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Research article

On generalized parameter-dependent Newton-type inequalities for proportional Caputo-hybrid operators

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Abstract: In this paper, we provide a thorough analysis of integral inequalities in the context of proportional Caputo-hybrid operators. Using these fractional operators, we first prove a new generalized parameter-dependent identity. We obtain a number of new error bounds for functions whose first and second derivatives are s-convex in the second sense using this identity as an auxiliary result. Our main results are significant because they are general; we recover the classical Simpson's $3/8$ rule, the corrected Simpson's $3/8$ rule, and other related inequalities as special cases by appropriately adjusting the parameters η and β. Furthermore, to demonstrate the validity and robustness of our theoretical findings, we provide numerical examples and graphical representations comparing the obtained bounds. These visualizations confirm the accuracy of the proposed estimates for varying fractional orders and convexity parameters.

Keywords: proportional Caputo-hybrid operators; integral inequalities; s-convex functions; Simpson's $3/8$ rule; error bounds
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1. Introduction

Convexity is one of the most important and useful ideas in mathematical analysis. It connects geometry, topology, and optimization theory; see [1, 2]. Its importance is especially clear in the area of integral inequalities, where convex functions make it easier to find exact error bounds for different approximation methods. Since the discovery of the classical Hermite-Hadamard inequality, which provides a double estimation for the mean value of a convex function, researchers have extensively utilized convexity to establish bounds for quadrature rules [3–5]. These inequalities are not merely theoretical constructs; they play a pivotal role in numerical analysis by quantifying the error incurred when approximating definite integrals using discrete data points.

Among the various numerical integration techniques, Newton-Cotes quadrature formulas are arguably the most widely adopted due to their simplicity and effectiveness. These formulas approximate the integral of a function over an interval by evaluating the function at equally spaced nodes. While the trapezoidal rule (2 nodes) and Simpson's 1/3 rule (3 nodes) are standard staples in calculus curricula, Simpson's 3/8 rule offers a higher degree of accuracy for sufficiently smooth functions. This rule utilizes a cubic interpolation polynomial passing through four equally spaced points. In classical analysis, the error estimation for Newton's 3/8 rule typically relies on the fourth derivative of the integrand. But in real-world applications, where functions might not be smooth or have limited differentiability, this reliance on high-order regularity is frequently a limiting condition. As a result, the use of convexity and generalized convexity to obtain error estimates involving only first or second derivatives has become more prevalent in contemporary research, expanding the scope of these quadrature rules.

The field of fractional calculus has experienced rapid expansion in recent decades, alongside major developments in inequality theory. Fractional calculus, whose origins can be traced back to the 1695 correspondence between L'Hôpital and Leibniz on the meaning of a half-order derivative, extends the classical notions of differentiation and integration to arbitrary non-integer orders. In contrast to classical local operators, fractional operators are inherently nonlocal, meaning that the present state of a system depends on its past history. Owing to this distinctive memory effect, fractional calculus has become an essential framework for modeling complex phenomena in many areas of science and engineering. In particular, fractional partial differential equations have been successfully applied in fluid dynamics to describe viscoelastic behavior, anomalous transport, and other hereditary processes [6]. They also play an important role in image processing, especially in image denoising and super-resolution, where fractional models provide enhanced capability for preserving edges and fine textures while reducing noise [7, 8]. These diverse applications further demonstrate the broad applicability and effectiveness of fractional-order models in capturing real-world phenomena that cannot be adequately described by classical integer-order approaches.

Over the years, numerous definitions of fractional operators have been proposed to better describe specific physical problems. These include the Riemann-Liouville, Caputo [9], Hadamard [10], Conformable [11], and Katugampola [12] fractional derivatives, among others [13–15]. Recently, a novel class of operators known as "proportional fractional operators" or "generalized proportional operators" has attracted significant attention. These operators incorporate exponential kernels, allowing for a more refined control over the decay of memory effects. Within this context, the hybridization of proportional derivatives with the Caputo setting has led to the development of "proportional Caputo-hybrid operators". Unlike the classical Caputo operators, which are defined solely through the fractional integral of the first-order derivative, the proportional Caputo-hybrid operators incorporate an additional hybrid structure, typically involving an extra kernel. Therefore, they provide a more flexible memory representation than the standard Caputo operators and should be viewed as a generalized framework rather than a mere reformulation of the classical Caputo derivative.

The theoretical foundation for these operators was laid out in recent literature. As defined in [16], the general form is given as follows:

Definition 1.1. *[16] Let* $\mathscr{A} \in L^1(\mathfrak{e}_1, \mathfrak{e}_2)$. *The proportional Caputo-operator of order* $\varrho \in [0, 1)$ *is defined by*

$$ {}^{PC}_{0}\mathbf{D}^{\varrho}_{\varpi}\mathscr{A}(\varpi) = \frac{1}{\Gamma(1-\varrho)} \int_0^{\varpi} \left[\mathscr{G}_1(\varrho, \mathfrak{w})\mathscr{A}(\mathfrak{w}) + \mathscr{G}_0(\varrho, \mathfrak{w})\mathscr{A}'(\mathfrak{w})\right] (\varpi - \mathfrak{w})^{-\varrho} d\mathfrak{w}, $$

where $\Gamma(\cdot)$ *is the Gamma function, and the kernels* $\mathscr{G}_0$ *and* $\mathscr{G}_1$ *satisfy specific limit conditions as* $\varrho \to 0^+$ *and* $\varrho \to 1^-$.

Building upon this foundation, Sarikaya [17] formalized the left and right proportional Caputo-hybrid operators for a specific choice of kernels, explicitly defined as follows:

Definition 1.2. *[17] For* $\varrho \in [0, 1]$, *the left-sided and right-sided proportional Caputo-hybrid operators are given, respectively, by*

$$ {}^{PC}_{\mathfrak{e}_1^+}\mathbf{D}^{\varrho}_{\mathfrak{e}_2}\mathscr{A}(\mathfrak{e}_2) = \frac{1}{\Gamma(1-\varrho)} \int_{\mathfrak{e}_1}^{\mathfrak{e}_2} \left(\mathscr{G}_1(\varrho, \mathfrak{e}_2 - \mathfrak{w}) \mathscr{A}(\mathfrak{w}) + \mathscr{G}_0(\varrho, \mathfrak{e}_2 - \mathfrak{w}) \mathscr{A}'(\mathfrak{w})\right) (\mathfrak{e}_2 - \mathfrak{w})^{-\varrho} d\mathfrak{w}, $$

and

$$ {}^{PC}_{\mathfrak{e}_2^-}\mathbf{D}^{\varrho}_{\mathfrak{e}_1}\mathscr{A}(\mathfrak{e}_1) = \frac{1}{\Gamma(1-\varrho)} \int_{\mathfrak{e}_1}^{\mathfrak{e}_2} \left(\mathscr{G}_1(\varrho, \mathfrak{w} - \mathfrak{e}_1) \mathscr{A}(\mathfrak{w}) + \mathscr{G}_0(\varrho, \mathfrak{w} - \mathfrak{e}_1) \mathscr{A}'(\mathfrak{w})\right) (\mathfrak{w} - \mathfrak{e}_1)^{-\varrho} d\mathfrak{w}, $$

where $\mathscr{G}_0(\varrho, \mathfrak{w}) = (1-\varrho)^2 \mathfrak{w}^{1-\varrho}$ *and* $\mathscr{G}_1(\varrho, \mathfrak{w}) = \varrho^2 \mathfrak{w}^{\varrho}$.

In the same work, Sarikaya [17] established the Hermite-Hadamard and trapezium-type inequalities for convex functions as follows:

Theorem 1.1. *Let* $\mathscr{A}[\mathfrak{e}_1, \mathfrak{e}_2] \to \mathbb{R}$, *with* $0 < \mathfrak{e}_1 < \mathfrak{e}_2$. *If* $\mathscr{A}$ *and* $\mathscr{A}'$ *are convex, then*

$$ \begin{aligned} &\varrho^2(\mathfrak{e}_2 - \mathfrak{e}_1)^{\varrho}\mathscr{A}\left(\frac{\mathfrak{e}_1 + \mathfrak{e}_2}{2}\right) + \frac{1-\varrho}{2}(\mathfrak{e}_2 - \mathfrak{e}_1)^{1-\varrho}\mathscr{A}'\left(\frac{\mathfrak{e}_1 + \mathfrak{e}_2}{2}\right) \\ &\le \frac{\Gamma(1-\varrho)}{2(\mathfrak{e}_2 - \mathfrak{e}_1)^{1-\varrho}}\left[{}^{P}\mathbf{D}^{\varrho}_{\mathfrak{e}_1+}\mathscr{A}(\mathfrak{e}_1) + {}^{P}\mathbf{D}^{\varrho}_{\mathfrak{e}_2-}\mathscr{A}(\mathfrak{e}_2)\right] \\ &\le \varrho^2(\mathfrak{e}_2 - \mathfrak{e}_1)^{\varrho}\left(\frac{\mathscr{A}'(\mathfrak{e}_1) + \mathscr{A}'(\mathfrak{e}_2)}{2}\right) + (1-\varrho)(\mathfrak{e}_2 - \mathfrak{e}_1)^{1-\varrho}\left(\frac{\mathscr{A}''(\mathfrak{e}_1) + \mathscr{A}''(\mathfrak{e}_2)}{2}\right). \end{aligned} $$

Theorem 1.2. *Let* $\mathscr{A}[\mathfrak{e}_1, \mathfrak{e}_2] \to \mathbb{R}$, *with* $0 < \mathfrak{e}_1 < \mathfrak{e}_2$. *If* $|\mathscr{A}'|$ *and* $|\mathscr{A}''|$ *are convex, then*

$$ \begin{aligned} &\Bigg|\varrho^2(\mathfrak{e}_2 - \mathfrak{e}_1)^{\varrho}\left(\frac{\mathscr{A}'(\mathfrak{e}_1) + \mathscr{A}'(\mathfrak{e}_2)}{2}\right) + (1-\varrho)(\mathfrak{e}_2 - \mathfrak{e}_1)^{1-\varrho}\left(\frac{\mathscr{A}''(\mathfrak{e}_1) + \mathscr{A}''(\mathfrak{e}_2)}{2}\right) \\ &-\frac{\Gamma(1-\varrho)}{2(\mathfrak{e}_2 - \mathfrak{e}_1)^{1-\varrho}}\left[{}^{P}\mathbf{D}^{\varrho}_{\mathfrak{e}_1+}\mathscr{A}(\mathfrak{e}_1) + {}^{P}\mathbf{D}^{\varrho}_{\mathfrak{e}_2-}\mathscr{A}(\mathfrak{e}_2)\right]\Bigg| \\ &\le \frac{\varrho^2(\mathfrak{e}_2 - \mathfrak{e}_1)^{\varrho+1}}{2}\left(\frac{|\mathscr{A}'(\mathfrak{e}_1)| + |\mathscr{A}'(\mathfrak{e}_2)|}{4}\right) + \frac{(1-\varrho)(\mathfrak{e}_2 - \mathfrak{e}_1)^{2-\varrho}}{(3-2\varrho)^2}\left[1 - \frac{1}{2^{3-2\varrho}}\right]\left(\frac{|\mathscr{A}''(\mathfrak{e}_1)| + |\mathscr{A}''(\mathfrak{e}_2)|}{2}\right). \end{aligned} $$

The formalization of proportional Caputo-hybrid operators has catalyzed a significant body of research dedicated to extending classical integral inequalities to this fractional framework. The literature regarding these specific operators has expanded rapidly, with a particular focus on establishing error bounds that mirror classical quadrature rules. Pioneering this direction, Sarikaya [18] derived Simpson-type inequalities, effectively bounding the error for functions with convex first- and sencond-order derivatives. Expanding the scope to higher-order quadrature rules, Demir [19] recently investigated Milne-type inequalities via proportional Caputo-hybrid operators. These foundational works highlight the versatility of hybrid fractional operators in approximation theory and set the stage for further generalizations.

In [20], Demir and Tunç established the following Newton-type inequalities for functions whose first- and second-order derivatives are convex:

$$\begin{aligned}&\left|\frac{\varrho^2(\mathrm{e}_2-\mathrm{e}_1)^{\varrho}}{8}\left(\mathscr{A}(\mathrm{e}_1)+3\mathscr{A}\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right)+3\mathscr{A}\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right)+\mathscr{A}(\mathrm{e}_2)\right)\right.\\&+\frac{(1-\varrho)}{16(\mathrm{e}_2-\mathrm{e}_1)^{\varrho-1}}\left(\mathscr{A}(\mathrm{e}_1)+3\mathscr{A}\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right)+3\mathscr{A}\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right)+\mathscr{A}(\mathrm{e}_2)\right)\\&\left.-\frac{\Gamma(1-\varrho)}{2(\mathrm{e}_2-\mathrm{e}_1)^{1-\varrho}}\left[{}^{P}\mathbf{D}^{\varrho}_{\mathrm{e}_1+}\mathscr{A}(\mathrm{e}_1)+{}^{P}\mathbf{D}^{\varrho}_{\mathrm{e}_2-}\mathscr{A}(\mathrm{e}_2)\right]\right|\\\leq\;&\frac{25\varrho^2(\mathrm{e}_2-\mathrm{e}_1)^{1+\varrho}}{576}\left(\frac{|\mathscr{A}'(\mathrm{e}_1)|+|\mathscr{A}'(\mathrm{e}_2)|}{2}\right)+(1-\varrho)(\mathrm{e}_2-\mathrm{e}_1)^{2-\varrho}\Delta(\varrho)\left(|\mathscr{A}''(\mathrm{e}_1)|+|\mathscr{A}''(\mathrm{e}_2)|\right),\end{aligned}$$

where the coefficient $\Delta(\varrho)$ is defined as

$$\Delta(\varrho)=\begin{cases}2\left(\frac{7}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}+\frac{1-2\left(\frac{7}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}}{3-2\varrho}-\frac{5}{4}, & \text{if}\quad 0<\varrho\leq\frac{\ln(3)-\ln(2\sqrt{2})}{\ln(3)},\\ \frac{4-4\varrho}{3-2\varrho}\left(\left(\frac{1}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}+\left(\frac{7}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}\right)+\frac{2\left(\frac{1}{3}\right)^{3-2\varrho}+1}{3-2\varrho}-\frac{3}{4}, & \text{if}\quad \frac{\ln(3)-\ln(2\sqrt{2})}{\ln(3)}<\varrho\leq\frac{\ln(3)-\ln(2\sqrt{2})}{\ln(3)-\ln(2)},\\ \frac{4-4\varrho}{3-2\varrho}\left(\left(\frac{1}{2}\right)^{\frac{3-2\varrho}{2-2\varrho}}+\left(\frac{1}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}+\left(\frac{7}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}\right) & \\ \quad+\frac{2\left(\frac{1}{3}\right)^{3-2\varrho}+2\left(\frac{2}{3}\right)^{3-2\varrho}+1}{3-2\varrho}-2, & \text{if}\quad \frac{\ln(3)-\ln(2\sqrt{2})}{\ln(3)-\ln(2)}<\varrho\leq\frac{\ln(3)-\ln(\sqrt{2})}{\ln(3)},\\ \frac{4-4\varrho}{3-2\varrho}\left(\left(\frac{1}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}+\left(\frac{7}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}\right)+\frac{2\left(\frac{2}{3}\right)^{3-2\varrho}+1}{3-2\varrho}-\frac{5}{3}, & \text{if}\quad \frac{\ln(3)-\ln(\sqrt{2})}{\ln(3)}<\varrho\leq\frac{\ln(3\sqrt{7})-\ln(4\sqrt{2})}{\ln(3)-\ln(2)},\\ \frac{4-4\varrho}{3-2\varrho}\left(\frac{1}{8}\right)^{\frac{3-2\varrho}{2-2\varrho}}+\frac{1}{3-2\varrho}-\frac{1}{2}, & \text{if}\quad \frac{\ln(3\sqrt{7})-\ln(4\sqrt{2})}{\ln(3)-\ln(2)}<\varrho\leq 1.\end{cases}$$

Beyond these results, the literature has been enriched by Demir [19, 21], who derived Milne-type inequalities for various classes of functions. Furthermore, Mehtab et al. [22] expanded the scope by investigating corrected Maclaurin-type inequalities utilizing the same proportional Caputo-hybrid operators. More recently, Al-Hazmy et al. [23] established a new version of Milne-type inequalities via convexity, while Al-Anzy et al. [24] conducted a parametrized study on three-point Newton-Cotes-type inequalities for functions with s-convex first- and second-order derivatives.

