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Research article

Local face metric dimension: a new resolvability parameter for Planar graphs

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Abstract: Resolving sets and metric-based parameters capture the ability to distinguish elements of a graph through distances. While the classical *metric dimension* focuses on distinguishing vertices, *planar graphs* naturally suggest an analogous question at the level of faces. In this work we introduce the *local face metric dimension*, a new parameter that measures the least number of vertices needed so that every two *adjacent faces* have distinct distance representation. This concept extends the scope of resolvability theory from vertex-sets to the facial structure of *planar graphs*. We develop the basic theory of *local face metric dimension* for connected planar graphs, focusing on its structural properties and the conditions under which it is well defined and finite. The parameter reflects a localized notion of *resolvability* and highlights the interaction between graph distances and the facial structure of planar embeddings. Its behavior is explored for several standard graph families to illustrate its variability and dependence on graph structure. The *local face metric dimension* thus opens a new direction in the study of graph resolvability, highlighting the interplay between metric properties and planar embeddings, and suggesting further structural, extremal, and algorithmic investigations.

Keywords: face resolving set; face metric dimension; local face resolving set; local face metric dimension
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1. Introduction

Graph theory has emerged as a central area of modern mathematics due to its wide range of applications in computer science, communication networks, chemistry, and cryptography. Among the many parameters studied in graph theory, resolvability parameters are fundamentally related to the notion of distance, as they measure how effectively the elements of a graph (vertices, edges, and faces) can be distinguished by their distance representations with respect to a selected set of vertices. A fundamental notion in this context is the *metric dimension* (MD), which was introduced independently by Chartrand, which studies the least number of vertices required so that every vertex in the graph is uniquely determined by its vector of distances to this set [1]. The concept of MD has numerous applications across diverse fields of science and technology. In network design, resolving set (RS) are used to optimize routing strategies and ensure fault-tolerant communication [2]. In robotics, RS play a crucial role in sensor placement, enabling efficient robot navigation, accurate mapping of environments, and precise localization [3, 4]. In molecular chemistry, RS provide a framework for modeling chemical structures and are particularly useful in distinguishing molecular isomers [5]. Moreover, in image processing, RS have been applied in object recognition and feature extraction, where graphs represent pixel connectivity and structural patterns [6].

Further applications arise in bioinformatics, where RS contribute to phylogenetic analysis, and in social network analysis, where they assist in detecting central or influential nodes [7]. In pharmaceutical sciences, MD has been employed to identify structural similarities among a wide range of medications [1, 8]. Other significant applications include computer network design [9] and combinatorial optimization and problems [10, 11]. These diverse applications highlight the fundamental role of MD in bridging graph theory with practical domains. In addition to applications, extensive research has been devoted to computing the MD of various graph families and network structures. For instance, the MD of the heptagonal snake graph has been investigated in [12], while the octagonal nanotube structure has been analyzed in [13, 14]. More generally, Nawaz [15] studied circulant networks, Haider [16] investigated bicyclic graphs, and Sikander [17] explored nanotubes. Further works include Klein's study of line graphs [18] and Manuel's analysis of torus networks [19]. Hallaway [20] addressed permutation graphs, and Imran [21] worked on convex polytopes. These investigations collectively highlight the breadth of research on MD, ranging from classical graph families to complex chemical and nanostructure models. Further contributions to the study of chemical and nanostructure-related graphs include the MD of crystal cubic carbon structures [22], the MD of hollow coronoids [23,24], and the computation of upper bounds for the MD of cellulose networks [25]. The concept of *fractional metric dimension* for *generalized Jahangir graphs* was studied in [26].

Since then, several variants of the MD have been proposed, such as the *local metric dimension (LMD)*, *edge metric dimension (EMD)*, and *face metric dimension (FMD)*, each motivated by particular structural or applied considerations. In particular, when graphs are embedded in the plane, the study of distances between faces leads to interesting *planar graph* parameters.

The *LFMD* is driven by both theoretical and practical reasons. Theoretically, resolvability parameters have been analyzed in a natural extension: the classical MD distinguishing all pairs of vertices, to local extensions, such as the *LMD* [27], which limits to adjacent vertices, and to face extensions, such as the *FMD* [28], which generalizes resolvability to the facial structure of planar graphs. The *LFMD* fills this picture by jointly considering both localizations but only to face pairs that are adjacent to each other. This is not just a weakening, but a very different structural property, as the face-adjacency graph of a planar embedding is a topological property of how the regions of the plane are structured around the edges. The knowledge of which sets of vertices suffice to solve this adjacency structure locally leads to a more in-depth insight into the interaction between metric and topological properties of planar graphs. On the applied side, the *LFMD* is driven by the issues of monitoring and distinguishability in planar layouts of networks. Regions or zones (corresponding to faces) are the basic objects of interest in many real-world applications, such as floor plan analysis, road and utility network design, circuit board layout, and image segmentation on grid-like structures, instead of junctions or nodes. A practical demand in such environments is that sensors or monitoring stations (located at vertices) must be able to tell any two adjacent regions apart using their proximity signatures. An example is in a planar distribution network, two adjacent service zones are identifiable by the distances observed at the monitoring stations in a unique way so that a fault or anomaly would be assigned to the appropriate zone. The local restriction of only distinguishing adjacent faces, is natural here, since in practice it is the confusion of neighboring zones which leads to operational errors, whereas distant zones are often distinguishable by other methods. This philosophy is similar to the motivation behind the *LMD* proposed by Okamoto et al. [27] and proposes that the *LFMD* is the face-analogue of the *LMD* in the same way that the *FMD* [28] is the face-analogue of the classical *MD*.

