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Research article

Statistical analysis for the truncated unit exponentiated Ailamujia distribution under progressive Type–II censoring

Faten S. Alamri[1], Ahlam H. Tolba[2,*], Hana S. Jabarah[2], Ahmed R. El-Saeed[3], and Ahmed T. Ramadan[4]

[1] Department of Mathematical Sciences, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia; Fsalamri@pnu.edu.sa

[2] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 33516, Egypt; dr_ahamdy156@mans.edu.eg, hanajbarah1@gmail.com

[3] Department of Mathematics and Statistics, Faculty of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia; aesaleh@imamu.edu.sa

[4] Department of Mathematics, Faculty of Science, Galala University, Suez 43713, Egypt; Ahmed_Taha@Gu.edu.eg

* **Correspondence:** Email: dr_ahamdy156@mans.edu.eg; Tel: +201008782195.

Abstract: This study introduces and investigates the truncated unit exponentiated Ailamujia (TUEA) distribution within the framework of a progressive Type–II censoring scheme. By incorporating truncation on the unit interval, the proposed model extends the unit exponentiated Ailamujia distribution, significantly enhancing its flexibility for modeling bounded lifetime data. We derive the fundamental mathematical properties of the TUEA model, including the probability density function, the cumulative distribution function, reliability measures, and hazard rate functions. Statistical inference for the model parameters is developed within both frequentist and Bayesian frameworks using progressive Type–II censored data. The maximum likelihood estimates are computed through the Newton–Raphson iterative algorithm, whereas Bayesian inference is carried out under symmetric squared error and asymmetric LINEX loss functions. Point estimates and highest posterior density credible intervals are obtained via Markov chain Monte Carlo (MCMC) sampling procedures. In addition, a comprehensive Monte Carlo simulation study is conducted to investigate the finite sample performance of the new estimators in terms of bias, mean square error, and confidence interval coverage probability. In addition, the practical usefulness of the TUEA distribution is shown via an analysis of a real-life data sets, where the new model exhibits a better fit than competing models.

Keywords: truncated unit exponentiated Ailamujia distribution; progressive Type–II censoring; maximum likelihood estimation; Bayesian inference
Mathematics Subject Classification: 62-XX

1. Introduction

The modeling of lifetime and reliability data through statistics is an essential foundation for various fields in engineering and science such as quality control, survival analysis in medical sciences, and biological sciences [1]. In many real-life situations involving life testing experiments, full data on failure times cannot be realized due to strict financial, ethical, or time-related considerations. As such, there is a need for an effective censorship process to maximize inferential insights from incomplete observations. One such effective process that has gained prominence is that of progressive Type–II censoring (PT-IIC) [2]. Unlike traditional censoring techniques, such as Type–I and Type–II, where censoring occurs at specific points, PT-IIC offers the flexibility to remove some of the surviving units at the point of intermediate failures [3, 4].

Traditional families, such as the Beta and Kumaraswamy distributions, may exhibit certain limitations when modeling complex lifetime data. Specifically, they often lack the necessary flexibility to accurately capture diverse hazard rate shapes and the behavior of the distribution tails. In fact, in several real-life applications, lifetime observations are intrinsically confined within a bounded range due to physical constraints, biological restrictions, or standardization purposes. This is the case, for example, of proportions, reliability indices, and standardized lifetime variables that fall within the unit interval $(0, 1)$; see [5].

Hence, truncation within the unit interval cannot be seen only as a simplification in mathematics but as a crucial part of modeling, which brings the statistical model closer to the actual data-generation process. The use of truncation within the exponential Ailamujia distribution results in obtaining the truncated unit exponentiated Ailamujia (TUEA) distribution, which possesses important features of the original distribution, adapted to the bounded support; see [6–8].

Furthermore, the TUEA distribution proves to be more adaptable than any other currently existing unit interval distribution because it shows various types of behavior in the form of its probability density function and hazard function, which can be monotonic increasing, monotonic decreasing, and unimodal functions. This flexibility in behavior becomes a useful feature in reliability and survival analysis because, depending on the underlying mechanism of failure, different types of risk development are possible.

There have been many developments in inferential methods for such models. Contributions have included the use of likelihood in exponential models [9], reliability analysis in the Kumaraswamy-G family [10], and an examination of optimal censoring in the Nadarajah–Haghighi model [11]. Other developments have covered optimal progressive censoring in U-shaped hazard rate models [12], and sampling methods for Kumaraswamy distribution [13]. These developments have come after pioneering work in Bayesian life testing [14], optimal experimental design [15], and a comparative study between classical and Bayesian approaches [16]. There have also been computational developments, including simulation methods [17] and exact inference under Type–I censoring [18], that have further expanded the applicability of these methods. In addition to a study on the Rayleigh distribution of progressively type II censored two-parameters, presented by [19], and [20] discussed an analysis of competing risk data from progressive type II with applications.

Motivated by the necessity of developing flexible models with bounded support, this paper introduces an extensive inferential approach for the TUEA distribution under progressive Type–II censoring. In order to illustrate reproducibility, some supplementary materials have been provided in connection with the proposed research findings. The materials presented include codes for data generation, implementation of progressive Type–II censoring, maximum likelihood estimation based on the Newton–Raphson technique, and Bayesian analysis using MCMC algorithms.

The organization of the paper is as follows: Section 2 introduces the TUEA distribution along with the progressive Type–II censoring scheme; the classical and Bayesian estimation methodologies are developed in Sections 3 and 4, respectively; an extensive Monte Carlo simulation study is reported in Section 5; Section 6 demonstrates the applicability of the proposed model using two real data sets; and concluding remarks are provided in Section 7.

2. Model description

This section introduces the progressive Type–II censoring scheme and formally defines the truncated unit exponentiated Ailamujia (TUEA) distribution [6], reminding us of its main distributional properties that are required for subsequent inference.

2.1. Progressive Type–II censoring scheme

The progressive Type–II Censoring (PT-IIC) scheme is widely used in life-testing experiments due to its flexibility and efficient use of experimental units. Consider a life-testing experiment in which n independent and identical units are put under observation [21].

Let $x_{(1)} < x_{(2)} < \cdots < x_{(m)}$, where $1 \leq m \leq n$ represents the ordered observed failure times. At the instant of the first observed failure time $x_{(1)}$, the surviving R_1 units are randomly withdrawn from the remaining $n-1$ units. Upon the occurrence of the second failure in $x_{(2)}$, additional R_2 units are randomly removed from the $n - R_1 - 2$ units still under test. This process proceeds sequentially until the mth failure is observed, at which stage the remaining pieces are discarded:

$$R_m = n - m - \sum_{i=1}^{m-1} R_i,$$

units are removed, and the experiment is terminated. The vector $\mathbf{R} = (R_1, R_2, \ldots, R_m)$ fully characterizes the progressive Type–II censoring scheme.

It is important to note that the classical Type–II censoring scheme arises as a special case of the PT-IIC framework when $R_1 = R_2 = \cdots = R_{m-1} = 0$. A schematic illustration of the PT-IIC mechanism is presented in Figure 1.

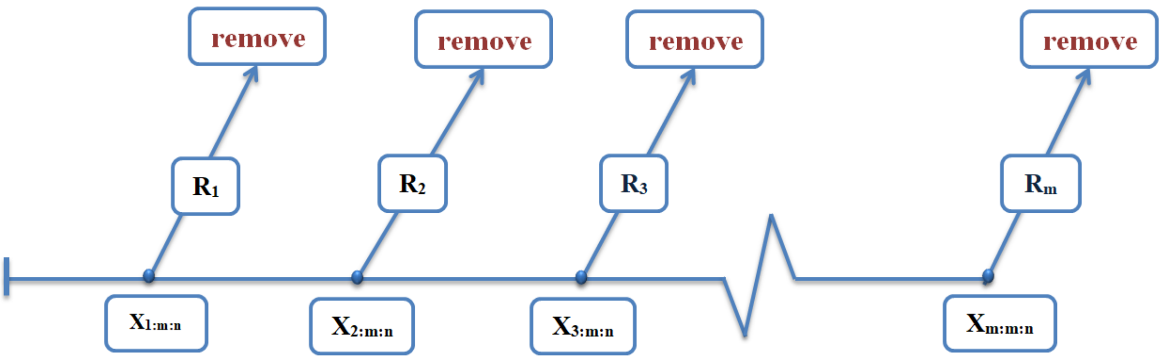

Figure 1. Schematic representation of PT-IIC.

Let $x_{(1)}, x_{(2)}, \ldots, x_{(m)}$ represent a progressively Type–II censored sample drawn from a continuous distribution with probability density function $f(\cdot)$ and cumulative distribution function $F(\cdot)$. The corresponding joint likelihood function of the observed data is expressed as

$$L(x_{(1)}, \ldots, x_{(m)}) = C \prod_{i=1}^{m} f(x_{(i)}) \left[1 - F(x_{(i)})\right]^{R_i}, \tag{2.1}$$

where the normalizing constant C is defined as

$$C = n\,(n - R_1 - 1)\,(n - R_1 - R_2 - 2) \cdots \left(n - \sum_{i=1}^{m-1} R_i - (m-1)\right). \tag{2.2}$$

2.2. Exponentiated and truncated unit exponentiated Ailamujia distribution

The probability density and the cumulative distribution functions corresponding to a random variable Y following the exponentiated Ailamujia (EA) distribution can be expressed as

$$f_{EA}(y, \alpha, \theta) = 4\alpha\theta^2 y e^{-2\theta y}\left(1 - (1 + 2\theta y)e^{-2\theta y}\right)^{\alpha - 1}, \quad y > 0,\ \theta > 0,\ \alpha > 0, \tag{2.3}$$

and

$$F_{EA}(y, \alpha, \theta) = \left(1 - (1 + 2\theta y)e^{-2\theta y}\right)^{\alpha}, \quad y > 0,\ \theta > 0,\ \alpha > 0. \tag{2.4}$$

Similarly, a random variable X is said to follow the right-truncated unit exponentiated Ailamujia (TUEA) distribution if its probability density function (PDF) is expressed as

$$f_{TUEA}(x, \alpha, \theta) = \frac{f_{EA}(x, \alpha, \theta)}{\int_0^1 f_{EA}(x, \alpha, \theta)\,dx}. \tag{2.5}$$

Note that

$$\int_0^1 f_{EA}(x, \alpha, \theta)\,dx = \int_0^1 4\alpha\theta^2 x e^{-2\theta x}\left(1 - (1 + 2\theta x)e^{-2\theta x}\right)^{\alpha - 1} dx.$$

Using integration by parts, where $u = 1 - (1 + 2\theta x)e^{-2\theta x}$, we obtain the following:

$$\begin{aligned}\int_0^1 f_{EA}(x, \alpha, \theta)\,dx &= \int_0^{1-(1+2\theta)e^{-2\theta}} \alpha u^{\alpha-1} du \\ &= \left(1 - (1 + 2\theta)e^{-2\theta}\right)^{\alpha}.\end{aligned}$$

As a result, we have

$$f_{TUEA}(x, \alpha, \theta) = \frac{4\alpha\theta^2 x e^{-2\theta x}\left(1 - (1 + 2\theta x)e^{-2\theta x}\right)^{\alpha - 1}}{(1 - (1 + 2\theta)e^{-2\theta})^{\alpha}}, \tag{2.6}$$

where $\alpha > 0$ and $\theta > 0$ are the shape and scale parameters, respectively, and $0 < x < 1$. The corresponding cumulative distribution function (CDF) is given by

$$F_{TUEA}(x,\alpha,\theta) = \frac{F(x,\alpha,\theta) - F(0,\alpha,\theta)}{F(1,\alpha,\theta) - F(0,\alpha,\theta)} = \frac{\left(1-(1+2\theta x)e^{-2\theta x}\right)^{\alpha}}{\left(1-(1+2\theta)e^{-2\theta}\right)^{\alpha}}. \tag{2.7}$$

The survival function of the TUEA distribution is therefore given by

$$S(x) = 1 - F_{TUEA}(x;\alpha,\theta) = 1 - \frac{\left[1-(1+2\theta x)e^{-2\theta x}\right]^{\alpha}}{\left[1-(1+2\theta)e^{-2\theta}\right]^{\alpha}}, \tag{2.8}$$

and the corresponding hazard rate function takes the form

$$h(x) = \frac{4\alpha\theta^2 x e^{-2\theta x}\left[1-(1+2\theta x)e^{-2\theta x}\right]^{\alpha-1}}{\left[1-(1+2\theta)e^{-2\theta}\right]^{\alpha} - \left[1-(1+2\theta x)e^{-2\theta x}\right]^{\alpha}}. \tag{2.9}$$

The primary advantage of the TUEA model lies in its hazard rate function $h(x)$ and PDF, which can take various forms, including increasing, decreasing, or unimodal shapes, depending on the parameter vector (α,θ); see Figure 2.