Despite these advancements, there remains a need to investigate parameterized versions of these inequalities to provide greater flexibility and tighter error bounds. In this paper, motivated by the utility of the Newton's 3/8 rule and the flexibility of parameterized identities, we aim to establish new parametrized Newton-type inequalities for proportional Caputo-hybrid operators. By introducing a new integral identity and leveraging the properties of convexity, we derive error estimates that generalize and improve upon existing results in the literature.

2. New parametrized identity involving proportional Caputo-hybrid operators

In this section, we introduce a new identity involving proportional Caputo-hybrid operators indispensable for the derivation of our results.

Lemma 2.1. *Let* $\mathscr{A} : I \to \mathbb{R}$ *be a twice differentiable function on* I° *(the interior of* I*), where* $\mathrm{e}_1, \mathrm{e}_2 \in I^\circ$ *satisfies* $0 < \mathrm{e}_1 < \mathrm{e}_2$ *and let* $\mathscr{A}, \mathscr{A}', \mathscr{A}'' \in L[\mathrm{e}_1, \mathrm{e}_2]$*, where* $L[\mathrm{e}_1, \mathrm{e}_2]$ *denotes the space of all Lebesgue-integrable real-valued functions defined on* $[\mathrm{e}_1, \mathrm{e}_2]$*. Then, the equality*

$$
\begin{aligned}
&\varrho^2 (\mathrm{e}_2 - \mathrm{e}_1)^{\varrho} \left(\frac{\eta \mathscr{A}(\mathrm{e}_1) + \beta \mathscr{A}\left(\frac{2\mathrm{e}_1 + \mathrm{e}_2}{3}\right) + \beta \mathscr{A}\left(\frac{\mathrm{e}_1 + 2\mathrm{e}_2}{3}\right) + \eta \mathscr{A}(\mathrm{e}_2)}{2(\eta + \beta)} \right) \\
&+ \frac{1-\varrho}{2(\mathrm{e}_2 - \mathrm{e}_1)^{\varrho - 1}} \left(\frac{\eta \mathscr{A}'(\mathrm{e}_1) + \beta \mathscr{A}'\left(\frac{2\mathrm{e}_1 + \mathrm{e}_2}{3}\right) + \beta \mathscr{A}'\left(\frac{\mathrm{e}_1 + 2\mathrm{e}_2}{3}\right) + \eta \mathscr{A}'(\mathrm{e}_2)}{2(\eta + \beta)} \right) \\
&- \frac{\Gamma(1-\varrho)}{2(\mathrm{e}_2 - \mathrm{e}_1)^{1-\varrho}} \left({}^{PC}_{\mathrm{e}_2^-}\mathbf{D}^{\varrho}_{\mathrm{e}_1} \mathscr{A}(\mathrm{e}_1) + {}^{PC}_{\mathrm{e}_1^+}\mathbf{D}^{\varrho}_{\mathrm{e}_2} \mathscr{A}(\mathrm{e}_2) \right) \\
&= \frac{\varrho^2 (\mathrm{e}_2 - \mathrm{e}_1)^{1+\varrho}}{2} \left(\int_0^{\frac{1}{3}} \left(\nu - \frac{\eta}{2(\eta+\beta)} \right) \left(\mathscr{A}'((1-\nu)\mathrm{e}_1 + \nu \mathrm{e}_2) - \mathscr{A}'(\nu \mathrm{e}_1 + (1-\nu)\mathrm{e}_2) \right) \mathrm{d}\nu \right. \\
&+ \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\nu - \frac{1}{2} \right) \left(\mathscr{A}'((1-\nu)\mathrm{e}_1 + \nu \mathrm{e}_2) - \mathscr{A}'(\nu \mathrm{e}_1 + (1-\nu)\mathrm{e}_2) \right) \mathrm{d}\nu \\
&+ \left. \int_{\frac{2}{3}}^{1} \left(\nu - \frac{\eta + 2\beta}{2(\eta+\beta)} \right) \left(\mathscr{A}'((1-\nu)\mathrm{e}_1 + \nu \mathrm{e}_2) - \mathscr{A}'(\nu \mathrm{e}_1 + (1-\nu)\mathrm{e}_2) \right) \mathrm{d}\nu \right) \\
&+ \frac{(1-\varrho)(\mathrm{e}_2 - \mathrm{e}_1)^{2-\varrho}}{4} \left(\int_0^{\frac{1}{3}} \left(\nu^{2-2\varrho} - \frac{\eta}{2(\eta+\beta)} \right) \left(\mathscr{A}''((1-\nu)\mathrm{e}_1 + \nu \mathrm{e}_2) - \mathscr{A}''(\nu \mathrm{e}_1 + (1-\nu)\mathrm{e}_2) \right) \mathrm{d}\nu \right. \\
&+ \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\nu^{2-2\varrho} - \frac{1}{2} \right) \left(\mathscr{A}''((1-\nu)\mathrm{e}_1 + \nu \mathrm{e}_2) - \mathscr{A}''(\nu \mathrm{e}_1 + (1-\nu)\mathrm{e}_2) \right) \mathrm{d}\nu
\end{aligned}
$$

$$+\int_{\frac{2}{3}}^{1}\left(\nu^{2-2\varrho}-\frac{\eta+2\beta}{2(\eta+\beta)}\right)(\mathscr{A}''((1-\nu)e_1+\nu e_2)-\mathscr{A}''(\nu e_1+(1-\nu)e_2))\,d\nu\Bigg],$$

holds, where η and β are two non-negative parameters and $\varrho \in [0,1)$.

Proof. Let

$$\mathscr{N}_1=\int_{0}^{\frac{1}{3}}\left(\nu-\frac{\eta}{2(\eta+\beta)}\right)(\mathscr{A}'((1-\nu)e_1+\nu e_2)-\mathscr{A}'(\nu e_1+(1-\nu)e_2))\,d\nu,$$

$$\mathscr{N}_2=\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\nu-\frac{1}{2}\right)(\mathscr{A}'((1-\nu)e_1+\nu e_2)-\mathscr{A}'(\nu e_1+(1-\nu)e_2))\,d\nu,$$

$$\mathscr{N}_3=\int_{\frac{2}{3}}^{1}\left(\nu-\frac{\eta+2\beta}{2(\eta+\beta)}\right)(\mathscr{A}'((1-\nu)e_1+\nu e_2)-\mathscr{A}'(\nu e_1+(1-\nu)e_2))\,d\nu,$$

$$\mathscr{M}_1=\int_{0}^{\frac{1}{3}}\left(\nu^{2-2\varrho}-\frac{\eta}{2(\eta+\beta)}\right)(\mathscr{A}''((1-\nu)e_1+\nu e_2)-\mathscr{A}''(\nu e_1+(1-\nu)e_2))\,d\nu,$$

$$\mathscr{M}_2=\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\nu^{2-2\varrho}-\frac{1}{2}\right)(\mathscr{A}''((1-\nu)e_1+\nu e_2)-\mathscr{A}''(\nu e_1+(1-\nu)e_2))\,d\nu,$$

and

$$\mathscr{M}_3=\int_{\frac{2}{3}}^{1}\left(\nu^{2-2\varrho}-\frac{\eta+2\beta}{2(\eta+\beta)}\right)(\mathscr{A}''((1-\nu)e_1+\nu e_2)-\mathscr{A}''(\nu e_1+(1-\nu)e_2))\,d\nu.$$

Using the integration by parts, $\mathscr{N}_1$ gives

$$\begin{aligned}\mathscr{N}_1&=\frac{1}{e_2-e_1}\left(\nu-\frac{\eta}{2(\eta+\beta)}\right)(\mathscr{A}((1-\nu)e_1+\nu e_2)+\mathscr{A}(\nu e_1+(1-\nu)e_2))\Big|_{0}^{\frac{1}{3}}\\&\quad-\frac{1}{e_2-e_1}\int_{0}^{\frac{1}{3}}(\mathscr{A}((1-\nu)e_1+\nu e_2)+\mathscr{A}(\nu e_1+(1-\nu)e_2))\,d\nu\\&=\frac{1}{e_2-e_1}\left(\frac{1}{3}-\frac{\eta}{2(\eta+\beta)}\right)\left(\mathscr{A}\left(\frac{2e_1+e_2}{3}\right)+\mathscr{A}\left(\frac{e_1+2e_2}{3}\right)\right)+\frac{1}{e_2-e_1}\left(\frac{\eta}{2(\eta+\beta)}\right)(\mathscr{A}(e_1)+\mathscr{A}(e_2))\end{aligned}$$

$$-\frac{1}{e_2-e_1}\int_0^{\frac{1}{3}}\left(\mathscr{A}\left(\left(1-\nu\right)e_1+\nu e_2\right)+\mathscr{A}\left(\nu e_1+\left(1-\nu\right)e_2\right)\right)d\nu. \tag{2.1}$$

Similarly, we obtain

$$\begin{aligned}\mathscr{N}_2&=\frac{1}{e_2-e_1}\left(\nu-\frac{1}{2}\right)\left(\mathscr{A}\left(\left(1-\nu\right)e_1+\nu e_2\right)+\mathscr{A}\left(\nu e_1+\left(1-\nu\right)e_2\right)\right)\Big|_{\frac{1}{3}}^{\frac{2}{3}}\\&\quad-\frac{1}{e_2-e_1}\int_{\frac{1}{3}}^{\frac{1}{2}}\left(\mathscr{A}\left(\left(1-\nu\right)e_1+\nu e_2\right)+\mathscr{A}\left(\nu e_1+\left(1-\nu\right)e_2\right)\right)d\nu\\&=\frac{1}{3\left(e_2-e_1\right)}\left(\mathscr{A}\left(\frac{2e_1+e_2}{3}\right)+\mathscr{A}\left(\frac{e_1+2e_2}{3}\right)\right)\\&\quad-\frac{1}{e_2-e_1}\int_{\frac{1}{3}}^{\frac{2}{3}}\left(\mathscr{A}\left(\left(1-\nu\right)e_1+\nu e_2\right)+\mathscr{A}\left(\nu e_1+\left(1-\nu\right)e_2\right)\right)d\nu,\end{aligned} \tag{2.2}$$

$$\begin{aligned}\mathscr{N}_3&=\frac{1}{e_2-e_1}\left(\nu-\frac{\eta+2\beta}{2\left(\eta+\beta\right)}\right)\left(\mathscr{A}\left(\left(1-\nu\right)e_1+\nu e_2\right)+\mathscr{A}\left(\nu e_1+\left(1-\nu\right)e_2\right)\right)\Big|_{\frac{2}{3}}^{1}\\&\quad-\frac{1}{e_2-e_1}\int_{\frac{2}{3}}^{1}\left(\mathscr{A}\left(\left(1-\nu\right)e_1+\nu e_2\right)+\mathscr{A}\left(\nu e_1+\left(1-\nu\right)e_2\right)\right)d\nu\\&=\frac{1}{e_2-e_1}\left(1-\frac{\eta+2\beta}{2\left(\eta+\beta\right)}\right)\left(\mathscr{A}\left(e_1\right)+\mathscr{A}\left(e_2\right)\right)\\&\quad-\frac{1}{e_2-e_1}\left(\frac{2}{3}-\frac{\eta+2\beta}{2\left(\eta+\beta\right)}\right)\left(\mathscr{A}\left(\frac{e_1+2e_2}{3}\right)+\mathscr{A}\left(\frac{2e_1+e_2}{3}\right)\right)\\&\quad-\frac{1}{e_2-e_1}\int_{\frac{2}{3}}^{1}\left(\mathscr{A}\left(\left(1-\nu\right)e_1+\nu e_2\right)+\mathscr{A}\left(\nu e_1+\left(1-\nu\right)e_2\right)\right)d\nu,\end{aligned} \tag{2.3}$$

$$\begin{aligned}\mathscr{M}_1&=\frac{1}{e_2-e_1}\left(\nu^{2-2\varrho}-\frac{\eta}{2\left(\eta+\beta\right)}\right)\left(\mathscr{A}'\left(\left(1-\nu\right)e_1+\nu e_2\right)+\mathscr{A}'\left(\nu e_1+\left(1-\nu\right)e_2\right)\right)\Big|_{0}^{\frac{1}{3}}\\&\quad-\frac{2-2\varrho}{e_2-e_1}\int_0^{\frac{1}{3}}\nu^{1-2\varrho}\left(\mathscr{A}'\left(\left(1-\nu\right)e_1+\nu e_2\right)+\mathscr{A}'\left(\nu e_1+\left(1-\nu\right)e_2\right)\right)d\nu\end{aligned}$$

$$= \frac{1}{e_2 - e_1}\left(\left(\frac{1}{3}\right)^{2-2\varrho} - \frac{\eta}{2(\eta+\beta)}\right)\left(\mathscr{A}'\left(\frac{2e_1+e_2}{3}\right) + \mathscr{A}'\left(\frac{e_1+2e_2}{3}\right)\right)$$
$$+ \frac{1}{e_2 - e_1}\frac{\eta}{2(\eta+\beta)}\left(\mathscr{A}'(e_1) + \mathscr{A}'(e_2)\right)$$
$$- \frac{2-2\varrho}{e_2 - e_1}\int_0^{\frac{1}{3}} \nu^{1-2\varrho}\left(\mathscr{A}'((1-\nu)e_1 + \nu e_2) + \mathscr{A}'(\nu e_1 + (1-\nu)e_2)\right)d\nu, \tag{2.4}$$

$$\mathscr{M}_2 = \frac{1}{e_2 - e_1}\left(\nu^{2-2\varrho} - \frac{1}{2}\right)\left(\mathscr{A}'((1-\nu)e_1 + \nu e_2) + \mathscr{A}'(\nu e_1 + (1-\nu)e_2)\right)\Big|_{\frac{1}{3}}^{\frac{2}{3}}$$
$$- \frac{2-2\varrho}{e_2 - e_1}\int_{\frac{1}{3}}^{\frac{2}{3}} \nu^{1-2\varrho}\left(\mathscr{A}'((1-\nu)e_1 + \nu e_2) + \mathscr{A}'(\nu e_1 + (1-\nu)e_2)\right)d\nu$$
$$= \frac{1}{e_2 - e_1}\left(\left(\frac{2}{3}\right)^{2-2\varrho} - \left(\frac{1}{3}\right)^{2-2\varrho}\right)\left(\mathscr{A}\left(\frac{2e_1+e_2}{3}\right) + \mathscr{A}\left(\frac{e_1+2e_2}{3}\right)\right)$$
$$- \frac{2-2\varrho}{e_2 - e_1}\int_{\frac{1}{3}}^{\frac{2}{3}} \nu^{1-2\varrho}\left(\mathscr{A}'((1-\nu)e_1 + \nu e_2) + \mathscr{A}'(\nu e_1 + (1-\nu)e_2)\right)d\nu, \tag{2.5}$$

and

$$\mathscr{M}_3 = \frac{1}{e_2 - e_1}\left(\nu^{2-2\varrho} - \frac{\eta+2\beta}{2(\eta+\beta)}\right)\left(\mathscr{A}'((1-\nu)e_1 + \nu e_2) + \mathscr{A}'(\nu e_1 + (1-\nu)e_2)\right)\Big|_{\frac{2}{3}}^{1}$$
$$- \frac{1}{e_2 - e_1}\int_{\frac{2}{3}}^{1} \nu^{1-2\varrho}\left(\mathscr{A}'((1-\nu)e_1 + \nu e_2) + \mathscr{A}'(\nu e_1 + (1-\nu)e_2)\right)d\nu$$
$$= \frac{1}{e_2 - e_1}\left(1 - \frac{\eta+2\beta}{2(\eta+\beta)}\right)\left(\mathscr{A}(e_1) + \mathscr{A}(e_2)\right)$$
$$- \frac{1}{e_2 - e_1}\left(\left(\frac{2}{3}\right)^{2-2\varrho} - \frac{\eta+2\beta}{2(\eta+\beta)}\right)\left(\mathscr{A}\left(\frac{e_1+2e_2}{3}\right) + \mathscr{A}\left(\frac{2e_1+e_2}{3}\right)\right)$$
$$- \frac{2-2\varrho}{e_2 - e_1}\int_{\frac{e_2 - \frac{2e_1+e_2}{3}}{e_2-e_1}}^{1} \nu^{1-2\varrho}\left(\mathscr{A}'((1-\nu)e_1 + \nu e_2) + \mathscr{A}'(\nu e_1 + (1-\nu)e_2)\right)d\nu. \tag{2.6}$$