In this article, we define and examine the local face resolving set ($LFRS$) and *LFMD* of *planar graphs*. This parameter is inspired by both the *LMD* and the FMD and is defined in terms of resolving *adjacent faces* of a plane graph. Our aim is to establish its value for some fundamental families of *planar graphs*, thereby providing initial insights into its structure and behavior. We focus on cycles, grids, and fan graphs, as these families form the building blocks of *planar graph* theory.

To place our work in context, Table 1 summarizes the principal resolvability parameters studied in the literature, together with their motivating idea and classical references. The continuous development of these parameters demonstrates the richness of resolvability theory in graphs; our work extends this family by focusing on the local resolvability of faces in *planar graphs*.

We call a finite graph *planar* if it can be drawn in the plane so that its edges do not cross, except at common endpoints of adjacent edges. A drawing of this type is known as a *plane embedding*. Every such embedding naturally partitions the plane into distinct regions, called *faces*. One of these regions extends without bound and is referred to as the *outer face*, while the remaining ones are enclosed and are called inner faces. The theory of *planar graphs* has been widely explored because of their elegant structure and diverse applications. A fundamental property of every connected *planar graph* G is expressed by Euler's formula:

$$|\mathcal{F}(G)| = |E(G)| - |V(G)| + 2,$$

where $|V(G)|$, $|E(G)|$, and $|\mathcal{F}(G)|$ denote the number of vertices, edges, and faces of a plane embedding of G, respectively. This equality demonstrates the precise balance between vertices, edges, and faces in *planar graphs* and serves as a starting point for many classical results in topological graph theory. A deeper structural characterization of *planar graphs* is given by *Kuratowski's theorem*, which states that a finite graph is planar if and only if it does not contain homeomorphic copies of K_5 (the complete graph on five vertices) or $K_{3,3}$ (the complete bipartite graph with partitions of size three) as a subgraph. This result provides a purely combinatorial criterion for planarity, independent of drawings [29]. *Planar graphs* also play a vital role in a variety of real-world applications. In computer science, planar embeddings are important for graph drawing, data visualization, and network layout optimization. In electrical and civil engineering, *planar graphs* naturally describe circuit layouts, road maps, and water or power distribution systems. In chemistry, they are used to model molecular structures such as aromatic hydrocarbons, where atoms and bonds form planar embeddings. Furthermore, algorithms exploiting planarity often achieve faster runtimes than their general-graph counterparts, making *planar* *graph* theory essential in algorithm design. The study of *planar graphs* is particularly relevant to our work because the concept of *faces* is central to defining parameters, such as the *local face metric dimension*. By analyzing how vertices relate to faces in a planar embedding, we can construct efficient resolving sets that distinguish adjacent regions of the graph, which directly connects theoretical graph parameters to practical applications such as monitoring zones in distribution networks.

Table 1. Summary of resolvability parameters.

Parameter	Description	Reference
Metric dimension (MD)	Minimum number of vertices needed to uniquely determine all vertices of a graph by their distance vectors.	Chartrand et al. [1]
Local metric dimension (LMD)	Resolves only *adjacent* vertices of a graph.	Okamoto et al. [27]
Edge metric dimension (EMD)	Resolves edges of a graph by their distance vectors to vertices.	Kelenc et al. [30]
Face metric dimension (FMD)	Resolves all faces of a planar graph using vertices.	Kamran et al. [28]
Local edge metric dimension (LEMD)	Resolves only *adjacent edges* of a graph.	Adawiyah et al. [31]
Local face metric dimension (LFMD)	Resolves only *adjacent faces* of a planar graph.	Furqan et al. (new parameter)
Mixed metric dimension (MMD)	Resolves both vertices and edges simultaneously.	Kelenc et al. [32]
Fault-Tolerant metric dimension (FTMD)	Resolves all vertices with redundancy so that removing one landmark still preserves resolvability.	Hernando et al. [33]
Fault-Tolerant edge metric dimension (FTEMD)	Resolves edges with redundancy so that the resolvability holds even if one landmark is removed.	Ahsan et al. [34]
Strong metric dimension (SMD)	Uses the concept of strong resolving sets, requiring vertices to strongly distinguish others.	Oellermann & Peters-Fransen [35]
Local strong metric dimension (LSMD)	Strong variant of the local metric dimension, distinguishing only adjacent vertices but with strong resolving sets.	Amalia et al. [36]
k-metric dimension (k-MD)	Generalization where at least k vertices are required to distinguish any pair of vertices.	Estrada-Moreno et al. [37]
Truncated metric dimension (TMD)	A resolvability parameter based on truncated distance, where a vertex set is used to uniquely distinguish all vertices of a graph. The smallest such set defines the truncated metric dimension.	Frongillo et al. [38]

2. Preliminaries

This section provides essential definitions and establishes the notation used throughout the paper. All graphs considered here are simple, undirected, and connected.

The study of resolvability in graphs was initiated with the concept of the MD. A subset $W \subseteq V(G)$

is called a RS if every two distinct vertices $s, t \in V(G)$ have distinct representation vectors with respect to W, defined as

$$r(s|W) = (d(s, w_1), d(s, w_2), \ldots, d(s, w_k)),$$

where $W = \{w_1, w_2, \ldots, w_k\}$. The minimum size of a RS is called the MD of G, denoted by $\dim(G)$. Variants, such as the LMD, the EMD, and the FMD, have been developed in recent years [28, 30].

In the context of *planar graphs*, resolving faces rather than vertices leads to the notion of the FMD. A further restriction gives rise to the $LFMD$, where only adjacent faces are required to be distinguished. This parameter, introduced and studied in this paper, combines aspects of both local and face-based resolvability, and is defined formally in the next section.