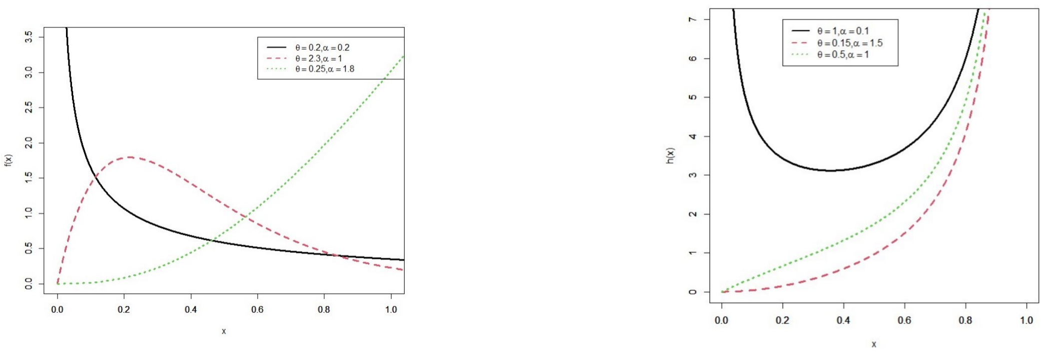

Figure 2. Graphical representations of the PDF and hazard rate function of the TUEA distribution for selected parameter configurations (α,θ).

Figure 2 illustrates how the parameters α and θ influence the structural form of the TUEA distribution. It can be seen that changes in α mainly modify the skewness and height of the density curve, whereas θ governs the rate at which the distribution decays. Moreover, the hazard rate function displays considerable variability, taking forms such as increasing and unimodal patterns depending on the chosen parameter values. This range of behaviors underscores the flexibility of the TUEA model in capturing diverse failure mechanisms, which makes it suitable for the analysis of data of a bounded lifetime with complex characteristics.

3. Maximum likelihood estimation

Within the framework of progressive Type–II censoring (PT-IIC), the likelihood function is formulated by gathering the information from both observed failure times and censored units. In particular, the contribution of each observed failure is represented by the probability density function, whereas the units removed during the experiment are accounted for through the survival function.

Let $x_{(1)}, x_{(2)}, \ldots, x_{(m)}$ represent a progressively Type–II censored sample (PT-IIC) of size m drawn from identical units n subjected to a predetermined censoring plan $\mathbf{R} = (R_1, R_2, \ldots, R_m)$. Assume that the lifetime of each unit follows the TUEA distribution with parameters $\alpha > 0$ and $\theta > 0$, whose PDF and CDF are given in Eqs (2.6) and (2.7), respectively. After ignoring multiplicative constants that do not depend on the unknown parameters, the likelihood function of the observed data can be expressed as

$$L(\alpha, \theta \mid \mathbf{x}) = C \prod_{i=1}^{m} \frac{4\alpha\theta^2 x_{(i)} e^{-2\theta x_{(i)}} \left[1 - (1 + 2\theta x_{(i)}) e^{-2\theta x_{(i)}}\right]^{\alpha-1}}{\left[1 - (1 + 2\theta) e^{-2\theta}\right]^{\alpha}} \times \left[1 - \frac{\left[1 - (1 + 2\theta x_{(i)}) e^{-2\theta x_{(i)}}\right]^{\alpha}}{\left[1 - (1 + 2\theta) e^{-2\theta}\right]^{\alpha}}\right]^{R_i}, \tag{3.1}$$

where C is the normalization constant associated with the PT-IIC scheme, as defined in Section 2. It is noted that, in Eq (3.1), the initial product term represents the contribution of the observed failure times through the probability density function, and the second component captures the effect of the progressively censored observations by incorporating the survival function raised to the powers R_i.

Applying the natural logarithm to the likelihood function produces the log-likelihood function

$$\begin{aligned} \ell(\alpha, \theta) &= m \log(4) + m \log \alpha + 2m \log \theta - m\alpha \log\left[1 - (1 + 2\theta) e^{-2\theta}\right] \\ &+ \sum_{i=1}^{m} \log x_{(i)} - 2\theta \sum_{i=1}^{m} x_{(i)} + (\alpha - 1) \sum_{i=1}^{m} \log\left[1 - (1 + 2\theta x_{(i)}) e^{-2\theta x_{(i)}}\right] \\ &+ \sum_{i=1}^{m} R_i \log\left\{1 - \frac{\left[1 - (1 + 2\theta x_{(i)}) e^{-2\theta x_{(i)}}\right]^{\alpha}}{\left[1 - (1 + 2\theta) e^{-2\theta}\right]^{\alpha}}\right\}. \end{aligned} \tag{3.2}$$

The estimators of α and θ are computed by resolving the nonlinear system of score equations arising from the first-order partial derivatives of the log-likelihood function with respect to the model parameters.

The score function corresponding to α is expressed as

$$\begin{aligned} \frac{\partial \ell}{\partial \alpha} &= \frac{m}{\alpha} - m \log\left[1 - (1 + 2\theta) e^{-2\theta}\right] + \sum_{i=1}^{m} \log\left[1 - (1 + 2\theta x_{(i)}) e^{-2\theta x_{(i)}}\right] \\ &- \sum_{i=1}^{m} R_i \frac{\left[1 - (1 + 2\theta x_{(i)}) e^{-2\theta x_{(i)}}\right]^{\alpha} \log\left[1 - (1 + 2\theta x_{(i)}) e^{-2\theta x_{(i)}}\right]}{\left[1 - (1 + 2\theta) e^{-2\theta}\right]^{\alpha} - \left[1 - (1 + 2\theta x_{(i)}) e^{-2\theta x_{(i)}}\right]^{\alpha}}. \end{aligned} \tag{3.3}$$

Similarly, the score function for θ is obtained as

$$
\begin{aligned}
\frac{\partial \ell}{\partial \theta} = \frac{2m}{\theta} - 2\sum_{i=1}^{m} x_{(i)} - m\alpha \frac{\partial}{\partial \theta}\log\Big[1-(1+2\theta)e^{-2\theta}\Big] \\
+ (\alpha-1)\sum_{i=1}^{m}\frac{\partial}{\partial \theta}\log\Big[1-(1+2\theta x_{(i)})e^{-2\theta x_{(i)}}\Big] \\
- \sum_{i=1}^{m} R_i \frac{\partial}{\partial \theta}\log\left\{1-\frac{\left[1-(1+2\theta x_{(i)})e^{-2\theta x_{(i)}}\right]^{\alpha}}{\left[1-(1+2\theta)e^{-2\theta}\right]^{\alpha}}\right\}.
\end{aligned}
\tag{3.4}
$$

Because the likelihood equations do not admit closed-form solutions, the numerical optimization method is employed through the Newton–Raphson iterative scheme. The algorithm is initiated with appropriately chosen starting values for (α, θ), selected to promote numerical stability and reliable convergence. At each iteration, updated estimates are obtained using the score vector and the observed Fisher information matrix.

The iterative procedure is repeated until a convergence is declared, specifically when the absolute difference between successive parameter estimates is less than 10^{-6}. Furthermore, to reduce the risk of convergence to local maxima and to enhance robustness, multiple sets of initial values were investigated in preliminary numerical experiments.

3.1. Asymptotic confidence intervals

Under regularity conditions, the maximum likelihood estimators of the parameters α and θ are asymptotically normally distributed; that is,

$$\hat{\alpha} \sim N(\alpha, Var(\hat{\alpha})), \hat{\theta} \sim N(\theta, Var(\hat{\theta})).$$

At a confidence level of $100(1-\alpha)\%$, the confidence intervals for α and θ are for 95% confidence, given by

$$CI_{\hat{\alpha}} : \hat{\alpha} \pm Z_{\gamma\backslash 2}\sqrt{Var(\hat{\alpha})}.$$

$$CI_{\hat{\theta}} : \hat{\theta} \pm Z_{\gamma\backslash 2}\sqrt{Var(\hat{\theta})}.$$

3.2. Computational procedure and implementation

To enhance reproducibility, R computations were implemented in the following steps:

Step 1. A random sample is generated from the TUEA distribution using the inverse transform method. The progressive Type II censoring scheme is then applied by specifying the censoring vector $(R_1, R_2, \ldots, R_m)$, where R_i denotes the number of units removed at the ith failure time.

Step 2. Given the observed failure times $x_1, x_2, \ldots, x_m$, the likelihood function is constructed as a product of the probability density function evaluated at failure times and the survival function raised to the censoring counts. This formulation reflects the contribution of both observed and censored units under the PT-IIC scheme.

Step 3. The log-likelihood function is numerically maximized using the `optim` function in R, subject to the positivity constraints on the parameters. Standard errors and confidence intervals are obtained using the observed Fisher information matrix.

Step 4. Posterior samples are generated using a Metropolis Hastings algorithm with log-normal proposals. Independent gamma priors are assigned to the model parameters. After discarding an appropriate burn-in period, posterior summaries such as means and credible intervals are computed.
Step 5. For the simulation study, statistical measures including bias, mean squared error (MSE), and coverage probabilities are calculated based on repeated samples. These quantities are summarized in tabular form for comparison purposes.

To ensure reproducibility and scientific reliability, the computational procedure has been carefully structured and fully documented. The implementation follows a clear workflow that includes data generation under the progressive Type–II censoring scheme, likelihood construction based on both observed and censored units, parameter estimation using maximum likelihood and Bayesian approaches, and the computation of performance measures such as bias, MSE, and interval estimates.

4. Bayesian estimation

In this section, we develop a Bayesian inferential framework for the unknown parameters of the TUEA distribution under the progressive Type-II censoring (PT-IIC) mechanism. Bayesian estimation offers a distinct advantage by integrating likelihood information with prior subjective or historical knowledge regarding the model parameters. Given the analytical intractability of the resulting posterior density, we implement a numerical strategy based on Markov chain Monte Carlo (MCMC) techniques to facilitate the computation of point and interval estimators.

4.1. Prior specifications and posterior formulation

In the absence of detailed informative priors for (α, θ), we treat the parameters as stochastically independent and assign them gamma-type conjugate priors, specifically $\alpha \sim \text{gamma}(a_1, b_1)$ and $\theta \sim \text{gamma}(a_2, b_2)$, where a_j and b_j $(j = 1, 2)$ denote strictly positive shape and rate hyperparameters, respectively.

The choice of gamma priors is due to their flexibility and their ability to represent both informative and weakly informative prior beliefs depending on the choice of hyperparameters.

In our study, we set (a_1, b_1) and (a_2, b_2) to values that reflect limited prior knowledge while avoiding excessive influence on the posterior distribution. This choice provides a balance between prior regularization and data-driven inference, particularly in the presence of censoring. Furthermore, the use of gamma priors facilitates computational efficiency in the MCMC algorithm and contributes to stable posterior sampling. Sensitivity analyses with alternative hyperparameter settings are also examined and yielded consistent results, indicating that the main conclusions are robust to reasonable prior variations.