Summing (2.1)–(2.3), then using the change of variable, we get

$$\sum_{k=1}^{3}\mathscr{N}_k = \frac{2}{e_2 - e_1}\left(\frac{\eta\mathscr{A}(e_1) + \beta\mathscr{A}\left(\frac{2e_1+e_2}{3}\right) + \beta\mathscr{A}\left(\frac{e_1+2e_2}{3}\right) + \eta\mathscr{A}(e_2)}{2(\eta+\beta)}\right) \tag{2.7}$$

$$-\frac{1}{e_2-e_1}\int_0^1\left(\mathscr{A}\left(\left(1-\nu\right)e_1+\nu e_2\right)+\mathscr{A}\left(\nu e_1+\left(1-\nu\right)e_2\right)\right)d\nu$$
$$=\frac{2}{e_2-e_1}\left(\frac{\eta\mathscr{A}\left(e_1\right)+\beta\mathscr{A}\left(\frac{2e_1+e_2}{3}\right)+\beta\mathscr{A}\left(\frac{e_1+2e_2}{3}\right)+\eta\mathscr{A}\left(e_2\right)}{2\left(\eta+\beta\right)}\right)-\frac{2}{\left(e_2-e_1\right)^2}\int_{e_1}^{e_2}\mathscr{A}\left(\mathfrak{w}\right)d\mathfrak{w}.$$

Now, adding (2.4)–(2.6), then using the change of variable, we get

$$\sum_{k=1}^{3}\mathscr{M}_k=\frac{2}{e_2-e_1}\left(\frac{\eta\mathscr{A}'\left(e_1\right)+\beta\mathscr{A}'\left(\frac{2e_1+e_2}{3}\right)+\beta\mathscr{A}'\left(e_1+e_2-\frac{2e_1+e_2}{3}\right)+\eta\mathscr{A}'\left(e_2\right)}{2\left(\eta+\beta\right)}\right)\tag{2.8}$$
$$-\frac{2-2\varrho}{e_2-e_1}\int_0^1\nu^{1-2\varrho}\left(\mathscr{A}'\left(\left(1-\nu\right)e_1+\nu e_2\right)+\mathscr{A}'\left(\nu e_1+\left(1-\nu\right)e_2\right)\right)d\nu$$
$$=\frac{2}{e_2-e_1}\left(\frac{\eta\mathscr{A}'\left(e_1\right)+\beta\mathscr{A}'\left(\frac{2e_1+e_2}{3}\right)+\beta\mathscr{A}'\left(\frac{e_1+2e_2}{3}\right)+\eta\mathscr{A}'\left(e_2\right)}{2\left(\eta+\beta\right)}\right)$$
$$-\frac{2-2\varrho}{\left(e_2-e_1\right)^{3-2\varrho}}\left(\int_{e_1}^{e_2}\left(\mathfrak{w}-e_1\right)^{1-2\varrho}\mathscr{A}'\left(\mathfrak{w}\right)d\mathfrak{w}+\int_{e_1}^{e_2}\left(e_2-\mathfrak{w}\right)^{1-2\varrho}\mathscr{A}'\left(\mathfrak{w}\right)d\mathfrak{w}\right).$$

Multiplying (2.7) by $\frac{\varrho^2(e_2-e_1)^{1+\varrho}}{2}$ and (2.8) by $\frac{(1-\varrho)(e_2-e_1)^{2-\varrho}}{4}$, and summing the resulting equalities, we obtain

$$\frac{\varrho^2\left(e_2-e_1\right)^{1+\varrho}}{2}\sum_{k=1}^{3}\mathscr{N}_k+\frac{\left(1-\varrho\right)\left(e_2-e_1\right)^{2-\varrho}}{3}\sum_{k=1}^{4}\mathscr{M}_k$$
$$=\varrho^2\left(e_2-e_1\right)^{\varrho}\left(\frac{\eta\mathscr{A}\left(e_1\right)+\beta\mathscr{A}\left(\frac{2e_1+e_2}{3}\right)+\beta\mathscr{A}\left(\frac{e_1+2e_2}{3}\right)+\eta\mathscr{A}\left(e_2\right)}{2\left(\eta+\beta\right)}\right)$$
$$+\frac{1-\varrho}{2\left(e_2-e_1\right)^{\varrho-1}}\left(\frac{\eta\mathscr{A}'\left(e_1\right)+\beta\mathscr{A}'\left(\frac{2e_1+e_2}{3}\right)+\beta\mathscr{A}'\left(\frac{e_1+2e_2}{3}\right)+\eta\mathscr{A}'\left(e_2\right)}{2\left(\eta+\beta\right)}\right)-\frac{\varrho^2}{\left(e_2-e_1\right)^{1-\varrho}}\int_{e_1}^{e_2}\mathscr{A}\left(\mathfrak{w}\right)d\mathfrak{w}$$
$$-\frac{\left(1-\varrho\right)^2}{2\left(e_2-e_1\right)^{1-\varrho}}\left(\int_{e_1}^{e_2}\left(\mathfrak{w}-e_1\right)^{1-2\varrho}\mathscr{A}'\left(\mathfrak{w}\right)d\mathfrak{w}+\int_{e_1}^{e_2}\left(e_2-\mathfrak{w}\right)^{1-2\varrho}\mathscr{A}'\left(\mathfrak{w}\right)d\mathfrak{w}\right)$$
$$=\varrho^2\left(e_2-e_1\right)^{\varrho}\left(\frac{\eta\mathscr{A}\left(e_1\right)+\beta\mathscr{A}\left(\frac{2e_1+e_2}{3}\right)+\beta\mathscr{A}\left(\frac{e_1+2e_2}{3}\right)+\eta\mathscr{A}\left(e_2\right)}{2\left(\eta+\beta\right)}\right)$$
$$+\frac{1-\varrho}{2\left(e_2-e_1\right)^{\varrho-1}}\left(\frac{\eta\mathscr{A}'\left(e_1\right)+\beta\mathscr{A}'\left(\frac{2e_1+e_2}{3}\right)+\beta\mathscr{A}'\left(\frac{e_1+2e_2}{3}\right)+\eta\mathscr{A}'\left(e_2\right)}{2\left(\eta+\beta\right)}\right)$$
$$-\frac{1}{2\left(e_2-e_1\right)^{1-\varrho}}\left(\int_{e_1}^{e_2}\left(\varrho^2\left(\mathfrak{w}-e_1\right)^{\varrho}\mathscr{A}\left(\mathfrak{w}\right)+\left(1-\varrho\right)^2\left(\mathfrak{w}-e_1\right)^{1-\varrho}\mathscr{A}'\left(\mathfrak{w}\right)\right)\left(\mathfrak{w}-e_1\right)^{-\varrho}d\mathfrak{w}\right.$$

$$+\int_{e_1}^{e_2}\left(\varrho^2(e_2-\mathfrak{w})^{\varrho}\mathscr{A}(\mathfrak{w})+(1-\varrho)^2(e_2-\mathfrak{w})^{1-\varrho}\mathscr{A}'(\mathfrak{w})\right)(e_2-\mathfrak{w})^{-\varrho}\,d\mathfrak{w}\Bigg)$$

$$=\varrho^2(e_2-e_1)^{\varrho}\left(\frac{\eta\mathscr{A}(e_1)+\beta\mathscr{A}\left(\frac{2e_1+e_2}{3}\right)+\beta\mathscr{A}\left(\frac{e_1+2e_2}{3}\right)+\eta\mathscr{A}(e_2)}{2(\eta+\beta)}\right)$$
$$+\frac{1-\varrho}{2(e_2-e_1)^{\varrho-1}}\left(\frac{\eta\mathscr{A}'(e_1)+\beta\mathscr{A}'\left(\frac{2e_1+e_2}{3}\right)+\beta\mathscr{A}'\left(\frac{e_1+2e_2}{3}\right)+\eta\mathscr{A}'(e_2)}{2(\eta+\beta)}\right)$$
$$-\frac{\Gamma(1-\varrho)}{2(e_2-e_1)^{1-\varrho}}\left({}^{PC}_{e_2^-}\mathbf{D}^{\varrho}_{e_1}\mathscr{A}(e_1)+{}^{PC}_{e_1^+}\mathbf{D}^{\varrho}_{e_2}\mathscr{A}(e_2)\right),$$

which is the desired result.

Remark 2.1. *It is important to note that Lemma 2.1 generalizes several well-known results found in the literature. By selecting specific values for the parameters η and β, we recover the following identities:*

- *By setting $\beta=0$, we get*

$$\frac{\varrho^2(e_2-e_1)^{\varrho}}{2}(\mathscr{A}(e_1)+\mathscr{A}(e_2))+\frac{1-\varrho}{4(e_2-e_1)^{\varrho-1}}(\mathscr{A}'(e_1)+\mathscr{A}'(e_2))$$
$$-\frac{\Gamma(1-\varrho)}{2(e_2-e_1)^{1-\varrho}}\left({}^{PC}_{e_2^-}\mathbf{D}^{\varrho}_{e_1}\mathscr{A}(e_1)+{}^{PC}_{e_1^+}\mathbf{D}^{\varrho}_{e_2}\mathscr{A}(e_2)\right)$$
$$=\frac{\varrho^2(e_2-e_1)^{1+\varrho}}{2}\int_0^1\left(\nu-\frac{1}{2}\right)(\mathscr{A}'((1-\nu)e_1+\nu e_2)-\mathscr{A}'(\nu e_1+(1-\nu)e_2))\,d\nu$$
$$+\frac{(1-\varrho)(e_2-e_1)^{2-\varrho}}{4}\int_0^1\left(\nu^{2-2\varrho}-\frac{1}{2}\right)(\mathscr{A}''((1-\nu)e_1+\nu e_2)-\mathscr{A}''(\nu e_1+(1-\nu)e_2))\,d\nu$$
$$=\frac{\varrho^2(e_2-e_1)^{1+\varrho}}{2}\int_0^1(2\nu-1)\mathscr{A}'((1-\nu)e_1+\nu e_2)\,d\nu$$
$$+\frac{(1-\varrho)(e_2-e_1)^{2-\varrho}}{4}\int_0^1\left(\nu^{2-2\varrho}-(1-\nu)^{2-2\varrho}\right)\mathscr{A}''((1-\nu)e_1+\nu e_2)\,d\nu,$$

  which was provided by Sarıkaya in [17, Lemma 2.2].
- *By choosing $\eta=1$ and $\beta=3$, we obtain*

$$\varrho^2(e_2-e_1)^{\varrho}\left(\frac{\mathscr{A}(e_1)+3\mathscr{A}\left(\frac{2e_1+e_2}{3}\right)+3\mathscr{A}\left(\frac{e_1+2e_2}{3}\right)+\mathscr{A}(e_2)}{8}\right)$$
$$+\frac{1-\varrho}{2(e_2-e_1)^{\varrho-1}}\left(\frac{\mathscr{A}'(e_1)+3\mathscr{A}'\left(\frac{2e_1+e_2}{3}\right)+3\mathscr{A}'\left(\frac{e_1+2e_2}{3}\right)+\mathscr{A}'(e_2)}{8}\right)$$
$$-\frac{\Gamma(1-\varrho)}{2(e_2-e_1)^{1-\varrho}}\left({}^{PC}_{e_2^-}\mathbf{D}^{\varrho}_{e_1}\mathscr{A}(e_1)+{}^{PC}_{e_1^+}\mathbf{D}^{\varrho}_{e_2}\mathscr{A}(e_2)\right)$$

$$= \frac{\varrho^2 (\mathrm{e}_2 - \mathrm{e}_1)^{1+\varrho}}{2} \left(\int_0^{\frac{1}{3}} \left(\nu - \frac{1}{8} \right) (\mathscr{A}' ((1-\nu) \mathrm{e}_1 + \nu \mathrm{e}_2) - \mathscr{A}' (\nu \mathrm{e}_1 + (1-\nu) \mathrm{e}_2)) \mathrm{d}\nu \right.$$

$$+ \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\nu - \frac{1}{2} \right) (\mathscr{A}' ((1-\nu) \mathrm{e}_1 + \nu \mathrm{e}_2) - \mathscr{A}' (\nu \mathrm{e}_1 + (1-\nu) \mathrm{e}_2)) \mathrm{d}\nu$$

$$\left. + \int_{\frac{2}{3}}^{1} \left(\nu - \frac{7}{8} \right) (\mathscr{A}' ((1-\nu) \mathrm{e}_1 + \nu \mathrm{e}_2) - \mathscr{A}' (\nu \mathrm{e}_1 + (1-\nu) \mathrm{e}_2)) \mathrm{d}\nu \right)$$

$$+ \frac{(1-\varrho)(\mathrm{e}_2 - \mathrm{e}_1)^{2-\varrho}}{4} \left(\int_0^{\frac{1}{3}} \left(\nu^{2-2\varrho} - \frac{1}{8} \right) (\mathscr{A}'' ((1-\nu) \mathrm{e}_1 + \nu \mathrm{e}_2) - \mathscr{A}'' (\nu \mathrm{e}_1 + (1-\nu) \mathrm{e}_2)) \mathrm{d}\nu \right.$$

$$+ \int_{\frac{1}{3}}^{\frac{2}{3}} \left(\nu^{2-2\varrho} - \frac{1}{2} \right) (\mathscr{A}'' ((1-\nu) \mathrm{e}_1 + \nu \mathrm{e}_2) - \mathscr{A}'' (\nu \mathrm{e}_1 + (1-\nu) \mathrm{e}_2)) \mathrm{d}\nu$$

$$\left. + \int_{\frac{2}{3}}^{1} \left(\nu^{2-2\varrho} - \frac{7}{8} \right) (\mathscr{A}'' ((1-\nu) \mathrm{e}_1 + \nu \mathrm{e}_2) - \mathscr{A}'' (\nu \mathrm{e}_1 + (1-\nu) \mathrm{e}_2)) \mathrm{d}\nu \right).$$

Multiplying the above equality by $(\mathrm{e}_2 - \mathrm{e}_1)^{1-2\varrho}$*, we obtain Lemma 2.1 from [20].*

3. Parametrized Newton-type inequalities via proportional Caputo-hybrid operators

This section presents the main outcomes in the form of a parametrized Newton-type inequalities via s-convexity for proportional Caputo-hybrid operators.

Theorem 3.1. *Let* $\mathscr{A} : I \to \mathbb{R}$ *be a twice differentiable function on* I°*, where* $\mathrm{e}_1, \mathrm{e}_2 \in I^\circ$ *(the interior of I) satisfies* $0 < \mathrm{e}_1 < \mathrm{e}_2$ *and let* $\mathscr{A}, \mathscr{A}', \mathscr{A}'' \in L[\mathrm{e}_1, \mathrm{e}_2]$*. If* $|\mathscr{A}'|$ *and* $|\mathscr{A}''|$ *are s-convex on* $[\mathrm{e}_1, \mathrm{e}_2]$*, then the following inequality holds:*

$$\left| \varrho^2 (\mathrm{e}_2 - \mathrm{e}_1)^{\varrho} \left(\frac{\eta \mathscr{A}(\mathrm{e}_1) + \beta \mathscr{A}\left(\frac{2\mathrm{e}_1 + \mathrm{e}_2}{3}\right) + \beta \mathscr{A}\left(\frac{\mathrm{e}_1 + 2\mathrm{e}_2}{3}\right) + \eta \mathscr{A}(\mathrm{e}_2)}{2(\eta + \beta)} \right) \right.$$

$$+ \frac{1-\varrho}{2(\mathrm{e}_2 - \mathrm{e}_1)^{\varrho-1}} \left(\frac{\eta \mathscr{A}'(\mathrm{e}_1) + \beta \mathscr{A}'\left(\frac{2\mathrm{e}_1 + \mathrm{e}_2}{3}\right) + \beta \mathscr{A}'\left(\frac{\mathrm{e}_1 + 2\mathrm{e}_2}{3}\right) + \eta \mathscr{A}'(\mathrm{e}_2)}{2(\eta + \beta)} \right)$$

$$\left. - \frac{\Gamma(1-\varrho)}{2(\mathrm{e}_2 - \mathrm{e}_1)^{1-\varrho}} \left({}^{PC}_{\mathrm{e}_2^-}\mathbf{D}^{\varrho}_{\mathrm{e}_1} \mathscr{A}(\mathrm{e}_1) + {}^{PC}_{\mathrm{e}_1^+}\mathbf{D}^{\varrho}_{\mathrm{e}_2} \mathscr{A}(\mathrm{e}_2) \right) \right|$$

$$\leq \frac{\varrho^2 (e_2 - e_1)^{1+\varrho}}{2} (2C_1(s,\eta,\beta) + C_2(s,\eta,\beta)) (|\mathscr{A}'(e_1)| + |\mathscr{A}'(e_2)|) + \frac{(1-\varrho)(e_2 - e_1)^{2-\varrho}}{4} \left(\sum_{j=1}^{3} \mathcal{E}_j(\varrho, s, \eta, \beta)\right) (|\mathscr{A}''(e_1)| + |\mathscr{A}''(e_2)|),$$