Definition 2.1. *A graph G is said to be planar when it admits a drawing in the plane in which edges do not cross, except at common endpoints. When G is given together with a fixed embedding in the plane, it is referred to as a plane graph. For a connected plane graph G, we denote by $\mathcal{F}(G)$ the collection of faces arising from its planar embedding, where the outer unbounded region is also regarded as a face.*

Definition 2.2. *Let G be a connected planar graph, and let $\mathcal{F}(G)$ denote the set of faces arising from its embedding. Two faces $g_1, g_2 \in \mathcal{F}(G)$ are said to be adjacent if they share a common edge in the planar embedding of G. Equivalently, g_1 and g_2 are adjacent whenever $V(g_1) \cap V(g_2)$ contains exactly the two endpoints of some edge of G.*

Definition 2.3 (*Distance from a vertex to a face*)**.** *For a vertex $s \in V(G)$ and a face $g \in \mathcal{F}(G)$, the* distance *between s and g is given by*

$$d(s, g) = \min\{\, d(s, t) : t \in V(g)\,\},$$

where $V(g)$ denotes the set of vertices lying on the boundary of g, and $d(s, t)$ represents the standard shortest-path distance between the vertices s and t in G.

Definition 2.4. *Consider a connected planar graph G with face set $\mathcal{F}(G)$. A subset $W \subseteq V(G)$ is said to be a $LFRS$ if for every pair of adjacent faces $g_1, g_2 \in \mathcal{F}(G)$, there exists a vertex $s \in W$ satisfying*

$$d(s, g_1) \neq d(s, g_2).$$

In other words, W forms a $LFRS$ whenever all neighboring faces can be distinguished through their distance representations relative to W.

Definition 2.5. *The $LFMD$ of a planar graph G, written as* $\mathrm{lfmdim}(G)$*, is defined as the smallest cardinality of a LFRS in G, where the empty set $W = \emptyset$ is admitted as a $LFRS$ wheneverno adjacent face pairs exist. More precisely,*

$$\mathrm{lfmdim}(G) = \min\{\, |W| : W \subseteq V(G),\ W \text{ is a } LFRS \text{ of } G\,\}.$$

Remark 1 (Embedding dependence)**.** *The $LFMD$ is defined with respect to a fixed planar embedding of G and may in principle vary across non-equivalent embeddings of the same graph. For the graph families studied in this paper, however, this issue does not arise, and the parameter is well-defined independently of the choice of embedding.*

3. Main results

In this section, we present the main results related to the newly introduced parameter, namely the *local face metric dimension (LFMD)* of planar graphs. We first establish fundamental properties and general bounds for $\mathrm{lfmdim}(G)$, including its relation with the face metric dimension. In particular, we characterize connected planar graphs satisfying $\mathrm{lfmdim}(G) = 1$ and prove that $\mathrm{lfmdim}(T) = 0$ for every tree. Furthermore, we determine exact values of $\mathrm{lfmdim}(G)$ for several important graph families, including *complete graphs, square grid graphs, fan graphs, and double-fan graphs*. We also derive sharp upper and lower bounds for connected planar graphs with finite *local face metric dimension*. These results provide deeper insight into the interaction between *local face resolvability*, graph structure, and planarity.

Proposition 3.1. *For any connected plane graph G, the following inequality holds:*

$$\mathrm{lfmdim}(G) \leq \mathrm{fmd}(G).$$

Proof. Recall that a face resolving set $W \subseteq V(G)$ distinguishes every pair of distinct faces of G, for every two faces g_1, g_2 there exists $w \in W$ such that

$$d(g_1, w) \neq d(g_2, w).$$

On the other hand, a *LFRS* requires this property only for pairs of adjacent faces (faces that share an edge).

Clearly, any face resolving set is also a *LFRS*, since if all face pairs are distinguished then in particular all adjacent pairs are distinguished. Thus, the minimum cardinality of a *LFRS* cannot exceed that of a face resolving set:

$$\mathrm{lfmdim}(G) \leq \mathrm{fmd}(G).$$

□

Theorem 3.1. *Let G be a connected planar graph. If there exist two adjacent faces $g_1, g_2 \in \mathcal{F}(G)$ such that $V(g_1) = V(g_2)$, then* $\mathrm{lfmdim}(G) = \infty$.

Proof. For any vertex $w \in V(G)$, we have

$$d(w, g_1) = \min_{v \in V(g_1)} d(w, v) = \min_{v \in V(g_2)} d(w, v) = d(w, g_2),$$

since $V(g_1) = V(g_2)$. Hence, no vertex can distinguish the pair (g_1, g_2), and therefore no finite *LFRS* exists. Hence, $\mathrm{lfmdim}(G) = \infty$. □

Example 3.1. *The planar embedding of cycle graph C_n consists of exactly two faces: the bounded (inner) face g_{in} and the unbounded (outer) face g_{out}. Since every vertex of C_n lies on the boundary of both g_{in} and g_{out}, we have $V(g_{\mathrm{in}}) = V(g_{\mathrm{out}}) = V(C_n)$. This directly satisfies the hypothesis of Theorem 3.1, which only requires $V(g_1) = V(g_2)$, so we conclude that* $\mathrm{lfmdim}(C_n) = \infty$.

Theorem 3.2. *Let T be any tree. Then,* $\mathrm{lfmdim}(T) = 0$.