The joint prior density function is then given by

$$\pi(\alpha, \theta) \propto \alpha^{a_1 - 1} \theta^{a_2 - 1} \exp\left(-b_1 \alpha - b_2 \theta\right), \quad \alpha, \theta > 0. \tag{4.1}$$

By applying Bayes' theorem to combine the joint prior with the likelihood function $L(\alpha, \theta \mid \mathbf{x})$ derived under the PT-IIC scheme in Section 3, the joint posterior density is formulated as:

$$\pi^*(\alpha, \theta \mid \mathbf{x}) = \frac{\pi(\alpha, \theta) L(\alpha, \theta \mid \mathbf{x})}{\int_0^\infty \int_0^\infty \pi(\alpha, \theta) L(\alpha, \theta \mid \mathbf{x}) d\alpha d\theta}. \tag{4.2}$$

Substituting the explicit structural forms, the kernel of the joint posterior density becomes

$$\begin{aligned}\pi^*(\alpha,\theta \mid \mathbf{x}) \propto \alpha^{a_1+m-1}\theta^{a_2+2m-1} \exp\left[-b_1\alpha - \theta\left(b_2 + 2\sum_{i=1}^{m} x_{(i)}\right)\right] \prod_{i=1}^{m} x_{(i)} \\ \times \prod_{i=1}^{m} \left[1-(1+2\theta x_{(i)})e^{-2\theta x_{(i)}}\right]^{\alpha-1} \left[1-(1+2\theta)e^{-2\theta}\right]^{-\alpha m} \\ \times \prod_{i=1}^{m} \left\{\left[1-(1+2\theta)e^{-2\theta}\right]^{\alpha} - \left[1-(1+2\theta x_{(i)})e^{-2\theta x_{(i)}}\right]^{\alpha}\right\}^{R_i}.\end{aligned} \tag{4.3}$$

4.2. Loss functions and point estimation

The selection of the right loss function in the context of Bayesian inference is crucial because it controls the penalties for estimation mistakes. The squared error loss (SEL) is the conventional loss function because it is easy to apply. However, it assumes that over- and under-estimations are equally weighted. In reliability engineering and life data analysis, the implications of making mistakes are highly asymmetrical. For example, overestimating the reliability of a part could be disastrous, and underestimating it leads to higher maintenance costs. This problem will be investigated using both the SEL and the LINEX asymmetric loss functions. The loss function can be expressed as

$$L(e) = c\left(e^{ae} - ae - 1\right), \quad a \neq 0,$$

where $e = \hat{\theta} - \theta$ represents the estimation error, and a serves as a shape parameter governing both the direction and magnitude of asymmetry.

In the present work, the LINEX loss function is employed to accommodate asymmetries in estimation errors, thereby yielding Bayesian estimates that are more robust and better aligned with practical considerations in reliability applications than pervious schemes.

Under the SEL function, the Bayes estimators $\widehat{\alpha}_{\text{SEL}}$ and $\widehat{\theta}_{\text{SEL}}$ are given by posterior means.

$$\widehat{\phi}_{\text{SEL}} = \int_0^\infty \int_0^\infty \phi\, \pi^*(\alpha,\theta \mid \mathbf{x})\, d\alpha d\theta, \quad \text{for } \phi \in \{\alpha,\theta\}. \tag{4.4}$$

The LINEX loss function, defined as $L(\Delta) \propto e^{c\Delta} - c\Delta - 1$, where $\Delta = \widehat{\phi} - \phi$, offers greater flexibility. For a given shape parameter c, the Bayes estimator under LINEX is obtained as:

$$\widehat{\phi}_{\text{LINEX}} = -\frac{1}{c} \ln\left[\mathbb{E}_{\pi^*}(e^{-c\phi} \mid \mathbf{x})\right]. \tag{4.5}$$

4.3. Computational implementation: MCMC approach

To generate samples from the posterior distributions of the model parameters, we use a Markov chain Monte Carlo (MCMC) procedure based on the Metropolis–Hastings (MH) algorithm.

The MCMC simulation is conducted over $N = 20{,}000$ iterations, with the first 5000 iterations discarded as burn-in to mitigate the influence of initial values. Posterior inference is then carried out using the remaining 15,000 samples.

For each parameter, a random-walk MH scheme is adopted, employing normal proposal distributions of the form

$$\theta^{(t+1)} = \theta^{(t)} + \epsilon, \quad \epsilon \sim \mathcal{N}(0,\sigma^2),$$

where σ^2 denotes a tuning parameter selected to achieve a suitable acceptance rate.

The MH algorithm consists of the following fundamental steps:
1. Initialize the parameter vector with an initial value or set of values.
2. Draw candidate parameter values from a proposal distribution, which determines how the next state is generated based on the current state. The proposal distribution should be chosen to ensure efficient exploration and adequate mixing of the parameter space.
3. Calculate the acceptance probability for the proposed parameters, defined as the ratio of the posterior density of the target at the proposed values to that at the current values as follows:

$$R = \frac{g(M^\star/data)p(M_{t-1}|M^\star)}{g(M_{t-1}/data)p(M^\star|M_{t-1})}.$$

4. Accept the values of the proposed parameters with probability equal to the accepted probability calculated. If accepted, update the current state to the proposed values; if rejected, retain the current state.
5. Repeat Steps 2–4 for a predetermined number of iterations or until convergence criteria are satisfied.

4.4. Credible interval

In the Bayesian framework, given prior distributions $p(\alpha), p(\theta)$ and posterior distributions $p(\alpha \mid \text{data}), p(\theta \mid \text{data})$, a credible interval $100(1-\gamma)\%$ is defined as

$$CrI_\alpha^{Bayes} = [\alpha_L, \alpha_U] \quad \text{such that} \quad \Pr(\alpha_L \le \alpha \le \alpha_U \mid \text{data}) = 1-\gamma,$$

$$CrI_\theta^{Bayes} = [\theta_L, \theta_U] \quad \text{such that} \quad \Pr(\theta_L \le \theta \le \theta_U \mid \text{data}) = 1-\gamma,$$

which facilitates Bayesian inference and the implementation of MCMC methods such as the MH algorithm.

5. Simulation study

In this section, a comprehensive Monte Carlo simulation study is conducted to investigate the finite-sample performance of the proposed estimation procedures for the parameters of the TUEA distribution under the PT-IIC scheme. The performance of the maximum likelihood estimators and the Bayesian estimators is evaluated in various experimental settings, with particular emphasis on the effects of sample size and censoring severity.

5.1. Data generation algorithm

To evaluate the performance of the proposed estimators, a Monte Carlo simulation study is carried out. Random samples are drawn from the proposed distribution using the inverse transform sampling technique. The data generation procedure proceeds as follows:

1. Generate $U_i \sim \text{Uniform}(0,1)$ for $i = 1, 2, \ldots, n$.

2. Obtain X_i by solving the equation
$$F(X_i; \boldsymbol{\theta}) = U_i,$$
where $\boldsymbol{\theta}$ represents the vector of the model parameters.
3. When a closed-form expression for the quantile function $F^{-1}(\cdot)$ is available, compute
$$X_i = F^{-1}(U_i; \boldsymbol{\theta}).$$
If not, the equation is solved numerically using methods such as the Newton–Raphson algorithm.
4. Repeat the preceding steps to generate a random sample of size n.

In this study, the above procedure is repeated 20,000 times to generate independent samples. For each generated sample, the model parameters are estimated using the proposed estimation methods, and the corresponding performance measures, such as bias, and MSE are computed.

Progressively Type–II censored samples are generated from the TUEA distribution for several combinations of the model parameters (α, θ), initial sample sizes n, effective sample sizes m, and censoring schemes $\mathbf{R} = (R_1, R_2, \ldots, R_m)$.

The progressive Type II censoring schemes considered in this study are carefully selected to represent different practical censoring scenarios commonly encountered in reliability and life-testing experiments. In particular, the schemes are designed to reflect early, uniform and late censoring patterns.

Under early censoring schemes, a considerable number of experimental units are withdrawn at the initial failure stages due to financial or temporal constraints. On the other hand, late censoring schemes involve the removal of a greater proportion of units during the later stages of failure, capturing situations where most units remain under observation for an extended period before being withdrawn. Approximately uniform censoring schemes, in contrast, distribute removals more evenly across failure times, reflecting experimental designs that aim for balance.

With the introduction of the aforementioned censoring patterns into the simulation study, an evaluation of the suggested estimators' performance under different degrees and timings of missing data will be provided. The estimation procedure involves calculation of maximum likelihood estimates for the parameters α and θ by numerically optimizing the log-likelihood function presented in Section 3. Furthermore, Bayesian inference will be conducted through the use of the MH algorithm explained in Section 4 with gamma priors used for estimation. Posterior samples obtained after removing a sufficient number of initial values will be used for calculating point and credible interval estimates of the parameters. The performance of the estimators will be examined in terms of standard statistics such as bias and MSE of an estimator $\hat{\psi}$, where $\psi \in \{\alpha, \theta\}$, are calculated as

$$\text{Bias}(\hat{\psi}) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\psi}_j - \psi), \qquad \text{MSE}(\hat{\psi}) = \frac{1}{N}\sum_{j=1}^{N}(\hat{\psi}_j - \psi)^2,$$

where $\hat{\psi}_j$ denotes the estimate obtained from the jth Monte Carlo replication, and N is the total number of replications.

The simulated outcomes are presented using the means, biases, and MSEs of the estimators for each case, as shown in Tables 1–8. In general, it can be observed that both the maximum likelihood and the Bayesian estimators perform well. As the number of effective samples increases, the accuracy of the estimate increases, whereas the proportion of censoring decreases. Additionally, the Bayesian estimates have lower MSEs in small sample cases owing to the stabilizing effect of prior data, especially, in highly progressive censoring situations.

Table 1. Average estimate values and MSE under PT-IIC schemes at $\alpha = 0.5$, $\theta = 0.5$ ($n = 20$, $m = 10$).

Scheme			Estimator					
(n, m)	CS	Measure	MLE	Ln1	Ln2	Ge1	Ge2	Sel
(20, 10)	I	Mean α	0.6988	0.5243	0.5121	0.5079	0.4883	0.5181
		Mean θ	1.1113	0.4224	0.3478	0.3216	0.2189	0.3810
		MSE α	0.3316	0.0200	0.0172	0.0172	0.0155	0.0185
		MSE θ	1.5955	0.1698	0.1226	0.1346	0.1462	0.1394
		RBias α	0.1987	0.0243	0.0121	0.0093	0.0117	0.0181
		RBias θ	0.6113	0.0776	0.1522	0.1781	0.2810	0.1190
(20, 10)	II	Mean α	0.7233	0.5247	0.5129	0.5092	0.4909	0.5186
		Mean θ	1.1329	0.4309	0.3555	0.3284	0.2238	0.3892
		MSE α	0.2439	0.0226	0.0191	0.0192	0.0170	0.0206
		MSE θ	1.4950	0.1589	0.1186	0.1292	0.1432	0.1331
		RBias α	0.2232	0.0247	0.0129	0.0092	0.0091	0.0186
		RBias θ	0.6329	0.0691	0.1445	0.1716	0.2762	0.1108
(20, 10)	III	Mean α	0.7338	0.5222	0.5116	0.5080	0.4910	0.5167
		Mean θ	1.2769	0.4507	0.3695	0.3419	0.2338	0.4054
		MSE α	0.2274	0.0171	0.0147	0.0148	0.0132	0.0158
		MSE θ	1.9007	0.1626	0.1134	0.1254	0.1378	0.1304
		RBias α	0.2338	0.0222	0.0116	0.0080	0.0089	0.0167
		RBias θ	0.7769	0.0429	0.1305	0.1581	0.2662	0.0946
(20, 10)	IV	Mean α	0.7535	0.5226	0.5119	0.5083	0.4915	0.5171
		Mean θ	1.2740	0.4562	0.3756	0.3480	0.2879	0.4114
		MSE α	0.2706	0.0181	0.0155	0.0154	0.0138	0.0166
		MSE θ	1.9920	0.1591	0.1123	0.1234	0.1359	0.1287
		RBias α	0.2535	0.0226	0.0119	0.0083	0.0085	0.0171
		RBias θ	0.7740	0.0437	0.1244	0.1519	0.2612	0.0886

Table 2. Average estimate values and MSE under PT-IIC schemes at $\alpha = 0.5$, $\theta = 0.5$ ($n = 50$, $m = 30$).