where $C_1(s,\eta,\beta)$, $C_2(s,\eta,\beta)$, and $\mathcal{E}_j(\varrho, s,\eta,\beta)$ for $j = 1$ to 3, are expressed as

$$C_1(s,\eta,\beta) = \int_0^{\frac{1}{3}} \left|\nu - \frac{\eta}{2(\eta+\beta)}\right| (\nu^s + (1-\nu)^s)\,d\nu = \int_{\frac{2}{3}}^{1} \left|\nu - \frac{\eta+2\beta}{2(\eta+\beta)}\right| (\nu^s + (1-\nu)^s)\,d\nu \tag{3.1}$$

$$= \frac{1}{(s+1)(s+2)} \begin{cases} 2\left(\frac{\eta}{2(\beta+\eta)}\right)^{s+2} + 2\left(\frac{2\beta+\eta}{2(\beta+\eta)}\right)^{s+2} + \frac{\eta(s+2) - 2(\beta+\eta)}{2(\beta+\eta)} \\ + \frac{2(\beta+\eta)(s+1) - 3\eta(s+2)}{6(\beta+\eta)}\left(\frac{1}{3}\right)^{s+1} \\ - \frac{(2\beta-\eta)(s+2) + 4(\beta+\eta)}{6(\beta+\eta)}\left(\frac{2}{3}\right)^{s+1}, & \text{if } \eta < 2\beta, \\ \frac{3\eta(s+2) - 2(\beta+\eta)(s+1)}{6(\beta+\eta)}\left(\frac{1}{3}\right)^{s+1} + \frac{(2\beta-\eta)(s+2) + 4(\beta+\eta)}{6(\beta+\eta)}\left(\frac{2}{3}\right)^{s+1} \\ + \frac{\eta(s+2) - 2(\beta+\eta)}{2(\beta+\eta)}, & \text{if } \eta \geq 2\beta, \end{cases}$$

$$C_2(s,\eta,\beta) = \frac{1}{(s+1)(s+2)} \left[\left(\frac{1}{2}\right)^s - (s+4)\left(\frac{1}{3}\right)^{s+2} + \frac{s-2}{2}\left(\frac{2}{3}\right)^{s+2}\right], \tag{3.2}$$

$$\mathcal{E}_1(\varrho, s, \eta, \beta) = \int_0^{\frac{1}{3}} \left|\nu^{2-2\varrho} - \frac{\eta}{2(\eta+\beta)}\right| (\nu^s + (1-\nu)^s)\,d\nu \tag{3.3}$$

$$= \begin{cases} \frac{4-4\varrho}{(s+1)(3-2\varrho+s)}\left(\frac{\eta}{2(\beta+\eta)}\right)^{1+\frac{s+1}{2-2\varrho}} - \frac{\eta}{(\beta+\eta)(s+1)}\left(1 - \left(\frac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2-2\varrho}}\right)^{s+1} \\ + \frac{\eta}{2(\beta+\eta)(s+1)} + \frac{1}{3-2\varrho+s}\left(\frac{1}{3}\right)^{3-2\varrho+s} + \frac{\eta}{2(\beta+\eta)(s+1)}\left(\frac{2^{s+1}-1}{3^{s+1}}\right) \\ + \mathcal{B}_{\frac{1}{3}}(3-2\varrho, s+1) - 2\mathcal{B}_{\left(\frac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho, s+1), & \text{if } \left(\frac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2-2\varrho}} < \frac{1}{3}, \\ \frac{\eta}{2(\beta+\eta)(s+1)}\left(\frac{1-2^{s+1}}{3^{s+1}}\right) - \frac{1}{3-2\varrho+s}\left(\frac{1}{3}\right)^{3-2\varrho+s} \\ + \frac{\eta}{2(\beta+\eta)(s+1)} - \mathcal{B}_{\frac{1}{3}}(3-2\varrho, s+1), & \text{if } \left(\frac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2-2\varrho}} \geq \frac{1}{3}, \end{cases}$$

$$\mathcal{E}_2(\varrho, s) = \int_{\frac{1}{3}}^{\frac{2}{3}} \left| \nu^{2-2\varrho} - \frac{1}{2} \right| (\nu^s + (1-\nu)^s) \, d\nu \tag{3.4}$$

$$= \begin{cases} \dfrac{2^{3-2\varrho+s}-1}{3^{3-2\varrho+s}(3-2\varrho+s)} - \dfrac{2^{s+1}-1}{3^{s+1}(s+1)} \\ +\mathcal{B}_{\frac{2}{3}}(3-2\varrho, s+1) - \mathcal{B}_{\frac{1}{3}}(3-2\varrho, s+1), & \text{if } \left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}} < \frac{1}{3}, \\ \dfrac{2-2\varrho}{(s+1)(3-2\varrho+s)} \left(\dfrac{1}{2}\right)^{\frac{s+1}{2-2\varrho}} - \dfrac{1}{s+1}\left(1-\left(\dfrac{1}{2}\right)^{\frac{1}{2-2\varrho}}\right)^{s+1} + \mathcal{B}_{\frac{1}{3}}(3-2\varrho, s+1) \\ +\dfrac{1+2^{3-2\varrho+s}}{3^{3-2\varrho+s}(3-2\varrho+s)} + \mathcal{B}_{\frac{2}{3}}(3-2\varrho, s+1) - 2\mathcal{B}_{\left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho, s+1), & \text{if } \frac{1}{3} \le \left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}} \le \frac{2}{3}, \\ \dfrac{2^{s+1}-1}{3^{s+1}(s+1)} + \dfrac{1-2^{3-2\varrho+s}}{3^{3-2\varrho+s}(3-2\varrho+s)} - \mathcal{B}_{\frac{2}{3}}(3-2\varrho, s+1) + \mathcal{B}_{\frac{1}{3}}(3-2\varrho, s+1), & \text{if } \left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}} > \frac{2}{3}, \end{cases}$$

and

$$\mathcal{E}_3(\varrho, s, \eta, \beta) = \int_{\frac{2}{3}}^{1} \left| \nu^{2-2\varrho} - \frac{\eta+2\beta}{2(\eta+\beta)} \right| (\nu^s + (1-\nu)^s) \, d\nu \tag{3.5}$$

$$= \begin{cases} \dfrac{1-\left(\frac{2}{3}\right)^{3-2\varrho+s}}{3-2\varrho+s} - \dfrac{\eta+2\beta}{2(\eta+\beta)(s+1)}\left[1-\dfrac{2^{s+1}-1}{3^{s+1}}\right] \\ +\mathcal{B}(3-2\varrho, s+1) - \mathcal{B}_{\frac{2}{3}}(3-2\varrho, s+1), & \text{if } \left(\dfrac{\eta+2\beta}{2(\eta+\beta)}\right)^{\frac{1}{2-2\varrho}} < \dfrac{2}{3}, \\ \dfrac{4-4\varrho}{(s+1)(3-2\varrho+s)}\left(\dfrac{\eta+2\beta}{2(\eta+\beta)}\right)^{1+\frac{s+1}{2-2\varrho}} + \dfrac{1+\left(\frac{2}{3}\right)^{3-2\varrho+s}}{3-2\varrho+s} + \mathcal{B}(3-2\varrho, s+1) \\ +\mathcal{B}_{\frac{2}{3}}(3-2\varrho, s+1) - 2\mathcal{B}_{\left(\frac{\eta+2\beta}{2(\eta+\beta)}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho, s+1) - \dfrac{\eta+2\beta}{2(\eta+\beta)(s+1)}\left[1+\dfrac{2^{s+1}-1}{3^{s+1}}\right] \\ -\dfrac{\eta+2\beta}{(\eta+\beta)(s+1)}\left[1-\left(\dfrac{\eta+2\beta}{2(\eta+\beta)}\right)^{\frac{1}{2-2\varrho}}\right]^{s+1}, & \text{if } \dfrac{2}{3} \le \left(\dfrac{\eta+2\beta}{2(\eta+\beta)}\right)^{\frac{1}{2-2\varrho}} \le 1. \end{cases}$$

Proof. From Lemma 2.1, absolute value, and s-convexity of $|\mathscr{A}'|$ and $|\mathscr{A}''|$, we have

$$\Bigg| \varrho^2 (e_2 - e_1)^{\varrho} \left(\frac{\eta\mathscr{A}(e_1) + \beta\mathscr{A}\left(\frac{2e_1+e_2}{3}\right) + \beta\mathscr{A}\left(\frac{e_1+2e_2}{3}\right) + \eta\mathscr{A}(e_2)}{2(\eta+\beta)} \right)$$

$$+ \frac{1-\varrho}{2(e_2-e_1)^{\varrho-1}} \left(\frac{\eta\mathscr{A}'(e_1) + \beta\mathscr{A}'\left(\frac{2e_1+e_2}{3}\right) + \beta\mathscr{A}'\left(\frac{e_1+2e_2}{3}\right) + \eta\mathscr{A}'(e_2)}{2(\eta+\beta)} \right)$$

$$
\begin{aligned}
&-\frac{\Gamma(1-\varrho)}{2(\mathrm{e}_2-\mathrm{e}_1)^{1-\varrho}}\left({}^{PC}_{\mathrm{e}_2^-}\mathbf{D}^{\varrho}_{\mathrm{e}_1}\mathscr{A}(\mathrm{e}_1)+{}^{PC}_{\mathrm{e}_1^+}\mathbf{D}^{\varrho}_{\mathrm{e}_2}\mathscr{A}(\mathrm{e}_2)\right)\Bigg| \\
&\le \frac{\varrho^2(\mathrm{e}_2-\mathrm{e}_1)^{1+\varrho}}{2}\left(\int_0^{\frac{1}{3}}\left|\nu-\frac{\eta}{2(\eta+\beta)}\right|\left(|\mathscr{A}'((1-\nu)\mathrm{e}_1+\nu\mathrm{e}_2)|+|\mathscr{A}'(\nu\mathrm{e}_1+(1-\nu)\mathrm{e}_2)|\right)\mathrm{d}\nu\right. \\
&+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\nu-\frac{1}{2}\right|\left(|\mathscr{A}'((1-\nu)\mathrm{e}_1+\nu\mathrm{e}_2)|+|\mathscr{A}'(\nu\mathrm{e}_1+(1-\nu)\mathrm{e}_2)|\right)\mathrm{d}\nu \\
&+\left.\int_{\frac{2}{3}}^{1}\left|\nu-\frac{\eta+2\beta}{2(\eta+\beta)}\right|\left(|\mathscr{A}'((1-\nu)\mathrm{e}_1+\nu\mathrm{e}_2)|+|\mathscr{A}'(\nu\mathrm{e}_1+(1-\nu)\mathrm{e}_2)|\right)\mathrm{d}\nu\right) \\
&+\frac{(1-\varrho)(\mathrm{e}_2-\mathrm{e}_1)^{2-\varrho}}{4}\left(\int_0^{\frac{1}{3}}\left|\nu^{2-2\varrho}-\frac{\eta}{2(\eta+\beta)}\right|\left(|\mathscr{A}''((1-\nu)\mathrm{e}_1+\nu\mathrm{e}_2)|+|\mathscr{A}''(\nu\mathrm{e}_1+(1-\nu)\mathrm{e}_2)|\right)\mathrm{d}\nu\right. \\
&+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\nu^{2-2\varrho}-\frac{1}{2}\right|\left(|\mathscr{A}''((1-\nu)\mathrm{e}_1+\nu\mathrm{e}_2)|+|\mathscr{A}''(\nu\mathrm{e}_1+(1-\nu)\mathrm{e}_2)|\right)\mathrm{d}\nu \\
&+\left.\int_{\frac{2}{3}}^{1}\left|\nu^{2-2\varrho}-\frac{\eta+2\beta}{2(\eta+\beta)}\right|\left(|\mathscr{A}''((1-\nu)\mathrm{e}_1+\nu\mathrm{e}_2)|+|\mathscr{A}''(\nu\mathrm{e}_1+(1-\nu)\mathrm{e}_2)|\right)\mathrm{d}\nu\right) \\
&\le \frac{\varrho^2(\mathrm{e}_2-\mathrm{e}_1)^{1+\varrho}}{2}\left(\int_0^{\frac{1}{3}}\left|\nu-\frac{\eta}{2(\eta+\beta)}\right|(\nu^s+(1-\nu)^s)\left(|\mathscr{A}'(\mathrm{e}_1)|+|\mathscr{A}'(\mathrm{e}_2)|\right)\mathrm{d}\nu\right. \\
&+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\nu-\frac{1}{2}\right|(\nu^s+(1-\nu)^s)\left(|\mathscr{A}'(\mathrm{e}_1)|+|\mathscr{A}'(\mathrm{e}_2)|\right)\mathrm{d}\nu \\
&+\left.\int_{\frac{2}{3}}^{1}\left|\nu-\frac{\eta+2\beta}{2(\eta+\beta)}\right|(\nu^s+(1-\nu)^s)\left(|\mathscr{A}'(\mathrm{e}_1)|+|\mathscr{A}'(\mathrm{e}_2)|\right)\mathrm{d}\nu\right) \\
&+\frac{(1-\varrho)(\mathrm{e}_2-\mathrm{e}_1)^{2-\varrho}}{4}\left(\int_0^{\frac{1}{3}}\left|\nu^{2-2\varrho}-\frac{\eta}{2(\eta+\beta)}\right|(\nu^s+(1-\nu)^s)\left(|\mathscr{A}''(\mathrm{e}_1)|+|\mathscr{A}''(\mathrm{e}_2)|\right)\mathrm{d}\nu\right.
\end{aligned}
$$

$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\nu^{2-2\varrho}-\frac{1}{2}\right|\left(\nu^{s}+(1-\nu)^{s}\right)\left(\left|\mathscr{A}''\left(\mathrm{e}_1\right)\right|+\left|\mathscr{A}''\left(\mathrm{e}_2\right)\right|\right)\mathrm{d}\nu$$

$$\left.+\int_{\frac{2}{3}}^{1}\left|\nu^{2-2\varrho}-\frac{\eta+2\beta}{2\left(\eta+\beta\right)}\right|\left(\nu^{s}+(1-\nu)^{s}\right)\left(\left|\mathscr{A}''\left(\mathrm{e}_1\right)\right|+\left|\mathscr{A}''\left(\mathrm{e}_2\right)\right|\right)\mathrm{d}\nu\right)$$

$$=\frac{\varrho^2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{1+\varrho}}{2}\left(2C_1\left(s,\eta,\beta\right)+C_2\left(s,\eta,\beta\right)\right)\left(\left|\mathscr{A}'\left(\mathrm{e}_1\right)\right|+\left|\mathscr{A}'\left(\mathrm{e}_2\right)\right|\right)$$

$$+\frac{\left(1-\varrho\right)\left(\mathrm{e}_2-\mathrm{e}_1\right)^{2-\varrho}}{4}\left(\sum_{j=1}^{3}\mathcal{E}_j\left(\varrho,s,\eta,\beta\right)\right)\left(\left|\mathscr{A}''\left(\mathrm{e}_1\right)\right|+\left|\mathscr{A}''\left(\mathrm{e}_2\right)\right|\right),$$

where we have used (3.1)–(3.5).