Proof. Every planar embedding of a tree T contains exactly one face, namely the unbounded outer face, since trees contain no cycle and therefore enclose no bounded regions. Consequently, no adjacent face pairs exist in T, and the empty set $\emptyset$ vacuously satisfies the condition of Definition 2.4: there is no adjacent pair of faces requiring distinction. Since $|\emptyset| = 0$ and no smaller set exists, we conclude by Definition 2.5 that $\mathrm{lfmdim}(T) = 0$. □

Corollary 3.1. *Let P_n be the path on $n \geq 2$ vertices, and let S_n be the star on $n+1$ vertices (one central vertex and n leaves, $n \geq 1$). Then,*

$$\mathrm{lfmdim}(P_n) = \mathrm{lfmdim}(S_n) = 0.$$

Proof. Both P_n and S_n are trees, so the results follows directly from Theorem 3.2. □

Theorem 3.3. *Let K_n be the complete graph with $n \geq 2$. Then, LFMD of K_n is given by*

$$\mathrm{lfmdim}(K_n) = \begin{cases} 0, & n = 2, \\ \infty, & n = 3, \\ 3, & n = 4. \end{cases}$$

Proof. We treat the three cases separately.
Case 1: $n = 2$**.** The $K_2 = P_2$ is a tree (a path on two vertices). Therefore, by Theorem 3.2, it follows that

$$\mathrm{lfmdim}(K_2) = 0.$$

Case 2: $n = 3$**.** The graph $K_3 = C_3$ is a cycle on three vertices. Since every vertex of C_n lies on the boundary of both g_{in} and g_{out}, we have $V(g_{\text{in}}) = V(g_{\text{out}}) = V(C_3)$. This directly satisfies the hypothesis of Theorem 3.1, which only requires $V(g_1) = V(g_2)$, so we conclude that $\mathrm{lfmdim}(K_3) = \infty$.
Case 3: $n = 4$**.** The K_4 is planar and admits the standard embedding with three bounded triangular faces and one outer triangular face. Label the three outer-cycle vertices v_1, v_2, v_3, and let v_4 denote the interior vertex; denote the faces by

$$g_0 = \text{outer face with } \{v_1, v_2, v_3\}, \quad g_1 = \{v_4, v_1, v_2\}, \quad g_2 = \{v_4, v_2, v_3\}, \quad g_3 = \{v_4, v_3, v_1\}.$$

In K_4 the distance between any two distinct vertices is 1, so for any vertex x and any face f we have

$$d(x, f) = \begin{cases} 0, & x \in V(f), \\ 1, & x \notin V(f). \end{cases}$$

Thus, a vertex x distinguishes two faces f and g precisely when $x \in V(f) \triangle V(g)$ (the symmetric difference of their boundary vertex-sets). Consequently, a set $W \subseteq V(K_4)$ is a *LFRS* if and only if W intersects $V(f) \triangle V(g)$ for every pair of adjacent faces (f, g).

Listing the symmetric differences for all adjacent face pairs yields:

$$V(g_0) \triangle V(g_1) = \{v_3, v_4\}, \quad V(g_0) \triangle V(g_2) = \{v_2, v_4\}, \quad V(g_0) \triangle V(g_3) = \{v_1, v_4\},$$
$$V(g_1) \triangle V(g_2) = \{v_1, v_3\}, \quad V(g_2) \triangle V(g_3) = \{v_1, v_2\}, \quad V(g_3) \triangle V(g_1) = \{v_2, v_3\}.$$

No 2-element subset of $V(K_4)$ meets all six sets simultaneously: any such subset omits at least one vertex of $\{v_1, v_2, v_3, v_4\}$ and therefore fails to hit the corresponding symmetric-difference set. Hence no *LFRS* of size 1 or 2 exists. On the other hand, every 3-element subset of $V(K_4)$ meets all six sets. In particular $\{v_1, v_2, v_4\}$ intersects each set listed above, so it is a valid *LFRS*. Therefore, lfmdim(K_4)=3. □

Remark 2. *For $n \geq 5$, the complete graph K_n is non-planar; therefore, the LFMD is not applicable in this case. Hence, the corresponding theorem is restricted to planar cases only, $n = 2, 3, 4$.*

Theorem 3.4. *Let G be a connected planar graph that is not a tree. Then,* lfmdim$(G) = 1$ *if and only if there exists a vertex $w \in V(G)$ satisfying the following:*

(i) every edge incident to w is a bridge (equivalently, w lies on exactly one face of G), and
(ii) $d(w, g_i) \neq d(w, g_j)$ for every pair of adjacent faces $g_i, g_j \in \mathcal{F}(G)$ with $g_i \neq g_{\text{out}}$ and $g_j \neq g_{\text{out}}$.

Proof. We first establish the equivalence used in condition (i): *w lies on exactly one face of G if and only if every edge incident to w is a bridge.* In a planar embedding, going around vertex w in cyclic order, the faces in consecutive sectors between edges $\{w, x_i\}$ and $\{w, x_{i+1}\}$ are identical if and only if $\{w, x_i\}$ is a bridge (since a bridge has the same face on both sides). Therefore, all sectors around w belong to a single face precisely when every edge incident to w is a bridge.

($\Leftarrow$) Suppose conditions (i) and (ii) hold. Set $W = \{w\}$. Since every edge incident to w is a bridge, each such edge has the same face on both sides, and in a connected *planar graph* this unique face must be the unbounded outer face g_{out}. Hence $w \notin V(g_i)$ for any bounded face g_i, and therefore $d(w, g_i) \geq 1$ for every bounded face g_i.

We verify that W is a local face resolving set by considering all adjacent face pairs in $\mathcal{F}(G)$:

- For every adjacent pair (g_i, g_{out}): we have $d(w, g_{\text{out}}) = 0$ and $d(w, g_i) \geq 1$, so w distinguishes the pair (g_i, g_{out}).
- For every adjacent pair (g_i, g_j) of bounded faces: $d(w, g_i) \neq d(w, g_j)$ holds directly by condition (ii).

Hence, $W = \{w\}$ is a local face resolving set. Since G is not a tree, it has at least one bounded face, so lfmdim$(G) \geq 1$. Therefore, lfmdim$(G) = 1$.

