{w}_i=1$. Then, the cubic circular picture fuzzy weighted Bonferroni mean operator CuCPFWBMO is a map $\mathrm{CuCPFBMO}:\mathcal{M}_i{}^n\to\mathcal{M}_i$, demonstrated as

$$\mathrm{CuCPFWBMO}(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\ldots,\mathcal{M}_n)=\oplus_{i=1}^{n}\mathbb{w}_i.\mathcal{M}_i,$$

$$\mathrm{CuCPFWBMO}(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\dots,\mathcal{M}_n)$$

$$=\left(\begin{array}{c}\left(\begin{array}{c}\left(\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{u}_i^{\mathbb{p}}.\mathrm{u}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{v}_i^{\mathbb{p}}.\mathrm{v}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\varepsilon_i^{\mathbb{p}}.\varepsilon_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right),\\ \left(\begin{array}{c}\left[\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\bar{\mathrm{a}}_i^{\mathbb{p}}.\bar{\mathrm{a}}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{b}_i^{\mathbb{p}}.\mathrm{b}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\\ \left[\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{c}_i^{\mathbb{p}}.\mathrm{c}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{d}_i^{\mathbb{p}}.\mathrm{d}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\\ \left[\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\varrho_i^{\mathbb{p}}.\varrho_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{f}_i^{\mathbb{p}}.\mathrm{f}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right]\end{array}\right)\end{array}\right);\\ \left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{r}_i^{\mathbb{p}}.\mathrm{r}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\end{array}\right),$$

where for each $\mathbb{v}\in\mathbb{V}$:

i) $\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{u}_i^{\mathbb{p}}.\mathrm{u}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}$ and $\left[\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\bar{\mathrm{a}}_i^{\mathbb{p}}.\bar{\mathrm{a}}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{b}_i^{\mathbb{p}}.\mathrm{b}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right]$ reflect the positive MS part of picture fuzzy information and interval-valued picture fuzzy information, respectively;

ii) $\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{v}_i^{\mathbb{p}}.\mathrm{v}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}$ and $\left[\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{c}_i^{\mathbb{p}}.\mathrm{c}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{d}_i^{\mathbb{p}}.\mathrm{d}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right]$ reflect the neutral MS part of picture fuzzy information and interval-valued picture fuzzy information, respectively;

iii) $\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\varepsilon_i^{\mathbb{p}}.\varepsilon_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}$ and $\left[\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\varrho_i^{\mathbb{p}}.\varrho_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{f}_i^{\mathbb{p}}.\mathrm{f}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right]$ reflect the negative MS part of picture fuzzy information and interval-valued picture fuzzy information, respectively;

iv) $\left(\frac{1}{1-\sum_{i=1}^{n}\mathbb{w}_i^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{w}_i\mathbb{w}_j\mathrm{r}_i^{\mathbb{p}}.\mathrm{r}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}$ represents radius uncertainty;

v) the parameters $\mathbb{p}$ and $\mathbb{q}$ indicate control parameters; and

vi) $\mathbb{w}_i$ denotes positive weight vectors such that $\sum_{i=1}^{n}\mathbb{w}_i=1$.

Theorem 8. Let $\mathcal{M}_i=\left(\left(\begin{array}{c}\langle\mathrm{u}_i,\mathrm{v}_i,\varepsilon_i\rangle,\\ \langle[\bar{\mathrm{a}}_i,\mathrm{b}_i],[\mathrm{c}_i,\mathrm{d}_i],[\varrho_i,\mathrm{f}_i]\rangle\end{array}\right);\mathrm{r}_i\right);i=1,2,3,\dots,n,$ reflect a collection of n CuCP-FNs and $\mathbb{w}_i>0$ shows any real number such that $\sum_{i=1}^{n}\mathbb{w}_i=1$. Then

i) $\mathrm{CuCPFBMO}(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\dots,\mathcal{M}_n)$ is a CuCP-FN;

ii) $\mathrm{CuCPFWBMO}(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\dots,\mathcal{M}_n)$ is a CuCP-FN.

Proof. i) The principle of mathematical induction is employed for the proof of this theorem. This principle states that the result is true for all n iff

ii) The given statement is valid for $n=2$.

iii) By assumption, it is valid for $n = k$.

iv) The result is valid for $n = k + 1$.

Initially, we check the base condition. For $n = 2$, we have

$$\mathrm{CuCPFWBMO}(\mathcal{M}_1, \mathcal{M}_2)$$

$$= \left(\left(\begin{array}{c} \langle (u_1{}^{\mathbb{p}} . u_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (v_1{}^{\mathbb{p}} . v_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (\varepsilon_1{}^{\mathbb{p}} . \varepsilon_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \rangle, \\ \langle \left[(\bar{a}_1{}^{\mathbb{p}} . \bar{a}_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (ɓ_1{}^{\mathbb{p}} . ɓ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \left[(ɕ_1{}^{\mathbb{p}} . ɕ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (ɗ_1{}^{\mathbb{p}} . ɗ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \left[(\varrho_1{}^{\mathbb{p}} . \varrho_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}, (ʄ_1{}^{\mathbb{p}} . ʄ_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}{}_1 \right] \rangle \end{array} \right); \\ (\mathbb{r}_1{}^{\mathbb{p}} . \mathbb{r}_2{}^{\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right)$$

$$= \mathcal{M}_1 \oplus \mathcal{M}_2,$$

which is true by part (i) of Theorem 4.

Suppose the statement is valid for $n = k$, that is, $\mathrm{CuCPFBMO}(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \dots, \mathcal{M}_k)$ is a CuCP-FN.

Now, $n = k + 1$ implies that

$$\mathrm{CuCPFBMO}(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \dots, \mathcal{M}_{k+1}) = \mathrm{CuCPFBMO}(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \dots, \mathcal{M}_k) \oplus \mathcal{M}_{k+1},$$

$$\mathrm{CuCPFBMO}(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \dots, \mathcal{M}_{k+1})$$

$$= \left(\left(\begin{array}{c} \left(\left(\left(\frac{1}{k(k-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} u_i{}^{\mathbb{p}} . u_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{k(k-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} v_i{}^{\mathbb{p}} . v_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{k(k-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \varepsilon_i{}^{\mathbb{p}} . \varepsilon_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right), \right. \\ \left(\begin{array}{c} \left[\left(\frac{1}{k(k-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \bar{a}_i{}^{\mathbb{p}} . \bar{a}_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{k(k-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} ɓ_i{}^{\mathbb{p}} . ɓ_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \\ \left[\left(\frac{1}{k(k-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} ɕ_i{}^{\mathbb{p}} . ɕ_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{k(k-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} ɗ_i{}^{\mathbb{p}} . ɗ_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \\ \left[\left(\frac{1}{k(k-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \varrho_i{}^{\mathbb{p}} . \varrho_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{k(k-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} ʄ_i{}^{\mathbb{p}} . ʄ_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right] \end{array} \right) \end{array} \right); \\ \left(\frac{1}{k(k-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \mathbb{r}_i{}^{\mathbb{p}} . \mathbb{r}_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right),$$

$$\oplus\left(\left(\begin{matrix}\langle \mathfrak{u}_{k+1}, \mathrm{v}_{k+1}, \varepsilon_{k+1}\rangle,\\ \langle[\bar{\mathring{\mathrm{a}}}_{k+1}, \mathfrak{b}_{k+1}], [\mathfrak{c}_{k+1}, \mathfrak{d}_{k+1}], [\varrho_{k+1}, \mathfrak{f}_{k+1}]\rangle\end{matrix}\right); \mathbb{r}_{k+1}\right)$$

$$=\left(\begin{matrix}\left(\begin{matrix}\left(\left(\frac{1}{k(k+1)}\sum_{\substack{i,j=1\\ i\neq j}}^{k+1} \mathfrak{u}_i^{\mathbb{p}}.\mathfrak{u}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{k(k+1)}\sum_{\substack{i,j=1\\ i\neq j}}^{k+1} \mathrm{v}_i^{\mathbb{p}}.\mathrm{v}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{k(k+1)}\sum_{\substack{i,j=1\\ i\neq j}}^{k+1} \varepsilon_i^{\mathbb{p}}.\varepsilon_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right),\\ \left(\begin{matrix}\left[\left(\frac{1}{k(k+1)}\sum_{\substack{i,j=1\\ i\neq j}}^{k+1} \bar{\mathring{\mathrm{a}}}_i^{\mathbb{p}}.\bar{\mathring{\mathrm{a}}}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{k(k+1)}\sum_{\substack{i,j=1\\ i\neq j}}^{k+1} \mathfrak{b}_i^{\mathbb{p}}.\mathfrak{b}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\\ \left[\left(\frac{1}{k(k+1)}\sum_{\substack{i,j=1\\ i\neq j}}^{k+1} \mathfrak{c}_i^{\mathbb{p}}.\mathfrak{c}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{k(k+1)}\sum_{\substack{i,j=1\\ i\neq j}}^{k+1} \mathfrak{d}_i^{\mathbb{p}}.\mathfrak{d}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\\ \left[\left(\frac{1}{k(k+1)}\sum_{\substack{i,j=1\\ i\neq j}}^{k+1} \varrho_i^{\mathbb{p}}.\varrho_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{k(k+1)}\sum_{\substack{i,j=1\\ i\neq j}}^{k+1} \mathfrak{f}_i^{\mathbb{p}}.\mathfrak{f}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right]\end{matrix}\right)\end{matrix}\right);\\ \left(\frac{1}{k(k+1)}\sum_{\substack{i,j=1\\ i\neq j}}^{k+1} \mathbb{r}_i^{\mathbb{p}}.\mathbb{r}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\end{matrix}\right).$$

Using the induction hypothesis and closure properties of the operator, the defining condition is preserved. Therefore, the third condition is also valid for $n = k + 1$. Thus, the result holds for all n and hence, $\mathrm{CuCPFBMO}(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \dots, \mathcal{M}_n)$ is a CuCP-FN.

v) The proof of (ii) is the same and hence is omitted.