Scheme			Estimator					
(n, m)	CS	Measure	MLE	Ln1	Ln2	Ge1	Ge2	Sel
(50, 30)	I	Mean α	0.5848	0.5130	0.5070	0.5049	0.4949	0.5099
		Mean θ	0.7894	0.4377	0.3813	0.3550	0.2594	0.4077
		MSE α	0.0416	0.0125	0.0112	0.0111	0.0102	0.0118
		MSE θ	0.4627	0.1329	0.1116	0.1212	0.1351	0.1204
		RBias α	0.0847	0.0130	0.0070	0.0049	0.0050	0.0099
		RBias θ	0.0894	0.0623	0.1187	0.1450	0.2406	0.0923
(50, 30)	II	Mean α	0.5792	0.5076	0.5024	0.5004	0.4917	0.5050
		Mean θ	0.7757	0.4357	0.3821	0.3558	0.2636	0.4072
		MSE α	0.0321	0.0095	0.0088	0.0087	0.0081	0.0091
		MSE θ	0.4098	0.1224	0.1047	0.1144	0.1301	0.1119
		RBias α	0.0792	0.0075	0.0024	0.0044	0.0083	0.0049
		RBias θ	0.2757	0.0643	0.1179	0.1442	0.2364	0.0928
(50, 30)	III	Mean α	0.5877	0.5122	0.5073	0.5055	0.4172	0.5097
		Mean θ	0.7976	0.4378	0.3818	0.3553	0.2623	0.4078
		MSE α	0.0367	0.0099	0.0091	0.0090	0.0083	0.0095
		MSE θ	0.4939	0.1287	0.1091	0.1190	0.1353	0.1169
		RBias α	0.0878	0.0122	0.0073	0.0055	0.0028	0.0097
		RBias θ	0.2976	0.0622	0.1182	0.1447	0.2377	0.0923
(50, 30)	IV	Mean α	0.5807	0.5000	0.5019	0.5001	0.4920	0.5144
		Mean θ	0.7871	0.4382	0.3846	0.3585	0.2658	0.4097
		MSE α	0.0308	0.0089	0.0082	0.0032	0.0076	0.0086
		MSE θ	0.4057	0.1185	0.1009	0.1101	0.1267	0.1080
		RBias α	0.0807	0.0068	0.0019	0.0001	0.0080	0.0043
		RBias θ	0.2871	0.0618	0.1154	0.1415	0.2342	0.0903

Table 3. Average estimate values and MSE under PT-IIC schemes at $\alpha = 0.5$, $\theta = 0.5$ (n=90, m=50).

(n, m)	CS	Measure	Parameter	MLE	Ln1	Ln2	Ge1	Ge2	Sel
(90,50)	I	Mean	α	0.5519	0.5021	0.4985	0.4970	0.4906	0.5003
			θ	0.7039	0.4254	0.3797	0.3532	0.2658	0.4013
		MSE	α	0.0158	0.0068	0.0065	0.0064	0.0062	0.0067
			θ	0.2689	0.1076	0.0983	0.1079	0.1297	0.1020
		RBias	α	0.0520	0.0021	0.0015	0.0030	0.0094	0.0031
			θ	0.2039	0.0746	0.1203	0.1468	0.2342	0.0987
(90,50)	II	Mean	α	0.5487	0.4989	0.4957	0.4943	0.4887	0.4973
			θ	0.6939	0.4249	0.3816	0.3556	0.2691	0.4022
		2*MSE	α	0.0142	0.0062	0.0059	0.0059	0.0056	0.0061
			θ	0.2265	0.0949	0.0873	0.0964	0.1173	0.0937
		RBias	α	0.0487	0.0011	0.0043	0.0057	0.0113	0.0027
			θ	0.1939	0.0751	0.1184	0.1444	0.2309	0.0977
(90,50)	III	Mean	α	0.5559	0.5019	0.4989	0.4977	0.4924	0.5004
			θ	0.7443	0.4430	0.3948	0.3689	0.2798	0.4176
		MSE	α	0.0150	0.0057	0.0054	0.0053	0.0051	0.0055
			θ	0.2867	0.1046	0.0927	0.1015	0.1195	0.0976
		RBias	α	0.0559	0.0019	0.0011	0.0023	0.0076	0.0037
			θ	0.2443	0.0570	0.1052	0.1311	0.2202	0.0824
(90,50)	IV	Mean	α	0.5477	0.4968	0.4939	0.4928	0.4877	0.4954
			θ	0.7030	0.4308	0.3875	0.3622	0.2768	0.4081
		MSE	α	0.0135	0.0055	0.0053	0.0052	0.0050	0.0053
			θ	0.2294	0.0963	0.0875	0.0962	0.1157	0.0911
		RBias	α	0.0477	0.0032	0.0060	0.0072	0.0123	0.0046
			θ	0.2030	0.0692	0.1125	0.1378	0.2232	0.0919

Table 4. Average estimate values and MSE under PT-IIC schemes at $\alpha = 0.5$, $\theta = 0.5$ ($n = 120$, $m = 90$).

Scheme			Estimator					
(n, m)	CS	Measure	MLE	Ln1	Ln2	Ge1	Ge2	Sel
(120, 90)	I	Mean α	0.5325	0.4947	0.4923	0.4912	0.4868	0.4935
		Mean θ	0.6153	0.4057	0.3710	0.3468	0.2698	0.3876
		MSE α	0.0094	0.0051	0.0050	0.0049	0.0048	0.0050
		MSE θ	0.1486	0.0840	0.0811	0.0899	0.1103	0.0836
		RBias α	0.0325	0.0053	0.0076	0.0088	0.0132	0.0065
		RBias θ	0.1153	0.0943	0.1289	0.1532	0.2302	0.1124
(120, 90)	II	Mean α	0.5275	0.4908	0.4886	0.4876	0.4836	0.4896
		Mean θ	0.6112	0.4078	0.3750	0.3515	0.2762	0.3909
		MSE α	0.0083	0.0049	0.0047	0.0048	0.0047	0.0048
		MSE θ	0.1355	0.0844	0.0817	0.0904	0.1118	0.0828
		RBias α	0.0275	0.0092	0.0114	0.0123	0.0164	0.0103
		RBias θ	0.1112	0.0921	0.1250	0.1485	0.2238	0.1091
(120, 90)	III	Mean α	0.5362	0.4971	0.4949	0.4939	0.4899	0.4960
		Mean θ	0.6367	0.4221	0.3863	0.3622	0.2840	0.4035
		MSE α	0.0096	0.0048	0.0047	0.0046	0.0045	0.0049
		MSE θ	0.1569	0.0847	0.0808	0.0893	0.1101	0.0823
		RBias α	0.0362	0.0029	0.0051	0.0061	0.0101	0.0040
		RBias θ	0.1367	0.0779	0.1137	0.1378	0.2160	0.0965
(120, 90)	IV	Mean α	0.5313	0.4960	0.4939	0.4929	0.4889	0.4949
		Mean θ	0.6163	0.4231	0.3899	0.3668	0.2913	0.4060
		MSE α	0.0075	0.0042	0.0041	0.0040	0.0039	0.0041
		MSE θ	0.1353	0.0816	0.0783	0.0866	0.1068	0.0796
		RBias α	0.0313	0.0040	0.0061	0.0071	0.0110	0.0050
		RBias θ	0.1163	0.0769	0.1100	0.1332	0.2087	0.0940

Table 5. Average estimate values and MSE under PT-IIC schemes at $\alpha = 0.25$, $\theta = 0.5$ ($n = 20$, $m = 10$).

Scheme			Estimator					
(n, m)	CS	Measure	MLE	Ln1	Ln2	Ge1	Ge2	Sel
(20, 10)	I	Mean α	0.3318	0.2830	0.2798	0.2763	0.2663	0.2814
		Mean θ	1.4474	0.5617	0.4494	0.4172	0.2775	0.4986
		MSE α	0.0180	0.0054	0.0050	0.0047	0.0040	0.0052
		MSE θ	2.2762	0.1636	0.0918	0.1022	0.1122	0.1465
		RBias α	0.0817	0.0330	0.0298	0.0263	0.0163	0.0314
		RBias θ	0.9474	0.0617	0.0505	0.0827	0.2224	0.0013
(20, 10)	II	Mean α	0.3468	0.2818	0.2789	0.2758	0.2668	0.2803
		Mean θ	1.6179	0.5852	0.4656	0.4329	0.2882	0.5178
		MSE α	0.0226	0.0050	0.0046	0.0043	0.0036	0.0048
		MSE θ	3.1360	0.1643	0.0884	0.0978	0.1277	0.1141
		RBias α	0.0968	0.0318	0.0289	0.0258	0.0168	0.0303
		RBias θ	1.1179	0.0852	0.0343	0.0670	0.2117	0.0178
(20, 10)	III	Mean α	0.3466	0.2757	0.2732	0.2703	0.2621	0.2744
		Mean θ	1.7612	0.5901	0.4651	0.4326	0.2873	0.5189
		MSE α	0.0270	0.0049	0.0046	0.0044	0.0038	0.0047
		MSE θ	3.6910	0.1678	0.1182	0.1290	0.1222	0.1143
		RBias α	0.0966	0.0257	0.0232	0.0203	0.0121	0.0244
		RBias θ	1.2612	0.0901	0.0348	0.0678	0.2126	0.0189
(20, 10)	IV	Mean α	0.3734	0.2785	0.2758	0.2728	0.2645	0.2771
		Mean θ	2.0615	0.6197	0.4815	0.4494	0.2963	0.5403
		MSE α	0.0448	0.0052	0.0048	0.0046	0.0038	0.0050
		MSE θ	3.7256	0.1877	0.0990	0.0995	0.1056	0.1224
		RBias α	0.1234	0.0285	0.0258	0.0228	0.0145	0.0271
		RBias θ	1.5615	0.1197	0.0184	0.0505	0.2036	0.0403

Table 6. Average estimate values and MSE under PT-IIC schemes at $\alpha = 0.25$, $\theta = 0.5$ ($n = 50$, $m = 30$).

Scheme			Estimator					
(n, m)	CS	Measure	MLE	Ln1	Ln2	Ge1	Ge2	Sel
(50, 30)	I	Mean α	0.2977	0.2719	0.2706	0.2690	0.2645	0.2713
		Mean θ	1.0093	0.5716	0.4881	0.4614	0.3402	0.5268
		MSE α	0.0059	0.0023	0.0022	0.0021	0.0018	0.0022
		MSE θ	0.7110	0.1644	0.0997	0.1011	0.1104	0.1155
		RBias α	0.0477	0.0219	0.0206	0.0190	0.0145	0.0213
		RBias θ	0.5093	0.0716	0.0118	0.0385	0.1597	0.0268
(50, 30)	II	Mean α	0.2990	0.2700	0.2689	0.2674	0.2635	0.2694
		Mean θ	1.0174	0.5584	0.4778	0.4497	0.3282	0.5151
		MSE α	0.0057	0.0021	0.0020	0.0019	0.0017	0.0021
		MSE θ	0.6236	0.1309	0.0886	0.0978	0.1222	0.1068
		RBias α	0.0490	0.0200	0.0189	0.0174	0.0135	0.0194
		RBias θ	0.5174	0.0584	0.0221	0.0502	0.1717	0.0151
(50, 30)	III	Mean α	0.2975	0.2673	0.2663	0.2650	0.2614	0.2668
		Mean θ	1.0972	0.5768	0.4932	0.4669	0.3472	0.5318
		MSE α	0.0055	0.0019	0.0018	0.0017	0.0015	0.0018
		MSE θ	0.8180	0.1643	0.1183	0.1299	0.1220	0.1117
		RBias α	0.0475	0.0173	0.0163	0.0150	0.0114	0.0168
		RBias θ	0.5972	0.0768	0.0067	0.0330	0.1527	0.0318
(50, 30)	IV	Mean α	0.3024	0.2693	0.2682	0.2669	0.2632	0.2687
		Mean θ	1.0944	0.5798	0.4940	0.4668	0.3436	0.5335
		MSE α	0.0061	0.0020	0.0019	0.0018	0.0016	0.0019
		MSE θ	0.7760	0.1497	0.0986	0.0975	0.1034	0.1225
		RBias α	0.0524	0.0193	0.0182	0.0169	0.0132	0.0187
		RBias θ	0.5944	0.0798	0.0059	0.0331	0.1563	0.0335

Table 7. Average estimate values and MSE under PT-IIC schemes at $\alpha = 0.25$, $\theta = 0.5$ (n=90, m=50).