($\Rightarrow$) Suppose lfmdim$(G) = 1$, so there exists $W = \{w\}$ resolving all adjacent face pairs. Let $\{u, v\}$ be any non-bridge edge of G. The two faces g_1, g_2 on either side of $\{u, v\}$ (which may include g_{out}) are distinct and both satisfy $\{u, v\} \subseteq V(g_1) \cap V(g_2)$. If $w \in \{u, v\}$, then $d(w, g_1) = d(w, g_2) = 0$, so w cannot distinguish the pair (g_1, g_2), contradicting that W is a local face resolving set. Hence, $w \notin \{u, v\}$ for every non-bridge edge $\{u, v\}$, which means every edge incident to w is a bridge. By the equivalence established above, w lies on exactly one face, so condition (i) holds. Condition (ii) holds since W resolves all adjacent bounded face pairs. □

Corollary 3.2. *Let G be a connected planar graph (not a tree) with a pendant vertex w attached to vertex x. If $d(x, g_i) \neq d(x, g_j)$ for every pair of adjacent bounded faces $g_i, g_j \in \mathcal{F}(G)$, then* lfmdim$(G) = 1$.

Proof. Since w has degree one, its unique incident edge $\{w, x\}$ is a bridge, so w satisfies condition (i) of Theorem 3.4. Moreover, since w is a pendant vertex, it lies only on the outer face g_{out}, so $w \notin V(g_i)$ for any bounded face g_i. Since x is the unique neighbor of w, for any bounded face g_i we have

$$d(w,\, g_i) = \min_{v \in V(g_i)} d(w, v) = 1 + \min_{v \in V(g_i)} d(x, v) = 1 + d(x,\, g_i).$$

Therefore, for every pair of adjacent bounded faces g_i, g_j,

$$d(w,\, g_i) \neq d(w,\, g_j) \iff d(x,\, g_i) \neq d(x,\, g_j),$$

which holds by hypothesis. Hence, condition (ii) of Theorem 3.4 is satisfied, and we conclude $\mathrm{lfmdim}(G) = 1$. □

Example 3.2. *Consider the graph G formed by two triangles sharing the edge $\{b, c\}$, with vertex set $\{a, b, c, d\}$, edge set $\{ab, bc, ca, db, dc\}$, and a pendant vertex e attached to a. The bounded faces are $g_1 = \{a, b, c\}$ and $g_2 = \{d, b, c\}$, which are adjacent along edge $\{b, c\}$. Taking $w = e$ and $x = a$:*

$$d(a,\, g_1) = 0, \quad d(a,\, g_2) = 1 \quad \Rightarrow \quad d(e,\, g_1) = 1 \neq 2 = d(e,\, g_2).$$

Hence, by Corollary 3.2, $\mathrm{lfmdim}(G) = 1$.

Proposition 3.2. *Let G be a connected planar graph on n vertices that is not a tree. If* $\mathrm{lfmdim}(G)$ *is finite, then*

$$1 \;\leq\; \mathrm{lfmdim}(G) \;\leq\; n - 1.$$

Both bounds are tight.

Proof. Lower bound. Since G is not a tree, it has at least one bounded face g_i. The pair $(g_i,\, g_{\text{out}})$ is adjacent, and any *LFRS* must contain at least one vertex to distinguish them; hence $\mathrm{lfmdim}(G) \geq 1$. The value $\mathrm{lfmdim}(G) = 1$ is achieved, for example, by any lollipop graph, as shown in Corollary 3.2.

Upper bound. Since G is not a tree, it contains at least one cycle, and hence at least one adjacent face pair (g_i, g_j) sharing an edge $\{u, v\}$. For any such pair, both endpoints satisfy $u, v \in V(g_i) \cap V(g_j)$, so

$$d(u,\, g_i) = d(u,\, g_j) = 0 \quad \text{and} \quad d(v,\, g_i) = d(v,\, g_j) = 0.$$

Hence, neither u nor v can resolve the pair (g_i, g_j), and therefore no *LFRS* needs to contain both u and v simultaneously for this purpose. In particular, fix any vertex $w \in V(G)$ that lies on the boundary of every face it borders together with at least one neighbor such a vertex always exists and is never the sole resolver of any adjacent face pair, since every adjacent pair (g_i, g_j) has at least two vertices in $V(g_i) \cap V(g_j)$. Therefore $V(G) \setminus \{w\}$ remains a valid *LFRS*, and since $|V(G)| = n$, we conclude $\mathrm{lfmdim}(G) \leq n - 1$.

Tightness of upper bound. The complete graph K_4 (which is planar) has $n = 4$ vertices and requires $\mathrm{lfmdim}(K_4) = 3 = n - 1$, showing the bound is sharp. □

The square grid graph: The Cartesian product of two graphs G_1 and G_2, written as $G_1 \square G_2$, is obtained by taking the Cartesian product of their vertex sets $V_1 \times V_2$. Two vertices (u_1, v_1) and (u_2, v_2) are connected by an edge whenever either their first components coincide and the second components are adjacent in G_2, or their second components coincide and the first components are adjacent in G_1. An important special case is the Cartesian product of two paths, $P_m \square P_n$, which yields the $m \times n$ square grid graph, denoted by $G_{m,n}$. This graph plays a central role in *planar graph* theory, lattice structures, and applications in computer science and networks. It has mn vertices arranged in m rows and n columns. Each vertex is labeled $a_{i,j}$, where $1 \le i \le m$ and $1 \le j \le n$, and it is adjacent to its horizontal and vertical neighbors whenever they exist. The structure of the grid is illustrated in Figure 1. Formally, the vertex set, edge set, and face set of $G_{m,n}$ are given by