Theorem 9. Let $\mathcal{M}_i = \left(\left(\begin{matrix}\langle \mathfrak{u}_i, \mathrm{v}_i, \varepsilon_i\rangle,\\ \langle[\bar{\mathring{\mathrm{a}}}_i, \mathfrak{b}_i], [\mathfrak{c}_i, \mathfrak{d}_i], [\varrho_i, \mathfrak{f}_i]\rangle\end{matrix}\right); \mathbb{r}_i\right); i = 1,2,3, \dots, n$, reflect a family of n CuCP-FNs demonstrated on $\mathbb{V}$ such that for all $\mathcal{M}_i = \mathcal{M}$, $\mathbb{w}_{\boldsymbol{i}} > 0$ is any real number with $\sum_{i=1}^{n} \mathbb{w}_{\boldsymbol{i}} = 1$. Then

i) $\mathrm{CuCPFBMO}(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \dots, \mathcal{M}_n) = \mathcal{M}$;

ii) $\mathrm{CuCPFWBMO}(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \dots, \mathcal{M}_n) = \mathcal{M}$.

Proof. i) Let $\mathcal{M}_i = \left(\left(\begin{matrix}\langle \mathfrak{u}_i, \mathrm{v}_i, \varepsilon_i\rangle,\\ \langle[\bar{\mathring{\mathrm{a}}}_i, \mathfrak{b}_i], [\mathfrak{c}_i, \mathfrak{d}_i], [\varrho_i, \mathfrak{f}_i]\rangle\end{matrix}\right); \mathbb{r}_i\right)$; $i = 1,2,3, \dots, n$, reflect a family of n CuCP-FNs demonstrated on $\mathbb{V}$ such that for all $\mathcal{M}_i = \mathcal{M}$, $\mathbb{w}_{\boldsymbol{i}} > 0$ is any real number with $\sum_{i=1}^{n} \mathbb{w}_{\boldsymbol{i}} = 1$. Then

$$\mathrm{CuCPFBMO}(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\dots,\mathcal{M}_n)$$

$$=\left(\begin{array}{c}\left(\begin{array}{c}\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathfrak{u}_i^{\mathbb{p}}.\mathfrak{u}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathrm{v}_i^{\mathbb{p}}.\mathrm{v}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon_i^{\mathbb{p}}.\varepsilon_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right),\\ \left(\begin{array}{c}\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\bar{\mathrm{a}}_i^{\mathbb{p}}.\bar{\mathrm{a}}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathfrak{b}_i^{\mathbb{p}}.\mathfrak{b}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\\ \left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathfrak{c}_i^{\mathbb{p}}.\mathfrak{c}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathrm{d}_i^{\mathbb{p}}.\mathrm{d}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\\ \left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varrho_i^{\mathbb{p}}.\varrho_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathfrak{f}_i^{\mathbb{p}}.\mathfrak{f}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right]\end{array}\right)\end{array}\right);\\ \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{r}_i^{\mathbb{p}}.\mathbb{r}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\end{array}\right),$$

$$\mathrm{CuCPFBMO}(\mathcal{M},\mathcal{M},\mathcal{M},\dots,\mathcal{M})$$

$$=\left(\begin{array}{c}\left(\begin{array}{c}\left(\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathfrak{u}^{\mathbb{p}}.\mathfrak{u}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathrm{v}^{\mathbb{p}}.\mathrm{v}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varepsilon^{\mathbb{p}}.\varepsilon^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right),\\ \left(\begin{array}{c}\left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\bar{\mathrm{a}}^{\mathbb{p}}.\bar{\mathrm{a}}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathfrak{b}^{\mathbb{p}}.\mathfrak{b}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\\ \left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathfrak{c}^{\mathbb{p}}.\mathfrak{c}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathrm{d}^{\mathbb{p}}.\mathrm{d}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\\ \left[\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\varrho^{\mathbb{p}}.\varrho^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathfrak{f}_i^{\mathbb{p}}.\mathfrak{f}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right]\end{array}\right)\end{array}\right);\\ \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\mathbb{r}^{\mathbb{p}}.\mathbb{r}^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\end{array}\right),$$

$$\mathrm{CuCPFBMO}(\mathcal{M},\mathcal{M},\mathcal{M},\ldots,\mathcal{M})$$

$$=\left(\begin{pmatrix}\left(\left(\frac{1}{n(n-1)}.n(n-1)\mathfrak{u}^{\mathbb{p}+\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}.n(n-1)\mathrm{v}^{\mathbb{p}+\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}.n(n-1)\varepsilon^{\mathbb{p}+\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right),\\ \begin{pmatrix}\left[\left(\frac{1}{n(n-1)}.n(n-1)\bar{\dot{a}}^{\mathbb{p}+\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}.n(n-1)\mathrm{б}^{\mathbb{p}+\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\\ \left[\left(\frac{1}{n(n-1)}.n(n-1)\mathfrak{c}^{\mathbb{p}+\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}.n(n-1)\mathrm{d}^{\mathbb{p}+\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\\ \left[\left(\frac{1}{n(n-1)}.n(n-1)\varrho^{\mathbb{p}+\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}},\left(\frac{1}{n(n-1)}.n(n-1)\mathfrak{f}^{\mathbb{p}+\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right]\end{pmatrix}\end{pmatrix};\\ \left(\frac{1}{n(n-1)}.n(n-1)\mathbb{r}^{\mathbb{p}+\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right)$$

$$=\left(\begin{pmatrix}\left((\mathfrak{u}^{\mathbb{p}+\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}},(\mathrm{v}^{\mathbb{p}+\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}},(\varepsilon^{\mathbb{p}+\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right),\\ \left(\left[(\bar{\dot{a}}^{\mathbb{p}+\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}},(\mathrm{б}^{\mathbb{p}+\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\left[(\mathfrak{c}^{\mathbb{p}+\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}},(\mathrm{d}^{\mathbb{p}+\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right],\left[(\varrho^{\mathbb{p}+\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}},(\mathfrak{f}^{\mathbb{p}+\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right]\right)\end{pmatrix};\\ (\mathbb{r}^{\mathbb{p}+\mathbb{q}})^{\frac{1}{\mathbb{p}+\mathbb{q}}}\right)$$

$$=\left(\begin{pmatrix}(\mathfrak{u},\mathrm{v},\varepsilon),\\ ([\bar{\dot{a}},\mathrm{б}],[\mathfrak{c},\mathrm{d}],[\varrho,\mathfrak{f}])\end{pmatrix};\\ \mathbb{r}\right).$$

Thus,

$$\mathrm{CuCPFBMO}(\mathcal{M},\mathcal{M},\mathcal{M},\ldots,\mathcal{M})=\mathcal{M}.$$

ii) The proof of (ii) is the same and hence is omitted.

Theorem 10. Let $\mathcal{M}_i=\left(\begin{pmatrix}\langle\mathfrak{u}_i,\mathrm{v}_i,\varepsilon_i\rangle,\\ \langle[\bar{\dot{a}}_i,\mathrm{б}_i],[\mathfrak{c}_i,\mathrm{d}_i],[\varrho_i,\mathfrak{f}_i]\rangle\end{pmatrix};\mathbb{r}_i\right);\ i=1,2,3,\ldots,n,$ and $G_i=\left(\begin{pmatrix}\langle\mathbb{T}_i,N_i,\mathbb{F}_i\rangle,\\ \langle[\mathbb{a}_i,\mathbb{b}_i],[\mathbb{c}_i,\mathbb{d}_i],[\mathbb{v}_i,\mathbb{f}_i]\rangle\end{pmatrix};\mathbb{h}_i\right);\ i=1,2,3,\ldots,n$ be two collections of CuCP-FNs defined over $\mathbb{V}$ and $\mathbb{w}_i>0$ such that $\mathfrak{u}_i\le\mathbb{T}_i,\ \mathrm{v}_i\le N_i,\varepsilon_i\ge\mathbb{F}_i,\bar{\dot{a}}_i\le\mathbb{a}_i,\mathrm{б}_i\le\mathbb{b}_i,\mathfrak{c}_i\le\mathbb{c}_i,\mathrm{d}_i\le\mathbb{d}_i,\varrho_i\ge\mathbb{v}_i,\mathfrak{f}_i\ge\mathbb{f}_i$, and $\mathbb{r}_i\le h_i$ for all i. Then

i) $CuCPFBMO(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\ldots,\mathcal{M}_n)\le CuCPFBMO(G_1,G_2,G_3,\ldots,G_n)$;

ii) $CuCPFWBMO(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\ldots,\mathcal{M}_n)\le CuCPFWBMO(G_1,G_2,G_3,\ldots,G_n)$.

Proof. i) Let $\mathcal{M}_i=\left(\begin{pmatrix}\langle\mathfrak{u}_i,\mathrm{v}_i,\varepsilon_i\rangle,\\ \langle[\bar{\dot{a}}_i,\mathrm{б}_i],[\mathfrak{c}_i,\mathrm{d}_i],[\varrho_i,\mathfrak{f}_i]\rangle\end{pmatrix};\mathbb{r}_i\right);\ i=1,2,3,\ldots,n,$ and $G_i=\left(\begin{pmatrix}\langle\mathbb{T}_i,N_i,\mathbb{F}_i\rangle,\\ \langle[\mathbb{a}_i,\mathbb{b}_i],[\mathbb{c}_i,\mathbb{d}_i],[\mathbb{v}_i,\mathbb{f}_i]\rangle\end{pmatrix};\mathbb{h}_i\right);\ i=1,2,3,\ldots,n$ be two collections of CuCP-FNs defined over $\mathbb{V}$ and $\mathbb{w}_i>0$ such that $\mathfrak{u}_i\le\mathbb{T}_i,\ \mathrm{v}_i\le N_i,\varepsilon_i\ge\mathbb{F}_i,\bar{\dot{a}}_i\le\mathbb{a}_i,\mathrm{б}_i\le\mathbb{b}_i,\mathfrak{c}_i\le\mathbb{c}_i,\mathrm{d}_i\le\mathbb{d}_i,\varrho_i\ge\mathbb{v}_i,\mathfrak{f}_i\ge\mathbb{f}_i$, and $\mathbb{r}_i\le h_i$ for all i. Now, $\mathfrak{u}_i\le\mathbb{T}_i$ implies that $\mathfrak{u}_i{}^{\mathbb{p}}.\mathfrak{u}_j{}^{\mathbb{q}}\le\mathbb{T}_i{}^{\mathbb{p}}.\mathbb{T}_j{}^{\mathbb{q}}$ and $\sum_{\substack{i,j=1\\ i\ne j}}^{n}\mathfrak{u}_i{}^{\mathbb{p}}.\mathfrak{u}_j{}^{\mathbb{q}}\le$

$\sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathbb{T}_i^{\mathbb{p}}.\mathbb{T}_j^{\mathbb{q}}$. Thus, $\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} u_i^{\mathbb{p}}.u_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \leq \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathbb{T}_i^{\mathbb{p}}.\mathbb{T}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}$. Similarly, the inequalities $v_i \leq N_i$ and $\varepsilon_i \geq \mathbb{F}_i$ imply that $\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} v_i^{\mathbb{p}}.v_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \leq \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} N_i^{\mathbb{p}}.N_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}$ and $\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathbb{F}_i^{\mathbb{p}}.\mathbb{F}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \leq \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \varepsilon_i^{\mathbb{p}}.\varepsilon_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}$. The same procedure can be followed for IVP-F information.