(n, m)	CS	Measure	Parameter	MLE	Ln1	Ln2	Ge1	Ge2	Sel
(90,50)	I	Mean	α	0.2833	0.2648	0.2640	0.2629	0.2601	0.2644
			θ	0.9217	0.5746	0.5079	0.4817	0.3690	0.5395
		MSE	α	0.0029	0.0016	0.0015	0.0015	0.0014	0.0016
			θ	0.3736	0.1522	0.0995	0.0996	0.1102	0.1059
		RBias	α	0.0333	0.0148	0.0140	0.0129	0.0101	0.0144
			θ	0.4217	0.0746	0.0079	0.0182	0.1309	0.0395
(90,50)	II	Mean	α	0.2812	0.2629	0.2622	0.2613	0.2589	0.2625
			θ	0.8877	0.5690	0.5053	0.4791	0.3697	0.5355
		MSE	α	0.0031	0.0016	0.0016	0.0015	0.0014	0.0016
			θ	0.4029	0.1308	0.0884	0.0977	0.1200	0.1087
		RBias	α	0.0312	0.0129	0.0122	0.0113	0.0089	0.0125
			θ	0.3877	0.0690	0.0053	0.0208	0.1302	0.0355
(90,50)	III	Mean	α	0.2772	0.2583	0.2577	0.2569	0.2548	0.2580
			θ	0.9517	0.5867	0.5145	0.4899	0.3769	0.5484
		MSE	α	0.0023	0.0012	0.0011	0.0011	0.0010	0.0012
			θ	0.4765	0.1596	0.1181	0.1257	0.1220	0.1115
		RBias	α	0.0272	0.0083	0.0077	0.0069	0.0048	0.0080
			θ	0.4517	0.0867	0.0145	0.0100	0.1230	0.0484
(90,50)	IV	Mean	α	0.2812	0.2613	0.2607	0.2599	0.2577	0.2610
			θ	0.9219	0.5809	0.5108	0.4839	0.3689	0.5437
		MSE	α	0.0031	0.0015	0.0015	0.0014	0.0013	0.0015
			θ	0.4943	0.1440	0.0980	0.0974	0.1013	0.1221
		RBias	α	0.0312	0.0113	0.0107	0.0099	0.0077	0.0110
			θ	0.4219	0.0809	0.0108	0.0160	0.1310	0.0437

Table 8. Average estimate values and MSE under PT-IIC schemes at $\alpha = 0.25$, $\theta = 0.5$ ($n = 120$, $m = 90$).

Scheme			Estimator					
(n, m)	CS	Measure	MLE	Ln1	Ln2	Ge1	Ge2	Sel
(120, 90)	I	Mean α	0.2804	0.2664	0.2658	0.2651	0.2632	0.2661
		Mean θ	0.7862	0.5550	0.5031	0.4786	0.3781	0.5281
		MSE α	0.0023	0.0013	0.0012	0.0012	0.0011	0.0012
		MSE θ	0.2386	0.1168	0.0952	0.0995	0.1100	0.1008
		RBias α	0.0304	0.0164	0.0158	0.0151	0.0132	0.0161
		RBias θ	0.2862	0.0550	0.0031	0.0213	0.1218	0.0281
(120, 90)	II	Mean α	0.2784	0.2645	0.2640	0.2634	0.2616	0.2643
		Mean θ	0.7651	0.5380	0.4909	0.4687	0.3784	0.5137
		MSE α	0.0020	0.0011	0.0010	0.0010	0.0010	0.0011
		MSE θ	0.2254	0.1216	0.0880	0.0900	0.1174	0.1011
		RBias α	0.0284	0.0145	0.0140	0.0134	0.0116	0.0143
		RBias θ	0.2651	0.0380	0.0090	0.0312	0.1215	0.0137
(120, 90)	III	Mean α	0.2772	0.2628	0.2623	0.2617	0.2601	0.2625
		Mean θ	0.7972	0.5529	0.5029	0.4792	0.3834	0.5269
		MSE α	0.0020	0.0010	0.0010	0.0010	0.0009	0.0010
		MSE θ	0.2519	0.1211	0.1018	0.1087	0.1175	0.1104
		RBias α	0.0272	0.0128	0.0123	0.0117	0.0101	0.0125
		RBias θ	0.2972	0.0529	0.0029	0.0207	0.1165	0.0269
(120, 90)	IV	Mean α	0.2773	0.2633	0.2628	0.2622	0.2606	0.2631
		Mean θ	0.7717	0.5463	0.4980	0.4749	0.3817	0.5213
		MSE α	0.0019	0.0010	0.0010	0.0009	0.0009	0.0010
		MSE θ	0.2310	0.1136	0.0957	0.0912	0.1005	0.1036
		RBias α	0.0273	0.0133	0.0128	0.0122	0.0106	0.0131
		RBias θ	0.2717	0.0463	0.0019	0.0250	0.1182	0.0213

Figure 3 shows the following:

1. The trace plots for the parameters α and θ have good mixing properties and are oscillating randomly around the mean values. The lack of any discernible trend behavior pattern ensures that the MCMC method has converged to the stationary distribution successfully.
2. Both density plots have the same smooth and unimodal behavior. The symmetry of the posterior densities provides an excellent validation to the Bayesian estimation technique and its accuracy.

The selection of a censoring scheme directly influences the efficiency with which parameters can be estimated. In other words, early censoring schemes tend to diminish estimation efficiency, as a substantial fraction of units are removed during the initial failure stages, thereby yielding limited insight into the distribution's tail behavior. This often results in increased bias and MSE.

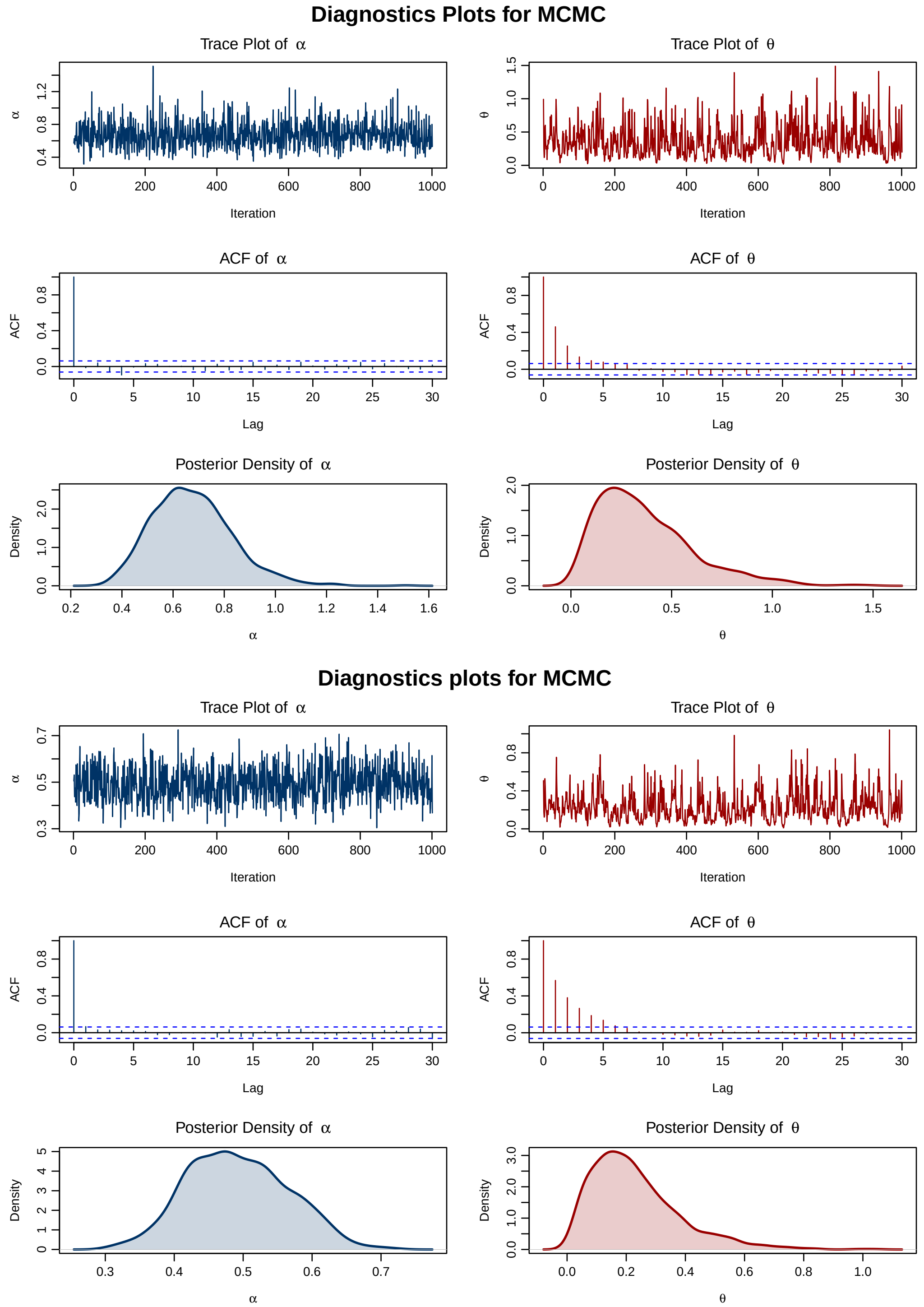

Figure 3. The diagnostic plots for MCMC at $n = 20$, $m = 10$ and $n = 50$, $m = 30$.

The choice of censoring scheme plays a key role in determining the efficiency of parameter estimation. Specifically, in the case of early censoring schemes, the efficiency of estimation tends to be low because of the elimination of many units at the early stages of failure. This makes it difficult to gain much knowledge about the behavior of the tail part of the distribution, resulting in high bias and MSE. However, late censoring schemes provide relatively more information from the lifetimes of units because most of the units are retained until later failure points. In this way, these schemes generally provide more efficient estimates with relatively smaller bias and MSE. The uniform censoring schemes represent a middle ground between the two. The simulation results reported in Tables 1–8 confirm these conclusions, as the estimators corresponding to late censoring schemes have been found to outperform the estimators of early censoring schemes.

In addition, confidence intervals are formed in Tables 9 and 10 based on the asymptotic distribution of the maximum likelihood estimates (MLEs), and their empirical coverage probabilities are calculated. It appears that the empirical coverage probabilities obtained for the various PT-IIC sampling plans are close to the nominal value. Also, Tables 11 and 12 show the lower and upper bounds of credible intervals (CrI) and coverage probabilities (CPs) for the parameters and indicate that the coverage probabilities are generally close to the nominal level, demonstrating satisfactory inferential performance of the MLE-based intervals across different PT-IIC schemes.

Overall, the simulation study corroborates the robustness and reliability of the proposed classical and Bayesian estimation procedures for the TUEA distribution under progressive Type–II censoring, confirming their practical utility in reliability and survival analysis applications. All simulations were performed using the statistical software R. The simulation results are presented in a structured format, including parameter estimates, bias, MSE, and confidence or credible intervals, to facilitate a comprehensive evaluation of estimator performance.

Table 9. Lower and upper bounds of confidence intervals and coverage probabilities (CPs) for the parameter α, θ under different PT-IIC schemes with $\alpha = 0.5$ and $\theta = 0.5$.