$$V(G_{m,n}) = \{ a_{i,j} \ : \ 1 \le i \le m, \ 1 \le j \le n \},$$

$$E(G_{m,n}) = \{ a_{i,j}a_{i+1,j} \ : \ 1 \le i \le m-1, \ 1 \le j \le n \} \cup \{ a_{i,j}a_{i,j+1} \ : \ 1 \le i \le m, \ 1 \le j \le n-1 \},$$

$$\mathcal{F}(G_{m,n}) = \{ g_{i,j} \ : \ 1 \le i \le m-1, \ 1 \le j \le n-1 \} \ \cup \ \{g_{\text{out}}\},$$

where each bounded face $g_{i,j}$ corresponds to the 4-cycle with vertex set

$$V(g_{i,j}) = \{ a_{i,j}, \ a_{i+1,j}, \ a_{i,j+1}, \ a_{i+1,j+1} \},$$

and g_{out} denotes the unbounded outer face. Therefore,

$$|V(G_{m,n})| = mn, \qquad |E(G_{m,n})| = 2mn - m - n, \qquad |\mathcal{F}(G_{m,n})| = (m-1)(n-1) + 1.$$

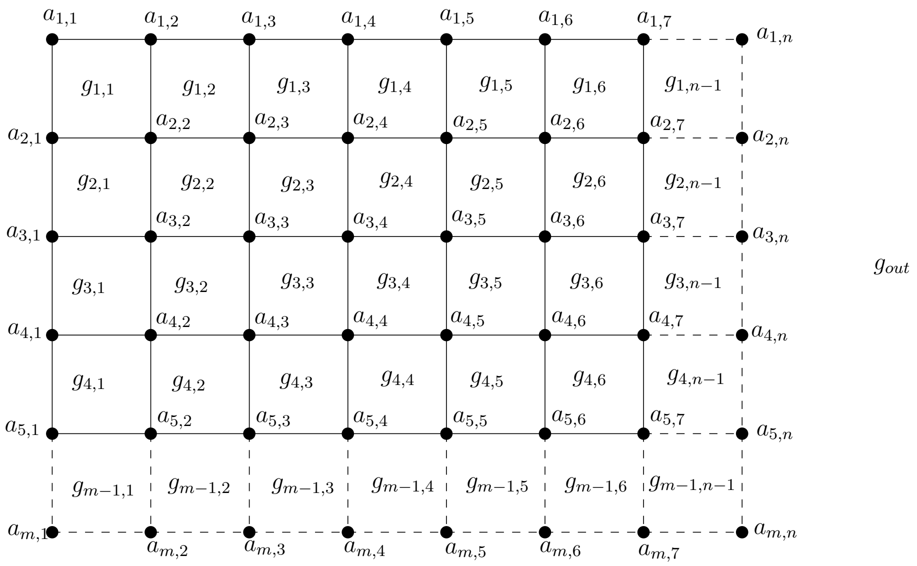

Figure 1. The square grid graph $G_{m,n}$ with vertex and face labeling.

Theorem 3.5. *Let $G_{m,n}$ be the square grid graph with $m \ge 2$, and $n \ge 3$. Then,*

$$\text{lfmdim}(G_{m,n}) = 2.$$

Proof. We show (i) no single vertex resolves all adjacent face-pairs (hence $\text{lfmdim}(G_{m,n}) \ge 2$), and (ii) the set $W = \{a_{1,1}, a_{1,n}\}$ resolves every adjacent face-pair (hence $\text{lfmdim}(G_{m,n}) \le 2$).
(i) Lower bound. Let $v = a_{p,q}$ be any vertex of the grid. If v is an interior vertex then v is incident with four bounded faces; two of those faces share an edge incident to v, so the two faces both have

distance 0 from v. If v is a boundary vertex, then v is incident with at least two faces (one bounded face and the outer face or two bounded faces if on a non-corner boundary), and again there exists a pair of *adjacent faces* that both contain v; hence, both have distance 0 to v. In every case a single vertex cannot distinguish every adjacent face-pair. Thus, $\operatorname{lfmdim}(G_{m,n}) \geq 2$.

(ii) The representations with respect to W. For a grid vertex (x, y) we have the lattice (Manhattan) distance

$$d(a_{1,1}, (x, y)) = (x-1) + (y-1) = x + y - 2, \qquad d(a_{1,n}, (x, y)) = (x-1) + (n-y).$$

A bounded face $g_{i,j}$ has vertex set $\{a_{i,j}, a_{i+1,j}, a_{i,j+1}, a_{i+1,j+1}\}$. Taking the minimum distance over those four vertices yields

$$d(a_{1,1}, g_{i,j}) = \min\{x + y - 2 : (x, y) \in V(g_{i,j})\} = i + j - 2,$$

and

$$d(a_{1,n}, g_{i,j}) = \min\{(x-1) + (n-y) : (x, y) \in V(g_{i,j})\} = i + n - j - 2.$$

Thus, the representation of the face $g_{i,j}$ with respect to W is the ordered pair

$$r(g_{i,j} \mid W) \;=\; (i + j - 2,\; i + n - j - 2).$$

For the outer face g_{out} both $a_{1,1}$ and $a_{1,n}$ lie on its boundary, so

$$r(g_{\text{out}} \mid W) = (0, 0).$$

(iii) Uniqueness of the face representations. Suppose two bounded faces have the same representation:

$$r(g_{i,j} \mid W) = r(g_{p,q} \mid W).$$

Then,

$$i + j - 2 = p + q - 2 \quad \text{and} \quad i + n - j - 2 = p + n - q - 2.$$

Subtracting the two equalities gives

$$(i + j) - (i + n - j) = (p + q) - (p + n - q) \quad \Longrightarrow \quad 2j - n = 2q - n.$$

Hence $j = q$. Substituting back gives $i = p$. Therefore $(i, j) = (p, q)$, so the representations are injective: distinct bounded faces have distinct ordered pairs. In particular every pair of adjacent bounded faces has different representations (so they are distinguished by at least one vertex of W).