Corollary 3.1. *By setting $s=1$, Theorem 3.1 yields*

$$\left|\varrho^2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{\varrho}\left(\frac{\eta\mathscr{A}\left(\mathrm{e}_1\right)+\beta\mathscr{A}\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right)+\beta\mathscr{A}\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right)+\eta\mathscr{A}\left(\mathrm{e}_2\right)}{2\left(\eta+\beta\right)}\right)\right.$$

$$+\frac{1-\varrho}{2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{\varrho-1}}\left(\frac{\eta\mathscr{A}'\left(\mathrm{e}_1\right)+\beta\mathscr{A}'\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right)+\beta\mathscr{A}'\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right)+\eta\mathscr{A}'\left(\mathrm{e}_2\right)}{2\left(\eta+\beta\right)}\right)$$

$$\left.-\frac{\Gamma\left(1-\varrho\right)}{2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{1-\varrho}}\left({}^{PC}_{\mathrm{e}_2^-}\mathbf{D}^{\varrho}_{\mathrm{e}_1}\mathscr{A}\left(\mathrm{e}_1\right)+{}^{PC}_{\mathrm{e}_1^+}\mathbf{D}^{\varrho}_{\mathrm{e}_2}\mathscr{A}\left(\mathrm{e}_2\right)\right)\right|$$

$$\leq\frac{\varrho^2\left(\mathrm{e}_2-\mathrm{e}_1\right)^{1+\varrho}}{2}\left(2C_1\left(1,\eta,\beta\right)+C_2\left(1,\eta,\beta\right)\right)\left(\left|\mathscr{A}'\left(\mathrm{e}_1\right)\right|+\left|\mathscr{A}'\left(\mathrm{e}_2\right)\right|\right)$$

$$+\frac{\left(1-\varrho\right)\left(\mathrm{e}_2-\mathrm{e}_1\right)^{2-\varrho}}{4}\left(\sum_{j=1}^{3}\mathcal{E}_j\left(\varrho,1,\eta,\beta\right)\right)\left(\left|\mathscr{A}''\left(\mathrm{e}_1\right)\right|+\left|\mathscr{A}''\left(\mathrm{e}_2\right)\right|\right),$$

where $C_1(1,\eta,\beta)$, $C_2(\varrho,1,\eta,\beta)$, and $\mathcal{E}_j(\varrho,1,\eta,\beta)$ for $j=1$ to 3, are expressed as

$$C_1(1,\eta,\beta)=\begin{cases}\dfrac{\eta^3+(2\beta+\eta)^3}{24\left(\beta+\eta\right)^3}-\dfrac{5\beta-\eta}{18\left(\beta+\eta\right)}, & \text{if}\quad \eta<2\beta,\\[2mm] \dfrac{2\eta-\beta}{18\left(\beta+\eta\right)}, & \text{if}\quad \eta\geq 2\beta,\end{cases}$$

$$C_2(1,\eta,\beta)=\frac{1}{36},$$

$$\mathcal{E}_1(\varrho,1,\eta,\beta)$$
$$=\begin{cases}\frac{1-\varrho}{2-\varrho}\left(\frac{\eta}{2(\beta+\eta)}\right)^{\frac{2-\varrho}{1-\varrho}}-\frac{\eta}{2(\beta+\eta)}\left(1-\left(\frac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2-2\varrho}}\right)^2+\frac{\eta}{3(\beta+\eta)} & \\ +\frac{1}{4-2\varrho}\left(\frac{1}{3}\right)^{4-2\varrho}+\mathcal{B}_{\frac{1}{3}}(3-2\varrho,2)-2\mathcal{B}_{\left(\frac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho,2), & \text{if } \left(\frac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2-2\varrho}}<\frac{1}{3},\\ \frac{\eta}{6(\beta+\eta)}-\frac{1}{4-2\varrho}\left(\frac{1}{3}\right)^{4-2\varrho}-\mathcal{B}_{\frac{1}{3}}(3-2\varrho,2), & \text{if } \left(\frac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2-2\varrho}}\geq\frac{1}{3},\end{cases}$$

$$\mathcal{E}_2(\varrho,1)=\begin{cases}\frac{2^{4-2\varrho}-1}{3^{4-2\varrho}(4-2\varrho)}-\frac{1}{6}+\mathcal{B}_{\frac{2}{3}}(3-2\varrho,2)-\mathcal{B}_{\frac{1}{3}}(3-2\varrho,2), & \text{if } \left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}<\frac{1}{3},\\ \frac{1-\varrho}{2(2-\varrho)}\left(\frac{1}{2}\right)^{\frac{1}{1-\varrho}}-\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}\right)^2+\mathcal{B}_{\frac{1}{3}}(3-2\varrho,2) & \\ +\frac{1+2^{4-2\varrho}}{3^{4-2\varrho}(4-2\varrho)}+\mathcal{B}_{\frac{2}{3}}(3-2\varrho,2)-2\mathcal{B}_{\left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho,2), & \text{if } \frac{1}{3}\leq\left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}\leq\frac{2}{3},\\ \frac{1}{6}+\frac{1-2^{4-2\varrho}}{3^{4-2\varrho}(4-2\varrho)}-\mathcal{B}_{\frac{2}{3}}(3-2\varrho,2)+\mathcal{B}_{\frac{1}{3}}(3-2\varrho,2), & \text{if } \left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}>\frac{2}{3},\end{cases} \tag{3.6}$$

and

$$\mathcal{E}_3(\varrho,1,\eta,\beta)=\begin{cases}\frac{1-\left(\frac{2}{3}\right)^{4-2\varrho}}{4-2\varrho}+\mathcal{B}(3-2\varrho,2)-\mathcal{B}_{\frac{2}{3}}(3-2\varrho,2)-\frac{\eta+2\beta}{6(\eta+\beta)}, & \text{if } \left(\frac{\eta+2\beta}{2(\beta+\eta)}\right)^{\frac{1}{2-2\varrho}}<\frac{2}{3},\\ \frac{4-4\varrho}{2(4-2\varrho)}\left(\frac{\eta+2\beta}{2(\eta+\beta)}\right)^{1+\frac{2}{2-2\varrho}}+\frac{1+\left(\frac{2}{3}\right)^{4-2\varrho}}{4-2\varrho} & \\ +\mathcal{B}(3-2\varrho,2)+\mathcal{B}_{\frac{2}{3}}(3-2\varrho,2)-2\mathcal{B}_{\left(\frac{\eta+2\beta}{2(\eta+\beta)}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho,2) & \\ -\frac{\eta+2\beta}{3(\eta+\beta)}-\frac{\eta+2\beta}{2(\eta+\beta)}\left[1-\left(\frac{\eta+2\beta}{2(\eta+\beta)}\right)^{\frac{1}{2-2\varrho}}\right]^2, & \text{if } \frac{2}{3}\leq\left(\frac{\eta+2\beta}{2(\beta+\eta)}\right)^{\frac{1}{2-2\varrho}}\leq 1.\end{cases}$$

Corollary 3.2. *If we attempt to tend* ϱ *to* 1*, Theorem 3.1 gives*

$$\left|\frac{\eta\mathscr{A}(\mathfrak{e}_1)+\beta\mathscr{A}\left(\frac{2\mathfrak{e}_1+\mathfrak{e}_2}{3}\right)+\beta\mathscr{A}\left(\frac{\mathfrak{e}_1+2\mathfrak{e}_2}{3}\right)+\eta\mathscr{A}(\mathfrak{e}_2)}{2(\eta+\beta)}-\frac{1}{\mathfrak{e}_2-\mathfrak{e}_1}\int_{\mathfrak{e}_1}^{\mathfrak{e}_2}\mathscr{A}(\mathfrak{w})\,d\mathfrak{w}\right|$$
$$\leq\frac{\mathfrak{e}_2-\mathfrak{e}_1}{2}\left(2C_1(s,\eta,\beta)+C_2(s,\eta,\beta)\right)\left(|\mathscr{A}'(\mathfrak{e}_1)|+|\mathscr{A}'(\mathfrak{e}_2)|\right),$$

where $C_1(s,\eta,\beta)$ *and* $C_2(s,\eta,\beta)$ *are defined as in* (3.1) *and* (3.2).

Moreover, by setting $s=1$*, we get*

$$\left|\frac{\eta\mathscr{A}(\mathrm{e}_1)+\beta\mathscr{A}\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right)+\beta\mathscr{A}\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right)+\eta\mathscr{A}(\mathrm{e}_2)}{2(\eta+\beta)}-\frac{1}{\mathrm{e}_2-\mathrm{e}_1}\int_{\mathrm{e}_1}^{\mathrm{e}_2}\mathscr{A}(\mathfrak{w})\,d\mathfrak{w}\right|$$
$$\leq\frac{\mathrm{e}_2-\mathrm{e}_1}{2}\left(2C_1(1,\eta,\beta)+C_2(1,\eta,\beta)\right)\left(|\mathscr{A}'(\mathrm{e}_1)|+|\mathscr{A}'(\mathrm{e}_2)|\right),$$

where

$$C_1(1,\eta,\beta)=\begin{cases}\dfrac{\eta^3+(2\beta+\eta)^3}{24(\beta+\eta)^3}+\dfrac{\eta-2\beta}{12(\beta+\eta)}+\dfrac{4\beta-5\eta}{324(\beta+\eta)}-\dfrac{10\beta+\eta}{81(\beta+\eta)}, & \text{if}\quad \eta<2\beta,\\ \dfrac{5\eta-4\beta}{324(\beta+\eta)}+\dfrac{10\beta+\eta}{81(\beta+\eta)}+\dfrac{\eta-2\beta}{12(\beta+\eta)}, & \text{if}\quad \eta\geq 2\beta,\end{cases}$$

and

$$C_2(1,\eta,\beta)=\frac{1}{36}.$$

Corollary 3.3. *If we attempt to take* $\varrho=0$*, Theorem 3.1 yields*

$$\left|\frac{\eta\mathscr{A}'(\mathrm{e}_1)+\beta\mathscr{A}'\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right)+\beta\mathscr{A}'\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right)+\eta\mathscr{A}'(\mathrm{e}_2)}{2(\eta+\beta)}-\frac{1}{\mathrm{e}_2-\mathrm{e}_1}\left(\mathscr{A}(\mathrm{e}_2)-\mathscr{A}(\mathrm{e}_1)\right)\right|$$
$$\leq\frac{\mathrm{e}_2-\mathrm{e}_1}{2}\left(\sum_{j=1}^{3}\mathcal{E}_j(0,s,\eta,\beta)\right)\left(|\mathscr{A}''(\mathrm{e}_1)|+|\mathscr{A}''(\mathrm{e}_2)|\right),$$

where

$$\mathcal{E}_1(0,s,\eta,\beta)$$
$$=\begin{cases}\dfrac{4}{(s+1)(s+3)}\left(\dfrac{\eta}{2(\beta+\eta)}\right)^{1+\frac{s+1}{2}}-\dfrac{\eta}{(\beta+\eta)(s+1)}\left(1-\left(\dfrac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2}}\right)^{s+1}\\ +\dfrac{\eta}{2(\beta+\eta)(s+1)}+\dfrac{1}{s+3}\left(\dfrac{1}{3}\right)^{3+s}+\dfrac{\eta}{2(\beta+\eta)(s+1)}\left(\dfrac{2^{s+1}-1}{3^{s+1}}\right)\\ +\mathcal{B}_{\frac{1}{3}}(3,s+1)-2\mathcal{B}_{\left(\frac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2}}}(3,s+1), & \text{if}\left(\dfrac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2}}<\dfrac{1}{3},\\ \dfrac{\eta}{2(\beta+\eta)(s+1)}\left(\dfrac{1-2^{s+1}}{3^{s+1}}\right)-\dfrac{1}{s+3}\left(\dfrac{1}{3}\right)^{s+3}\\ +\dfrac{\eta}{2(\beta+\eta)(s+1)}-\mathcal{B}_{\frac{1}{3}}(3,s+1), & \text{if}\left(\dfrac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2}}\geq\dfrac{1}{3},\end{cases}$$

$$\mathcal{E}_2(0,s)=\frac{2^{s+1}-1}{3^{s+1}(s+1)}+\frac{1-2^{s+3}}{3^{s+3}(s+3)}-\mathcal{B}_{\frac{2}{3}}(3,s+1)+\mathcal{B}_{\frac{1}{3}}(3,s+1),\tag{3.7}$$

$$
\mathcal{E}_3(0,s,\eta,\beta) = \begin{cases} \dfrac{2^{s+3}-1}{3^{s+3}(s+3)} - \dfrac{\eta+2\beta}{(\beta+\eta)(s+1)}\left(\dfrac{2^{s+1}-1}{3^{s+1}}\right) \\ +\mathcal{B}_{\frac{2}{3}}(3,s+1) - \mathcal{B}_{\frac{1}{3}}(3,s+1), & if \left(\dfrac{\eta+2\beta}{2(\beta+\eta)}\right)^{\frac{1}{2}} < \dfrac{2}{3}, \\ \dfrac{4}{(s+1)(s+3)}\left(\dfrac{\eta+2\beta}{2(\beta+\eta)}\right)^{1+\frac{s+1}{2}} - \dfrac{\eta+2\beta}{(\beta+\eta)(s+1)}\left(1-\left(\dfrac{\eta+2\beta}{2(\beta+\eta)}\right)^{\frac{1}{2}}\right)^{s+1} \\ +\dfrac{\eta+2\beta}{(\beta+\eta)(s+1)}\left(\dfrac{2^{s+1}-1}{3^{s+1}}\right) + \dfrac{1+2^{s+3}}{3^{s+3}(s+3)} + \mathcal{B}_{\frac{1}{3}}(3,s+1) \\ +\mathcal{B}_{\frac{2}{3}}(3,s+1) - 2\mathcal{B}_{\left(\frac{\eta+2\beta}{2(\beta+\eta)}\right)^{\frac{1}{2}}}(3,s+1), & if \dfrac{2}{3} \le \left(\dfrac{\eta+2\beta}{2(\beta+\eta)}\right)^{\frac{1}{2}} \le 1. \end{cases}
$$

Moreover, by choosing $s = 1$, we get

$$
\left|\frac{\eta\mathscr{A}'(\mathrm{e}_1) + \beta\mathscr{A}'\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right) + \beta\mathscr{A}'\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right) + \eta\mathscr{A}'(\mathrm{e}_2)}{2(\eta+\beta)} - \frac{1}{\mathrm{e}_2-\mathrm{e}_1}(\mathscr{A}(\mathrm{e}_2) - \mathscr{A}(\mathrm{e}_1))\right|
$$
$$
\le \frac{\mathrm{e}_2-\mathrm{e}_1}{2}\left(\sum_{j=1}^{3}\mathcal{E}_j(0,1,\eta,\beta)\right)(|\mathscr{A}''(\mathrm{e}_1)| + |\mathscr{A}''(\mathrm{e}_2)|),
$$

where

$$
\mathcal{E}_1(0,1,\eta,\beta) = \begin{cases} \dfrac{1}{81} - \dfrac{1}{3}\left(\dfrac{\eta}{2(\beta+\eta)}\right) + \dfrac{4}{3}\left(\dfrac{\eta}{2(\beta+\eta)}\right)^{\frac{3}{2}} - \dfrac{3}{8}\left(\dfrac{\eta}{2(\beta+\eta)}\right)^{2}, & if \quad \left(\dfrac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2}} < \dfrac{1}{3}, \\ \dfrac{1}{3}\left(\dfrac{\eta}{2(\beta+\eta)}\right) - \dfrac{1}{81}, & if \quad \left(\dfrac{\eta}{2(\beta+\eta)}\right)^{\frac{1}{2}} \ge \dfrac{1}{3}, \end{cases}
$$
$$
\mathcal{E}_2(0,1) = \frac{13}{162},
$$
$$
\mathcal{E}_3(0,1,\eta,\beta) = \begin{cases} \dfrac{7}{81} - \dfrac{1}{3}\left(\dfrac{\eta+2\beta}{2(\beta+\eta)}\right), & if \quad \left(\dfrac{\eta+2\beta}{2(\beta+\eta)}\right)^{\frac{1}{2}} < \dfrac{2}{3}, \\ \dfrac{1}{15} - \dfrac{2}{3}\left(\dfrac{\eta+2\beta}{2(\beta+\eta)}\right) + \dfrac{4}{3}\left(\dfrac{\eta+2\beta}{2(\beta+\eta)}\right)^{\frac{3}{2}}, & if \quad \dfrac{2}{3} \le \left(\dfrac{\eta+2\beta}{2(\beta+\eta)}\right)^{\frac{1}{2}} \le 1. \end{cases}
$$

4. Specific cases

In this section, by fixing specific values of the parameters, we discuss some specific cases.