(n, m)	Censoring scheme	Estimation method		Lower bound	Upper bound	CP (%)
(20, 10)	I	MLE	α	0.2797	1.3488	95.0
			θ	0.0000	2.9799	96.0
	II	MLE	α	0.2925	1.5711	95.0
			θ	0.0000	3.0948	95.5
	III	MLE	α	0.2936	1.4701	95.5
			θ	0.0000	3.3900	95.0
	IV	MLE	α	0.3064	1.5373	92.0
			θ	0.0000	3.5086	97.0
(50, 30)	I	MLE	α	0.3406	0.9442	92.5
			θ	0.0000	1.9027	95.0
	II	MLE	α	0.3331	0.9023	94.0
			θ	0.0000	1.7581	94.5
	III	MLE	α	0.3390	0.9067	95.0
			θ	0.0000	1.9389	94.5
	IV	MLE	α	0.3522	0.9137	96.5
			θ	0.0000	1.8008	95.5
(90, 50)	I	MLE	α	0.3551	0.7746	92.5
			θ	0.0000	1.4864	94.5
	II	MLE	α	0.3792	0.7652	95.0
			θ	0.0000	1.4512	96.0
	III	MLE	α	0.3753	0.7608	93.0
			θ	0.0000	1.5703	96.5
	IV	MLE	α	0.3781	0.7422	96.0
			θ	0.0000	1.4719	94.5
(120, 90)	I	MLE	α	0.3816	0.7092	95.5
			θ	0.0000	1.2573	94.5
	II	MLE	α	0.3917	0.7065	96.5
			θ	0.0000	1.2241	94.5
	III	MLE	α	0.3824	0.6993	97.0
			θ	0.0000	1.2899	95.5
	IV	MLE	α	0.3845	0.6885	94.5
			θ	0.0000	1.2334	95.5

Table 10. Lower and upper bounds of confidence intervals and coverage probabilities (CPs) for the parameter α, θ under different PT-IIC schemes with $\alpha = 0.25$ and $\theta = 0.5$.

(n, m)	Censoring scheme	Estimation method		Lower bound	Upper bound	CP (%)
(20, 10)	I	MLE	α	0.1725	0.5334	94.0
			θ	0.0000	3.2205	94.0
	II	MLE	α	0.1666	0.5242	95.0
			θ	0.0000	3.6606	95.5
	III	MLE	α	0.1683	0.6221	95.5
			θ	0.0000	4.1625	95.0
	IV	MLE	α	0.1638	0.6099	92.0
			θ	0.0005	5.5958	94.0
(50, 30)	I	MLE	α	0.1943	0.3957	92.5
			θ	0.0019	2.2311	95.0
	II	MLE	α	0.2011	0.3969	94.0
			θ	0.0549	2.0098	94.5
	III	MLE	α	0.2108	0.4219	95.0
			θ	0.0507	2.4808	94.5
	IV	MLE	α	0.3522	0.3996	96.5
			θ	0.0363	2.1710	93.5
(90, 50)	I	MLE	α	0.2062	0.3617	92.5
			θ	0.1768	1.8361	94.5
	II	MLE	α	0.2140	0.3732	95.0
			θ	0.0378	1.8560	96.0
	III	MLE	α	0.2125	0.3634	93.0
			θ	0.0646	1.9266	96.5
	IV	MLE	α	0.2151	0.3821	96.0
			θ	0.0231	2.0714	94.5
(120, 90)	I	MLE	α	0.2204	0.3462	95.5
			θ	0.0853	1.5348	94.5
	II	MLE	α	0.2248	0.3429	96.5
			θ	0.0304	1.5810	94.5
	III	MLE	α	0.2235	0.3402	97.0
			θ	0.0328	1.5787	93.5
	IV	MLE	α	0.2233	0.3450	94.5
			θ	0.0820	1.6209	95.5

Table 11. Lower and upper bounds of credible intervals (CrI) and coverage probabilities (CPs) for the parameters α, θ under different PT-IIC schemes with $\alpha = 0.5$ and $\theta = 0.5$.

(n, m)	CS	ME		Ln1	Ln2	Ge1	Ge2	Sel	CP (%)
(20, 10)	I	BE	α	(0.297,0.813)	(0.315,0.808)	(0.312,0.806)	(0.263,0.736)	(0.295,0.799)	95.0
			θ	(0.000,1.163)	(0.000,0.963)	(0.000,0.959)	(0.000,0.783)	(0.000,1.044)	96.0
	II	BE	α	(0.303,0.839)	(0.300,0.802)	(0.296,0.799)	(0.285,0.765)	(0.311,0.832)	95.0
			θ	(0.000,1.125)	(0.000,0.906)	(0.000,0.885)	(0.000,0.719)	(0.000,0.989)	95.5
	III	BE	α	(0.324,0.799)	(0.304,0.759)	(0.302,0.756)	(0.294,0.725)	(0.323,0.788)	95.5
			θ	(0.000,1.206)	(0.000,0.966)	(0.000,0.964)	(0.000,0.763)	(0.000,1.064)	95.0
	IV	BE	α	(0.321,0.797)	(0.318,0.770)	(0.315,0.767)	(0.306,0.741)	(0.319,0.784)	92.0
			θ	(0.000,1.197)	(0.000,0.964)	(0.000,0.962)	(0.000,0.783)	(0.000,1.073)	97.0
(50, 30)	I	BE	α	(0.328,0.717)	(0.326,0.702)	(0.324,0.699)	(0.320,980)	(0.327,0.709)	95.0
			θ	(0.000,1.088)	(0.000,0.959)	(0.000,0.953)	(0.000,0.789)	(0.000,1.023)	96.0
	II	BE	α	(0.352,0.717)	(0.345,0.693)	(0.344,0.691)	(0.329,0.667)	(0.353,0.711)	95.0
			θ	(0.000,1.030)	(0.000,0.901)	(0.000,0.899)	(0.000,0.768)	(0.000,0.951)	95.5
	III	BE	α	(0.348,0.696)	(0.345,0.684)	(0.344,0.682)	(0.339,0.670)	(0.346,688)	95.5
			θ	(0.000,1.115)	(0.000,0.952)	(0.000,0.949)	(0.000,0.804)	(0.000,1.026)	95.0
	IV	BE	α	(0.357,0.795)	(0.352,0.690)	(0.355,0.691)	(0.348,0.671)	(0.353,0.695)	92.0
			θ	(0.000,1.044)	(0.000,0.922)	(0.000,0.914)	(0.000,0.779)	(0.000,0.981)	97.0
(90, 50)	I	BE	α	(0.354,0.654)	(0.349,0.641)	(0.348,0.639)	(0.349,0.632)	(0.349,0.645)	95.0
			θ	(0.000,1.009)	(0.000,0.927)	(0.000,0.924)	(0.000,0.828)	(0.000,0.959)	96.0
	II	BE	α	(0.377,0.658)	(0.366,0.642)	(0.366,0.640)	(0.336,0.632)	(0.367,0.646)	95.0
			θ	(0.000,0.987)	(0.000,0.899)	(0.000,895)	(0.000,0.771)	(0.000,0.941)	95.5
	III	BE	α	(0.375,0.653)	(0.374,0.646)	(0.373,0.644)	(0.374,0.638)	(0.374,0.649)	95.5
			θ	(0.000,1.007)	(0.000,0.974)	(0.000,0.907)	(0.000,0.793)	(0.000,0.966)	95.0
	IV	BE	α	(0.376,0.633)	(0.375,0.624)	(0.374,0.622)	(0.371,0.615)	(0.378,0.630)	92.0
			θ	(0.000,0.972)	(0.000,0.883)	(0.000,0.880)	(0.000,0.757)	(0.000,0.914)	97.0
(120, 90)	I	BE	α	(0.383,0.635)	(0.383,0.630)	(0.382,0.629)	(0.380,0.619)	(0.383,0.633)	95.0
			θ	(0.000,0,906)	(0.000,0.832)	(0.000,0.824)	(0.000,0.754)	(0.000,0.873)	96.0
	II	BE	α	(0.390,0.632)	(0.389,0.628)	(0.388,0.627)	(0.371,0.605)	(0.389,0.630)	95.0
			θ	(0.000,0.907)	(0.000,0.843)	(0.000,0.836)	(0.000,0.754)	(0.000,0.870)	95.5
	III	BE	α	(0.377,0.622)	(0.376,0.619)	(0.375,0.618)	(0.371,0.609)	(0.376,0.621)	95.5
			θ	(0.000,0.944)	(0.000,0.893)	(0.000,0.890)	(0.000,0.804)	(0.000,0.913)	95.0
	IV	BE	α	(0.366,0.610)	(0.375,0.615)	(0.374,0.613)	(0.376,0.608)	(0.375,0.617)	92.0
			θ	(0.000,0.954)	(0.000,0.884)	(0.000,0.881)	(0.000,0.782)	(0.000,0.918)	97.0

Table 12. Lower and upper bounds of credible intervals (CrI) and coverage probabilities (CPs) for the parameter α, θ under different PT-IIC schemes with $\alpha = 0.25$ and $\theta = 0.5$.

(n, m)	CS	ME		Ln1	Ln2	Ge1	Ge2	Sel	CP (%)
(20, 10)	I	BE	α	(0.166,0.413)	(0.161,0.400)	(0.159,0.395)	(0.154,0.376)	(0.162,0.405)	95.0
			θ	(0.000,1.291)	(0.000,1.003)	(0.000,1.004)	(0.000,0.816)	(0.000,1.102)	96.0
	II	BE	α	(0.162,0.400)	(0.161,0.392)	(0.159,0.388)	(0.155,0.373)	(0.162,0.395)	95.0
			θ	(0.006,1.231)	(0.006,0.985)	(0.004,0.983)	(0.001,0.796)	(0.006,1.094)	95.5
	III	BE	α	(0.165,0.395)	(0.164,0.390)	(0.162,0.386)	(0.168,0.385)	(0.165,0.392)	95.5
			θ	(0.000,1.260)	(0.000,1.005)	(0.000,1.018)	(0.000,0.792)	(0.000,1.118)	95.0
	IV	BE	α	(0.172,0.408)	(0.171,0.401)	(0.170,0.396)	(0.165,0.380)	(0.172,0.404)	92.0
			θ	(0.002,1.314)	(0.002,1.049)	(0.002,1.053)	(0.001,0.807)	(0.002,1.156)	97.0
(50, 30)	I	BE	α	(0.193,0.338)	(0.193,0.336)	(0.192,0.334)	(0.199,0.329)	(0.193,0.337)	95.0
			θ	(0.003,1.452)	(0.003,1.276)	(0.002,1.289)	(0.009,1.002)	(0.003,1.340)	96.0
	II	BE	α	(0.194,0.349)	(0.194,0.346)	(0.193,0.344)	(0.191,0.337)	(0.194,0.347)	95.0
			θ	(0.020,1.232)	(0.019,1.089)	(0.009,1.095)	(0.003,0.857)	(0.020,1.157)	95.5
	III	BE	α	(0.195,0.350)	(0.195,0.347)	(0.195,0.346)	(0.192,0.339)	(0.195,0.349)	95.5
			θ	(0.019,1.412)	(0.018,1.203)	(0.011,1.210)	(0.003,1.009)	(0.018,1.282)	95.0
	IV	BE	α	(0.357,0.345)	(0.352,0.343)	(0.355,0.341)	(0.348,0.336)	(0.353,0.344)	92.0
			θ	(0.019,1.305)	(0.018,1.154)	(0.014,1.159)	(0.004,0.932)	(0.019,1.238)	97.0
(90, 50)	I	BE	α	(0.210,0.346)	(0.210,0.345)	(0.210,0.343)	(0.207,0.339)	(0.210,0.345)	95.0
			θ	(0.005,1.103)	(0.037,1.005)	(0.033,1.002)	(0.009,0.845)	(0.005,1.036)	96.0
	II	BE	α	(0.207,0.337)	(0.207,0.336)	(0.206,0.334)	(0.204,0.331)	(0.207,0.336)	95.0
			θ	(0.011,1.152)	(0.011,1.053)	(0.009,1.051)	(0.005,0.925)	(0.011,1.102)	95.5
	III	BE	α	(0.208,0.328)	(0.209,0.327)	(0.207,0.326)	(0.206,0.322)	(0.208,0.328)	95.5
			θ	(0.002,1.300)	(0.002,1.114)	(0.001,1.110)	(0.001,0.933)	(0.002,1.205)	95.0
	IV	BE	α	(0.204,0.336)	(0.204,0.335)	(0.204,0.333)	(0.202,0.330)	(0.204,0.335)	92.0
			θ	(0.013,1.247)	(0.013,1.065)	(0.011,1.052)	(0.002,0.910)	(0.013,1.164)	97.0
(120, 90)	I	BE	α	(0.212,0.313)	(0.212,0.312)	(0.211,0.311)	(0.210,0.308)	(0.212,0.313)	95.0
			θ	(0.051,1,237)	(0.049,1.123)	(0.032,1.125)	(0.010,0.990)	(0.050,1.184)	96.0
	II	BE	α	(0.210,0.310)	(0.210,0.310)	(0.209,0.308)	(0.208,0.305)	(0.210,0.310)	95.0
			θ	(0.026,1.251)	(0.046,1.117)	(0.023,1.160)	(0.006,1.042)	(0,047,1.214)	95.5
	III	BE	α	(0.213,0.312)	(0.213,0.311)	(0.216,0.310)	(0.212,0.308)	(0.213,0.312)	95.5
			θ	(0.031,1.299)	(0.050,1.190)	(0.031,1.197)	(0.021,1.134)	(0.051,1.252)	95.0
	IV	BE	α	(0.211,0.310)	(0.211,0.309)	(0.210,0.309)	(0.210,0.306)	(0.211,0.310)	92.0
			θ	(0.024,1.242)	(0.023,1.167)	(0.020,1.170)	(0.010,1.105)	(0.023,1.193)	97.0