Also, $\mathbb{r}_i \leq h_i$ implies that $\mathbb{r}_i^{\mathbb{p}}.\mathbb{r}_j^{\mathbb{q}} \leq h_i^{\mathbb{p}}.h_j^{\mathbb{q}}$ and $\sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathbb{r}_i^{\mathbb{p}}.\mathbb{r}_j^{\mathbb{q}} \leq \sum_{\substack{i,j=1\\ i\neq j}}^{n} h_i^{\mathbb{p}}.h_j^{\mathbb{q}}$. Thus, $\left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \mathbb{r}_i^{\mathbb{p}}.\mathbb{r}_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \leq \left(\frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\neq j}}^{n} h_i^{\mathbb{p}}.h_j^{\mathbb{q}}\right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}$.

The inequalities conclude that

$$CuCPFBMO(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\dots,\mathcal{M}_n) \leq CuCPFBMO(G_1,G_2,G_3,\dots,G_n).$$

ii) The proof of (ii) is the same and hence is omitted.

Theorem 11. Let $\mathcal{M}^- = \left(\left(\begin{matrix}(u^-,v^-,\varepsilon^-),\\ \langle[\dot{\bar{a}}^-,\text{ɓ}^-],[\text{ɕ}^-,\text{ɗ}^-],[\varrho^-,\text{ʄ}^-]\rangle\end{matrix}\right);\mathbb{r}^-\right)$ and $\mathcal{M}^+ = \left(\left(\begin{matrix}(u^+,v^+,\varepsilon^+),\\ \langle[\dot{\bar{a}}^+,\text{ɓ}^+],[\text{ɕ}^+,\text{ɗ}^+],[\varrho^+,\text{ʄ}^+]\rangle\end{matrix}\right);\mathbb{r}^+\right)$ be collections of CuCP-FNs, where $u^- = \min(u_i)$, $v^- = \min(v_i)$, $\varepsilon^- = \max(\varepsilon_i)$, $\dot{\bar{a}}^- = \min(\dot{\bar{a}}_i)$, $\text{ɓ}^- = \min(\text{ɓ}_i)$, $\text{ɕ}^- = \min(\text{ɕ}_i)$, $\text{ɗ}^- = \min(\text{ɗ}_i)$, $\varrho^- = \max(\varrho_i)$, $\text{ʄ}^- = \max(\text{ʄ}_i)$, $\mathbb{r}^- = \min(\mathbb{r}_i)$ and $u^+ = \max(u_i)$, $v^+ = \max(v_i)$, $\varepsilon^+ = \min(\varepsilon_i)$, $\dot{\bar{a}}^+ = \max(\dot{\bar{a}}_i)$, $\text{ɓ}^+ = \max(\text{ɓ}_i)$, $\text{ɕ}^+ = \max(\text{ɕ}_i)$, $\text{ɗ}^+ = \max(\text{ɗ}_i)$, $\varrho^+ = \min(\varrho_i)$, $\text{ʄ}^+ = \min(\text{ʄ}_i)$, and $\mathbb{r}^+ = \max(\mathbb{r}_i)$. Then

i) $\mathcal{M}^- \leq CuCPFBMO(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\dots,\mathcal{M}_n) \leq \mathcal{M}^+$;

ii) $\mathcal{M}^- \leq CuCPFWBMO(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\dots,\mathcal{M}_n) \leq \mathcal{M}^+$.

Proof. Since $u^- = \min(u_i) \leq \max(u_i) \leq u^+$, $v^- = \min(v_i) \leq \max(v_i) \leq v^+$, $\varepsilon^+ = \min(\varepsilon_i) \leq \max(\varepsilon_i) \leq \varepsilon^-$, $\dot{\bar{a}}^- = \min(\dot{\bar{a}}_i) \leq \max(\dot{\bar{a}}_i) \leq \dot{\bar{a}}^+$, $\text{ɓ}^- = \min(\text{ɓ}_i) \leq \max(\text{ɓ}_i) \leq \text{ɓ}^+$, $\text{ɕ}^- = \min(\text{ɕ}_i) \leq \max(\text{ɕ}_i) \leq \text{ɕ}^+$, $\text{ɗ}^- = \min(\text{ɗ}_i) \leq \max(\text{ɗ}_i) \leq \text{ɗ}^+$, $\varrho^+ = \min(\varrho_i) \leq \max(\varrho_i) \leq \varrho^-$, $\text{ʄ}^+ = \min(\text{ʄ}_i) \leq \max(\text{ʄ}_i) \leq \text{ʄ}^-$, and $\mathbb{r}^- = \min(\mathbb{r}_i) \leq \max(\mathbb{r}_i) \leq \mathbb{r}^+$, then by the monotonicity property, we have

$$\mathcal{M}^- \leq CuCPFBMO(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\dots,\mathcal{M}_n) \leq \mathcal{M}^+;$$

$$\mathcal{M}^- \leq CuCPFWBMO(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3,\dots,\mathcal{M}_n) \leq \mathcal{M}^+.$$

5. Multi-criteria decision-making approach within the cubic circular picture fuzzy set environments

In this section, we construct an MCDM approach based on the proposed BMOs and score functions of CuCP-FSs. We initially demonstrate the significance of MCDM techniques in daily problems.

5.1. Significance of the MCDM approach in real-world applications

MCDM techniques play a significant role in solving daily life problems because real-life decision applications involve assessing many competing criteria collectively. In practical applications, policy-makers and organizations often make decisions using a single attribute rather than involving multiple factors, enhancing the reliability and robustness of the decision. The framework of MCDM contributes a systematic approach to analyze problems involving several competing options. In many practical situations, people often face situations that need to balance distinct criteria. In case of purchasing, a consumer may assess products based on brand reputation, performance, price, and durability.   Similarly, selecting a job opportunity or choosing a university often require multiple criteria simultaneously. MCDM approaches are critical for modeling these complex decisions by considering multiple criteria, assigning their importance, and systematically ranking the possible alternatives. Therefore, MCDM techniques have been widely employed in modern decision science, allowing organizations and individuals to handle increasingly complex DM problems in a systematic and effective manner.

5.2. Significance of CuCP-FS model in MCDM

The proposed model combines P-FSs, IVP-FSs, and a radius parameter within a unified single framework. The proposed approach has significant importance in MCDM because modern decision problems involve multilayered uncertain information and varying levels of confidence in expert opinions. The CuCP-FS model improves the representation of complex information by allowing experts to describe their information through precise and interval-valued positive, neutral, and negative MS values, which already contribute richer evaluations than the traditional IF models. The proposed model is crucial in many cases where experts cannot offer precise information and instead prefer to describe their evaluations as intervals. The proposed approach enables MS values to be expressed as ranges by integrating IVP-FSs, and therefore capturing multilayered uncertainty present in expert assessments. Furthermore, by integrating a radius parameter, the proposed framework models the confidence levels or dispersion of expert opinions. As a result, the CuCP-FS model improves MCDM techniques by allowing more expressive uncertainty modeling, enhanced reliability analysis, and flexible integration of heterogeneous information.

5.3. Significance of the proposed AOs in MCDM

In MCDM problems, the alternatives are evaluated based on multiple decision criteria. The integration of P-FSs, IVP-FSs, and a circular radius enables the establishment of more reliable and robust AOs. The proposed BMOs can significantly combine multi-dimensional fuzzy evaluation while preserving the structure of information, resulting to more accurate selection of alternatives. Furthermore, the proposed BMOs are crucial for capturing the interrelationships or interactions among the decision criteria. Thus, the proposed BMOs provide comprehensive approaches for combining the information under different decision criterion.

5.4. Decision-making method for solving MCDM problems

This subsection develops the steps of the proposed MCDM approach within the environment of CuCP-FSs. The proposed BMOs and score functions are employed to construct the DM technique. The following Figure 3 demonstrates the algorithmic flowchart of the proposed method.