Corollary 4.1. *If we attempt to take $\beta = 0$, Theorem 3.1 yields the following trapezium-type inequality for proportional Caputo-hybrid operators via s-convexity:*

$$\begin{aligned}
&\left|\varrho^2\left(e_2-e_1\right)^{\varrho}\left(\frac{\mathscr{A}\left(e_1\right)+\mathscr{A}\left(e_2\right)}{2}\right)+\frac{1-\varrho}{2\left(e_2-e_1\right)^{\varrho-1}}\left(\frac{\mathscr{A}'\left(e_1\right)+\mathscr{A}'\left(e_2\right)}{2}\right)\right.\\
&\left.-\frac{\Gamma(1-\varrho)}{2\left(e_2-e_1\right)^{1-\varrho}}\left({}^{PC}_{e_2^-}\mathbf{D}{e_1}^{\varrho}\mathscr{A}\left(e_1\right)+{}^{PC}_{e_1^+}\mathbf{D}{e_2}^{\varrho}\mathscr{A}\left(e_2\right)\right)\right|\\
\leq&\ \frac{\varrho^2\left(e_2-e_1\right)^{1+\varrho}}{2}\left(2C_1(s)+C_2(s)\right)\left(\left|\mathscr{A}'\left(e_1\right)\right|+\left|\mathscr{A}'\left(e_2\right)\right|\right)\\
&+\frac{(1-\varrho)\left(e_2-e_1\right)^{2-\varrho}}{4}\left(\sum_{j=1}^{3}\mathcal{E}_j(\varrho,s)\right)\left(\left|\mathscr{A}''\left(e_1\right)\right|+\left|\mathscr{A}''\left(e_2\right)\right|\right),
\end{aligned}$$

where $C_1(s)$, $C_2(s)$, and $\mathcal{E}_j(\varrho,s)$ for $j=1$ to 3 are expressed as

$$C_1(s)=\frac{1}{(s+1)(s+2)}\left[\frac{s+4}{6}\left(\frac{1}{3}\right)^{s+1}+\frac{2-s}{6}\left(\frac{2}{3}\right)^{s+1}+\frac{s}{2}\right], \tag{4.1}$$

$$C_2(s)=\frac{1}{(s+1)(s+2)}\left[\left(\frac{1}{2}\right)^{s}-(s+4)\left(\frac{1}{3}\right)^{s+2}+\frac{s-2}{2}\left(\frac{2}{3}\right)^{s+2}\right], \tag{4.2}$$

$$\mathcal{E}_1(\varrho,s)=\begin{cases}
\frac{4-4\varrho}{(s+1)(3-2\varrho+s)}\left(\frac{1}{2}\right)^{1+\frac{s+1}{2-2\varrho}}-\frac{1}{s+1}\left(1-\left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}\right)^{s+1}\\
+\frac{1}{2(s+1)}+\frac{1}{3-2\varrho+s}\left(\frac{1}{3}\right)^{3-2\varrho+s}+\frac{1}{2(s+1)}\left(\frac{2^{s+1}-1}{3^{s+1}}\right)\\
+\mathcal{B}_{\frac{1}{3}}(3-2\varrho,s+1)-2\mathcal{B}_{\left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho,s+1), & \textit{if } \left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}<\frac{1}{3},\\
\frac{1}{2(s+1)}\left(\frac{1-2^{s+1}}{3^{s+1}}\right)-\frac{1}{3-2\varrho+s}\left(\frac{1}{3}\right)^{3-2\varrho+s}\\
+\frac{1}{2(s+1)}-\mathcal{B}_{\frac{1}{3}}(3-2\varrho,s+1), & \textit{if } \left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}\geq\frac{1}{3},
\end{cases} \tag{4.3}$$

$$\mathcal{E}_3(\varrho,s)=\begin{cases}
\frac{1-\left(\frac{2}{3}\right)^{3-2\varrho+s}}{3-2\varrho+s}-\frac{1}{2(s+1)}\left[1-\frac{2^{s+1}-1}{3^{s+1}}\right]\\
+\mathcal{B}(3-2\varrho,s+1)-\mathcal{B}_{\frac{2}{3}}(3-2\varrho,s+1), & \textit{if } \left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}<\frac{2}{3},\\
\frac{4-4\varrho}{(s+1)(3-2\varrho+s)}\left(\frac{1}{2}\right)^{1+\frac{s+1}{2-2\varrho}}+\frac{1+\left(\frac{2}{3}\right)^{3-2\varrho+s}}{3-2\varrho+s}\\
+\mathcal{B}(3-2\varrho,s+1)+\mathcal{B}_{\frac{2}{3}}(3-2\varrho,s+1)-2\mathcal{B}_{\left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho,s+1)\\
-\frac{1}{2(s+1)}\left[1+\frac{2^{s+1}-1}{3^{s+1}}\right]-\frac{1}{s+1}\left[1-\left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}\right]^{s+1}, & \textit{if } \left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}\geq\frac{2}{3},
\end{cases} \tag{4.4}$$

and $\mathcal{E}_2(\varrho, s)$ *is defined as in* (3.4).

Specifically, by setting $s = 1$, *we get*

$$\begin{aligned}&\left|\varrho^2(\mathrm{e}_2-\mathrm{e}_1)^{\varrho}\left(\frac{\mathscr{A}(\mathrm{e}_1)+\mathscr{A}(\mathrm{e}_2)}{2}\right)+\frac{1-\varrho}{2(\mathrm{e}_2-\mathrm{e}_1)^{\varrho-1}}\left(\frac{\mathscr{A}'(\mathrm{e}_1)+\mathscr{A}'(\mathrm{e}_2)}{2}\right)\right.\\&\left.-\frac{\Gamma(1-\varrho)}{2(\mathrm{e}_2-\mathrm{e}_1)^{1-\varrho}}\left({}^{PC}_{\mathrm{e}_2^-}\mathbf{D}^{\varrho}_{\mathrm{e}_1}\mathscr{A}(\mathrm{e}_1)+{}^{PC}_{\mathrm{e}_1^+}\mathbf{D}^{\varrho}_{\mathrm{e}_2}\mathscr{A}(\mathrm{e}_2)\right)\right|\\&\leq\frac{\varrho^2(\mathrm{e}_2-\mathrm{e}_1)^{1+\varrho}}{8}\left(|\mathscr{A}'(\mathrm{e}_1)|+|\mathscr{A}'(\mathrm{e}_2)|\right)+\frac{(1-\varrho)(\mathrm{e}_2-\mathrm{e}_1)^{2-\varrho}}{4}\left(\sum_{j=1}^{3}\mathcal{E}_j(\varrho,1)\right)\left(|\mathscr{A}''(\mathrm{e}_1)|+|\mathscr{A}''(\mathrm{e}_2)|\right),\end{aligned}$$

where

$$\mathcal{E}_1(\varrho,1)=\begin{cases}\dfrac{1-\varrho}{2-\varrho}\left(\dfrac{1}{2}\right)^{1+\frac{1}{1-\varrho}}-\dfrac{1}{2}\left(1-\left(\dfrac{1}{2}\right)^{\frac{1}{2-2\varrho}}\right)^2+\dfrac{1}{12}\\ +\dfrac{1}{4-2\varrho}\left(\dfrac{1}{3}\right)^{4-2\varrho}+\mathcal{B}_{\frac{1}{3}}(3-2\varrho,2)-2\mathcal{B}_{\left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho,2), & \text{if } \left(\dfrac{1}{2}\right)^{\frac{1}{2-2\varrho}}<\dfrac{1}{3},\\ \dfrac{1}{12}-\dfrac{1}{4-2\varrho}\left(\dfrac{1}{3}\right)^{4-2\varrho}-\mathcal{B}_{\frac{1}{3}}(3-2\varrho,2), & \text{if } \left(\dfrac{1}{2}\right)^{\frac{1}{2-2\varrho}}\geq\dfrac{1}{3},\end{cases}\tag{4.5}$$

$$\mathcal{E}_3(\varrho,1)=\begin{cases}\dfrac{1-\left(\frac{2}{3}\right)^{4-2\varrho}}{4-2\varrho}+\mathcal{B}(3-2\varrho,2)-\mathcal{B}_{\frac{2}{3}}(3-2\varrho,2)-\dfrac{1}{6}, & \text{if } \left(\dfrac{1}{2}\right)^{\frac{1}{2-2\varrho}}<\dfrac{2}{3},\\ \dfrac{4-4\varrho}{2(4-2\varrho)}\left(\dfrac{1}{2}\right)^{1+\dfrac{2}{2-2\varrho}}+\dfrac{1+\left(\frac{2}{3}\right)^{4-2\varrho}}{4-2\varrho}-\dfrac{1}{3}-\dfrac{1}{2}\left[1-\left(\dfrac{1}{2}\right)^{\frac{1}{2-2\varrho}}\right]^2\\ +\mathcal{B}(3-2\varrho,2)+\mathcal{B}_{\frac{2}{3}}(3-2\varrho,2)-2\mathcal{B}_{\left(\frac{1}{2}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho,2), & \text{if } \left(\dfrac{1}{2}\right)^{\frac{1}{2-2\varrho}}\geq\dfrac{2}{3},\end{cases}\tag{4.6}$$

and $\mathcal{E}_2(\varrho, 1)$ *is defined as in* (3.6).

Corollary 4.2. *If we tend* $\varrho \to 1$ *in Corollary 4.1, then the following trapezoid-type inequalities hold:*

(1) For any $s \in (0, 1]$,

$$\left|\frac{\mathscr{A}(\mathrm{e}_1)+\mathscr{A}(\mathrm{e}_2)}{2}-\frac{1}{\mathrm{e}_2-\mathrm{e}_1}\int_{\mathrm{e}_1}^{\mathrm{e}_2}\mathscr{A}(t)\mathrm{d}t\right|\leq\frac{\mathrm{e}_2-\mathrm{e}_1}{2}\left(2C_1(s,1,0)+C_2(s,1,0)\right)\left(|\mathscr{A}'(\mathrm{e}_1)|+|\mathscr{A}'(\mathrm{e}_2)|\right),$$

where $C_1(s)$ *and* $C_2(s)$ *are defined in* (4.1) *and* (4.2).

(2) For $s = 1$, *we recover the classical trapezoid inequality for first differentiable functions:*

$$\left|\frac{\mathscr{A}(\mathrm{e}_1)+\mathscr{A}(\mathrm{e}_2)}{2}-\frac{1}{\mathrm{e}_2-\mathrm{e}_1}\int_{\mathrm{e}_1}^{\mathrm{e}_2}\mathscr{A}(t)\mathrm{d}t\right|\leq\frac{\mathrm{e}_2-\mathrm{e}_1}{8}\left(|\mathscr{A}'(\mathrm{e}_1)|+|\mathscr{A}'(\mathrm{e}_2)|\right),$$

presented by Dragomir and Agarwal in [25].

Similarly, for $\varrho = 0$, *the following inequalities hold:*

- *For any* $s \in (0,1]$,

$$\left|\frac{e_2-e_1}{4}\left(\mathscr{A}'(e_1)+\mathscr{A}'(e_2)\right)-\frac{1}{2}\left(\mathscr{A}(e_2)-\mathscr{A}(e_1)\right)\right| \\ \leq \frac{(e_2-e_1)^2}{4}\mathcal{K}(s)\left(|\mathscr{A}''(e_1)|+|\mathscr{A}''(e_2)|\right),$$

where the constant $\mathcal{K}(s)$ *is explicitly given by*

$$\mathcal{K}(s) = \frac{1}{s+1}\left[\left(\frac{1}{\sqrt{2}}\right)^{s+1}-\left(1-\frac{1}{\sqrt{2}}\right)^{s+1}-\frac{1}{2}\right]+\frac{1}{s+3}\left[1-2\left(\frac{1}{\sqrt{2}}\right)^{s+3}\right] \\ +\mathcal{B}(3,s+1)-2\mathcal{B}_{\frac{1}{\sqrt{2}}}(3,s+1).$$

Here, $\mathcal{B}(x,y)$ *is the Beta function, and* $\mathcal{B}_z(x,y)$ *is the incomplete Beta function.*

- *For* $s=1$*, the inequality simplifies to*

$$\left|\frac{e_2-e_1}{4}\left(\mathscr{A}'(e_1)+\mathscr{A}'(e_2)\right)-\frac{1}{2}\left(\mathscr{A}(e_2)-\mathscr{A}(e_1)\right)\right| \\ \leq \frac{2\sqrt{2}-1}{24}(e_2-e_1)^2\left(|\mathscr{A}''(e_1)|+|\mathscr{A}''(e_2)|\right).$$

Corollary 4.3. *If we attempt to take* $\eta=1$ *and* $\beta=3$*, Theorem 3.1 yields the following Simpson* $3/8$ *inequality:*

$$\left|\varrho^2(e_2-e_1)^{\varrho}\left(\frac{\mathscr{A}(e_1)+3\mathscr{A}\left(\frac{2e_1+e_2}{3}\right)+3\mathscr{A}\left(\frac{e_1+2e_2}{3}\right)+\mathscr{A}(e_2)}{8}\right)\right. \\ +\frac{1-\varrho}{2(e_2-e_1)^{\varrho-1}}\left(\frac{\mathscr{A}'(e_1)+3\mathscr{A}'\left(\frac{2e_1+e_2}{3}\right)+3\mathscr{A}'\left(\frac{e_1+2e_2}{3}\right)+\mathscr{A}'(e_2)}{8}\right) \\ \left.-\frac{\Gamma(1-\varrho)}{2(e_2-e_1)^{1-\varrho}}\left({}^{PC}_{e_2^-}\mathbf{D}{e_1}^{\varrho}\mathscr{A}(e_1)+{}^{PC}_{e_1^+}\mathbf{D}{e_2}^{\varrho}\mathscr{A}(e_2)\right)\right| \\ \leq \frac{\varrho^2(e_2-e_1)^{1+\varrho}}{2}\left(2C_1(s,1,3)+C_2(\varrho,s,1,3)\right)\left(|\mathscr{A}'(e_1)|+|\mathscr{A}'(e_2)|\right) \\ +\frac{(1-\varrho)(e_2-e_1)^{2-\varrho}}{4}\left(\sum_{j=1}^{3}\mathcal{E}_j(\varrho,s,1,3)\right)\left(|\mathscr{A}''(e_1)|+|\mathscr{A}''(e_2)|\right),$$

where

$$C_1(s,1,3)=\frac{1}{(s+1)(s+2)}\times\left[2\left(\left(\frac{1}{8}\right)^{s+2}+\left(\frac{7}{8}\right)^{s+2}\right)+\frac{s-6}{8}+\frac{5s+2}{24}\left(\frac{1}{3}\right)^{s+1}-\frac{5s+26}{24}\left(\frac{2}{3}\right)^{s+1}\right], \tag{4.7}$$

$$C_2(\varrho,s,1,3)=\frac{1}{(s+1)(s+2)}\left[\frac{1}{2^s}+\frac{2^{s+1}(s-2)-(s+4)}{3^{s+2}}\right], \tag{4.8}$$

$$
\mathcal{E}_1(\varrho, s, 1, 3) \tag{4.9}
$$
$$
= \begin{cases} \dfrac{4-4\varrho}{(s+1)(3-2\varrho+s)}\left(\dfrac{1}{8}\right)^{1+\frac{s+1}{2-2\varrho}} - \dfrac{1}{4(s+1)}\left(1-\left(\dfrac{1}{8}\right)^{\frac{1}{2-2\varrho}}\right)^{s+1} + \dfrac{1}{8(s+1)} \\ +\dfrac{1}{3-2\varrho+s}\left(\dfrac{1}{3}\right)^{3-2\varrho+s} + \dfrac{1}{8(s+1)}\left(\dfrac{2^{s+1}-1}{3^{s+1}}\right) \\ +\mathcal{B}_{\frac{1}{3}}(3-2\varrho, s+1) - 2\mathcal{B}_{\left(\frac{1}{8}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho, s+1), & \textit{if } \left(\dfrac{1}{8}\right)^{\frac{1}{2-2\varrho}} < \dfrac{1}{3}, \\ \dfrac{1}{8(s+1)}\left(\dfrac{1-2^{s+1}}{3^{s+1}}\right) - \dfrac{1}{3-2\varrho+s}\left(\dfrac{1}{3}\right)^{3-2\varrho+s} \\ +\dfrac{1}{8(s+1)} - \mathcal{B}_{\frac{1}{3}}(3-2\varrho, s+1), & \textit{if } \left(\dfrac{1}{8}\right)^{\frac{1}{2-2\varrho}} \geq \dfrac{1}{3}, \end{cases}
$$

$$
\mathcal{E}_3(\varrho, s, 1, 3) \tag{4.10}
$$
$$
= \begin{cases} \dfrac{1-\left(\frac{2}{3}\right)^{3-2\varrho+s}}{3-2\varrho+s} - \dfrac{7}{8(s+1)}\left[1-\dfrac{2^{s+1}-1}{3^{s+1}}\right] \\ +\mathcal{B}(3-2\varrho, s+1) - \mathcal{B}_{\frac{2}{3}}(3-2\varrho, s+1), & \textit{if } \left(\dfrac{7}{8}\right)^{\frac{1}{2-2\varrho}} < \dfrac{2}{3}, \\ \dfrac{4-4\varrho}{(s+1)(3-2\varrho+s)}\left(\dfrac{7}{8}\right)^{1+\frac{s+1}{2-2\varrho}} + \dfrac{1+\left(\frac{2}{3}\right)^{3-2\varrho+s}}{3-2\varrho+s} \\ +\mathcal{B}(3-2\varrho, s+1) + \mathcal{B}_{\frac{2}{3}}(3-2\varrho, s+1) - 2\mathcal{B}_{\left(\frac{7}{8}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho, s+1) \\ -\dfrac{7}{8(s+1)}\left[1+\dfrac{2^{s+1}-1}{3^{s+1}}\right] - \dfrac{7}{4(s+1)}\left[1-\left(\dfrac{7}{8}\right)^{\frac{1}{2-2\varrho}}\right]^{s+1}, & \textit{if } \dfrac{2}{3} \leq \left(\dfrac{7}{8}\right)^{\frac{1}{2-2\varrho}} \leq 1, \end{cases}
$$

and $\mathcal{E}_2(\varrho, s)$ *is defined as in* (3.4).