6. Applications

In this section, the practical utility of the proposed estimation procedures for the TUEA distribution under progressive Type–II censoring is demonstrated using two real-world data sets. For each data set, the model parameters are estimated using both maximum likelihood and Bayesian methods, and the resulting estimates are compared across different censoring schemes.

Data Set I

The first data set, previously analyzed by Kumari et al. [10], comprises the failure times (in hours) of the air-conditioning systems from a fleet of Boeing 720 aircraft. The data correspond to observations from two aircraft, labeled 7913 and 7914, and have been widely used in the literature to assess lifetime models. The complete data set comprises $n = 25$ observations, given by

0.0046, 0.0184, 0.0507, 0.0737, 0.0829, 0.1106, 0.1429, 0.1797, 0.2120,
0.2350, 0.2488, 0.2903, 0.3134, 0.3548, 0.3687, 0.3779, 0.4470, 0.4885,
0.5115, 0.6498, 0.6544, 0.7512, 0.8802, 0.9493, 0.9954.

As in the previous case, two progressively Type–II censored samples are constructed using censoring schemes R_1 with $m = 15$ and $P = 0.5$, and R_2 with $m = 10$ and $P = 0.7$. The corresponding censored observations are summarized in Table 13.

Table 13. Progressively censored sample based on Data Set I.

R_1	7	5	0	1	1	0	1	0	0
X_1	0.0046	0.2120	0.3687	0.3779	0.4885	0.6498	0.6544	0.8802	0.9493
R_2	9	1	0	0	0	0	0	0	
X_2	0.5115	0.6514	0.6498	0.6544	0.7512	0.8802	0.9493	0.9954	

Figure 4 shows the empirical CDF, PDF, and the (P-P) plots for Data Set I for the TUEA distribution.

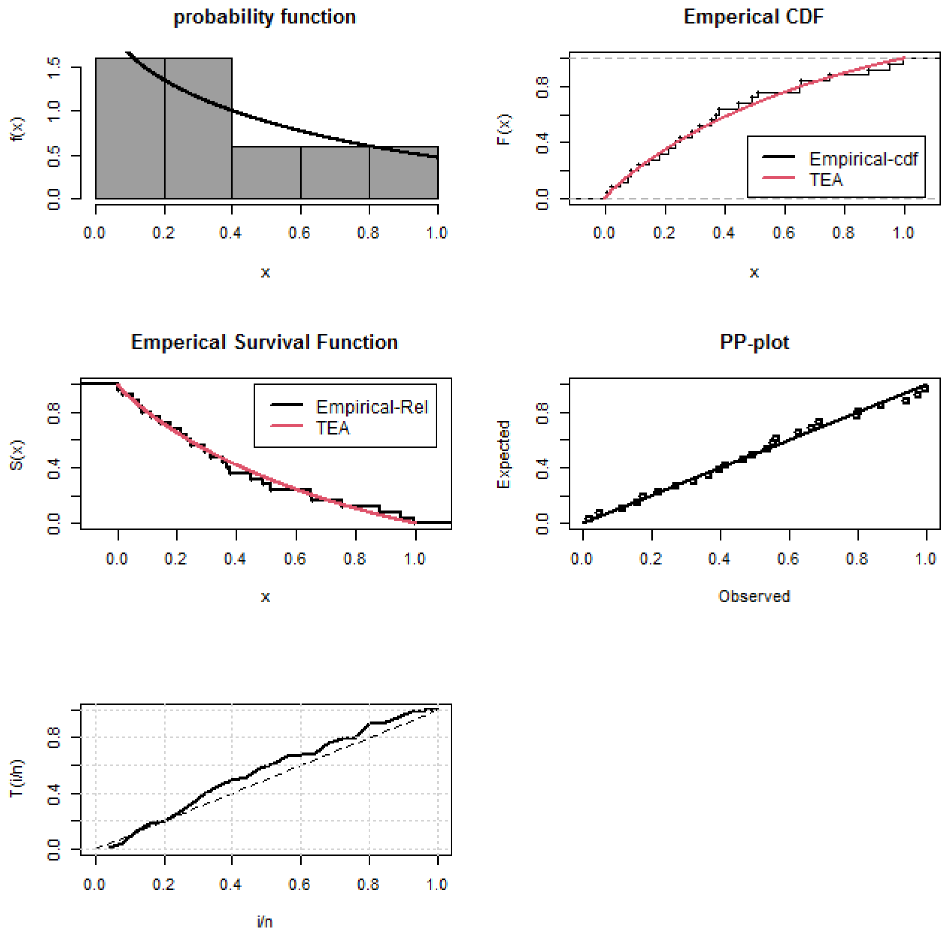

Figure 4. The empirical CDF, PDF, Sr, and (P-P) plot for Data Set I for TUEA distribution

Data Set II

The second data set comprises $n = 30$ observations of the tensile strength of polyester fibers, as reported by Mazuchli et al. [5]. The observed values are as follows:

0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216,

0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567,
0.642, 0.674, 0.752, 0.823, 0.887, 0.926.

As in the previous case, two progressively Type–II censored samples are constructed using censoring schemes R_1 with $m = 10$ and $P = 0.5$, and R_2 with $m = 15$ and $P = 0.7$. The corresponding censored observations are summarized in Table 14.

Table 14. Progressively censored sample based on Data Set II.

R_1	8	7	2	0	1	0	1	1	0	0
X_1	0.023	0.169	0.395	0.481	0.519	0.567	0.642	0.752	0.887	0.926
R_2	7	6	2	0	0	0	0	0	0	0
	0	0	0	0	0					
X_2	0.023	0.148	0.361	0.432	0.463	0.481	0.519	0.529	0.567	0.642
	0.674	0.752	0.823	0.887	0.926					

Figure 5 presents the empirical distributional plots for the TUEA model fitted to this data set, further supporting its flexibility in modeling bounded lifetime data.

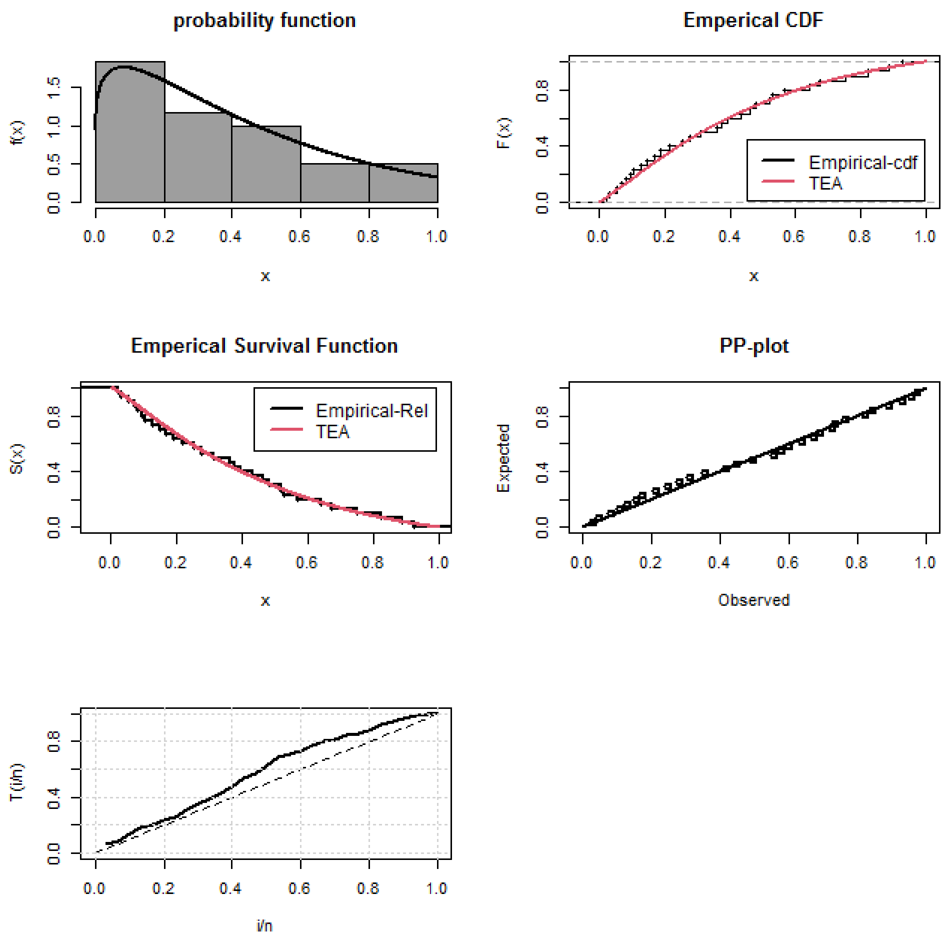

Figure 5. The empirical CDF, PDF, Sr, and (P-P) plot for Data Set II for TUEA distribution.

Estimation results

Table 15 gives the maximum likelihood estimates and Bayesian estimates of the parameters α and θ for both data sets according to the censoring schemes given. Bayesian estimates are computed using the squared error loss (SEL), LINEX, and entropy loss functions. It is observed from the results that, generally, the maximum likelihood estimates of parameters using SEL are higher than the estimates using LINEX and entropy loss functions, and the estimates using entropy loss function s are smaller because the entropy loss function is asymmetric in nature.

Table 15. MLE and Bayes estimations for real Data I and Data II under SEL and LINEX.

Data	Sc	Parameter	MLE	SEL	Linex	Entropy	CP(mle)	CP(Bayesian)
	R_1	α	0.6344	0.8239	0.8130	0.7717	0.9467	1.0000
		θ	0.0001	0.5993	0.5670	0.3570	0.8867	0.9000
Data I	R_2	α	0.5391	0.7530	0.7428	0.7010	0.9867	1.0000
		θ	0.0001	0.7969	0.7531	0.5268	0.8667	0.9000
	R_1	α	0.7710	1.0979	1.0772	1.0257	0.8533	1.0000
		θ	0.0001	0.8184	0.7767	0.5850	0.8833	0.9500
Data II	R_2	α	0.7126	1.0728	1.0495	0.9900	0.9767	0.9900
		θ	0.0648	0.9997	0.9384	0.5864	0.9433	0.9500

Table 16 provides the confidence intervals for MLEs, and the Bayesian credible intervals. Generally, Bayesian credible intervals are longer than asymptotic confidence intervals based on MLEs because Bayesian credible intervals provide a wider range of quantification. For both data sets, greater censoring makes longer intervals, especially for the scale parameter θ, highlighting the sensitivity of interval estimation to the degree of information loss.