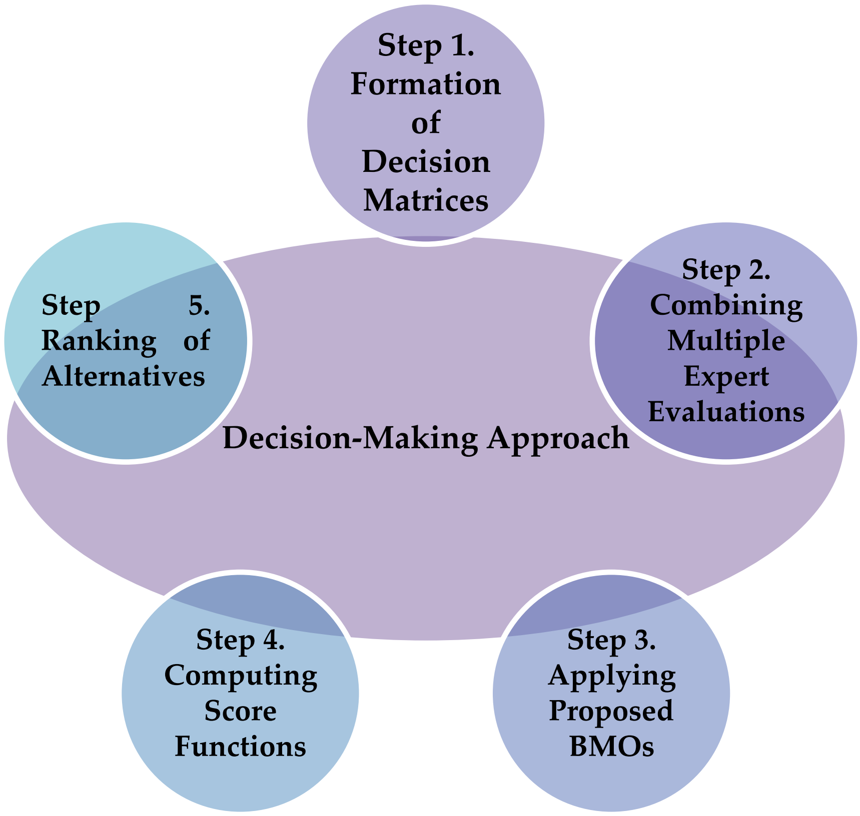

Figure 3. Algorithmic flowchart of the proposed method.

Below is a detailed step procedure for evaluation of the possible alternatives within the framework of CuCP-FSs.

Suppose a DM problem involves n alternatives, m decision attributes, and k experts. The key objective of this DM process is to choose a most preferable choice among n possible alternatives under uncertain information represented in the proposed framework. Suppose

i. $\mathbb{u} = \{\mathbb{u}_1, \mathbb{u}_2, \ldots, \mathbb{u}_n\}$ reflects a set of possible alternatives;
ii. $\mathbb{f} = \{\mathbb{f}_1, \mathbb{f}_2, \ldots, \mathbb{f}_m\}$ represents a set of decision criteria;
iii. $\mathbb{A} = \{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_k\}$ is the set of experts; and
iv. $\mathbb{w}_j, j \in \{1,2, \ldots, m\}$, denotes the relative importance of m decision criteria where $\mathbb{w}_j \in [0,1]$ and $\sum_{j=1}^{m} \mathbb{w}_j = 1$.

Step 1. In Step 1, we arrange the evaluations of alternatives provided by multiple experts in the shape of decision matrices. Each expert contributes assessments of possible alternatives under each criterion $\mathbb{f}_j, j \in \{1,2, \ldots, m\}$, using the P-FS and IVP-FS frameworks. Each expert $\mathbb{A}_t, t \in \{1,2, \ldots, k\}$, designs a decision matrix that can be written as:

$$\mathbb{D}^{(t)} = \left[\mathbb{d}^{(t)}{}_{ij}\right]_{n\times m} = \left[\left(\left(\begin{array}{c}\langle \mathfrak{u}^{(t)}{}_{ij}, \mathrm{v}^{(t)}{}_{ij}, \varepsilon^{(t)}{}_{ij}\rangle, \\ \langle[\bar{\mathring{a}}^{(t)}{}_{ij}, \mathrm{b}^{(t)}{}_{ij}], [\varsigma^{(t)}{}_{ij}, \mathrm{d}^{(t)}{}_{ij}], [\varrho^{(t)}{}_{ij}, \mathfrak{f}^{(t)}{}_{ij}]\rangle\end{array}\right)\right)\right]_{n\times m},$$

where n represents the number of alternatives, m represents the number of criterions, and $\mathbb{d}^{(t)}{}_{ij}$ reflects the evaluation of alternative L_i under criterion $\mathfrak{f}_j$ provided by the t-th expert. The proposed decision matrix $\mathbb{D}^{(t)}$ can be interpreted in tabular form as:

Table 2. Decision matrix constructed by expert $\mathbb{A}_t$.

Criteria($\mathfrak{f}_j$)/ Alternatives(L_i)	$\mathfrak{f}_1$	$\mathfrak{f}_2$	$\cdots$	$\mathfrak{f}_m$
L_1	$\mathbb{d}^{(t)}{}_{11}$	$\mathbb{d}^{(t)}{}_{12}$	$\cdots$	$\mathbb{d}^{(t)}{}_{1m}$
L_2	$\mathbb{d}^{(t)}{}_{21}$	$\mathbb{d}^{(t)}{}_{22}$	$\cdots$	$\mathbb{d}^{(t)}{}_{2m}$
.	.	.		.
.	.	.	$\cdots$	.
.	.	.		.
L_n	$\mathbb{d}^{(t)}{}_{n1}$	$\mathbb{d}^{(t)}{}_{n2}$	$\cdots$	$\mathbb{d}^{(t)}{}_{nn}$

Step 2. In Step 2, we employ an appropriate aggregation mechanism to combine the information of multiple experts. This step combines different decision matrices provided by various experts into a collective evaluation matrix. To compute the overall assessments of experts into a single evaluation matrix, the following aggregation mechanisms are applied.

$$\mathfrak{u} = \frac{\sum_{t=1}^{k} \mathfrak{u}_t}{k}, \mathrm{v} = \frac{\sum_{t=1}^{k} \mathrm{v}_t}{k}, \varepsilon = \frac{\sum_{t=1}^{k} \varepsilon_t}{k},$$

$$\bar{\mathring{a}} = \frac{\sum_{t=1}^{k} \bar{\mathring{a}}_t}{k}, \mathrm{b} = \frac{\sum_{t=1}^{k} \mathrm{b}_t}{k}, \varsigma = \frac{\sum_{t=1}^{k} \varsigma_t}{k}, \mathrm{d} = \frac{\sum_{t=1}^{k} \mathrm{d}_t}{k}, \varrho = \frac{\sum_{t=1}^{k} \varrho_t}{k}, \mathfrak{f} = \frac{\sum_{t=1}^{k} \mathfrak{f}_t}{k}, \text{ and}$$

$$\mathbb{r} = min\left\{max_{1\le t\le k}\sqrt{(\mathfrak{u}-\mathfrak{u}_t)^2 + (\mathrm{v}-\mathrm{v}_t)^2 + (\varepsilon-\varepsilon_t)^2 + (\bar{\mathring{a}}-\bar{\mathring{a}}_t)^2 + (\mathrm{b}-\mathrm{b}_t)^2 + (\varsigma-\varsigma_t)^2 + (\mathrm{d}-\mathrm{d}_t)^2 + (\varrho-\varrho_t)^2 + (\mathfrak{f}-\mathfrak{f}_t)^2}, 1\right\},$$

where $\mathfrak{u}_t, \mathrm{v}_t, \varepsilon_t, \bar{\mathring{a}}_t, \mathrm{b}_t, \varsigma_t, \mathrm{d}_t, \varrho_t,$ and $\mathfrak{f}_t$ denote the evaluation provided by the t-th expert, and k represents the total number of experts. After aggregation of the overall assessment, we obtain a single evaluation matrix, which can be demonstrated as

$$\mathbb{D} = \left[\mathbb{d}_{ij}\right]_{n\times m} = \left[\left(\left(\begin{array}{c}\langle \mathfrak{u}_{ij}, \mathrm{v}_{ij}, \varepsilon_{ij}\rangle, \\ \langle[\bar{\mathring{a}}_{ij}, \mathrm{b}_{ij}], [\varsigma_{ij}, \mathrm{d}_{ij}], [\varrho_{ij}, \mathfrak{f}_{ij}]\rangle\end{array}\right); \mathbb{r}_{ij}\right)\right]_{n\times m}.$$

Step 3. In Step 3, we aggregate the collective evaluation matrix $\mathbb{D}$ by employing the proposed BMOs. This step is crucial for converting the evaluations of alternatives under different decision criteria into a single CuCP-FN. To aggregate collective evaluation matrix $\mathbb{D}$, the following BMOs are employed.

$$\mathrm{CuCPFBMO}(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \dots, \mathcal{M}_n)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(\left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathfrak{u}_i{}^{\mathbb{p}} . \mathfrak{u}_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathrm{v}_i{}^{\mathbb{p}} . \mathrm{v}_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \varepsilon_i{}^{\mathbb{p}} . \varepsilon_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right), \\ \left(\begin{array}{c} \left[\left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \bar{\dot{a}}_i{}^{\mathbb{p}} . \bar{\dot{a}}_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} ɓ_i{}^{\mathbb{p}} . ɓ_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \\ \left[\left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} ɕ_i{}^{\mathbb{p}} . ɕ_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} ɗ_i{}^{\mathbb{p}} . ɗ_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \\ \left[\left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \varrho_i{}^{\mathbb{p}} . \varrho_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} ʄ_i{}^{\mathbb{p}} . ʄ_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right] \end{array} \right) \end{array} \right); \\ \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbb{r}_i{}^{\mathbb{p}} . \mathbb{r}_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \end{array} \right),$$

$$\mathrm{CuCPFWBMO}(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \dots, \mathcal{M}_n)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(\left(\frac{1}{1-\sum_{i=1}^{n} \mathbb{w}_i{}^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbb{w}_i \mathbb{w}_j \mathfrak{u}_i{}^{\mathbb{p}} . \mathfrak{u}_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{1-\sum_{i=1}^{n} \mathbb{w}_i{}^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbb{w}_i \mathbb{w}_j \mathrm{v}_i{}^{\mathbb{p}} . \mathrm{v}_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{1-\sum_{i=1}^{n} \mathbb{w}_i{}^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbb{w}_i \mathbb{w}_j \varepsilon_i{}^{\mathbb{p}} . \varepsilon_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right), \\ \left(\begin{array}{c} \left[\left(\frac{1}{1-\sum_{i=1}^{n} \mathbb{w}_i{}^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbb{w}_i \mathbb{w}_j \bar{\dot{a}}_i{}^{\mathbb{p}} . \bar{\dot{a}}_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{1-\sum_{i=1}^{n} \mathbb{w}_i{}^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbb{w}_i \mathbb{w}_j ɓ_i{}^{\mathbb{p}} . ɓ_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \\ \left[\left(\frac{1}{1-\sum_{i=1}^{n} \mathbb{w}_i{}^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbb{w}_i \mathbb{w}_j ɕ_i{}^{\mathbb{p}} . ɕ_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{1-\sum_{i=1}^{n} \mathbb{w}_i{}^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbb{w}_i \mathbb{w}_j ɗ_i{}^{\mathbb{p}} . ɗ_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right], \\ \left[\left(\frac{1}{1-\sum_{i=1}^{n} \mathbb{w}_i{}^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbb{w}_i \mathbb{w}_j \varrho_i{}^{\mathbb{p}} . \varrho_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}}, \left(\frac{1}{1-\sum_{i=1}^{n} \mathbb{w}_i{}^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbb{w}_i \mathbb{w}_j ʄ_i{}^{\mathbb{p}} . ʄ_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \right] \end{array} \right) \end{array} \right); \\ \left(\frac{1}{1-\sum_{i=1}^{n} \mathbb{w}_i{}^2} \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \mathbb{w}_i \mathbb{w}_j \mathbb{r}_i{}^{\mathbb{p}} . \mathbb{r}_j{}^{\mathbb{q}} \right)^{\frac{1}{\mathbb{p}+\mathbb{q}}} \end{array} \right).$$