Remark 4.1. *By choosing* $s = 1$*, Corollary 4.3 coincides with the result presented by Demir and Tunç in [20, Theorem 2.4].*

Corollary 4.4. *Setting* $\varrho = 1$ *in Corollary 4.3, we obtain*

$$
\left| \frac{\mathscr{A}(\mathfrak{e}_1) + 3\mathscr{A}\left(\frac{2\mathfrak{e}_1+\mathfrak{e}_2}{3}\right) + 3\mathscr{A}\left(\frac{\mathfrak{e}_1+2\mathfrak{e}_2}{3}\right) + \mathscr{A}(\mathfrak{e}_2)}{8} - \frac{1}{\mathfrak{e}_2-\mathfrak{e}_1}\int_{\mathfrak{e}_1}^{\mathfrak{e}_2} \mathscr{A}(\mathfrak{w})d\mathfrak{w} \right|
$$
$$
\leq \frac{\mathfrak{e}_2-\mathfrak{e}_1}{2}\left(2C_1(s,1,3) + C_2(s,1,3)\right)\left(|\mathscr{A}'(\mathfrak{e}_1)| + |\mathscr{A}'(\mathfrak{e}_2)|\right),
$$

where $C_1(s, 1, 3)$ *and* $C_2(s, 1, 3)$ *are defined as* (4.7) *and* (4.8)*, respectively.*

Specifically for $s = 1$, this reduces to

$$\left|\frac{\mathscr{A}(\mathrm{e}_1) + 3\mathscr{A}\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right) + 3\mathscr{A}\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right) + \mathscr{A}(\mathrm{e}_2)}{8} - \frac{1}{\mathrm{e}_2 - \mathrm{e}_1}\int_{\mathrm{e}_1}^{\mathrm{e}_2} \mathscr{A}(\mathfrak{w})d\mathfrak{w}\right|$$
$$\leq \frac{25(\mathrm{e}_2 - \mathrm{e}_1)}{576}\left(|\mathscr{A}'(\mathrm{e}_1)| + |\mathscr{A}'(\mathrm{e}_2)|\right),$$

which was obtained by Sitthiwirattham et al. in [26].

Similarly, setting $\varrho = 0$ in Corollary 4.3, we get

$$\left|\frac{\mathscr{A}'(\mathrm{e}_1) + 3\mathscr{A}'\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right) + 3\mathscr{A}'\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right) + \mathscr{A}'(\mathrm{e}_2)}{8} - \frac{\mathscr{A}(\mathrm{e}_2) - \mathscr{A}(\mathrm{e}_1)}{\mathrm{e}_2 - \mathrm{e}_1}\right|$$
$$\leq \frac{\mathrm{e}_2 - \mathrm{e}_1}{2}\left(\sum_{j=1}^{3} \mathcal{E}_j(0, s, 1, 3)\right)\left(|\mathscr{A}''(\mathrm{e}_1)| + |\mathscr{A}''(\mathrm{e}_2)|\right).$$

For $s = 1$, this simplifies to

$$\left|\frac{\mathscr{A}'(\mathrm{e}_1) + 3\mathscr{A}'\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right) + 3\mathscr{A}'\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right) + \mathscr{A}'(\mathrm{e}_2)}{8} - \frac{\mathscr{A}(\mathrm{e}_2) - \mathscr{A}(\mathrm{e}_1)}{\mathrm{e}_2 - \mathrm{e}_1}\right|$$
$$\leq (\mathrm{e}_2 - \mathrm{e}_1)\left(\frac{7\sqrt{14}}{48} - \frac{1319}{6480}\right)\left(|\mathscr{A}''(\mathrm{e}_1)| + |\mathscr{A}''(\mathrm{e}_2)|\right).$$

Corollary 4.5. *By setting $\eta = 13$ and $\beta = 27$, Theorem 3.1 yields the following corrected Simpson $3/8$ inequality via proportional Caputo-hybrid operators:*

$$\left|\varrho^2 (\mathrm{e}_2 - \mathrm{e}_1)^{\varrho}\left(\frac{13\mathscr{A}(\mathrm{e}_1) + 27\mathscr{A}\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right) + 27\mathscr{A}\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right) + 13\mathscr{A}(\mathrm{e}_2)}{80}\right)\right.$$
$$+ \frac{1-\varrho}{2(\mathrm{e}_2 - \mathrm{e}_1)^{\varrho-1}}\left(\frac{13\mathscr{A}'(\mathrm{e}_1) + 27\mathscr{A}'\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right) + 27\mathscr{A}'\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right) + 13\mathscr{A}'(\mathrm{e}_2)}{80}\right)$$
$$\left.-\frac{\Gamma(1-\varrho)}{2(\mathrm{e}_2 - \mathrm{e}_1)^{1-\varrho}}\left({}^{PC}_{\mathrm{e}_2^-}\mathbf{D}^{\varrho}_{\mathrm{e}_1}\mathscr{A}(\mathrm{e}_1) + {}^{PC}_{\mathrm{e}_1^+}\mathbf{D}^{\varrho}_{\mathrm{e}_2}\mathscr{A}(\mathrm{e}_2)\right)\right|$$
$$\leq \frac{\varrho^2 (\mathrm{e}_2 - \mathrm{e}_1)^{1+\varrho}}{2}\left(2C_1(s, 13, 27) + C_2(s, 13, 27)\right)\left(|\mathscr{A}'(\mathrm{e}_1)| + |\mathscr{A}'(\mathrm{e}_2)|\right)$$
$$+ \frac{(1-\varrho)(\mathrm{e}_2 - \mathrm{e}_1)^{2-\varrho}}{4}\left(\sum_{j=1}^{3}\mathcal{E}_j(\varrho, s, 13, 27)\right)\left(|\mathscr{A}''(\mathrm{e}_1)| + |\mathscr{A}''(\mathrm{e}_2)|\right),$$

where

$$C_1(s, 13, 27) = \frac{1}{(s+1)(s+2)}\left[2\left(\frac{13}{80}\right)^{s+2} + 2\left(\frac{67}{80}\right)^{s+2} + \frac{13s-54}{80} + \frac{41s+2}{240}\left(\frac{1}{3}\right)^{s+1} - \frac{41s+242}{240}\left(\frac{2}{3}\right)^{s+1}\right], \tag{4.11}$$

$$C_2(s,13,27)=\frac{1}{(s+1)(s+2)}\left[\left(\frac{1}{2}\right)^{s}-(s+4)\left(\frac{1}{3}\right)^{s+2}+\frac{s-2}{2}\left(\frac{2}{3}\right)^{s+2}\right], \tag{4.12}$$

$$\mathcal{E}_1(\varrho,s,13,27)=\begin{cases}\frac{4-4\varrho}{(s+1)(3-2\varrho+s)}\left(\frac{13}{80}\right)^{1+\frac{s+1}{2-2\varrho}}-\frac{13}{40(s+1)}\left(1-\left(\frac{13}{80}\right)^{\frac{1}{2-2\varrho}}\right)^{s+1} \\ +\frac{13}{80(s+1)}+\frac{1}{3-2\varrho+s}\left(\frac{1}{3}\right)^{3-2\varrho+s}+\frac{13}{80(s+1)}\left(\frac{2^{s+1}-1}{3^{s+1}}\right) \\ +\mathcal{B}_{\frac{1}{3}}(3-2\varrho,s+1)-2\mathcal{B}_{\left(\frac{13}{80}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho,s+1), & \text{if } \left(\frac{13}{80}\right)^{\frac{1}{2-2\varrho}}<\frac{1}{3}, \\ \frac{13}{80(s+1)}\left(\frac{1-2^{s+1}}{3^{s+1}}\right)-\frac{1}{3-2\varrho+s}\left(\frac{1}{3}\right)^{3-2\varrho+s} \\ +\frac{13}{80(s+1)}-\mathcal{B}_{\frac{1}{3}}(3-2\varrho,s+1), & \text{if } \left(\frac{13}{80}\right)^{\frac{1}{2-2\varrho}}\geq\frac{1}{3},\end{cases} \tag{4.13}$$

$$\mathcal{E}_3(\varrho,s,13,27) \tag{4.14}$$

$$=\begin{cases}\frac{1-\left(\frac{2}{3}\right)^{3-2\varrho+s}}{3-2\varrho+s}-\frac{67}{80(s+1)}\left[1-\frac{2^{s+1}-1}{3^{s+1}}\right] \\ +\mathcal{B}(3-2\varrho,s+1)-\mathcal{B}_{\frac{2}{3}}(3-2\varrho,s+1), & \text{if } \left(\frac{67}{80}\right)^{\frac{1}{2-2\varrho}}<\frac{2}{3}, \\ \frac{4-4\varrho}{(s+1)(3-2\varrho+s)}\left(\frac{67}{80}\right)^{1+\dfrac{s+1}{2-2\varrho}}+\frac{1+\left(\frac{2}{3}\right)^{3-2\varrho+s}}{3-2\varrho+s} \\ +\mathcal{B}(3-2\varrho,s+1)+\mathcal{B}_{\frac{2}{3}}(3-2\varrho,s+1)-2\mathcal{B}_{\left(\frac{67}{80}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho,s+1) \\ -\frac{67}{80(s+1)}\left[1+\frac{2^{s+1}-1}{3^{s+1}}\right]-\frac{67}{40(s+1)}\left[1-\left(\frac{67}{80}\right)^{\frac{1}{2-2\varrho}}\right]^{s+1}, & \text{if } \frac{2}{3}\leq\left(\frac{67}{80}\right)^{\frac{1}{2-2\varrho}}\leq 1,\end{cases}$$

and $\mathcal{E}_2(\varrho,s)$ *is defined as in* (3.4).

Moreover, by selecting $s=1$*, we get*

$$\begin{aligned}&\left|\varrho^2(e_2-e_1)^{\varrho}\left(\frac{13\mathscr{A}(e_1)+27\mathscr{A}\left(\frac{2e_1+e_2}{3}\right)+27\mathscr{A}\left(\frac{e_1+2e_2}{3}\right)+13\mathscr{A}(e_2)}{80}\right)\right. \\ &+\frac{1-\varrho}{2(e_2-e_1)^{\varrho-1}}\left(\frac{13\mathscr{A}'(e_1)+27\mathscr{A}'\left(\frac{2e_1+e_2}{3}\right)+27\mathscr{A}'\left(\frac{e_1+2e_2}{3}\right)+13\mathscr{A}'(e_2)}{80}\right) \\ &\left.-\frac{\Gamma(1-\varrho)}{2(e_2-e_1)^{1-\varrho}}\left({}^{PC}_{e_2^-}\mathbf{D}{e_1}^{\varrho}\mathscr{A}(e_1)+{}^{PC}_{e_1^+}\mathbf{D}{e_2}^{\varrho}\mathscr{A}(e_2)\right)\right| \\ \leq{}&\frac{2401}{57600}\varrho^2(e_2-e_1)^{1+\varrho}\left(|\mathscr{A}'(e_1)|+|\mathscr{A}'(e_2)|\right)\end{aligned}$$

$$+\frac{(1-\varrho)(\mathfrak{e}_2-\mathfrak{e}_1)^{2-\varrho}}{4}\left(\sum_{j=1}^{3}\mathcal{E}_i(\varrho,1,13,27)\right)\left(|\mathscr{A}''(\mathfrak{e}_1)|+|\mathscr{A}''(\mathfrak{e}_2)|\right),$$

with

$$\mathcal{E}_1(\varrho,1,13,27)=\begin{cases}\frac{1-\varrho}{2-\varrho}\left(\frac{13}{80}\right)^{\frac{2-\varrho}{1-\varrho}}-\frac{13}{80}\left(1-\left(\frac{13}{80}\right)^{\frac{1}{2-2\varrho}}\right)^2+\frac{13}{160}+\frac{1}{4-2\varrho}\left(\frac{1}{3}\right)^{4-2\varrho} & \\ +\frac{13}{160}\left(\frac{3}{9}\right)+\mathcal{B}_{\frac{1}{3}}(4-2\varrho,2)-2\mathcal{B}_{\left(\frac{13}{80}\right)^{\frac{1}{2-2\varrho}}}(4-2\varrho,2), & \text{if } \left(\frac{13}{80}\right)^{\frac{1}{2-2\varrho}}<\frac{1}{3},\\ \frac{13}{160}\left(-\frac{1}{3}\right)-\frac{1}{4-2\varrho}\left(\frac{1}{3}\right)^{4-2\varrho}+\frac{13}{160}-\mathcal{B}_{\frac{1}{3}}(4-2\varrho,2), & \text{if } \left(\frac{13}{80}\right)^{\frac{1}{2-2\varrho}}\geq\frac{1}{3},\end{cases}$$

$$\begin{aligned}&\mathcal{E}_3(\varrho,1,13,27)\\&=\begin{cases}\frac{1-\left(\frac{2}{3}\right)^{4-2\varrho}}{4-2\varrho}+\mathcal{B}(3-2\varrho,2)-\mathcal{B}_{\frac{2}{3}}(3-2\varrho,2)-\frac{67}{240} & \\ +\mathcal{B}_{\frac{2}{3}}(4-2\varrho,2)-\mathcal{B}_{\frac{1}{3}}(4-2\varrho,2), & \text{if } \left(\frac{67}{80}\right)^{\frac{1}{2-2\varrho}}<\frac{2}{3},\\ \frac{4-4\varrho}{2(4-2\varrho)}\left(\frac{67}{80}\right)^{1+\frac{2}{2-2\varrho}}+\frac{1+\left(\frac{2}{3}\right)^{4-2\varrho}}{4-2\varrho}-\frac{67}{120}-\frac{67}{80}\left[1-\left(\frac{67}{80}\right)^{\frac{1}{2-2\varrho}}\right]^2 & \\ +\mathcal{B}(3-2\varrho,2)+\mathcal{B}_{\frac{2}{3}}(3-2\varrho,2)-2\mathcal{B}_{\left(\frac{67}{80}\right)^{\frac{1}{2-2\varrho}}}(3-2\varrho,2), & \text{if } \frac{2}{3}\leq\left(\frac{67}{80}\right)^{\frac{1}{2-2\varrho}}\leq 1,\end{cases}\end{aligned}$$

and $\mathcal{E}_2(\varrho,1)$ *is defined as in* (3.6).