Table 16. Confidence intervals/credible intervals (CIs/CrIs) for real Data I and Data II under MLE and Bayes.

Data	Sc.	Parameter	CI	CrI
Data I	R1	α	(0.3311, 0.9378)	(0.4769, 1.2978)
		θ	(0.0000, 0.0001)	(0.0916, 1.4666)
	R2	α	(0.2810, 0.7973)	(0.4301,1.2304)
		θ	(0.0000, 0.0001)	(0.1598, 1.7640)
Data II	R1	α	(0.4265,1.1155)	(0.6364,1.7905)
		θ	(0.0000,0.0001)	(0.1789, 1.7997)
	R2	α	(0.0008,1.4245)	(0.5930, 1.8231)
		θ	(-2.4845, 2.6142)	(0.1553, 2.1143)

The trace plot and the posterior density estimate of the Bayesian estimation procedure under both censoring methods are illustrated in Figures 6–9. From the figures, it is evident that the Markov chain has converged well, implying that the obtained Bayesian estimates are reliable. In summary, the actual data analysis reveals that the suggested TUEA model, along with the progressive Type–II censoring scheme, offers a versatile tool for the study of bounded lifetime data. Inferences from both classical and Bayesian approaches offer consistent outcomes, further validating the utility of the suggested approach.

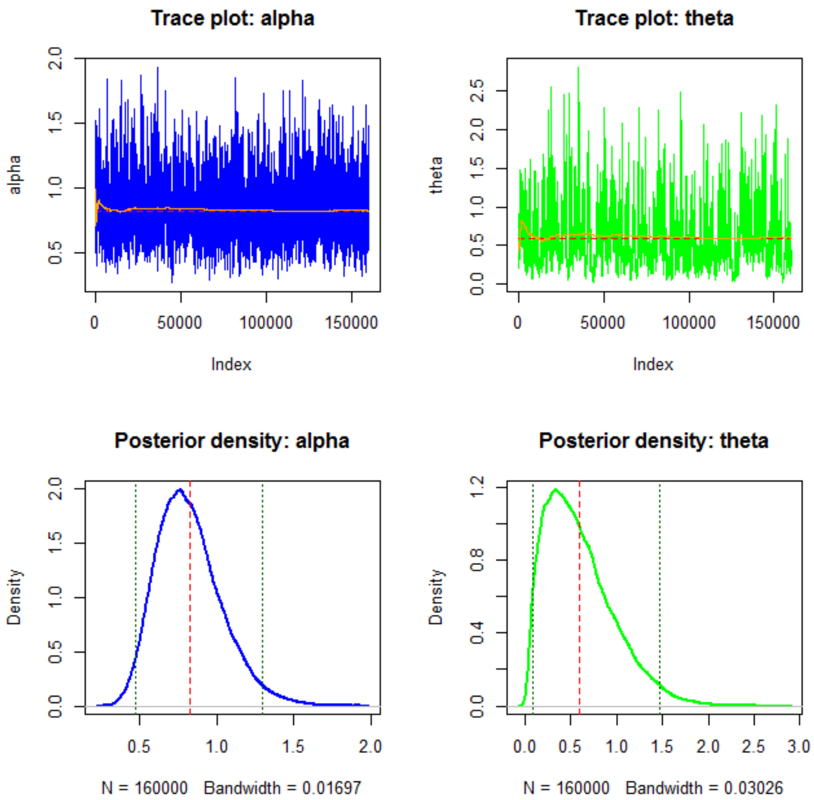

Figure 6. The trace and the density plots of Data I R_1.

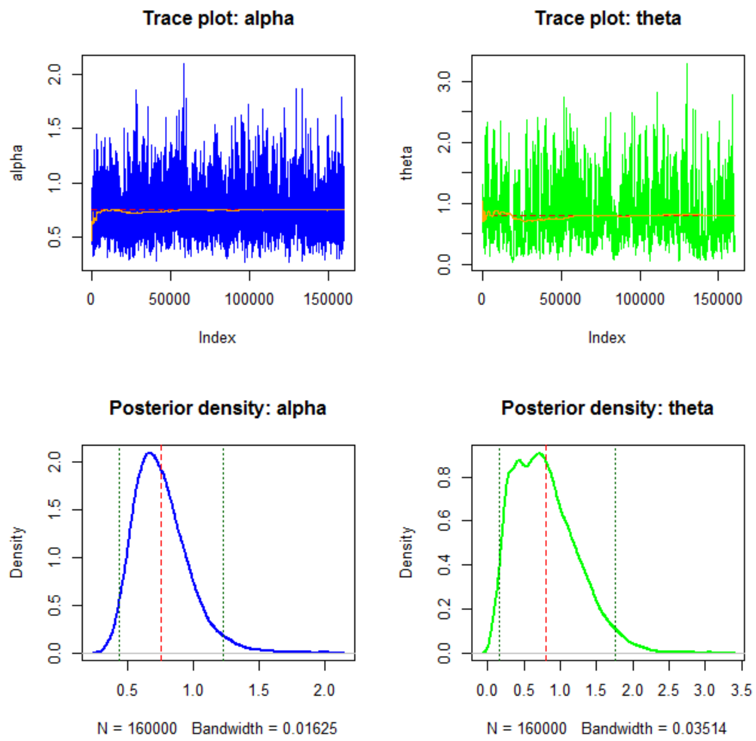

Figure 7. The trace and the density plots of Data I R_2.

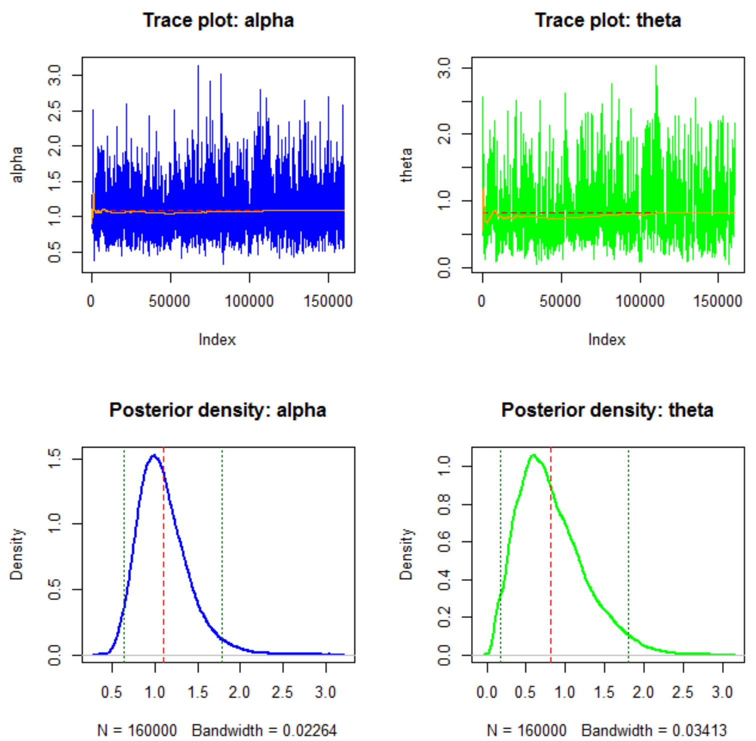

Figure 8. The trace and the density plots of Data II R_1.

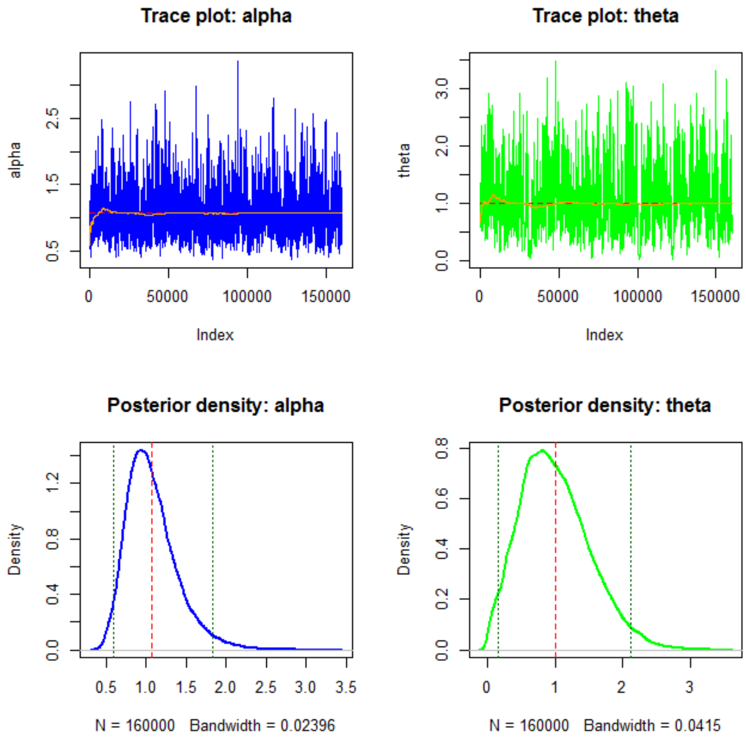

Figure 9. The trace and the density plots of Data II R_2.

In general, the real data analyzes demonstrate that the proposed TUEA distribution, combined with progressive Type–I censoring, provides a flexible and effective framework for modeling bounded lifetime data. Both classical and Bayesian inference procedures yield coherent and interpretable results, reinforcing the practical relevance of the proposed methodology.

7. Conclusions and future research

In this paper, a new lifetime model known as the truncated unit exponentiated Ailamujia (TUEA) distribution is proposed and studied under the scheme of progressive Type–II censoring. By truncating the unit exponentiated Ailamujia distribution, the new model can be used for modeling real-life situations where the lifetime data are constrained to take values on the unit interval. The unit interval is a popular choice in reliability and survival analysis and many other applications. Methods of estimation based on maximum likelihood and Bayes approaches are discussed to estimate the shape and scale parameters of the TUEA distribution. For maximum likelihood estimation, an iterative method is used to compute estimates of the unknown parameters. Similarly, asymptotic confidence intervals were obtained on the basis of maximum likelihood. From the Bayesian point of view, independent gamma distributions are considered as prior distributions for the model parameters. Bayes estimators are computed using squared error and LINEX loss functions. However, MCMC methods were used to obtain Bayes estimates due to the complexity of posterior distributions. A complete Monte Carlo simulation study is carried out to assess the performance of the estimators for small sample sizes and progressive Type–II censoring schemes. The results of the simulations show that more accurate estimates are achieved with larger sample sizes and lower levels of censorship. In addition, it is found that Bayesian estimators perform better than classical estimators in terms of bias, mean squared error, and confidence interval coverages, particularly when there was heavy censoring. Further evidence of the practical relevance of the suggested model is obtained by applying it to two actual-life data sets. It is revealed that both traditional and Bayesian techniques result in consistent and easily interpretable estimates, and that the TUEA model possesses adequate flexibility to properly reflect the key characteristics of censored lifetime observations. Potential areas for future research include generalization of the TUEA distribution to more complicated cases, such as regression modeling, the use of different priors, and censored data estimation under hybrid or adaptive censoring.

Author contributions

Faten S. Alamri: Resources, data curation, writing–review & editing, funding acquisition; Ahlam H. Tolba: Conceptualization, software, formal analysis, validation, methodology, investigation, writing–review & editing, visualization; Hana S. Jabarah: Methodology, formal analysis, resources; Ahmed R El-Saeed: Software, formal analysis, writing–review & editing, funding acquisition; Ahmed T. Ramadan: Methodology, writing–review & editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflict of interest in this paper.

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