Step 4. In Step 4, we compute the score functions of each alternative. The score functions play a significant role by transforming fuzzy evaluations into comparable numerical values. This step is crucial for the final ranking of alternatives. The proposed score function is given as follows:

$$S\mathrm{F}(\mathrm{Ɩ}_i) = \frac{3+\mathbb{r}_i+\mathfrak{u}_i+\mathrm{v}_i+\bar{\dot{a}}_i+ɓ_i+ɕ_i+ɗ_i-\varepsilon_i-\varrho_i-ʄ_i}{10}.$$

Step 5. In Step 5, we rank the alternatives based on the computed score functions. The order of alternatives could be established by $\mathrm{Ɩ}_i \succ \mathrm{Ɩ}_j$, where $S\mathrm{F}(\mathrm{Ɩ}_i) > S\mathrm{F}(\mathrm{Ɩ}_j)$. Thus, a larger score function represents a better alternative.

6. Case study

Here, the proposed technique is employed to address an illustrative example of an MCDM problem involving the selection of the most suitable quantum computing (QC) platforms under uncertain and imprecise information. The assessment data is illustrative in nature, demonstrating typical DM situations in the deficiency of standardized raw datasets.

The rapid development of QC has provided new opportunities for handling complex computational problems that are even impossible or difficult for classical computers. Technology companies and research institutions are progressively allocating resources toward quantum technology advanced breakthroughs in distinct areas such as artificial intelligence, optimization, material sciences, and cryptography. However, many conflicting QC platforms exist, each with different technological limitations and advantages. Experts frequently handle significant uncertainty when assessing distinct hardware architectures because QC is still an emerging and evolving field. Performance information may be insufficient, experimental results may differ across laboratories, and the reliability of technology varies significantly among platforms. Therefore, the selection of a QC platform becomes a crucial MCDM problem involving a higher level of uncertainty.

Suppose that $\text{ꭎ} = \{\text{ꭎ}_1, \text{ꭎ}_2, \text{ꭎ}_3, \text{ꭎ}_4\}$ represents a set of four major QC technologies.

i) Superconducting qubits "ꭎ_1": Currently, these qubits are one of the most advanced QC technologies. This QC technology depends on superconducting circuits operating at extensively low temperature to express quantum bits. Due to their compatibility with existing semiconductor manufacturing techniques, major technology companies have heavily invested in this platform.

ii) Topological qubits "ꭎ_2": This QC technology is based on exotic quantum states that are more inflexible to decoherence and noise. The significant advantages of this technology include strong theoretical scalability, potentially very low error rates, and high reliable to environmental disturbances.

iii) Photonic quantum computers "ꭎ_3": This QC technology utilizes photons as qubits, allowing information to be transmitted through optical systems. The key advantages of this technology include operating at room temperature, an existing optical communication system, and suitable quantum networks.

iv) Trapped Ion Qubits "ꭎ_4": This QC technology utilizes electrically charged atoms limited in electromagnetic fields as qubits. These systems facilitate very high operational accuracy, highly reliable quantum gates, and excellent coherence times.

In MCDM problems, the alternatives are evaluated under several conflicting criteria. The following criteria are used to assess these alternatives effectively.

i) Qubits stability "ꞙ_1": This criteria indicates the capability of a quantum system to support its quantum state without decoherence. More reliable quantum computations create a system with greater stability.

ii) Scalability "ꞙ_2": This criteria reflects the viability of increasing the number of qubits while supporting system efficiency. Thousands or even millions of qubits are required for large-scale quantum computers.

iii) Error rate "ꞙ_3": This criteria reflects the possibility of computational errors during quantum operations. For accurate quantum algorithms, lower error rates are required.

iv) Cost of deployment "$\mathfrak{f}_4$": Cost of deployment measures the financial investment needed to build and support quantum hardware, including infrastructure such as specialized laboratories and cooling systems.

v) Programming environment and software support "$\mathfrak{f}_5$": A mature programming ecosystem, including simulation tools, compilers, and development frameworks, is crucial for developers and researchers to construct quantum algorithms.

Let $\mathfrak{U} = \{\mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3, \mathfrak{U}_4\}$ represent a set of four alternatives (QC technologies) and $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3, \mathfrak{f}_4, \mathfrak{f}_5\}$ reflect a set of five attributes that are used to evaluate these four alternatives. Let $\mathbb{A} = \{\mathbb{A}_1, \mathbb{A}_2\}$ indicate a team of experts, which provides the assessments of each alternative based on their own experience. A team of experts assign the positive weight vectors to each criterion, which is demonstrated in Table 3.

Table 3. Positive weight vectors to each criterion.

Criteria	Weights
$\mathfrak{f}_1$	0.1
$\mathfrak{f}_2$	0.3
$\mathfrak{f}_3$	0.1
$\mathfrak{f}_4$	0.2
$\mathfrak{f}_5$	0.3

To select a best QC technology among four potential technologies, the proposed algorithm is employed. In this procedure, we employ CuCPFBMOs and CuCPFWBMOs to aggregate the information. The computational steps of this procedure are given as follows.

Step 1. Step 1 arranges the information of alternatives supported by multiple experts in the shape of decision matrices. Each expert contributes assessments of possible alternatives under each criterion $\mathfrak{f}_j, j = 1,2, \dots, 5,$ using the P-FS and IVP-FS frameworks. The decision matrices evaluated by each expert $\mathbb{A}_t, t = 1,2,$ are arranged in Tables 4 and 5:

Table 4. D-matrix provided by expert $\mathbb{A}_1$.

$\mathfrak{f}_j / \mathfrak{U}_i$	$\mathfrak{f}_1$	$\mathfrak{f}_2$	$\mathfrak{f}_3$	$\mathfrak{f}_4$	$\mathfrak{f}_5$
$\mathfrak{U}_1$	$\left[\begin{pmatrix}\langle 0.12,0.23,0.41\rangle,\\ [0.1,0.22],\\ \langle[0.14,0.25]\rangle\\ [0.2,0.45]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.32,0.06,0.17\rangle,\\ [0.3,0.4],\\ \langle [0.02,0.1] \rangle\\ [0.12,0.22]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.28,0.13,0.32\rangle,\\ [0.15,0.3],\\ \langle[0.08,0.23]\rangle\\ [0.1,0.35]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.08,0.3,0.25\rangle,\\ [0.05,0.2],\\ \langle[0.05,0.32]\rangle\\ [0.1,0.28]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.35,0.1,0.4\rangle,\\ [0.1,0.4],\\ \langle[0.08,0.15]\rangle\\ [0.25,0.43]\end{pmatrix}\right]$
$\mathfrak{U}_2$	$\left[\begin{pmatrix}\langle 0.3,0.2,0.15\rangle,\\ [0.23,0.32],\\ \langle[0.05,0.23]\rangle\\ [0.12,0.2]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.1,0.25,0.38\rangle,\\ [0.01,0.12],\\ \langle [0.08,0.3] \rangle\\ [0.22,0.4]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.4,0.13,0.05\rangle,\\ [0.25,0.43],\\ \langle[0.02,0.15]\rangle\\ [0.04,0.2]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.18,0.1,0.35\rangle,\\ [0.13,0.2],\\ \langle[0.08,0.14]\rangle\\ [0.1,0.4]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.4,0.2,0.24\rangle,\\ [0.3,0.43],\\ \langle[0.14,0.25]\rangle\\ [0.2,0.26]\end{pmatrix}\right]$
$\mathfrak{U}_3$	$\left[\begin{pmatrix}\langle 0.2,0.15,0.36\rangle,\\ [0.12,0.24],\\ \langle [0.12,0.2] \rangle\\ [0.2,0.38]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.3,0.2,0.06\rangle,\\ [0.28,0.35],\\ \langle[0.18,0.25]\rangle\\ [0.08,0.16]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.12,0.18,0.3\rangle,\\ [0.08,0.15],\\ \langle [0.1,0.25] \rangle\\ [0.14,0.35]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.02,0.05,0.12\rangle,\\ [0.01,0.08],\\ \langle [0.04,0.1] \rangle\\ [0.1,0.28]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.25,0.3,0.14\rangle,\\ [0.14,0.3],\\ \langle[0.26,0.33]\rangle\\ [0.1,0.2]\end{pmatrix}\right]$
$\mathfrak{U}_4$	$\left[\begin{pmatrix}\langle 0.05,0.23,0.32\rangle,\\ [0.02,0.12],\\ \langle [0.2,0.29] \rangle\\ [0.25,0.38]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.19,0.1,0.05\rangle,\\ [0.15,0.25],\\ \langle[0.08,0.22]\rangle\\ [0.02,0.15]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.33,0.13,0.2\rangle,\\ [0.25,0.38],\\ \langle [0.1,0.21] \rangle\\ [0.14,0.24]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.12,0.15,0.22\rangle,\\ [0.07,0.18],\\ \langle[0.07,0.19]\rangle\\ [0.14,0.3]\end{pmatrix}\right]$	$\left[\begin{pmatrix}\langle 0.2,0.1,0.4\rangle,\\ [0.14,0.25],\\ \langle[0.06,0.13]\rangle\\ [0.2,0.42]\end{pmatrix}\right]$

Table 5. D-matrix provided by expert $\mathbb{A}_2$.