Corollary 4.6. *Tending* ϱ *to* 1 *in Corollary 4.5, we obtain*

$$\begin{aligned}&\left|\frac{13\mathscr{A}(\mathfrak{e}_1)+27\mathscr{A}\left(\frac{2\mathfrak{e}_1+\mathfrak{e}_2}{3}\right)+27\mathscr{A}\left(\frac{\mathfrak{e}_1+2\mathfrak{e}_2}{3}\right)+13\mathscr{A}(\mathfrak{e}_2)}{80}-\frac{1}{\mathfrak{e}_2-\mathfrak{e}_1}\int_{\mathfrak{e}_1}^{\mathfrak{e}_2}\mathscr{A}(\mathfrak{w})d\mathfrak{w}\right|\\&\leq\frac{\mathfrak{e}_2-\mathfrak{e}_1}{2}\left(2C_1(s,13,27)+C_2(s,13,27)\right)\left(|\mathscr{A}'(\mathfrak{e}_1)|+|\mathscr{A}'(\mathfrak{e}_2)|\right),\end{aligned}$$

where $C_1(s,13,27)$ *and* $C_2(s,13,27)$ *are defined as in* (4.11) *and* (4.12)*, respectively.*

Specifically for $s=1$*, this reduces to*

$$\begin{aligned}&\left|\frac{13\mathscr{A}(\mathfrak{e}_1)+27\mathscr{A}\left(\frac{2\mathfrak{e}_1+\mathfrak{e}_2}{3}\right)+27\mathscr{A}\left(\frac{\mathfrak{e}_1+2\mathfrak{e}_2}{3}\right)+13\mathscr{A}(\mathfrak{e}_2)}{80}-\frac{1}{\mathfrak{e}_2-\mathfrak{e}_1}\int_{\mathfrak{e}_1}^{\mathfrak{e}_2}\mathscr{A}(\mathfrak{w})d\mathfrak{w}\right|\\&\leq\frac{2401(\mathfrak{e}_2-\mathfrak{e}_1)}{57600}\left(|\mathscr{A}'(\mathfrak{e}_1)|+|\mathscr{A}'(\mathfrak{e}_2)|\right).\end{aligned}$$

Similarly, setting $\varrho = 0$ in Corollary 4.5, we get

$$\left|\frac{13\mathscr{A}'(\mathrm{e}_1) + 27\mathscr{A}'\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right) + 27\mathscr{A}'\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right) + 13\mathscr{A}'(\mathrm{e}_2)}{80} - \frac{\mathscr{A}(\mathrm{e}_2) - \mathscr{A}(\mathrm{e}_1)}{\mathrm{e}_2 - \mathrm{e}_1}\right|$$
$$\leq \frac{\mathrm{e}_2 - \mathrm{e}_1}{2}\left(\sum_{j=1}^{3}\mathcal{E}_j(0, s, 13, 27)\right)\left(|\mathscr{A}''(\mathrm{e}_1)| + |\mathscr{A}''(\mathrm{e}_2)|\right),$$

where

$$\mathcal{E}_1(0, s, 1, 3) = \frac{1}{8(s+1)}\left(1 + \frac{1-2^{s+1}}{3^{s+1}}\right) - \frac{1}{(s+3)3^{s+3}} - \mathcal{B}_{\frac{1}{3}}(3, s+1),$$

$$\mathcal{E}_3(0, s, 1, 3) = \frac{4}{(s+1)(s+3)}\left(\frac{7}{8}\right)^{\frac{s+3}{2}} - \frac{7}{4(s+1)}\left(1 - \sqrt{\frac{7}{8}}\right)^{s+1} + \frac{7}{4(s+1)}\left(\frac{2^{s+1}-1}{3^{s+1}}\right) + \frac{1+2^{s+3}}{3^{s+3}(s+3)}$$
$$+ \mathcal{B}\frac{1}{3}(3, s+1) + \mathcal{B}_{\frac{2}{3}}(3, s+1) - 2\mathcal{B}_{\sqrt{\frac{7}{8}}}(3, s+1),$$

and $\mathcal{E}_2(0, s)$ is defined as in (3.7).

For $s = 1$, the constants simplify to

$$\left|\frac{13\mathscr{A}'(\mathrm{e}_1) + 27\mathscr{A}'\left(\frac{2\mathrm{e}_1+\mathrm{e}_2}{3}\right) + 27\mathscr{A}'\left(\frac{\mathrm{e}_1+2\mathrm{e}_2}{3}\right) + 13\mathscr{A}'(\mathrm{e}_2)}{80} - \frac{\mathscr{A}(\mathrm{e}_2) - \mathscr{A}(\mathrm{e}_1)}{\mathrm{e}_2 - \mathrm{e}_1}\right|$$
$$\leq (\mathrm{e}_2 - \mathrm{e}_1)\frac{1809\sqrt{335} - 11950}{64800}\left(|\mathscr{A}''(\mathrm{e}_1)| + |\mathscr{A}''(\mathrm{e}_2)|\right).$$

5. Graphical validation

To illustrate the validity of our findings, we present two numerical examples together with graphical representations that visually support and clarify the obtained results.

Example 5.1. Consider the mapping $\mathscr{A} : [0, 1) \to \mathbb{R}$ expressed as

$$\mathscr{A}(\mathfrak{w}) = \frac{\mathfrak{w}^{s+2}}{s+2}, \quad \text{for any } s \in (0, 1].$$

Verifying that this function meets the conditions of the main theorem is straightforward; simple differentiation confirms that $\mathscr{A}'(\mathfrak{w}) = \mathfrak{w}^{s+1}$ and $\mathscr{A}''(\mathfrak{w}) = (s+1)\mathfrak{w}^{s}$. These derivatives are well-known examples of s-convex functions on the unit interval. Thus, by employing Theorem 3.1, we obtain the subsequent bound:

$$\left|\frac{\varrho^2}{2(\eta+\beta)(s+2)}\left[\eta + \frac{\beta(1+2^{s+2})}{3^{s+2}}\right] + \frac{1-\varrho}{4(\eta+\beta)}\left[\eta + \frac{\beta(1+2^{s+1})}{3^{s+1}}\right]\right.$$
$$\left. - \frac{(1-\varrho)^2}{2}\left[\mathcal{B}(s+2, 2-2\varrho) + \frac{1}{s-2\varrho+3}\right] - \frac{\varrho^2}{3(s+3)}\right|$$

$$
\begin{aligned}
&\leq \frac{\varrho^2}{2}\left(2C_1(s,\eta,\beta)+C_2(s,\eta,\beta)\right) \\
&\quad+\frac{(1-\varrho)}{4}\left(\mathcal{E}_1\left(\varrho,s,\eta,\beta\right)+\mathcal{E}_2\left(\varrho,s\right)+\mathcal{E}_3\left(\varrho,s,\eta,\beta\right)\right).
\end{aligned}
\tag{5.1}
$$

Here, the parameters $C_j(s,\eta,\beta)$ $(i=1,2)$ and $\mathcal{E}_j(\varrho,s,\eta,\beta)$ follow the definitions given in (3.1)–(3.5).

In what follows, we examine two particular choices of the parameters η and β to recover well-known quadrature rules and provide a visual verification of the obtained bounds.

Case 1. By fixing $\eta=1$ and $\beta=3$, inequality (5.1) yields the following Simpson 3/8 inequality:

$$
\begin{aligned}
&\left|\frac{\varrho^2}{8(s+2)}\left[\frac{3^{s+1}+2^{s+2}+1}{3^{s+1}}\right]+\frac{1-\varrho}{16}\left[\frac{3^s+2^{s+1}+1}{3^s}\right]\right. \\
&\quad\left.-\frac{(1-\varrho)^2}{2}\left[\mathcal{B}(s+2,2-2\varrho)+\frac{1}{s-2\varrho+3}\right]-\frac{\varrho^2}{3(s+3)}\right| \\
&\leq \frac{\varrho^2}{2}\left(2C_1(s,1,3)+C_2(s,1,3)\right) \\
&\quad+\frac{(1-\varrho)}{4}\left(\mathcal{E}_1\left(\varrho,s,1,3\right)+\mathcal{E}_2\left(\varrho,s\right)+\mathcal{E}_3\left(\varrho,s,1,3\right)\right),
\end{aligned}
\tag{5.2}
$$

where $C_1(s,1,3)$, $C_2(s,1,3)$, $\mathcal{E}_1\left(\varrho,s,1,3\right)$, $\mathcal{E}_2\left(\varrho,s\right)$, and $\mathcal{E}_3\left(\varrho,s,1,3\right)$ are given by (4.7), (4.8), (4.9), (3.4), and (4.10), respectively.

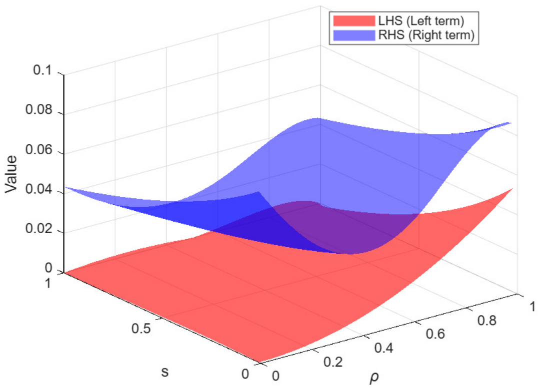
The validity of this inequality is visually demonstrated in Figure 1(a), which displays the surfaces of both the left- and right-hand sides of (5.2) for varying $\varrho\in[0,1)$ and $s\in(0,1]$.

Case 2. By fixing $\eta=13$ and $\beta=27$, inequality (5.1) yields the following corrected Simpson 3/8 inequality:

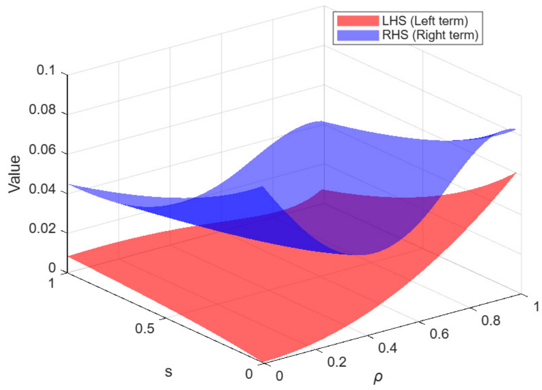
$$
\begin{aligned}
&\left|\frac{\varrho^2}{80(s+2)}\left[13+\frac{27(1+2^{s+2})}{3^{s+2}}\right]+\frac{1-\varrho}{160}\left[13+\frac{27(1+2^{s+1})}{3^{s+1}}\right]\right. \\
&\quad\left.-\frac{(1-\varrho)^2}{2}\left[\mathcal{B}(s+2,2-2\varrho)+\frac{1}{s-2\varrho+3}\right]-\frac{\varrho^2}{3(s+3)}\right| \\
&\leq \frac{\varrho^2}{2}\left(2C_1(s,13,27)+C_2(s,13,27)\right) \\
&\quad+\frac{(1-\varrho)}{4}\left(\mathcal{E}_1\left(\varrho,s,13,27\right)+\mathcal{E}_2\left(\varrho,s\right)+\mathcal{E}_3\left(\varrho,s,13,27\right)\right),
\end{aligned}
\tag{5.3}
$$

where $C_1(s,13,27)$, $C_2(s,13,27)$, $\mathcal{E}_1\left(\varrho,13,1,27\right)$, $\mathcal{E}_2\left(\varrho,s\right)$, and $\mathcal{E}_3\left(\varrho,s,13,27\right)$ are given by (4.11)–(4.13), (3.4), and (4.14), respectively.

Similarly, Figure 1(b) depicts the behavior of both sides of inequality (5.3), confirming the result for the specified range of parameters $\varrho\in[0,1)$ and $s\in(0,1]$.



(a) Graphical validation of Case 1.



(b) Graphical validation of Case 2.

Figure 1. Graphical illustration supporting the theoretical findings.

Example 5.2. Let $\mathscr{A} : [0,1] \to \mathbb{R}$ be defined by

$$\mathscr{A}(\mathfrak{w}) = e^{\mathfrak{w}}.$$

Then $\mathscr{A}'(\mathfrak{w}) = \mathscr{A}''(\mathfrak{w}) = e^{\mathfrak{w}}$. Since $e^{\mathfrak{w}}$ is convex on $[0,1]$, the assumptions of Corollary 3.1 are satisfied.

For $\mathfrak{e}_1 = 0$ and $\mathfrak{e}_2 = 1$, put

$$\begin{aligned}\mathcal{I}(\varrho) &= \int_0^1 \mathfrak{w}^{1-2\varrho} e^{\mathfrak{w}}\, d\mathfrak{w} + \int_0^1 (1-\mathfrak{w})^{1-2\varrho} e^{\mathfrak{w}}\, d\mathfrak{w} \\ &= \frac{1}{2-2\varrho}\left[{}_1F_1(2-2\varrho; 3-2\varrho; 1) + e\, {}_1F_1(2-2\varrho; 3-2\varrho; -1)\right],\end{aligned}$$

where ${}_1F_1(a;b;z)$ denotes the confluent hypergeometric function of the first kind, also known as Kummer's function.

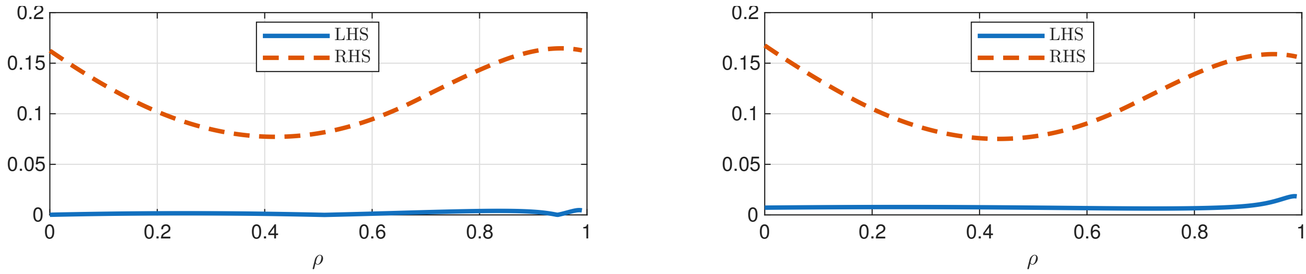
Case 1. For $\eta = 1$ and $\beta = 3$, Corollary 3.1 gives

$$\begin{aligned}&\left|\left(\varrho^2 + \frac{1-\varrho}{2}\right)\frac{1 + 3e^{1/3} + 3e^{2/3} + e}{8} - \varrho^2(e-1) - \frac{(1-\varrho)^2}{2}\mathcal{I}(\varrho)\right| \\ &\le (1+e)\left[\frac{25}{576}\varrho^2 + \frac{1-\varrho}{4}\left(\mathcal{E}_1(\varrho,1,1,3) + \mathcal{E}_2(\varrho,1) + \mathcal{E}_3(\varrho,1,1,3)\right)\right].\end{aligned}$$

Case 2. For $\eta = 13$ and $\beta = 27$, Corollary 3.1 gives

$$\begin{aligned}&\left|\left(\varrho^2 + \frac{1-\varrho}{2}\right)\frac{13 + 27e^{1/3} + 27e^{2/3} + 13e}{80} - \varrho^2(e-1) - \frac{(1-\varrho)^2}{2}\mathcal{I}(\varrho)\right| \\ &\le (1+e)\left[\frac{2401}{57600}\varrho^2 + \frac{1-\varrho}{4}\left(\mathcal{E}_1(\varrho,1,13,27) + \mathcal{E}_2(\varrho,1) + \mathcal{E}_3(\varrho,1,13,27)\right)\right].\end{aligned}$$

The graphical and numerical validations of the obtained inequalities are presented below. Figure 2 shows the behavior of both sides of the inequalities for $\varrho \in [0,1)$, while Table 1 provides some numerical values.



(a) Graphical validation for $\eta = 1$ and $\beta = 3$.

(b) Graphical validation for $\eta = 13$ and $\beta = 27$.

Figure 2. Comparison between the left- and right-hand sides for $\mathscr{A}(\mathfrak{w}) = e^{\mathfrak{w}}$.

Table 1. Numerical values of both sides for two choices of (η, β).

	$\eta = 1$ and $\beta = 3$			$\eta = 13$ and $\beta = 27$	
ϱ	Left term	Right term	ϱ	Left term	Right term
0.0	1.291625e-04	1.623495e-01	0.0	7.159200e-03	1.675561e-01
0.1	1.065915e-03	1.287377e-01	0.1	7.533550e-03	1.336537e-01
0.2	1.549494e-03	1.020481e-01	0.2	7.735928e-03	1.049403e-01
0.3	1.555230e-03	8.466026e-02	0.3	7.741663e-03	8.514250e-02
0.4	1.074011e-03	7.738751e-02	0.4	7.541646e-03	7.580128e-02
0.5	1.291625e-04	8.069188e-02	0.5	7.159200e-03	7.749648e-02
0.6	1.191030e-03	9.462174e-02	0.6	6.682613e-03	9.041283e-02
0.7	2.668646e-03	1.181063e-01	0.7	6.329803e-03	1.135960e-01
0.8	3.814570e-03	1.434236e-01	0.8	6.589887e-03	1.392180e-01
0.9	3.087164e-03	1.617066e-01	0.9	9.004501e-03	1.565306e-01

The graphical comparison of these two pairs of functions confirms the validity of Corollary 3.1 for the considered mapping.

6. Conclusions

We have effectively developed a new class of integral inequalities involving proportional Caputo-hybrid operators in this work. Our method was based on establishing a generalized parametric identity, as well as applying s-convexity properties to the first and second derivatives of the functions under consideration to obtain a family of parametrized Newton-type inequalities.

We demonstrated that our results generalize a number of well-known quadrature formulas, such as the standard and corrected Simpson's 3/8 rules, by choosing particular values for the parameters η and β. Lastly, a thorough numerical analysis confirmed the applicability of our main theorem. The tightness and accuracy of the established bounds were visually confirmed by graphical comparisons between the left- and right-hand sides of the inequalities, carried out for different values of ϱ and s.

This work creates new opportunities for the use of hybrid fractional operators in error estimation and numerical integration.

Author contributions

The authors collaborated equally on the conceptualization, analysis, and writing of this paper. All authors have reviewed the final version and consented to its submission.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this paper.

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Conflict of interest

The authors report no competing interests to declare.

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