Concretely, for adjacency:

- Horizontal neighbors: $r(g_{i+1,j}) = (i+1+j-2, i+1+n-j-2) = (i+j-2+1, i+n-j-2+1) = r(g_{i,j}) + (1, 1)$, so both coordinates change and the two faces are distinguished.
- Vertical neighbors: $r(g_{i,j+1}) = (i+j-1, i+n-(j+1)-2) = (i+j-2+1, i+n-j-2-1) = r(g_{i,j})+(1, -1)$, so again the representations differ.

For a boundary bounded face adjacent to the outer face (for example $g_{i,1}$ or $g_{i,n-1}$), the corresponding representation is never equal to $(0,0)$ when $n \geq 3$: e.g., $r(g_{i,1}) = (i-1, i+n-3)$ and for $n \geq 3$ this differs from $(0,0)$. Thus, each boundary bounded face is distinguished from the outer face by at least one coordinate. Hence, every adjacent pair (bounded-bounded or bounded-outer) is distinguished by W.

(iv) Minimality. We already argued in (i) that no single vertex can resolve all adjacent face-pairs. Therefore, the two-element set W is of minimum cardinality.

Combining (i)–(iv), we conclude that $W = \{a_{1,1}, a_{1,n}\}$ is a minimum-cardinality *LFRS* for $G_{m,n}$, and hence

$$\operatorname{lfmdim}(G_{m,n}) = 2.$$

□

Definition 3.1 (Generalized fan graph)**.** *For integers $m \geq 1$ and $n \geq 2$, the* generalized fan graph $F_{m,n}$ *is defined as the graph join $\overline{K_m} + P_n$, where $\overline{K_m}$ denotes the edgeless graph (complement of the complete graph) on m vertices $u_1, \ldots, u_m$, and P_n is the path $v_1 v_2 \cdots v_n$. Formally,*

$$V(F_{m,n}) = \{u_1, \ldots, u_m\} \cup \{v_1, \ldots, v_n\},$$

$$E(F_{m,n}) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_k v_j : 1 \leq k \leq m,\ 1 \leq j \leq n\}.$$

The vertices $u_1, \ldots, u_m$ are called apex vertices*, while $v_1, \ldots, v_n$ form the* path spine*. For $m = 1$, the graph $F_{1,n}$ is the classical fan graph (a single apex joined to P_n), and for $m = 2$, it is the double fan $F_{2,n}$. For $m \geq 3$, the graph $F_{m,n}$ contains a subdivision of $K_{3,3}$ as a subgraph and is therefore non-planar; consequently, the LFMD is undefined in this case.*

Theorem 3.6. *Let $F_{1,n}$ be the usual fan graph with $n \geq 2$. Then,*

$$\operatorname{lfmdim}(F_{1,n}) = \begin{cases} \infty, & n = 2, \\ \lceil \frac{n}{2} \rceil, & n \geq 3. \end{cases}$$

Proof. The case $n = 2$ is the triangle K_3: its planar embedding has two faces (inner and outer) and every vertex lies on both faces, so no finite *LFRS* exists; hence $\operatorname{lfmdim}(F_{1,2}) = \infty$.

Now, assume $n \geq 3$. Label the bounded triangular faces $g_i = \{u, v_i, v_{i+1}\}$ for $i = 1, \ldots, n-1$, and let g_{out} denote the outer face.

Upper bound. Let

$$W = \{v_1, v_3, v_5, \ldots\}$$

be the set of odd-indexed path vertices (so $|W| = \lceil n/2 \rceil$). We show W is a *LFRS*.

First, consider any two adjacent bounded faces g_i and g_{i+1}. The only vertices that can distinguish this pair are exactly v_i and v_{i+2}: indeed v_{i+1} and u are incident to both faces (distance 0 to each), and any other path vertex is at distance 1 to both faces. Since W contains every other path vertex, for each i at least one of v_i or v_{i+2} belongs to W; that vertex has distance 0 to one of g_i, g_{i+1} and distance 1 to the other, so W distinguishes the pair (g_i, g_{i+1}).

Next, consider a bounded face g_i and the outer face g_{out}. For any path vertex $w \in W \setminus \{v_i, v_{i+1}\}$ we have $d(w, g_{\text{out}}) = 0$ (every path vertex lies on the outer face) while $d(w, g_i) = 1$ (since $w \notin V(g_i)$ but adjacent to the apex), hence w distinguishes g_i and g_{out}. Because $n \geq 3$, the set W always contains at least one vertex not in $\{v_i, v_{i+1}\}$. Thus, every bounded face is distinguished from the outer face. Therefore W resolves all adjacent face pairs and

$$\text{lfmdim}(F_{1,n}) \leq \lceil n/2 \rceil.$$

Lower bound. Let W be any *LFRS*. Consider the adjacent bounded-face pairs (g_i, g_{i+1}) for $i = 1, \ldots, n-2$. As observed above, a vertex distinguishes (g_i, g_{i+1}) only if it is either v_i or v_{i+2} (neither u nor v_{i+1} distinguishes). Hence, for every $i = 1, \ldots, n-2$ the set W must contain at least one vertex from the pair $\{v_i, v_{i+2}\}$. Thus, S is a hitting set for the collection of overlapping 2-step windows $\{\{1,3\}, \{2,4\}, \ldots, \{n-2, n\}\}$ (indices interpreted in the natural way). Any such hitting set has size at least $\lceil n/2 \rceil$ (a straightforward covering/counting argument partition the index set $\{1, \ldots, n\}$ into $\lceil n/2 \rceil$ disjoint blocks of size at most 2). Therefore, $|W| \geq \lceil n/2 \rceil$, and consequently

$$\text{lfmdim}(F_{1,n}) \geq \lceil n/2 \rceil.$$

Combining upper and lower bounds yields $\text{lfmdim}(F_{1,n}) = \lceil n/2 \rceil$ for all $n \geq 3$, as required. □