ƒ$_j$/ɯ$_i$	ƒ$_1$	ƒ$_2$	ƒ$_3$	ƒ$_4$	ƒ$_5$
ɯ$_1$	[(⟨0.08,0.2,0.35⟩, [0.05,0.12], ⟨[0.18,0.3]⟩ [0.25,0.4])]	[(⟨0.27,0.12,0.2⟩, [0.22,0.32], ⟨[0.08,0.15]⟩ [0.16,0.26])]	[(⟨0.33,0.1,0.27⟩, [0.25,0.35], ⟨[0.08,0.13]⟩ [0.22,0.35])]	[(⟨0.16,0.23,0.28⟩, [0.12,0.23], ⟨[0.15,0.28]⟩ [0.21,0.33])]	[(⟨0.29,0.13,0.31⟩, [0.18,0.32], ⟨[0.1,0.17]⟩ [0.28,0.37])]
ɯ$_2$	[(⟨0.22,0.32,0.25⟩, [0.2,0.25], ⟨[0.3,0.37]⟩ [0.21,0.3])]	[(⟨0.05,0.2,0.28⟩, [0.01,0.1], ⟨[0.1,0.23]⟩ [0.25,0.34])]	[(⟨0.34,0.23,0.1⟩, [0.3,0.4], ⟨[0.2,0.25]⟩ [0.07,0.12])]	[(⟨0.23,0.12,0.28⟩, [0.17,0.29], ⟨[0.08,0.16]⟩ [0.23,0.37])]	[(⟨0.32,0.23,0.14⟩, [0.25,0.33], ⟨[0.18,0.29]⟩ [0.1,0.16])]
ɯ$_3$	[(⟨0.14,0.23,0.31⟩, [0.1,0.17], ⟨[0.2,0.27]⟩ [0.1,0.38])]	[(⟨0.24,0.26,0.16⟩, [0.2,0.3], ⟨[0.18,0.32]⟩ [0.12,0.2])]	[(⟨0.22,0.1,0.25⟩, [0.18,0.25], ⟨[0.08,0.2]⟩ [0.2,0.37])]	[(⟨0.1,0.2,0.22⟩, [0.05,0.18], ⟨[0.14,0.23]⟩ [0.1,0.3])]	[(⟨0.28,0.23,0.1⟩, [0.17,0.3], ⟨[0.2,0.32]⟩ [0.07,0.18])]
ɯ$_4$	[(⟨0.12,0.28,0.35⟩, [0.1,0.19], ⟨[0.23,0.32]⟩ [0.32,0.43])]	[(⟨0.26,0.11,0.15⟩, [0.19,0.3], ⟨[0.08,0.2]⟩ [0.12,0.25])]	[(⟨0.4,0.1,0.25⟩, [0.28,0.42], ⟨[0.13,0.18]⟩ [0.19,0.3])]	[(⟨0.17,0.2,0.3⟩, [0.15,0.24], ⟨[0.17,0.3]⟩ [0.24,0.35])]	[(⟨0.27,0.13,0.33⟩, [0.2,0.35], ⟨[0.06,0.2]⟩ [0.2,0.35])]

Step 2: In Step 2, we apply the aggregation mechanism given in Step 2 of the proposed algorithm to combine the information provided in Tables 4 and 5. This step combines these two decision matrices provided by experts $\mathbb{A}_1$ and $\mathbb{A}_2$ into a collective evaluation matrix. Table 6 represents the collective evaluation matrix.

Table 6. Aggregated values of overall assessements.

ƒ$_j$ / ɯ$_i$	ƒ$_1$	ƒ$_2$	ƒ$_3$	ƒ$_4$	ƒ$_5$
ɯ$_1$	[((⟨0.1,0.215,0.38⟩, [0.075,0.17], ⟨[0.16,0.275]⟩ [0.225,0.425]); 0.08)]	[((⟨0.295,0.09,0.185⟩, [0.26,0.36], ⟨[0.05,0.125]⟩ [0.14,0.24]); 0.09)]	[((⟨0.305,0.115,0.295⟩, [0.2,0.325], ⟨[0.08,0.18]⟩ [0.16,0.35]); 0.1)]	[((⟨0.12,0.265,0.265⟩, [0.085,0.215], ⟨[0.1,0.3]⟩ [0.155,0.305]); 0.11)]	[((⟨0.32,0.115,0.355⟩, [0.18,0.32], ⟨[0.1,0.17]⟩ [0.28,0.37]); 0.15)]
ɯ$_2$	[((⟨0.26,0.26,0.2⟩, [0.215,0.345], ⟨[0.115,0.3]⟩ [0.165,0.25]); 0.15)]	[((⟨0.075,0.225,0.33⟩, [0.01,0.11], ⟨[0.09,0.265]⟩ [0.235,0.37]); 0.08)]	[((⟨0.37,0.18,0.075⟩, [0.275,0.415], ⟨[0.11,0.2]⟩ [0.105,0.16]); 0.14)]	[((⟨0.205,0.11,0.315⟩, [0.15,0.215], ⟨[0.08,0.15]⟩ [0.165,0.385]); 0.08)]	[((⟨0.36,0.215,0.19⟩, [0.275,0.285], ⟨[0.16,0.27]⟩ [0.15,0.21]); 0.18)]
ɯ$_3$	[((⟨0.17,0.19,0.285⟩, [0.11,0.205], ⟨[0.16,0.235]⟩ [0.15,0.38]); 0.12)]	[((⟨0.27,0.23,0.11⟩, [0.24,0.325], ⟨[0.18,0.285]⟩ [0.1,0.18]); 0.09)]	[((⟨0.17,0.14,0.275⟩, [0.13,0.2], ⟨[0.09,0.225]⟩ [0.17,0.36]); 0.12)]	[((⟨0.06,0.125,0.17⟩, [0.03,0.13], ⟨[0.09,0.165]⟩ [0.1,0.29]); 0.14)]	[((⟨0.265,0.265,0.12⟩, [0.155,0.3], ⟨[0.23,0.325]⟩ [0.085,0.19]); 0.06)]
ɯ$_4$	[((⟨0.085,0.255,0.335⟩, [0.06,0.155], ⟨[0.215,0.305]⟩ [0.285,0.405]); 0.08)]	[((⟨0.225,0.105,0.1⟩, [0.17,0.275], ⟨[0.08,0.21]⟩ [0.07,0.2]); 0.11)]	[((⟨0.365,0.115,0.225⟩, [0.265,0.4], ⟨[0.115,0.195]⟩ [0.165,0.21]); 0.07)]	[((⟨0.145,0.175,0.26⟩, [0.11,0.21], ⟨[0.12,0.245]⟩ [0.19,0.325]); 0.12)]	[((⟨0.235,0.115,0.365⟩, [0.17,0.3], ⟨[0.06,0.165]⟩ [0.2,0.385]); 0.09)]

Step 3. In Step 3, we aggregate the information provided in Table 6 by applying proposed BMOs. The aggregation process is essential for transforming the overall assessments under distinct decision criteria into single CuCP-F information. Table 7 reflects the aggregated information by proposed BMOs.

Table 7. Aggregated information by proposed BMOs.

Criteria($ɟ_j$)/Alternatives ($Ա_i$)	Aggregated Information by CuCPFBMOs	Aggregated Information by CuCPFWBMOs
$Ա_1$	$\left[\left(\begin{pmatrix}\langle 0.22,0.16,0.29\rangle,\\ [0.156,0.276],\\ \langle [0.096,0.207]\rangle\\ [0.19,0.337]\end{pmatrix};\right)\right]$ 0.105	$\left[\left(\begin{pmatrix}\langle 0.239,0.147,0.282\rangle,\\ [0.168,0.29],\\ \langle [0.089,0.195]\rangle\\ [0.191,0.323]\end{pmatrix};\right)\right]$ 0.11
$Ա_2$	$\left[\left(\begin{pmatrix}\langle 0.248,0.196,0.217\rangle,\\ [0.178,0.269],\\ \langle [0.11,0.235]\rangle\\ [0.163,0.271]\end{pmatrix};\right)\right]$ 0.124	$\left[\left(\begin{pmatrix}\langle 0.227,0.196,0.239\rangle,\\ [0.154,0.239],\\ \langle [0.111,0.237]\rangle\\ [0.171,0.286]\end{pmatrix};\right)\right]$ 0.12
$Ա_3$	$\left[\left(\begin{pmatrix}\langle 0.183,0.188,0.188\rangle,\\ [0.129,0.229],\\ \langle [0.148,0.245]\rangle\\ [0.12,0.277]\end{pmatrix};\right)\right]$ 0.105	$\left[\left(\begin{pmatrix}\langle 0.195,0.2,0.163\rangle,\\ [0.138,0.244],\\ \langle [0.159,0.256]\rangle\\ [0.109,0.248]\end{pmatrix};\right)\right]$ 0.09
$Ա_4$	$\left[\left(\begin{pmatrix}\langle 0.206,0.15,0.253\rangle,\\ [0.151,0.265],\\ \langle [0.115,0.223]\rangle\\ [0.179,0.302]\end{pmatrix};\right)\right]$ 0.094	$\left[\left(\begin{pmatrix}\langle 0.21,0.14,0.241\rangle,\\ [0.154,0.268],\\ \langle [0.101,0.213]\rangle\\ [0.164,0.299]\end{pmatrix};\right)\right]$ 0.098

Step 4. The aggregated information provided in Table 7 is not comparable. Therefore, we transform this aggregated information into numerical values. Step 4 computes the score functions of each alternative using the information provided in Table 7. The score functions are crucial for the final ranking of alternatives. Table 8 lists the score values of each alternative.

Table 8. Score values of each alternative.

Criteria($ɟ_j$)/Alternative s($Ա_i$)	Score values Based on CuCPFBMOs	Score values Based on CuCPFWBMOs
$Ա_1$	0.3664	0.3442
$Ա_2$	0.3709	0.3588
$Ա_3$	0.3642	0.3762
$Ա_4$	0.3475	0.348

Figures 4 and 5 reflect the individual score values by CuCPFBMOs and CuCPFWBMOs.

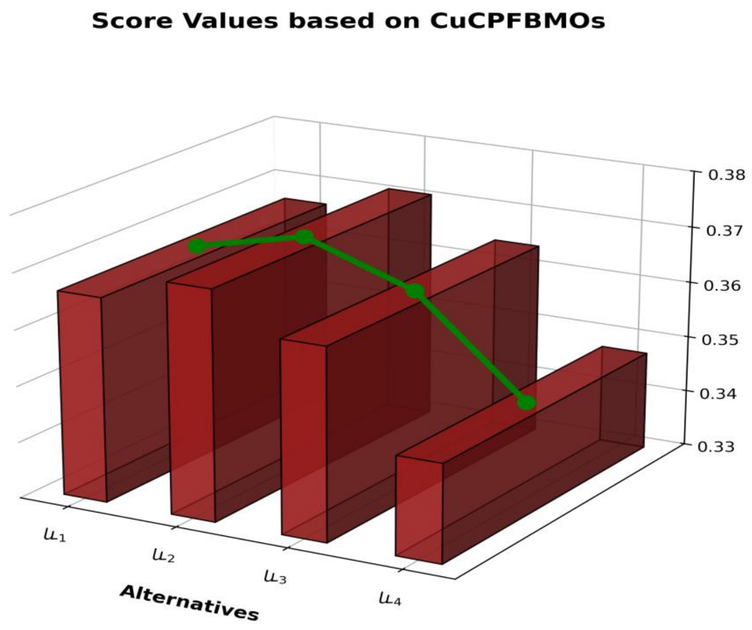

Figure 4. Score values by CuCPFBMOs.

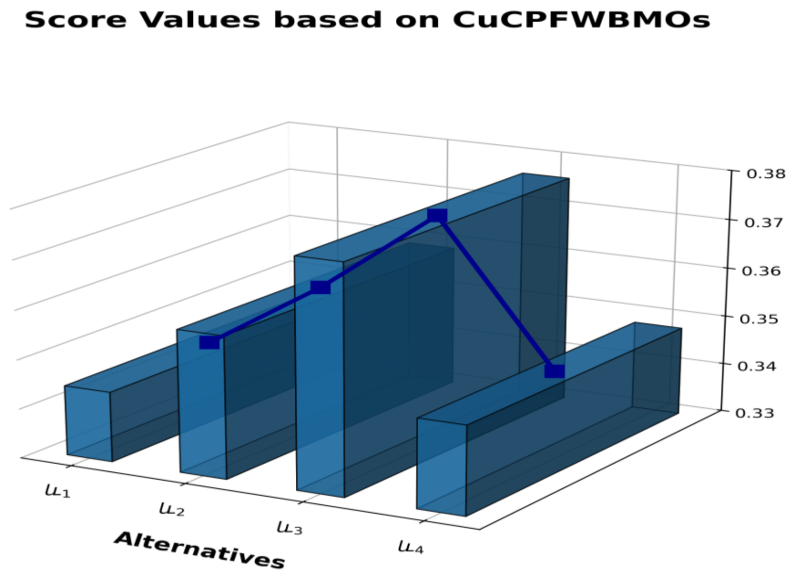

Figure 5. Score values by CuCPFWBMOs.

Step 5. In this final step, we rank the alternatives based on the computed score values provided in Table 8. Based on the score functions, the ranking of the decision alternatives are given in Table 9.

Table 9 Final ranking of the alternatives.

Aggregation Operators	Rank
CuCPFBMOs	$\mathfrak{u}_2 \succ \mathfrak{u}_1 \succ \mathfrak{u}_3 \succ \mathfrak{u}_4$
CuCPFWBMOs	$\mathfrak{u}_3 \succ \mathfrak{u}_2 \succ \mathfrak{u}_4 \succ \mathfrak{u}_1$

Thus, CuCPFBMOs and CuCPFWBMOs ranked $\mathbb{u}_2$ and $\mathbb{u}_3$ as the most preferable choices among a set of four potential alternatives. The proposed CuCPFBMOs and CuCPFWBMOs provide distinct aggregation mechanisms, thereby resulting in a difference in rankings. CuCPFBMOs model interdependence among criteria, whereas CuCPFWBMOs integrate the relative importance of criteria.

Figure 6 represents the comparative score values of CuCPFBMOs and CuCPFWBMOs.

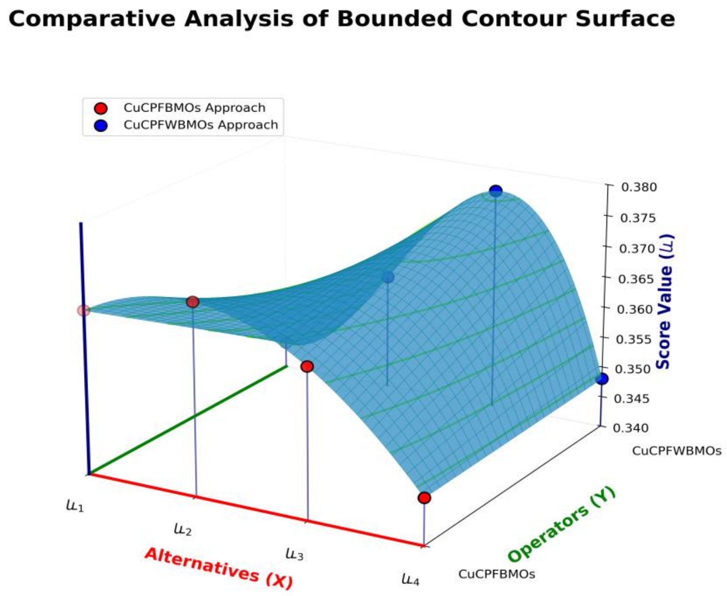

Figure 6. Comparative score values by CuCPFBMOs and CuCPFWBMOs.

6.1. Sensitivity analysis

This subsection demonstrates sensitivity analysis for BMOs. The sensitivity analysis is crucial because it investigates how variations in AO parameters impact the final decision outcomes. The proposed BMOs involve adjustable parameters and conducting sensitivity analysis verifies the reliability, robustness, and stability of the proposed DM approach. For the proposed BMOs, commonly the parameters $\mathbb{p}$ and $\mathbb{q}$ handle the interactions among decision criteria. Experts may verify whether the final rankings remain stable or change by varying these parameters systematically. The sensitivity analysis for the proposed BMOSs can be examined through in Table 10.

Table 10. Sensitivity analysis for the proposed BMOSs.

$(\mathbb{p}, \mathbb{q})$		Ranking	
(1,1)	CuCPFBMOs	0.3664, 0.3709, 0.3642, 0.3475	$\mathbb{u}_2 \succ \mathbb{u}_1 \succ \mathbb{u}_3 \succ \mathbb{u}_4$
	CuCPFWBMOs	0.3442, 0.3588, 0.3762, 0.3448	$\mathbb{u}_3 \succ \mathbb{u}_2 \succ \mathbb{u}_4 \succ \mathbb{u}_1$
(1,2)	CuCPFBMOs	0.3431, 0.3742, 0.3665, 0.3488	$\mathbb{u}_2 \succ \mathbb{u}_3 \succ \mathbb{u}_4 \succ \mathbb{u}_1$
	CuCPFWBMOs	0.3473, 0.3632, 0.3796, 0.3485	$\mathbb{u}_3 \succ \mathbb{u}_2 \succ \mathbb{u}_4 \succ \mathbb{u}_1$
(2,2)	CuCPFBMOs	0.3444, 0.3762, 0.3675, 0.3489	$\mathbb{u}_2 \succ \mathbb{u}_3 \succ \mathbb{u}_4 \succ \mathbb{u}_1$
	CuCPFWBMOs	0.3483, 0.3632, 0.3809, 0.3487	$\mathbb{u}_3 \succ \mathbb{u}_2 \succ \mathbb{u}_4 \succ \mathbb{u}_1$

Table 9 illustrates that by taking different values of parameters $\mathbb{p}$ and $\mathbb{q}$, the proposed approaches ranked $\mathbb{l}_2$ and $\mathbb{l}_3$ as the most preferable choices among a set of four potential alternatives. This indicates that the proposed DM approach is reliable and stable. Figure 7 illustrates the sensitivity analysis of the proposed BMOSs.

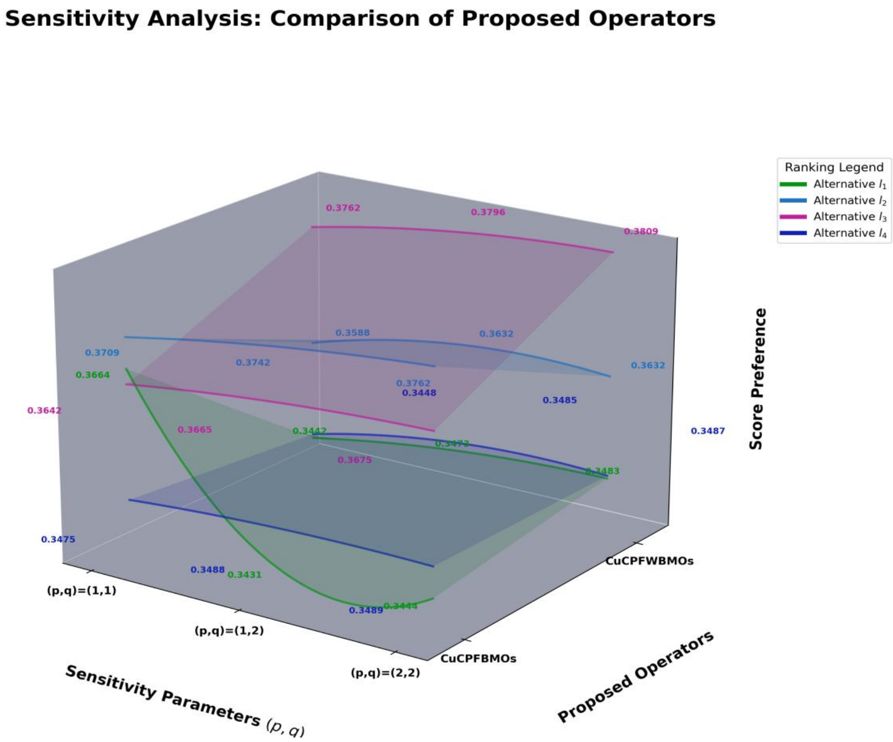

Figure 7. Sensitivity analysis of the proposed approaches.