Theorem 3.7. *Let $F_{2,n}$ be the double-fan graph with $n \geq 3$. Then,*

$$\text{lfmdim}(F_{2,n}) = \lceil \frac{n}{2} \rceil.$$

Proof. Embed $F_{2,n}$ in the plane with the path vertices $v_1, \ldots, v_n$ along the rim and the two apexes u, w on opposite sides of the path so that the bounded faces are the quadrilaterals

$$g_i = \{u,\ v_i,\ v_{i+1},\ w\}, \qquad i = 1, \ldots, n-1.$$

As before, let g_{out} denote the outer face.
Upper bound. Take

$$W = \{v_1, v_3, v_5, \ldots\},$$

the odd-indexed path vertices; $|W| = \lceil n/2 \rceil$. We show W resolves all adjacent face pairs.

First, consider two consecutive bounded faces g_i and g_{i+1}. An elementary distance check shows that a path vertex v_j distinguishes (g_i, g_{i+1}) precisely when $j \in \{i, i+2\}$, because v_{i+1} belongs to both faces and each apex u, w is incident to every bounded face (hence has equal distance 0 to both). Since W contains every second path vertex, for each i at least one of v_i or v_{i+2} lies in W and therefore distinguishes g_i and g_{i+1}.

Second, any bounded face g_i is adjacent to the outer face along the rim edge $v_i v_{i+1}$. For any $w_0 \in W \setminus \{v_i, v_{i+1}\}$ we have $d(w_0, g_{\text{out}}) = 0$ while $d(w_0, g_i) = 1$, so w_0 distinguishes g_i and g_{out}. Because $n \geq 3$ such a w_0 exists. Hence, W distinguishes every adjacent face-pair, and $\text{lfmdim}(F_{2,n}) \leq \lceil n/2 \rceil$.
Lower bound. Let S be any *LFRS*. For each $i = 1, \ldots, n-2$ the adjacent pair (g_i, g_{i+1}) must be distinguished; as argued above only v_i or v_{i+2} (among the path vertices) can do so. The two apexes cannot distinguish these pairs since each apex is incident to every bounded face. Hence, S must contain at least one vertex from each pair $\{v_i, v_{i+2}\}$, and by the same covering/counting argument as in the previous theorem any such hitting set has size at least $\lceil n/2 \rceil$. Therefore, $|S| \geq \lceil n/2 \rceil$, giving the lower bound. Combining the two bounds yields $\text{lfmdim}(F_{2,n}) = \lceil n/2 \rceil$ for $n \geq 3$. □

4. Conclusion and open problems

In this paper we introduced and studied the *LFMD* of a *planar graph*, a new parameter inspired by the classical notion of *MD* and its variants. The parameter is defined on the face-adjacency graph of a planar embedding and measures the minimum number of vertices needed to distinguish all pairs of *adjacent faces* via their distance representations. This framework captures a more localized view of resolvability in planar structures and provides a bridge between classical *MD* theory and the combinatorial properties of planar embeddings. We establish fundamental properties of the LFMD. In particular, we show that $\text{lfmdim}(G) = \infty$ whenever a planar graph contains two adjacent faces with identical vertex sets. We also characterize graphs with $\text{lfmdim}(G) = 1$ and provide sharp bounds $1 \leq \text{lfmdim}(G) \leq n - 1$ whenever the parameter is finite. For *complete graphs*, we determine exact values, showing that $\text{lfmdim}(K_2) = 0$, $\text{lfmdim}(K_3) = \infty$, and $\text{lfmdim}(K_4) = 3$. Furthermore, we compute LFMD for several important graph families. Trees attain the trivial value $\text{lfmdim}(T) = 0$, while grid graphs $G_{m,n}$ (with $m \geq 2$, $n \geq 3$) admit a *local face resolving set* of size 2. For *fan graphs*, we obtain exact formulas: $\text{lfmdim}(F_{1,n}) = \infty$ for $n = 2$ and $\text{lfmdim}(F_{1,n}) = \lceil n/2 \rceil$ for $n \geq 3$, while $\text{lfmdim}(F_{2,n}) = \lceil n/2 \rceil$ for all $n \geq 3$.

Several natural questions remain open.

- **Wheel graphs and prisms.** Determining $\text{lfmdim}(W_n)$ for *wheel graphs* and for *generalized prisms* $C_m \square P_n$ remains an interesting next step, as these families combine cycles with hub-like structures and polyhedral embeddings.
- **Computational complexity.** While the *MD* is known to be NP-hard in general, the complexity of computing the *LFMD* of a *planar graph* has not yet been established.
- **Applications.** Since resolvability parameters are closely linked with navigation, network design, and chemical graph theory, it would be interesting to investigate potential applications of the *LFMD* in areas, such as planar network routing, molecular embeddings, and image segmentation on planar grids.

We believe that the *LFMD* offers a promising new direction within resolvability theory, enriching the interplay between metric graph parameters and *planar graph* embeddings. Its study in broader classes of graphs, as well as its algorithmic and applied aspects, remains an open and fertile area for future research.

Author contributions

Amal S. Alali: Conceptualization, Data curation, Formal analysis, Writing-review & editing; Furqan Ahmad: Investigation, Methodology, Software, Visualization, Writing-original draft; Zubair Hafeez: Conceptualization, Methodology, Supervision, Validation, Writing-review & editing; Refah Alotaibi: Investigation, Resources, Validation, Writing-review & editing.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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