7. Comparison

In this section, we compare the proposed CuCP-FS approach with some existing fuzzy-based models to highlight its effectiveness and superiority. Classical fuzzy approaches including IFSs [4], P-FSs [11], IVP-FSs [16], CuP-FSs [3], and CP-FSs [33] are crucial for representing uncertain problems; however, they have some limitations to capture complex uncertainty. The IFS [4] approach is described by MS and NMS values such that their sum is at most one. This approach offers a more comprehensive representation of vague information but it lacks the capability to explicitly model neutral uncertainty in DM problems. P-FSs [11] overcome this limitation by associating neutral MS values in grade value. The P-FS approach fails when the information is in the shape of intervals. IVP-FSs [16] modify P-FSs by allowing interval-valued positive, neutral, and negative MS values. This model is very useful for capturing higher levels of uncertainty but lacks the ability to handle multilayered uncertainty. CuP-FSs [3] combine the frameworks of P-FSs and IVP-FSs within a unified framework to model multilayered uncertainty. However, the approach fails when the fluctuations or variability is present in expert opinions. CP-FSs [33] extend the P-FS approach by integrating a radius parameter. This radius parameter is significant for representing fluctuations around a reference assessment. The CP-FS approach is unable to model multilayered uncertainty.

The proposed comparative analysis especially considers two types of comparisons:

ranking-wise comparison and characteristic-wise comparison. In ranking-wise comparison, the final ranking of the proposed approaches and existing approaches are emphasized. But, in characteristic-wise comparison, the major characteristics of the proposed model and existing models are addressed. Table 11 provides the ranking-wise comparison of the proposed approaches and existing approaches.

Table 11. Ranking-wise comparison.

Methods	Score Values	Ranking Values
IFS Approach [4]	0.0, 0.0, 0.0, 0.0	Not evaluated
P-FS Approach [11]	0.0, 0.0, 0.0, 0.0	Not evaluated
IVP-FS Approach [16]	0.0, 0.0, 0.0, 0.0	Not evaluated
CuP-FS Approach [3]	0.0, 0.0, 0.0,0.0	Not evaluated
CP-FS Approach [33]	0.0, 0.0, 0.0,0.0	Not evaluated
Proposed Approach based on CuCPFBMOs	0.3664, 0.3709, 0.3642, 0.3475	$\text{ɥ}_2 \succ \text{ɥ}_1 \succ \text{ɥ}_3 \succ \text{ɥ}_4$
Proposed Approach based on CuCPFWBMOs	0.3442, 0.3588, 0.3762, 0.3448	$\text{ɥ}_3 \succ \text{ɥ}_2 \succ \text{ɥ}_4 \succ \text{ɥ}_1$

Table 11 reflects that the existing approaches fail to rank the alternatives due to its mathematical structure. These approaches cannot be employed to model the information provided in Tables 3 and 4. However, the proposed approaches ranked ɥ_2 and ɥ_3 as the most better choices, thereby providing a reliable ranking of decision alternatives. Figure 8 represents the graphical representation of the final ranking by proposed approaches and existing approaches.

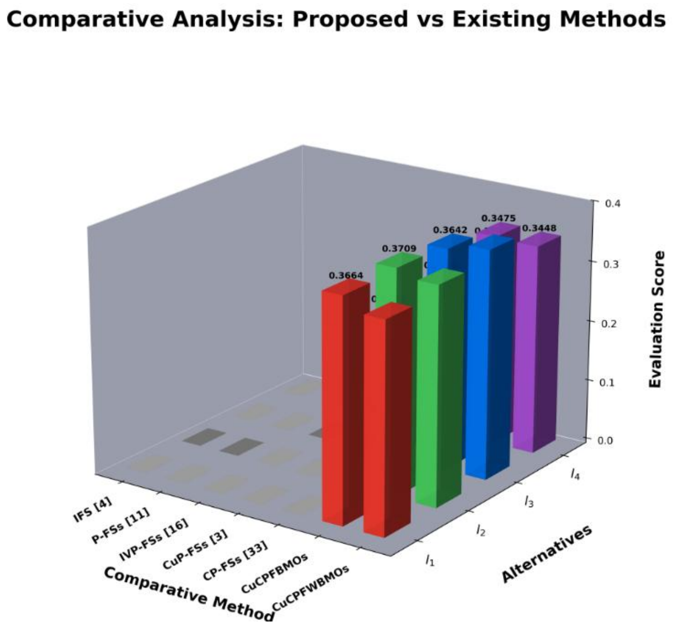

Figure 8. Graphical representation of the final ranking by proposed approaches and existing approaches.

Now, we examine the key characteristics such as positive MS value, neutral MS value, negative MS value, interval-valued representation, fluctuations or variability, and multilayered uncertainty of the proposed model and existing models. Table 12 lists these main characteristics of the proposed CuCP-FS model and existing models.

Table 12. Key Features of the proposed model and existing models.

Methods	Positive	Neutral	Negative	Interval-valued representation	Radius Parameter	Multilayered uncertainty
IFS Approach [4]	Yes	No	Yes	No	No	No
P-FS Approach [11]	Yes	Yes	Yes	No	No	No
IVP-FS Approach [16]	Yes	Yes	Yes	Yes	No	No
CuP-FS Approach [3]	Yes	Yes	Yes	Yes	No	Yes
CP-FS Approach [33]	Yes	Yes	Yes	No	Yes	No
Proposed CuCP-FS Approach	Yes	Yes	Yes	Yes	Yes	Yes

Table 12 shows that the proposed CuCP-FS model overcomes all the limitations of the existing approaches. However, the existing approaches have several limitations, which can be interpreted as follows.

i) The IFS [4] approach was proposed to extend traditional FSs for better representing of uncertainty in DM problems. Each element in the IFS approach is described by MS and NMS values such that their sum is at most one. IFSs provide a more flexible representation of vague information compared to classical FSs. However, this approach fails to model the neutral MS value, interval-valued information, multilayered uncertainty, and fluctuations around a reference assessment. Therefore, it fails to solve the expert evaluations present in Table 5.

ii) P-FSs further extend the IFS framework by integrating the neutral MS value such that the sum of positive, neutral, and negative MS values does not exceed 1. P-FSs contribute more flexible representation of uncertain information compared to IFSs. However, the P-FS model is unable to handle interval-valued information, multilayered uncertainty, and fluctuations around a reference assessment. Therefore, P-FSs cannot be employed to model the information provided in Table 5.

iii) IVP-FSs modify the framework of P-FSs by associating interval-valued positive, neutral, and negative MS values. This approach is very useful for handling higher levels of uncertain information. But, IVP-FSs fail to model multilayered uncertainty and fluctuations around a reference assessment. Therefore, IVP-FSs cannot be applied to solve the expert evaluations provided in Table 5.

iv) CuP-FSs combine the structures of P-FSs and IVP-FSs for capturing multilayered uncertainty. They extend both P-FS and IVP-FS frameworks by combining them within a unified single framework.

However, this approach lacks the ability to model fluctuations or variability around a reference assessment. Therefore, CuP-FSs are unable to model the information provided in Table 5.

v) CP-FSs further extend P-FSs by integrating the radius parameter to deal with fluctuations or tolerance around P-FS information. This approach is very useful for MCDM problems where fluctuations are present in expert opinions. However, this approach fails to model interval-valued information and multilayered uncertainty. Therefore, CP-FSs cannot be employed to evaluate the information provided in Table 5.

This comparative study reflects that the proposed CuCP-FS model extends these existing approaches by representing interval-valued information, multilayered uncertainty, and fluctuations around P-FS information. Thus, the proposed model is crucial for reliable and robust DM processes.

8. Conclusions

This paper addressed the challenge of handling complex uncertainty in DM environments. Existing fuzzy approaches lack the ability to model interval-valued information, multilayered uncertainty, and the reliability of evaluation information simultaneously. To handle these challenges, the proposed study presented a novel approach that integrated P-FSs, IVP-FSs, and a circular radius parameter into a unified model. The newly defined model improved the capability to represent uncertain, hesitant, and reliability-based information more significantly. Some basic operations for the proposed approach were designed to contribute mathematical manipulation of the newly defined structure. Moreover, BMOs were developed to aggregate evaluation information while considering the interactions among decision criteria. Based on proposed AOs and score functions, an MCDM technique was demonstrated within the proposed CuCP-FS model. To present its practical applicability, the DM approach was employed for the selection of quantum computing technology, and the final results illustrated its significance in handling complex uncertain data. Furthermore, a comparative analysis was established with existing models, which presented the enhanced flexibility of the proposed work.

8.1. Future research

The proposed study showed significant results. However, several promising directions remain open for further research. In the future, we may focus on validating the newly defined approaches using real-world raw datasets achieved from various life-time domains. Integrating high-scale empirical data would improve the applicability, reliability, and robustness of the proposed aggregation mechanism. Future research may explore the proposed model by integrating some other approaches such as the linguistic hypersoft set and linguistic super hypersoft set [13]. Based on the proposed approach, other advanced DM methods including TODIM, TOPSIS, and VIKOR may also be developed. To further improve its practical applications, the proposed model may be integrated to develop optimization and machine learning-based DM techniques. The proposed model can be employed to solve uncertainty problems in other real-world domains including risk assessment, renewable source energy, pattern recognition, artificial intelligence-based systems, and medical diagnosis. Moreover, the proposed approaches may be employed to increase the complexity of real-world systems, such as fractional-order epidemic models [5].

Author contributions

Haitham Qawaqneh: Conceptualization, data curation, methodology; Abdallah Shihadeh: Methodology, validation, formal analysis, visualization; Wael Mahmoud Mohammad Salameh: Software, validation, resources, visualization; Sultan Hussain: Methodology, formal analysis, visualization; Muhammad Zeeshan: Validation, software, conceptualization; Takaaki Fujita: Writing review, editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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